

The Difficult Point Conjecture for Graph Polynomials

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Graph polynomial project:

<http://www.cs.technion.ac.il/~janos/RESEARCH/gp-homepage.html>

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Graph polynomials

Graph polynomials

Graph polynomials are uniformly defined families of graph invariants which are (possibly) multivariate polynomials in some polynomial ring \mathcal{R} , usually \mathbb{Q} , \mathbb{R} or \mathbb{C} . We find

- Graph polynomials as generating functions;
- Graph polynomials as counting certain types of colorings;
- Graph polynomials defined by recurrence relations;
- Graph polynomials as counting weighted homomorphisms (=partition functions).

A general study addresses the following:

- Representability of graph polynomials;
- How to compare graph polynomials;
- The distinguishing power of graph polynomials;
- Universality properties of graph polynomials;

Prominent (classical) graph polynomials

- The **chromatic polynomial** (G. Birkhoff, 1912)
- The **Tutte polynomial** and its colored versions (W.T. Tutte 1954, B. Bollobas and O. Riordan, 1999);
- The **characteristic polynomial** (T.H. Wei 1952, L.M. Lihtenbaum 1956, L. Collatz and U. Sinogowitz 1957)
- The various **matching polynomials** (O.J. Heilman and E.J. Lieb, 1972)
- Various **clique** and **independent set polynomials** (I. Gutman and F. Harary 1983)
- The **Farrel polynomials** (E.J. Farrell, 1979)
- The **cover polynomials** for digraphs (F.R.K. Chung and R.L. Graham, 1995)
- The **interlace-polynomials** (M. Las Vergnas, 1983, R. Arratia, B. Bollobás and G. Sorkin, 2000)
- The various **knot polynomials** (of signed graphs) (Alexander polynomial, Jones polynomial, HOMFLY-PT polynomial, etc)

Applications of classical graph polynomials

There are plenty of applications of these graph polynomials in

- Graph theory proper and **knot theory**;
- Chemistry and biology;
- Statistical mechanics (Potts and Ising models)
- **Social networks** and **finance mathematics**;
- Quantum physics and quantum computing

And what about the **many** other graph polynomials?

Outline of this talk

- Evaluations of graph polynomials
- Turing complexity vs BSS complexity
- The chromatic and the Tutte polynomial: A case study
- The Difficult Point Property (DPP)
- The class SOLEVAL as the BSS-analog for $\#P$.
- The DPP Conjectures

Evaluations of graph polynomials

Evaluations of graph polynomials, I

Let $P(G; \bar{X})$ be a graph polynomial in the indeterminates X_1, \dots, X_n .

Let \mathcal{R} be a subfield of the complex numbers \mathbb{C} .

For $\bar{a} \in \mathcal{R}^n$, $P(-; \bar{a})$ is a graph invariant taking values in \mathcal{R} .

We could restrict the graphs to be from a class (graph property) \mathcal{C} of graphs.

What is the complexity of computing $P(-; \bar{a})$ for graphs from \mathcal{C} ?

- If for all graphs $G \in \mathcal{C}$ the value of $P(-; \bar{a})$ is a graph invariant taking values in \mathbb{N} , we can work in the **Turing model of computation**.
- Otherwise we identify the graph G with its **adjacency matrix** M_G , and we work in the **Blum-Schub-Smale (BSS) model of computation**.

Our goal

We want to discuss and **extend** the classical result of F. Jaeger and D.L. Vertigan and D.J.A. Welsh

on the complexity of evaluations of the Tutte polynomial. They show:

- either evaluation at a point $(a, b) \in \mathbb{C}^2$ is polynomial time computable in the Turing model, and a and b are integers,
- or some $\#\mathbf{P}$ -complete problem is reducible to the evaluation at $(a, b) \in \mathbb{C}^2$.
- To stay in the **Turing model** of computation, they assume that (a, b) is in some finite dimensional extension of the field \mathbb{Q} .

The proof of the second part is a **hybrid statement**:

The reduction is more naturally placed in the **BSS model** of computation, **However**, $\#\mathbf{P}$ -completeness has **no suitable counterpart** in the BSS model.

It seems to us **more natural** to work **entirely** in the BSS model of computation.

