Logical methods in combinatorics: Which numeric graph invariants are polynomials?

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Overview

- The chromatic polynomial: G. Birkhoff 1912
- Numeric graph invariants: More motivating examples
- Definability of numeric graph invariants
- Graph polynomials
- Main results

The (vertex) chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A λ-**vertex-coloring** is a map

$$c : V(G) \to [\lambda]$$

such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$.

We define $\chi(G, \lambda)$ to be the number of λ-vertex-colorings

Theorem: (G. Birkhoff, 1912)

$\chi(G, \lambda)$ is a polynomial in $\mathbb{Z}[\lambda]$.

Proof:

(i) $\chi(E_n) = \lambda^n$ where E_n consists of n isolated vertices.

(ii) For any edge $e = E(G)$ we have $\chi(G - e, \lambda) = \chi(G, \lambda) + \chi(G/e, \lambda)$.

The edge-chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A λ-**edge-coloring** is a map

$$c : V(G) \rightarrow [\lambda]$$

such that if $(e, f) \in E(G)$ have a common vertex then $c(e) \neq c(f)$.

We define $\chi_e(G, \lambda)$ to be the number of λ- edge-colorings

Question:

Is $\chi_e(G, \lambda)$ a polynomial in $\mathbb{Z}[\lambda]$?

Let $L(G)$ be the **line graph** of G.
$V(L(G)) = E(G)$ and $(e, f) \in E(L(G))$ iff e and f have a common vertex.

Observation: $\chi_e(G, \lambda) = \chi(L(G), \lambda)$

Conclusion: $\chi_e(G, \lambda)$ **is a polynomial** in $\mathbb{Z}[\lambda]$.

Variations on coloring, I

We can count other coloring functions $f : V \to [\lambda]$.

- Connected components

  If $(u, v) \in E$ then $f(u) = f(v)$.

- Pre-coloring extensions

  Given graph $G = (V, E)$ and an equivalence relation R on V, we require that if $(u, v) \in R$ they have the same color, and if $(u, v) \in E - R$ they have different colors.

- Hypergraph colorings

  Given hypergraph $G = (V, E)$ with $E \subset \wp(V)$, we require that if $u \in e$ for some $e \in E$ which is not a singleton, then there is $v \in E, u \neq v$ with $f(u) \neq f(v)$.

Question: Are the corresponding counting functions polynomials?

Variations on coloring, II

- Strong hypergraph colorings

  Given hypergraph $G = (V, E)$ with $E \subset \wp(V)$, for every $e \in E$, for every $u, v \in E, u \neq v$ we have $f(u) \neq f(v)$.

- Mixed hypergraph colorings

  Given hypergraph $G = (V, E, D)$ with $D, E \subset \wp(V)$, we require that

  - if $u \in e$ for some $e \in E$, which is not a singleton, then there is $v \in E, u \neq v$ with $f(u) \neq f(v)$.
  - if $u, v \in d$ for some $d \in D$, then $f(u) = f(v)$.

Question: Are the corresponding counting functions polynomials?

Bounded numeric invariants

In graph theory it is often customary to look at numeric invariants which bounded by a function $b : G \to \mathbb{N}$.

- $k(G)$: the number of connected components of G;
  $k(G, \lambda)$: the number of connected components of G of size λ.

- $cl(G)$: the number of cliques of G;
  $cl(G, \lambda)$: the number of cliques of G of size λ.

- $indep(G, \lambda)$: the number of independent sets of G of size λ.

- $v(G, \lambda)$: the number of vertex covers of G of size λ.

- $m(G, \lambda)$: the number of matchings of G of size λ.

Obviously, these functions are not polynomials in λ,
because they vanish for large enough λ.

Pngi's: Parametrized numeric graph invariants

Let $\mathcal{K}$ denote a class of finite (colored) graphs
(hypergraphs, or structures over some fixed vocabulary).

