Application of Logic to Generating Functions

Holonomic (P-recursive) Sequences

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Joint work with T. Kotek (Haifa)

Combinatorial Functions

We want to study recurrence relations of combinatorial functions.

- Fibonacci numbers: $f(n+2) = f(n+1) + f(n)$.
- Factorials: $n!$ with $(n+1)! = (n+1) \cdot n!$
- Bell numbers: $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$.
- Catalan numbers: $C(n) = \frac{1}{n+1} \cdot \binom{2n}{n}$.
- Derangement numbers: $D(n) = (n-1)(D(n-1) + D(n-2))$.
- Hermite numbers: $H(n) = -2(n-1)H(n-2)$.

More examples from the following books:

Monographs, I

There is a substantial theory of how to verify and prove (automatically) identities among the terms of $a(n)$:

- R. Graham, D. Knuth and O. Patashnik, Concrete Mathematics, 2nd edition, Addison-Wesley, 1994
- M. Petkovsek, H. Wilf and D. Zeilberger, A=B, AK Peters, 1996,
- R. Stanley, Enumerative Combinatorics, Cambridge University Press, volume I (1997), volume II (1999)

Monographs, II

Other relevant monographs are:

- L. Comtet, Advanced Combinatorics, Reidel 1974.
- I.P. Goulden and D.M. Jackson, Combinatorial Enumeration, Wiley & Sons, 1983
- G. Everest and A. van Porten and I. Shparlinski and T. Ward, Recurrence Sequences, Mathematical Surveys and Monographs, vol. 104, American Mathematical Society, 2003
- M. Aigner, A Course in Enumeration , Graduate Texts in Mathematics, vol. 238, Springer, 2007
- P. Flajolet and R. Sedgewick, Analytic Combinatorics, Cambridge University Press, 2009

Outline of the talk

- Recurrence relations of sequences $a(n)$
- Combinatorial interpretations of $a(n)$
- Logical interpretations of $a(n)$
- The Theorems of Chomsky-Schützenberger, Specker-Blatter, and Gessel
- Logical interpretations of holonomic (P-recursive) sequences

Recurrence Relations

Recurrence Relations over $\mathbb{Z}$

Given a sequence $a(n)$ of integers we say $a(n)$ is

(i) C-finite or rational if $a(n)$ can be written as

$$a(n+q) = \sum_{i=0}^{q-1} p_i \cdot a(n+i)$$

where each $p_i \in \mathbb{Z}$.

(ii) P-recursive or holonomic if it can be written as

$$p_q(n) \cdot a(n+q) = \sum_{i=0}^{q-1} p_i(n) \cdot a(n+i)$$

where each p_i is a polynomial in $\mathbb{Z}[X]$ and $p_q(n) \neq 0$.

We call it *simply P-recursive or SP-recursive*, if additionally $p_q(n) = 1$ for every $n \in \mathbb{Z}$.

Recurrence Relations over $\mathbb{Z}_m$

- MC-finite (modularly C-finite), if for every $m \in \mathbb{N}$ it can be written as

$$a(n+q(m)) = \sum_{i=0}^{q(m)-1} p_i(m)a(n+i) \mod m$$

  where $q(m)$ and $p_i(m)$ depend on m, and $p_i(m) \in \mathbb{Z}$.

Equivalently, $a(n)$ is MC-finite, if for all $m \in \mathbb{N}$ the sequence $a(n) \pmod{m}$ is ultimately periodic.

The terminology is due or inspired by R. Stanley's and D. Zeilberger's many papers.

Dictionary

- C-finite or rational =

  linear recurrence over $\mathbb{Z}$ with **constant** coefficients.

- P-recursive or holonomic =

  linear recurrence over $\mathbb{Z}$ with **polynomial** coefficients.

- SP-recursive or simple P-recursive =

  linear recurrence over $\mathbb{Z}$ with **polynomial** coefficients and leading coefficient $p_{n+q}(n) = 1$.