Evaluations of graph polynomials, II

- A **graph invariant** or **graph parameter** is a function $f : \bigcup_n \{0, 1\}^{n \times n} \rightarrow \mathcal{R}$ which is invariant under permutations of columns and rows of the input adjacency matrix.
- A **graph transformation** is a function $T : \bigcup_n \{0, 1\}^{n \times n} \rightarrow \bigcup_n \{0, 1\}^{n \times n}$ which is invariant under permutations of columns and rows of the input adjacency matrix.
- The **BSS-P-time computable functions** over \mathcal{R} , $P_{\mathcal{R}}$, are the functions $f : \{0, 1\}^{n \times n} \rightarrow \mathcal{R}$ BSS-computable in time $O(n^c)$ for some fixed $c \in \mathbb{N}$.
- Let f_1, f_2 be graph invariants. f_1 is **BSS-P-time reducible to f_2** , $f_1 \leq_P f_2$ if there are BSS-P-time computable functions T and F such that
 - (i) T is a graph transformation ;
 - (ii) For all graphs G with adjacency matrix M_G we have

$$f_1(M_G) = F(f_2(T(M_G)))$$
- two graph invariants f_1, f_2 are **BSS-P-time equivalent**, $f_1 \sim_{BSS-P} f_2$, if $f_1 \leq_{BSS-P} f_2$ and $f_2 \leq_{BSS-P} f_1$.

Evaluations of graph polynomials, III: Degrees and Cones

What are difficult graph parameters in the **BSS-model**?

Let g, g' be a graph parameters computable in exponential time in the **BSS-model**, i.e., $g, g' \in EXP_{BSS}$.

BSS-Degrees We denote by $[g]_{BSS}$ and $[g]_T$ the equivalence class (BSS-degree) of all graph parameters $g' \in EXP_{BSS}$ under the equivalence relation \sim_{BSS-P} .

BSS-Cones We denote by $\langle g \rangle_{BSS}$ the class (BSS-cone) $\{g' \in EXP_{BSS} : g \leq_{BSS-P} g'\}$.

NP-completeness There are BSS-NP-complete problems, and instead of specifying them, we consider NP to be a degree (which may vary with the choice of the Ring \mathcal{R}).

NP-hardness The cone of an NP-complete problem forms the NP-hard problems.

Decision problems, functions and graph parameters

- The BSS model deals traditionally with **decision problems** where the input is an \mathcal{R} -vector.
- A function f maps \mathcal{R} -vectors into \mathcal{R} .
 $f(\vec{X}) = a$ becomes a decision problem.
- There is **no** well developed theory of degrees and cones of functions in the BSS model.
- In the study of graph polynomials decision problems and functions have as input $(0, 1)$ -matrices and the decision problems and functions have to be **graph invariants**.

Evaluations of graph polynomials, IV

We work in **BSS model** over \mathcal{R} .

We define

$$\text{EASY}_{BSS}(P, \mathcal{C}) = \{\bar{a} \in \mathcal{R}^n : P(-; \bar{a}) \text{ is BSS-P-time computable} \}$$

and

$$\text{HARD}_{BSS}(P, \mathcal{C}) = \{\bar{a} \in \mathcal{R}^n : P(-; \bar{a}) \text{ is BSS-NP-hard} \}$$

We use $\text{EASY}_{BSS}(P)$ and $\text{HARD}_{BSS}(P)$ if \mathcal{C} is the class of all finite graphs.

How can we describe $\text{EASY}(P, \mathcal{C})$ and $\text{HARD}(P, \mathcal{C})$?

Turing Complexity vs BSS complexity

Problems with hybrid complexity, I

Let f_1, f_2 be two graph parameters taking values in \mathbb{N} as a subset of the ring \mathcal{R} .

We have *two kind of reductions*:

- **T-P-time Turing reductions** (via oracles) in the Turing model.
 $f_1 \leq_{T-P} f_2$ iff f_1 can be computed in T-P-Time using f_2 as an oracle.
- **BSS-P-time reductions** over the ring \mathcal{R} .
 $f_1 \leq_{BSS-P} f_2$ iff f_1 can be computed in BSS-P-Time using f_2 as an oracle.
- In the Turing model there is a **natural class** of problems $\#\mathbf{P}$ for **counting**, problems which contains many evaluation of graph polynomials.
However, $\#\mathbf{P}$ is **NOT CLOSED** under T-P-reductions.
- In the BSS model **no** corresponding class seems to **acomodate graph polynomials**.