A **parametrized numeric graph invariant (pngi)** is a function $\alpha(G, \lambda)$

$$\mathcal{K} \times \mathbb{N} \rightarrow \mathbb{N}$$

such that, for each $\lambda \in \mathbb{N}$ and G_1 isomorphic to G_2
we have that $\alpha(G_1, \lambda) = \alpha(G_2, \lambda)$.

Let $\alpha(G, \lambda)$ and Let $\beta(G, \lambda)$ be two pngi's.
Clearly, we can form new such invariants by forming

- $\alpha(G, \lambda) + \beta(G, \lambda)$, $\alpha(G, \lambda) \cdot \beta(G, \lambda)$, $2^{\alpha(G,\lambda)}$
- If $\alpha(G, \lambda) = 0$ for all large enough λ,

$$\beta(G, \lambda) = \sum_n \alpha(G, n)\lambda^n$$

If $\alpha(G, \lambda) \in \mathbb{Z}[\lambda]$ is a polynomial we speak of **graph polynomials**.

The behaviour of parametrized numeric graph invariants

The pngi's of the form $\alpha(G,\lambda)$ we have seen so far show the following behaviour:

- For each graph there is $b_G \in \mathbb{N}$ such that $\alpha(G,\lambda) \leq \lambda^{b_G}$.
- For each $n \in \mathbb{N}$ we have $\alpha(G,n) \in \mathbb{N}$.
- There is $n_G \in \mathbb{N}$ such that
  either $\alpha(G,n) = 0$ for all $n \geq n_G$
  or $\alpha(G,n)$ is not decreasing for all $n \geq n_G$.

Which parametrized numeric graph invariants are polynomials?

The following pngi's are polynomials:

- $\chi_{connected}(G, \lambda)$ is the number of functions $f : V \rightarrow [\lambda]$ such that if $(u, v) \in E(G)$ then $f(u) = f(v)$.

  Proof: $\chi_{connected}(G, \lambda) = \lambda^{k(G)}$ where $k(G)$ is the number of connected components of G.

- Given graph $G = (V, E)$ and an equivalence relation R on V, $\chi_{pre-coloring}(G, R, \lambda)$ is the number of functions $f : V \rightarrow [\lambda]$ such that if $(u, v) \in R$ then $f(u) = f(v)$ and if $(u, v) \in E - R$ then $f(u) \neq f(v)$.

  Proof: Here the proof depends on R.

- Given hypergraph $G = (V, E)$ with $E \subset \wp(V)$, $\chi_{hypergraph}(G, \lambda)$ is the number of hypergraph collorings.

  Proof: Mimick Birkhoff's proof for graphs.

More parametrized numeric graph invariants which are polynomials.

The following pngi's are polynomials:

- Strong hypergraph colorings.

  Proof: No direct proof in the literature.

- Mixed hypergraph colorings.

  Proof: This was shown in

  Vitaly I. Voloshin
  Coloring Mixed Hypergraphs: Theory, Algorithms and Applications
  AMS 2002

We shall give a **uniform proof** of these statements which uses **advanced model theory**, namely **stability theory**.

Enter logic: Model theory

Our framework is as follows:

- Let $\mathfrak{M}$ be a finite τ-structure with universe M.
- Let $k \in \mathbb{N}$ and $[k] = \{0, \ldots, k-1\}$.
- Let $\mathfrak{M}_k$ be the two-sorted τ' structure $\langle \mathfrak{M}, [k] \rangle$.
- Let F be an r-ary function symbol with interpretations in $\mathfrak{M}_k$ of the form $f : M_r \rightarrow [k]$.
- Let $\phi(F)$ be a second order $\tau' \cup \{F\}$-formula.

We denote by $\chi_\phi(\mathfrak{M}, k)$ the number of interpretations f of F such that

$$\langle \mathfrak{M}_k, f \rangle \models \phi(F)$$

Coloring properties, I

We denote relation symbols by bold-face letters, and their interpretation by the corresponding roman-face letter.