- MC-finite or modularly C-finite =

  linear recurrence over $\mathbb{Z}_m$ with **constant** coefficients, for every $m \in \mathbb{Z}$.

Examples, I

- The Fibonacci sequence is C-finite.
- The factorial $n!$ is not C-finite, but both SP-recursive and MC-finite.
- The function n^n is MC-finite, which is an easy consequence of Fermat's Little Theorem. S. Gerhold (2004) showed that n^n is not P-recursive.
- The function $\binom{2n}{n}$, central binomial coefficient, is P-recursive by

$$(n+1)^2 \cdot \binom{2(n+1)}{n+1} = 2 \cdot \binom{2(n+1)}{2} \cdot \binom{2n}{n}$$

  $\binom{2n}{n}$ is not to be MC-finite, hence it is not SP-recursive and not C-finite.
- The Catalan numbers are P-recursive by $C(n+1) = \frac{2(2n+1)}{n+2} \cdot C(n)$. They are not SP-recursive and not MC-finite.

Examples, II

- The derangement numbers $D(n)$ are usually defined by their combinatorial definition as the set of functions $f : [n] \to [n]$ such that f is bijective and for all $i \in [n]$ we have $f(i) \neq i$. Their explict definition is given by

  $$D(n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

  They are SP-recursive by

  $$D(n) = (n-1)(D(n-1) + D(n-2))$$

  with $D(0) = 1$ and $D(1) = 0$, hence MC-finite, but not C-finite, by a growth argument.

- The Bell numbers B_n count the number of partitions of an n-element set, and are MC-finite, by a theorem of C. Blatter and E. Specker (1981). They satisfy the recurrence relation

  $$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k,$$

  but $\binom{n}{k}$ is not a polynomial in n.
  Gerhold (2004) shows that they are not P-recursive.

How many sequences $a(n)$ are there?

Proposition:

(i) There are only countably many P-recursive sequences $a : \mathbb{N} \to \mathbb{Z}$.

(ii) There are continuum many MC-finite sequences.

Furthermore, let $a(n)$ be an MC-finite sequence such that $a(n) = 0 \pmod{m}$ for all m and $n \geq q(m)$.
For $A \subseteq \mathbb{N}$, let

$$a_A(n) = \begin{cases} a_A(n) & n \in A \\ 2 \cdot a_A(n) & n \notin A \end{cases}$$

Then $a_A(n) = 0 \pmod{m}$ for all m and $n \geq q(m)$.

Growth of recurrent sequences and their arithmetic closures

The following are well known, or from the literature.

Lemma:

(i) Let $a(n)$ be C-finite. Then there is a constant $c \in \mathbb{Z}$ such that $f(n) \leq 2^{cn}$.

(ii) Furthermore, for every holonomic sequence $a(n)$ there is a constant $\gamma \in \mathbb{N}$ such that $| a(n) | \leq n!^{\gamma}$ for all $n \geq 2$.

In general, this bound is best possible, since $a(n) = n!^{m}$ is easily seen to be holonomic for integer m, see Gerhold (2004).

(iii) The sets of C-finite, MC-finite, SP-recursive and P-recursive sequences are closed under addition, subtraction and point-wise multiplication.

Relationship between Recurrence Properties

Proposition:

Let $a(n)$ be a function $a : \mathbb{N} \to \mathbb{Z}$. The following implications are strict.

(i) If $a(n)$ is C-finite then $a(n)$ is SP-recursive.

(ii) If $a(n)$ is SP-recursive then $a(n)$ is P-recursive.

(iii) If $a(n)$ is SP-recursive then $a(n)$ is MC-finite.

Proof: The implications follow from the definitions. Strictness from the examples before:

(i) $n!$ is SP-recursive, but not C-finite, as, by Growth Lemma, it grows too fast.

(ii) $\binom{2n}{n}$ and the Catalan numbers are P-recursive but not SP-recursive.