Problems with hybrid complexity, II

- We shall propose a **new candidate**, the class $\text{SOLEVAL}_{\mathcal{R}}$ of evaluations of **SOL-polynomials**, the graph polynomials definable in Second Order Logic as described by T. Kotek, JAM, and B. Zilber (2008, 2011).
- The **main problem with hybrid complexity** is the **apparent incompatibility** of the two notions of polynomial reductions, $f_1 \leq_{T-P} f_2$ and $f_1 \leq_{BSS-P} f_2$ even in the case where f_1 and f_2 are both in $\#\mathbf{P}$.
- The number of 3-colorings of a graph, $\#\mathbf{3COL}$, and the number of acyclic orientations $\#\mathbf{ACYCLOR}$ are T-P-equivalent, and $\#\mathbf{P}$ -complete in the Turing model.
- In the BSS model we have $\#\mathbf{3COL} \leq_{BSS-P} \#\mathbf{ACYCLOR}$, but it is **open** whether $\#\mathbf{ACYCLOR} \leq_{BSS-P} \#\mathbf{3COL}$ holds.

Case study: The chromatic polynomial and the
Tutte polynomial

The (vertex) chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A **λ -vertex-coloring** is a map

$$c : V(G) \rightarrow [\lambda]$$

such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$.

We define $\chi(G, \lambda)$ to be the number of λ -vertex-colorings

Theorem: (G. Birkhoff, 1912)

$\chi(G, \lambda)$ is a polynomial in $\mathbb{Z}[\lambda]$.

Proof:

- (i) $\chi(E_n) = \lambda^n$ where E_n consists of n isolated vertices.
- (ii) For any edge $e \in E(G)$ we have $\chi(G - e, \lambda) = \chi(G, \lambda) - \chi(G/e, \lambda)$.

Interpretation of $\chi(G, \lambda)$ for $\lambda \notin \mathbb{N}$

What's the point in considering $\lambda \notin \mathbb{N}$?

Stanley, 1973 For simple graphs G , $|\chi(G, -1)|$ counts the number of **acyclic orientations** of G .

Stanley, 1973 There are also combinatorial interpretations of $\chi(G, -m)$ for each $m \in \mathbb{N}$, which are more complicated to state.

Open: What about $\chi(G, \lambda)$ for each $m \in \mathbb{R} - \mathbb{Z}$?

Open: What about $\chi(G, \lambda)$ for each $m \in \mathbb{C} - \mathbb{R}$?

The complexity of the chromatic polynomial, I

Theorem:

- $\chi(G, 0)$, $\chi(G, 1)$ and $\chi(G, 2)$ are P-time computable (Folklore)
- $\chi(G, 3)$ is $\#\mathbf{P}$ -complete (Valiant 1979).
- $\chi(G, -1)$ is $\#\mathbf{P}$ -complete (Linial 1986).

Question:

What is the complexity of computing $\chi(G, \lambda)$ for

$\lambda = \lambda_0 \in \mathbb{Q}$

or even for

$\lambda = \lambda_0 \in \mathbb{C}$?

The complexity of the chromatic polynomial, II

Let $G_1 \bowtie G_2$ denote the join of two graphs.

We observe that

$$\chi(G \bowtie K_n, \lambda) = (\lambda)^n \cdot \chi(G, \lambda - n) \quad (\star)$$

Hence we get

$$(i) \quad \chi(G \bowtie K_1, 4) = 4 \cdot \chi(G, 3)$$

$$(ii) \quad \chi(G \bowtie K_n, 3 + n) = (n + 3)^n \cdot \chi(G, 3)$$

hence for $n \in \mathbb{N}$ with $n \geq 3$ it is **#P-complete**.

This works in the Turing model of computation

for λ in some Turing-computable field extending \mathbb{Q} .