Let $\tau_R = \tau_1 \cup \{\mathbf{R}\}$, where is $\mathbf{R}$ is a two-sorted relation symbol of arity $r = s + t$. A class of τ_R- structures $\mathcal{P}$ is a **coloring property** if

(i) $\mathcal{P}$ is closed under τ_R-isomorphisms,

(ii) Let $\mathcal{M}$ be fixed. Then $\mathcal{M}_k$ is a substructure of $\mathcal{M}_n$ for each $n \geq k$. Let R_0 be a fixed relation on $\mathcal{M}_k$. If $\langle \mathcal{M}_k, R_0 \rangle \in \mathcal{P}$ and $n \geq k$ then also $\langle \mathcal{M}_n, R_0 \rangle \in \mathcal{P}$.

(iii) Let $R \subseteq M^s \times [k]^t$ be a fixed relation on $\mathcal{M}_k$. For π is a permutation of $[k]$, We define

$$R_\pi = \{(\bar{m}, \pi(\bar{a})) \in M^\times [k]^t : (\bar{m}, \bar{a}) \in R\}.$$

Then $\langle \mathcal{M}_k, R \rangle \in \mathcal{P}$ iff $\langle \mathcal{M}_k, R_\pi \rangle \in \mathcal{P}$.

We refer to $\mathbf{R}$ and its interpretations R as *coloring predicates*.

Coloring properties, II

(i) A coloring property is **bounded**, if for every $\mathcal{M}$ there is a number N_M such that for all $k \in \mathbb{N}$ the set of colors

$$\{x \in [k] : \exists \bar{y} \in M^m R(\bar{y}, x)\}$$

has size at most N_M.

(ii) A coloring property is **range bounded**, if its range is bounded in the following sense: There is a number $d \in \mathbb{N}$ such that for every $\mathcal{M}$ and $\bar{y} \in M^m$ the set $\{x \in [k] : R(\bar{y}, x)\}$ has at most d elements.

Clearly, if a coloring property is range bounded, it is also bounded.

Coloring properties, III

(i) A first order (or second order) formula $\phi(\mathbf{R})$ is a **coloring formula**, if the class of its models, which are of the form of the form $\langle \mathcal{M}, [k], R \rangle$, is a coloring property.

(ii) Let $\mathcal{P}$ be a bounded coloring property. A relation $R_M \subset M^m \times [k]$ is a **generalised** $k - \mathcal{P}$**-coloring** if $\langle \mathcal{M}_k, R \rangle \in \mathcal{P}$.
We denote by $\chi_{\mathcal{P}}(\mathcal{M}, k)$ the number of generalised $k - \mathcal{P}$-coloring R on $\mathcal{M}$. If $\mathcal{P}$ is definable by some formula $\phi(\mathbf{R})$ we also write $\chi_{\phi(R)}(\mathcal{M}, k)$.

Main result

THEOREM: If $\phi(\mathbf{R})$ is a second order formula and defines a bounded coloring property

$$\chi_\phi(\mathfrak{M}, k) \in \mathbb{Z}[k]$$

is indeed a polynomial in k.

We shall call polynomials obtained like this second order MT-polynomials.

MT-polynomial for model theoretic polynomial.

We can analogously define $\mathcal{L}$-polynomials for any other logic $\mathcal{L}$.

Corollary: This covers the three previous examples,
and allows us to construct **infinitely many more** MT-polynomials.

An elementary proof

We prove something a bit stronger:

THEOREM: For every $\mathcal{M}$ the number $\chi_{\phi(R)}(\mathcal{M},k)$ is a polynomial in k of the form

$$\sum_{j=0}^{d\cdot|M|^m} c_{\phi(R)}(\mathcal{M},j)\binom{k}{j}$$

where $c_{\phi(R)}(\mathcal{M},j)$ is the number of generalised $k-\phi$-colorings R with a fixed set of j colors.

In the light of this theorem we call $\chi_{\phi(R)}(\mathcal{M},k)$ a *generalised chromatic polynomial.*

Proof, suggested by A. Blass

We first observe that any generalised coloring R uses at most

$$N = d \cdot \mid M \mid^m$$

of the k colors.