(iii) The Bell numbers $B(n)$ are MC-finite, by a Theorem of C. Blatter and E. Specker, but not P-recursive, hence not SP-recursive

Q.E.D.

Combinatorial and Logical Interpretations

Combinatorial Interpretations of Sequences of Integers

Let $a(n)$ be a sequence of natural numbers.

We are interested in the case where $a(n)$ admits a

combinatorial
or a
logical interpretation,

i.e., $a(n)$ counts the number of some relations or functions on the set

$$[n] = \{1, \ldots, n\}$$

(with or without its natural order)

which have a certain property P,

possibly definable in some logical formalism.

We shall mostly deal with the logics

SOL, Second Order Logic, and **MSOL**, Monadic Second Order Logic.

Examples

(i) $n!$ has several combinatorial interpretations:

- It counts the number of functions $f : [n] \to [n]$ which are bijective.
- it counts the number of functions $f : [n] \to [n]$ such that $f(i) <_{nat} i+1$.

The first does not depend on the natural order of $[n]$, whereas the second does.

(ii) Bell numbers B_n count the number of partitions of an n-element set, or, equivalently,
they also count the number of equivalence relations on an n-element set.

Combinatorial Interpretations

A combinatorial interpretation $\mathcal{K}$ of $a(n)$ is given by

(i) a class of finite structures $\mathcal{K}$ over a vocabulary $\tau = \{R_1, \ldots R_r\} = \{\bar{R}\}$ or $\tau_{ord} = \{<_{nat}, \bar{R}\}$ with finite universe $[n] = \{1, \ldots, n\}$ and a relation symbol $<_{nat}$ for the natural order on $[n]$.

(ii) A density function $d_{\mathcal{K}}(n)$ which counts the number of relations

$$d_{\mathcal{K}}(n) = \mid \{\bar{R} \text{ on } [n] : \langle [n], <_{nat}, \bar{R} \rangle \in \mathcal{K}\} \mid$$

such that $d_{\mathcal{K}}(n) = a(n)$.

(iii) A combinatorial interpretation $\mathcal{K}$ is a *pure combinatorial interpretation* of $a(n)$ if $\mathcal{K}$ is closed under τ-isomorphisms.

In particular, if $\mathcal{K}$ does not depend on the natural order $<_{nat}$ on $[n]$, but only on τ.

Logical interpretations

(i) A combinatorial interpretation $\mathcal{K}$ of $a(n)$ is an

$\mathbf{SOL}$-interpretation ($\mathbf{MSOL}$-interpretation) of $a(n)$,

if $\mathcal{K}$ is definable in $\mathbf{SOL}(\tau_{ord})$ ($\mathbf{MSOL}(\tau_{ord})$).

(ii) Analogously,

Pure $\mathbf{SOL}$-interpretations ($\mathbf{MSOL}$-interpretation) of $a(n)$

are combinatorial interpretation $\mathcal{K}$ of $a(n)$ which do not depend on the natural order of $[n]$.

Remarks

(i) If $a(n)$ has a combinatorial interpretation then $a(n) \geq 0$ for all $n \in \mathbb{N}$.

(ii) There are only countably many sequences $a(n)$ which have an $\mathbf{SOL}$-interpretation.

(iii) Every sequence of non-negative integers which has an $\mathbf{SOL}$-interpretation is computable, and in fact it is in $\sharp \cdot \mathbf{PH}$, where $\mathbf{PH}$ is the polynomial hierarchy, hence computable in exponential time.

(iv) The sets of sequences with $\mathbf{SOL}$-interpretations is closed under addition, subtraction and point-wise multiplication.