The complexity of the chromatic polynomial, III

If we have an oracle for some $q \in \mathbb{Q} - \mathbb{N}$ which allows us to compute $\chi(G, q)$ we can compute $\chi(G, q')$ for any $q' \in \mathbb{Q}$ as follows:

Algorithm $A(q, q', |V(G)|)$:

- (i) Given G the degree of $\chi(G, q)$ is at most $n = |V(G)|$.
- (ii) Use the oracle and (\star) to compute $n + 1$ values of $\chi(G, \lambda)$.
- (iii) Using Lagrange interpolation we can compute $\chi(G, q')$ in polynomial time.

We note that this algorithm is purely algebraic and works for all graphs G , $q \in (F) - \mathbb{N}$ and $q' \in F$ for any field F extending \mathbb{Q} .

Hence we get that for all $q_1, q_2 \in \mathbb{C} - \mathbb{N}$ the graph parameters are **polynomially reducible to each other**.

Furthermore, for $3 \leq i \leq j \in \mathbb{N}$, $\chi(G, i)$ is reducible to $\chi(G, j)$.

This works in the BSS-model of computation.

The complexity of the chromatic polynomial, IV

We summarize the situation for the chromatic polynomial as follows:

- (i) $EASY_{BSS}(\chi) = \{0, 1, 2\}$ and $HARD_{BSS}(\chi) = \mathbb{C} - \{0, 1, 2\}$.
- (ii) $HARD_{BSS}(\chi)$ can be split into two sets:
 - (ii.a) $HARD_{\#P}(\chi)$: the graph parameters which are **counting functions** in $\#P$ in the sense of Valiant, with $\chi(-, 3) \leq_P \chi(-, j)$ for $j \in \mathbb{N}$ and $3 \leq j$.
All graph parameters in $HARD_{\#P}(\chi)$ are $\#P$ -complete in **the Turing model**.
 - (ii.b) $HARD_{BSS-NP}(\chi)$: the graph parameters which are **not counting functions**.
In the **BSS model** they are all **polynomially reducible to each other**, and all graph parameters in $HARD_{\#P}(\chi)$ are P-reducible to each of the graph parameters in $HARD_{BSS}(\chi)$.
 - (ii.c) In the **BSS-model** the graph parameter $\chi(-, 3)$ is P-reducible to all the parameters in $HARD_{BSS}(\chi)$.
 - (ii.d) Inside $HARD_{BSS}(\chi)$ we have:

$$\chi(-, 3) \leq_{BSS_P} \chi(-, 4) \leq_{BSS_P} \dots \chi(-, j) \dots \leq_{BSS-P} \chi(-, a) \sim_{BSS_P} \chi(-, -1)$$

with $j \in \mathbb{N} - \{0, 1, 2\}$ and $a \in \mathbb{C} - \mathbb{N}$.

The complexity of the chromatic polynomial, χ

We have a **Dichotomy Theorem** for the evaluations of $\chi(-, \lambda)$:

(i) $\text{EASY}_{BSS}(\chi) = \{0, 1, 2\}$

Over \mathbb{C} this is a **quasi-algebraic set** (a finite boolean combination of algebraic sets) of **dimension 0**.

(ii) All graph parameters in $\text{HARD}_{BSS}(\chi)$

are at least as difficult as $\chi(-, 3)$

(**via BSS-P-reductions**)

This is a **quasi-algebraic set of dimension 1**.

Evaluating the Tutte polynomial (Jaeger, Vertigan, Welsh)

The Tutte polynomial $T(G, X, Y)$ is a bivariate polynomial and $\chi(G, \lambda) \leq_P T(G, 1 - \lambda, 0)$.

We have the following **Dichotomy Theorem**:

- (i) $\text{EASY}_{BSS}(T) = \{(x, y) \in \mathbb{C}^2 : (x - 1)(y - 1) = 1\} \cup \text{Except}$, with
 $\text{Except} = \{(0, 0), (1, 1), (-1, -1), (0, -1), (-1, 0), (i, -i), (-i, i), (j, j^2), (j^2, j)\}$
 and $j = e^{\frac{2\pi i}{3}}$
 Over \mathbb{C} this is a **quasi-algebraic** set of **dimension 1**.
- (ii) All graph parameters in $\text{HARD}_{BSS}(T)$ are at least as hard as $T(G, 1 - \lambda, 0)$.
 This is a **quasi-algebraic** set of **dimension 2**.