For any $j \leq N$, let $c_{\phi(R)}(\mathcal{M}, j)$ be the number of colorings, with a fixed set of j colors, which are generalised vertex colorings and use all j of the colors.

Next we observe that any permutation of the set of colors used is also a coloring.

Therefore, given k colors, the number of vertex colorings that use exactly j of the k colors is the product of $c_{\phi(R)}(\mathcal{M}, j)$ and the binomial coefficient $\binom{k}{j}$. So

$$\chi_{\phi(R)}(\mathcal{M}, k) = \sum_{j \leq N} c_{\phi(R)}(\mathcal{M}, j) \binom{k}{j}$$

The right side here is a polynomial in k, because each of the binomial coefficients is. We also use that for $k \leq j$ we have $\binom{k}{j} = 0$. Q.E.D.

Bounded numeric invariants revisited, I

- The number of connected components $k(G)$ of the graph G is a constant, hence a polynomial.
- $\chi_{connected}(G, \lambda) = \lambda^{k(G)}$ is a polynomial which does fit our framework.
- The pngi $k(G, \lambda)$ is not a polynomial, hence does not fit our framework.
- The generating function

$$g_{conn}(G, X) = \sum_j k(G, j) \cdot X^j$$

is a polynomial in X.

We shall see that it does fit our framework.

Graph polynomials from pngi's

Let $\alpha(G, \lambda_1, \ldots, \lambda_m)$ be a numeric graph invariant paratmetrized with m parameters, such that for some N_G and all $n_1, \ldots, n_m \geq N_G$ $\alpha(G, n_1, \ldots, n_m) = 0$.

This gives rise to a generating function

$$A(G, X_1, \ldots, X_m) = \sum_{k_1, \ldots, k_m \in \mathbb{N}} \alpha(G, k, \ldots, k) \cdot \prod_{j=1}^{m} X_j^{k_j}$$

which is a polynomial.

Conversely, the coefficients of a pngi which is a polynomial
are themselves pngi's which ultimately vanish.

Examples of graph polynomials, I

Well studied graph polynomials are:

- The chromatic polynomial;
  (G. Birkhoff, 1912)

- The Tutte polynomial and its colored versions
  (W.T. Tutte 1954, B. Bollobas and O. Riordan, 1999);

- The characteristic polynomial
  (T.H. Wei 1952, L.M. Lihtenbaum 1956,
  L. Collatz and U. Sinogowitz 1957)

- The various matching polynomials;
  (O.J. Heilman and E.J. Lieb, 1972)

Examples of graph polynomials, II

- Various clique and independent set polynomials
  (I. Gutman and F. Harary 1983)

- The Farrel polynomials,
  (E.J. Farrell, 1979)

- The various knot polynomials,
  which can be viewed as graph polynomials of signed graphs
  (Alexander polynomial, Jones polynomial, HOMFLY-PT polynomial, etc)

- The various cover polynomials for digraphs
  (F.R.K. Chung and R.L. Graham, 1995)

- The interlace-polynomials,
  (R. Arratia, B. Bollobás and G. Sorkin, 2000)

Application of graph polynomials

There are plenty of applications of graph polynomials in

- Graph theory proper
- Knot theory
- Chemistry
- Statistical mechanics
- Quantum physics
- Quantum computing
- Biology

Enter Logic, II

Logic enters again here by looking at **SOL**-definable polynomials.

For this purpose we allow (full) Second Order Logic in the definition of the polynomials:

$$g(G, \bar{X}) = \sum_{R:\psi(R)} \prod_{\bar{v}:\bar{v}\in R} t_{\bar{v}}(\bar{X})$$

R now can be a relation variable of any fixed arity, and ψ any formula of Second Order Logic (**SOL**). We speak then of **SOL**-polynomials.

If $\mathcal{L}$ is a sublogic of **SOL** and ψ is an $\mathcal{L}$ formula, we speak of $\mathcal{L}$-polynomials.

In particular,

- Monadic Second Order Logic **MSOL** and
- Fixed Point Logic (**FPL**)

are of interest.