(v) The same holds for $\mathbf{MSOL}$-interpretations

Examples revisited, I

(i) $n!$ has several combinatorial interpretations:

(i) It counts the number of functions $f : [n] \to [n]$ which are bijective.
This interpretation is pure and $\mathbf{MSOL}$-definable,

(ii) it counts the number of functions $f : [n] \to [n]$ such that $f(i) <_{nat} i+1$.
This interpretation is not pure but $\mathbf{MSOL}$-definable,

(ii) Bell numbers B_n count the number of partitions of an n-element set,
or, equivalently,
they also count the number of equivalence relations on an n-element set,

This interpretation is pure and $\mathbf{MSOL}$-definable,

(iii) The derangement numbers $D(n)$ are defined as the set of functions $f : [n] \to [n]$ such that f is bijective and for all $i \in [n]$ we have $f(i) \neq i$.

This interpretation is not pure but $\mathbf{MSOL}$-definable,

Examples revisited, II

(iv) $\binom{2n}{n}$, central binomial coefficient has a pure $\mathbf{SOL}$-interpretation as the number of equivalence relations with **two equivalence classes of the same size**, but no pure $\mathbf{MSOL}$-interpretation. However, it has an $\mathbf{MSOL}$-interpretation using the natural order on $[n]$.

(v) Hermite numbers are SP-recursive by $H(n) = -2(n-1)H(n-2)$. They have no combinatorial interprepretation, as their sign alternates. But they can be written as the difference of two functions $d_1(n)$ and $d_2(n)$, which both have a combinatorial interpretation.

The Theorems of Chomsky-Schützenberger, Specker-Blatter, and Gessel

Logical interpretations and recurrence relations

There are three early results:

- The Chomsky-Schützenberger Theorem (1963)
- The Specker-Blatter Theorem (1981)
- Gessel's Theorem (1984)

They are all of the form

> If a sequence $a(n)$ has a (pure) combinatorial interpretation $\mathcal{K}$
> where $\mathcal{K}$ satisfies some definability or closure condition
> then $a(n)$ satisfies some recurrence relations.

Regular languages

N. Chomsky and M. Schützenberger (1963), proved:

Theorem: Let $d_L(n)$ be a density function of a regular language L.
Then $d_L(n)$ is C-finite.

The converse is not true.

However, for C-finite sequences with non-negative coefficients, it is easy to construct the required regular language.

Theorem (T. Kotek and J.A.M., 2009):
Let $a(n)$ be a function $a : \mathbb{N} \rightarrow \mathbb{Z}$ which is C-finite.
Then there are two regular languages L_1, L_2 with density functions $d_1(n), d_2(n)$ such that $a(n) = d_1(n) - d_2(n)$.

Pure $\mathbf{MSOL}$-definable interpretations

Theorem (C. Blatter and E. Specker, 1981):

Let $a(n)$ have a pure $\mathbf{MSOL}$-interpretation $\mathcal{K}$ over a fixed finite vocabulary which contains only relation symbols of arity at most two.
Then $a(n)$ is MC-finite.

Remarks:

(i) E. Fischer and J.A.M. (2003) showed that the theorem is not true for $\mathbf{MSOL}$-interpretations with order.

(ii) E. Fischer (2002) showed that it is also not true if one allows relation symbols of arity ≥ 4.

(iii) As there are continuum many sequences $a(n)$ which are MC-finite, there cannot be a converse of the theorem.

Combinatorial interpretations closed under components

A class of directed graphs $\mathcal{K}$ is component closed or a a Gessel class iff it is closed under disjoint unions and components.

Theorem(I. Gessel 1984):

If $\mathcal{C}$ is a Gessel class of directed graphs of degree at most d with density function $d_{\mathcal{C}}(n)$, then $d_{\mathcal{C}}(n)$ is MC-finite.

More precisely,

$$d_{\mathcal{C}}(m+n) \equiv d_{\mathcal{C}}(m) \cdot d_{\mathcal{C}}(n) \pmod{\frac{m}{\ell}}$$

where ℓ is the least common multiple of all divisors of m not greater than d.

In particular, $d_{\mathcal{C}}(n)$ satisfies for every $m \in \mathbb{N}$ the linear recurrence relation

$$d_{\mathcal{C}}(n) \equiv a^{(m)} d_{\mathcal{C}}(n - d!m) \pmod{m}$$

where $a^{(m)} = d_{\mathcal{C}}(d!m)$.