Evaluations of graph polynomials, V: How hard is $\#3\text{COL} = \chi(-, 3)$?

- It is known that 3-colorability of graphs can be phrased as problem of solvability of quadratic equations and therefore is $\text{NP}_{\mathbb{R}}$ -hard and $\text{NP}_{\mathbb{C}}$ -hard in the **BSS-model** (Hillar and Lim, 2010).
- For \mathbb{C} , Malajovich and Meer (2001) proved an analogue of Ladner's Theorem for the **BSS-model** over \mathbb{C} :
Assuming that $\text{P}_{\mathbb{C}} \neq \text{NP}_{\mathbb{C}}$ there are infinitely many different BSS-degrees between them.
- Although the problem $\chi(-, 3) \neq 0?$ is $\text{NP}_{\mathbb{C}}$ -hard we do not know whether there is $a \in \mathbb{C} - \mathbb{N}$ for which computing $\chi(-, a)$ is really harder!
- In particular, we know that $\chi(a, 3) \leq_{\text{BSS-P}} \chi(-, -1)$,
but we do not know whether

$$\chi(a, -1) \leq_{\text{BSS-P}} \chi(-, 3)$$

The Difficult Point Property (DPP)

Difficult Point Property, I

Given a graph polynomial $P(G, \bar{X})$ in n indeterminates X_1, \dots, X_n we are interested in the set $\text{HARD}_{BSS}(P)$.

- (i) We say that P has the **weak difficult point property (WDPP)** if there is a quasi-algebraic subset $D \subset \mathbb{C}^n$ of co-dimension $\leq n - 1$ which is contained in $\text{HARD}_{BSS}(P)$.
- (ii) We say that P has the **strong difficult point property (SDPP)** if there is a quasi-algebraic subset $D \subset \mathbb{C}^n$ of co-dimension $\leq n - 1$ such that $D = \text{HARD}_{BSS}(P)$ and $\mathbb{C} - D = \text{EASY}_{BSS}(P)$.

In both cases $\text{EASY}_{BSS}(P)$ is of dimension $\leq n - 1$, and for almost all points $\bar{a} \in \mathbb{C}^n$ the evaluation of $P(-, \bar{a})$ is BSS-NP-hard.

$\chi(G; \lambda)$ and $T(G; X, Y)$ both have the SDPP.

Difficult Point Property, II

We compare WDPP and SDPP to Dichotomy Properties.

- (i) We say that P has the **dichotomy property (DiP)** if $\text{HARD}_{BSS}(P) \cup \text{EASY}_{BSS}(P) = \mathbb{C}^n$.
Clearly, if $\mathbf{P}_{\mathbb{C}} \neq \mathbf{NP}_{\mathbb{C}}$, $\text{HARD}_{BSS}(P) \cap \text{EASY}_{BSS}(P) = \emptyset$.
- (ii) WDPP is **not** a dichotomy property, but **SDPP** a dichotomy property.
- (iii) The two versions of DPP have a **quantitative aspect**:

$\text{EASY}_{BSS}(P)$ is small.

The class $\text{SOLEVAL}_{\mathcal{R}}$ as the
BSS-analog for $\#P$.

Uniform definability of subset expansions

of graph polynomials

in (Monadic) Second Order Logic SOL (MSOL)

After: T. Kotek and J.A. Makowsky and B. Zilber,
On Counting Generalized Colorings,
In: Model Theoretic Methods in Finite Combinatorics,
M. Grohe and J.A. Makowsky, eds.,
Contemporary Mathematics, vol. 558 (2011), pp. 207-242
American Mathematical Society,

Simple (M)SOL-graph polynomials

Let $ind(G, i)$ denote the number of independent sets of size i of a graph G .

The graph polynomial $ind(G, X) = \sum_i ind(G, i) \cdot X^i$, can be written also as

$$ind(G, X) = \sum_{I \subseteq V(G)} \prod_{v \in I} X$$

where I ranges over all independent sets of G .