SOL-polynomials

All the polynomials we have encountered in the literature can be put into the framework of **SOL**-polynomials.

For certain polynomials this is obvious.

The matching polynomial can be written as

$$g(G,\lambda) = \sum_{\substack{M : M \subseteq E \\ M \text{ is a matching}}} \prod_{e : e \in M} \lambda$$

Sometimes this needs a twist or a non-trivial proof, which is the case for the

- For the characteristic polynomial we need an additional variable Y and obtain the ususal characteristic polynomial by evaluating Y at -1.
- For the Tutte polynomial we need its spanning tree expansion or its substitution instance of the Sokal polynomial.
- the interlace polynomial.

Using our framework: The matching polynomial

We want to show that the matching polynomial can be obtained in our framework.

- For a graph $G = (V, E)$ we form a 4-sorted structure

$$\mathfrak{M}(G) = \langle V, E, \wp(V), \wp(E), \in, R_G \rangle$$

  where $\in$ is the membership relation between elements of V and $\wp(V)$, and elements of E and $\wp(E)$ respectively,
  and R_G is the incidence relation between vertices and edges.

- $\mathfrak{M}(G)_k = \langle V, E, \wp(V), \wp(E), \in, R_G, [k] \rangle$

- The formula $\phi_{matching}(m, f)$ now says:

  (i) $m \in \wp(E)$ is a matching.

  (ii) f is a function $f : m \to [k]$.

Using our framework: The matching polynomial, contd

We replace k by λ.

Now we put $\bar{g}(G, \lambda)$ to be the number of pairs (m, f) such that

$$\langle \mathfrak{M}(G)_\lambda, m, f \rangle \models \phi_{matching}(m, f)$$

- For fixed m there are $\lambda^{|m|}$ many f's satisfying the formula $\phi_{matching}(m, f)$.
- For matchings m with $| m |= j$ we get $m(G, j)\lambda^j$ many such pairs.
- Hence we get

$$\bar{g}(G, \lambda) = \sum_j m(G, j)\lambda^j = \sum_{\substack{M : M \subseteq E \\ M \text{ is a matching}}} \prod_{e : e \in M} \lambda = g(G, \lambda)$$

MT-polynomials and **SOL**-polynomials

The definition of MT-polynomials is very flexible and can be easily extended to multivariate polynomials.

Theorem: The **SOL**-graph polynomials is exactly the second order MT-polynomials.

Remark: In the proof for the matching polynomial we we used the powersets of V and E as part of the structure $\mathfrak{M}(G)$. One can iterate this idea, hence also graph polynomials defined with higher order logic are MT-polynomials.

Why MT-polynomials?

MT-polynomials are useful because:

- They cover **all the examples** of graph polynomials encountered in the literature (so far).
- They provide us with a very **general method** for proving that certain parametrized numeric graph invariants are indeed polynomials.

Outline of the proof of the main theorem, I

The proof uses an encoding of the structures $\mathfrak{M}(G)_k$
in a countable model $\mathbb{M}(G)$.

The complete first order theory of $\mathbb{M}(G)$ is

- $\aleph_0$-categorical
- ω-stable
- of finite rank, more precisely,
  in the case of m-variate polynomials, it has m independent dimensions.

The formula $\phi(m, F)$ is translated into a formula $\psi(m, c_F)$ where the function symbol F becomes an indvidual constant c_F.

Outline of the proof of the main theorem, II

Now we can apply the structure theory for $\aleph_0$-categorical, ω-stable theories as developed in

- B. Zilber, Uncountably categorical theories, Translations of Mathematical Monographs, AMS 1993
- G. Cherlin and E. Hrushovski, Finite structures with few types, Annals of Mathematical Studies, Princeton University Press, 2003 G. Cherlin,

The key result we use, is that the size of definable sets, in finite approximations of models of $\mathbb{M}(G)$ with basis of size λ, is always a polynomial in λ.

In the example of the matching polynomial, this definable set is the set of pairs (m, c_F) such that $\mathbb{M}(G) \models \psi(m, c_F)$.

Thank you for your attention !