More on bounded degree

For a relational structure $\mathcal{A}$ we define the degree and connectedness via the Gaifman graph of $\mathcal{A}$.

Theorem (E. Fischer and J.A.M., 2002):

Let $\mathcal{P}$ be an $\mathbf{MSOL}$-definable class of structures of bounded degree d over a vocabulary with relations of arbitrary arity. Then

- $d_{\mathcal{P}}(n)$ satisfies a modular recurrence relation for every m, i.e., is MC-finite.
- Furthermore, if additionally all the models in $\mathcal{P}$ are connected, then for every $m \in \mathbb{N}$ there is an $n_0(m)$ such that for every $n_0(m) \leq n \in \mathbb{N}$

  $$d_{\mathcal{P}}(n) = 0 \pmod{m}$$

  i.e., $d_{\mathcal{P}}(n)$ is ultimately constant and equals 0 modulo m.

Logical Interpretations of

Holonomic (P-recursive) Sequences

Specker's question

Let $\mathcal{K}$ be a combinatorial or logical interpretation of $a(n)$.

In 1988 E. Specker asked whether one can formulate a definability condition on $\mathcal{K}$ which ensures that $a(n)$ is SP-recursive?

There are really two questions here:

Question A: Can one formulate a definability condition on combinatorial interpretations $\mathcal{K}$ of $a(n)$ which ensures that $a(n)$ is SP-recursive.

Question B: Can one formulate a definability condition on **pure** combinatorial interpretations $\mathcal{K}$ of $a(n)$ which ensures that $a(n)$ is SP-recursive.

We shall answer Question A in the affirmative, but Question B remains open.

We also note that for C-finite sequences the answer to Question A is affirmative, by the Chomsky-Schützenberger Theorem.

LP-interpretations

Let τ_t be a vocabulary which consists of a linear order relation and t unary relation symbols.
Let σ_r be a vocabulary which consists of r unary function relations only.
Let ϕ be an $\mathbf{MSOL}(\tau_t)$ and $\bar{s} = (s_1, \ldots, s_r)$ be a tuple elements from $[t]$.

Then $m_{\phi,\bar{s}}(n)$ is the function over the natural numbers which counts the number of $(\tau_t \cup \sigma_r)$-structures

$$\mathcal{A} = \langle [n], <_{nat}, U_1, \ldots, U_t, F_1, \ldots, F_r \rangle$$

where $<_{nat}$ is the natural order on n such that

(i) $\langle [n], <_{nat}, U_1, \ldots, U_t \rangle$ satisfies ϕ, and

(ii) for every $i \in [r]$ and $j \in [n]$, $F_i(j) \leq j$, and furthermore if $j \notin U_{s_i}$, $F_i(j) = 1$.

A sequence $a(n)$ has an LP-interpretation if $a(n)$ can be written as

$$a(n) = m_{\phi,\bar{s}}(n)$$

for some sentence $\phi \in \mathbf{MSOL}(\tau_t)$ with suitable choice of $\bar{s}$.

Characterizing holonomic sequences via LP-interpretations

Our first characterization can now be stated as follows.

Theorem (T. Kotek and J.A.M., 2010):

Let $a(n)$ be a function $a : \mathbb{N} \to \mathbb{Z}$.

(i) $a(n)$ is SP-recursive iff there are two sequences $d_1(n)$, $d_2(n)$ with LP-interpretations such that $a(n) = d_1(n) - d_2(n)$.

(ii) $a(n)$ is P-recursive with leading polynomial $p_q(n)$ iff there are two sequences $d_1(n), d_2(n)$ with LP-interpretations such that

$$a(n) = \frac{d_1(n) - d_2(n)}{\prod_{i=1}^{n} p_q(i)}$$

A proof by example: Catalan numbers

We illustrate part (ii) of the Theorem with the Catalan numbers.