To be an independent set is definable by a formula of Monadic Second Order Logic (MSOL) $\phi(I)$.

A **simple (M)SOL-definable graph polynomial** $p(G, X)$ is a polynomial of the form

$$p(G, X) = \sum_{A \subseteq V(G): \phi(A)} \prod_{v \in A} X$$

where A ranges over all subsets of $V(G)$ satisfying $\phi(A)$

and $\phi(A)$ is a (M)SOL-formula.

General (M)SOL-graph polynomials

For the general case

- One allows several indeterminates X_1, \dots, X_t .
- One gives an inductive definition.
- One allows an ordering of the vertices.
- One requires the definition to be **invariant under the ordering**, i.e., different orderings still give the same polynomial.
- This also allows to define the modular counting quantifiers $C_{m,q}$ "there are, modulo q exactly m elements..."

The general case includes the **chromatic polynomial** and the **Tutte polynomial** and its variations, and **virtually all graph polynomials from the literature**.

1, 2, many SOL-definable graph polynomials

It is now **easy** to define many (also non-prominent) graph polynomials without further combinatorial motivation.

- Let $c_{i,j,k}(G)$ denote the number of triples of subsets of $A, B, C \subseteq E(G)$ such that $|A| = i, |B| = j, |C| = k$ and $\langle G, A, B, C \rangle \models \phi(A, B, C)$ where ϕ is any SOL-formula. Then

$$f(G; X, Y, Z) = \sum_{i,j,k} c_{i,j,k}(G) X^i Y^j Z^k$$

is an SOL-definable graph polynomial.

- The **generalized chromatic polynomials** introduced by Kotek, JAM and Zilber (2008) are all SOL-definable provided the coloring condition is also SOL-definable.
- The **PhD theses** of my students **I. Averbouch** and **T. Kotek** contain detailed studies of new graph polynomials with

combinatorial motivations.

The class $\text{SOLEVAL}_{\mathcal{R}}$

We now propose an analog to Valiant's counting class $\#\mathbf{P}$ for the BSS-model.

Assume $\mathbb{N} \subset \mathcal{R}$.

A function $f : \bigcup_n \{0, 1\}^{n \times n} \rightarrow \mathcal{R}$ is in $\text{SOLEVAL}_{\mathcal{R}}$ if it is the evaluation of some SOL-definable graph polynomial.

Facts and questions:

Fact: If $f \in \text{SOLEVAL}_{\mathcal{R}}$ then it is in $\text{EXPTIME}_{\mathcal{R}}$ and even in $\text{PSPACE}_{\mathcal{R}}$.

Fact: Every graph parameter $f \in \#\mathbf{P}$ is in $\text{SOLEVAL}_{\mathcal{R}}$.

Question: Is $\text{SOLEVAL}_{\mathcal{R}}$ contained in a single BSS-degree?

In particular, is $[\chi(-, -1)]_{\text{BSS}}$ its **maximal** degree?

Question: What is the BSS-degree structure of $\text{SOLEVAL}_{\mathcal{R}}$?

The DPP conjectures

The DPP conjectures

We have conjectured the following:

J.A. Makowsky,

From a Zoo to a Zoology: Towards a general theory of graph polynomials,
Theory of Computing Systems, vol. 43 (2008), pp. 542-562.

Let P be an SOL-definable graph polynomial in n indeterminates.

Assume that for some $\bar{a} \in \mathbb{C}^n$ evaluation of $P(-, \bar{a})$ is BSS-NP-hard over \mathbb{C} .

Weak DPP Conjecture: Then P has the WDPP.

Strong DPP Conjecture: Then P has the SDPP.

In the following we present **more evidence** for these conjectures.

Counting weighted homomorphisms as graph polynomials

- Let $A \in \mathbb{C}^{n \times n}$ a **symmetric** and G be a graph. Let

$$Z_A(G) = \sum_{\sigma: V(G) \rightarrow [n]} \prod_{(v,w) \in E(G)} A_{\sigma(v), \sigma(w)}$$

Z_A is called a partition function.

- Let \mathbf{X} be the matrix $(X_{i,j})_{i,j \leq n}$ of indeterminates. Then $Z_{\mathbf{X}}$ is a **graph polynomial in n^2 indeterminates**, Z_A is an **evaluation** of $Z_{\mathbf{X}}$, and $Z_{\mathbf{X}}$ is **MSOL-definable**.
- [J. Cai, X. Chen and P. Lu \(2010\)](#). building on A. Bulatov and M. Grohe (2005), proved a dichotomy theorem for $Z_{\mathbf{X}}$ where $\mathcal{R} = \mathbb{C}$.
- Analyzing their proofs reveals:
 $Z_{\mathbf{X}}$ satisfies the SDPP for $\mathcal{R} = \mathbb{C}$.
- There are various generalizations of this to Hermitian matrices, [M. Thurley \(2009\)](#), and beyond.

More SOL-definable graph polynomials with the DPP, I

SDPP: the **cover polynomial** $C(G, x, y)$ introduced by Chung and Graham (1995)
by Bläser, Dell 2007, Bläser, Dell, Fouz 2011

SDPP: the bivariate **matching polynomial** for multigraphs,
by Averbouch and JAM, 2007

SDPP: the **harmonious chromatic polynomial**,
by Kotek and JAM, 2007

WDPP: the **Bollobás-Riordan polynomial**, generalizing the Tutte polynomial and introduced by Bollobás and Riordan (1999),
by Bläser, Dell and JAM 2008, 2010.

WDPP: the **interlace polynomial** (aka Martin polynomial) introduced by Martin (1977) and independently by Arratia, Bollobás and Sorkin (2000),
by Bläser and Hoffmann, 2007, 2008

Generalized chromatic polynomials

Let $f : V(G) \rightarrow [k]$ be a coloring of the vertices of $G = (V(G), E(G))$.

- (i) f is **proper** if $(uv) \in E(G)$ implies that $f(u) \neq f(v)$. In other words if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces an independent set.
- (ii) f is **convex** if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces a connected graph.
- (iii) f is **t -improper** if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces a graph of maximal degree t .
- (iv) f is **H -free** if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces an H -free graph.
- (v) f is **acyclic** if for every $i, j \in [k]$ the union $[f^{-1}(i)] \cup [f^{-1}(j)]$ induces an acyclic graph.

By Kotek, JAM, Zilber (2008), for all the above properties, counting the number of colorings is a polynomial in k .

More SOL-definable graph polynomials with the DPP, II

T. Kotek and JAM (2011) have shown

SDPP: The graph polynomial for convex colorings.

SDPP: The graph polynomial for t -improper colorings (for multigraphs).

SDPP: The bivariate chromatic polynomial introduced by Döhmen, Pönitz and Tittman in 2003.

WDPP: The graph polynomial for acyclic colorings.

C. Hoffmann's PhD thesis (written under M. Bläser, 2010) contains a general **sufficient criterion** which allows to establish the WDPP for a wide class of **(mostly non-prominent)** graph polynomials.

A good test problem: H -free colorings.

We look at the generalized chromatic polynomial $\chi_{H\text{-free}}(G; k)$, which, for $k \in \mathbb{N}$ counts the number of H -free colorings of G .

- For $H = K_2$, $\chi_{H\text{-free}}(G; k) = \chi(G; k)$, and we have the SDPP.
- For $H = K_3$, $\chi_{H\text{-free}}(G; k)$ counts the triangle free-colorings.
- From [ABCM98] it follows that $\chi_{H\text{-free}}(G; k)$ is #P-hard for every $k \geq 3$ and H of size at least 2.

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- In [Achlioptas97] it is shown that computing $\chi_{H\text{-free}}(G; 2)$ is NP-hard for every H of size at most 2.

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- **Characterize H for which $\chi_{H\text{-free}}(G; k)$ satisfies the SDPP (WDPP).**

Thank you for your attention !

Model Theoretic Methods in Finite Combinatorics

M. Grohe and J.A. Makowsky, eds.,
Contemporary Mathematics, vol. 558 (2011), pp. 207-242
American Mathematical Society,
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Especially the papers

- Application of Logic to Combinatorial Sequences and Their Recurrence Relations
E. Fischer, T. Kotek, and J. A. Makowsky
- On Counting Generalized Colorings
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