The Catalan numbers satisfy the recurrence relation

$$C(n+1) = \frac{2(2n+1)}{n+2} \cdot C(n) = \prod_{j=1}^{n} \frac{2(2j+1)}{j+2} = \frac{1}{\prod_{j=1}^{n}(j+2)} \cdot \prod_{j=1}^{n}(4j+2)$$

Let L_1 be language defined by $(a_1 \vee \ldots \vee a_k)^\star$. This allows us to write

$$\prod_{j=1}^{n}(A_1 + \ldots + A_k) = \sum_{\ell(w)=n, w \in L_1} \prod_{i=1}^{k} \prod_{w[j]=a_i} A_i \qquad (1)$$

We translate words $w \in \{a_1, \ldots, a_k\}^\star$ with $\ell(w) = n$ into structures $\mathcal{A}_w = \langle [n], U^w_{a_1}, \ldots, U^w_{a_k} \rangle$.

Using Büchi's Theorem, the correspondence between regular languages and $\mathbf{MSOL}$,
the language L_1 is defined by some formula $\phi_k \in \mathbf{MSOL}(\tau_k)$.

We write now Equation (1) as

$$\prod_{j=1}^{n}(A_1 + \ldots + A_k) = \sum_{\ell(w)=n, \mathcal{A}_w \models \phi_k} \prod_{i=1}^{k} \prod_{j \in U^w_{a_i}} A_i \qquad (2)$$

Proof, contd

We wrote Equation (1) as

$$\prod_{j=1}^{n}(A_1+\ldots+A_k) = \sum_{\ell(w)=n,\mathcal{A}_w\models\phi_k}\prod_{i=1}^{k}\prod_{j\in U_{a_i}^w} A_i \tag{2}$$

We apply Equation (2) to the term $4j+2 = j+j+j+j+1+1$ with $k=6$ and get

$$C(n+1) = \frac{1}{\prod_{j=1}^{n}(j+2)}\cdot\sum_{\ell(w)=n,\mathcal{A}_w\models\phi_6}\prod_{i=1}^{4}\prod_{j\in U_{a_i}^w} j\prod_{i=5}^{6}\prod_{j\in U_{a_i}^w} 1 =$$

$$= \frac{1}{\prod_{j=1}^{n}(j+2)}\cdot\sum_{\ell(w)=n,\mathcal{A}_w\models\phi_6}\prod_{i=1}^{4}\prod_{j\in U_{a_i}^w} j \tag{3}$$

Proof, contd

Equation (2) gave us

$$C(n+1) = \frac{1}{\prod_{j=1}^{n}(j+2)} \cdot \sum_{\ell(w)=n, \mathcal{A}_w \models \phi_6} \prod_{i=1}^{4} \prod_{j \in U_{a_i}^{w}} j \tag{3}$$

Now we use Definition of $m_{\phi,\bar{s}}(n)$ applied to ϕ_6 and $\bar{s} = (1,2,3,4)$ and we get

$$m_{\phi_6,(1,2,3,4)}(n) = \sum_{\ell(w)=n, \mathcal{A}_w \models \phi_6} \prod_{i=1}^{4} \prod_{j \in U_{a_i}^{w}} j$$

and

$$C(n+1) = \frac{m_{\phi_6,(1,2,3,4)}(n)}{\prod_{j=1}^{n}(j+2)}$$

Q.E.D.

Conclusions

- We have looked at three tipes of recurrence relations of sequences of natural numbers, C-finite, P-recursive (holonomic) and MC-finte.
- We have proposed the notion of Logical Interpretation of such sequences.
- We have reviewed three paradigmatic results, the Theorems of Chomsky-Schützenberger, Specker-Blatter, and Gessel.
- We have proven that every holonomic (P-recursive) sequence is the difference of two sequences which have an **MSOL**-interpretation.
- Th question, whether every holonomic (P-recursive) sequence has a pure **MSOL**-interpretation, remains open.

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Thank you for your attention!
