

Intriguing Graph Polynomials

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Reporting also recent work by [M. Freedman](#), [L. Lovász](#), [A. Schrijver](#) and [B. Szegedy](#)

Graph polynomial project:
<http://www.cs.technion.ac.il/~janos/RESEARCH/gp-homepage.html>

Overview

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- Evaluations of graph polynomials
- What we find intriguing
- Numeric graph invariants: Properties and guiding examples
- Connection matrices
- MSOL-definable graph polynomials
- Finite rank of connection matrices
- Applications of the Finite Rank Theorem
- Complexity of evaluations of graph polynomials
- Towards a dichotomy theorem

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Parametrized numeric graph invariants

The (vertex) chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A λ -**vertex-coloring** is a map

$$c : V(G) \rightarrow [\lambda]$$

such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$.

We define $\chi(G, \lambda)$ to be the number of λ -vertex-colorings

Theorem: (G. Birkhoff, 1912)

$\chi(G, \lambda)$ is a polynomial in $\mathbb{Z}[\lambda]$.

Proof:

- (i) $\chi(E_n) = \lambda^n$ where E_n consists of n isolated vertices.
- (ii) For any edge $e \in E(G)$ we have $\chi(G - e, \lambda) = \chi(G, \lambda) - \chi(G/e, \lambda)$.

Parametrized numeric graph invariants

A **parametrized numeric graph invariant** is a function

$$f : \mathcal{G} \times R \rightarrow \mathbb{R}$$

which is invariant under graph isomorphism.

Here R can be $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ or any ring.

Examples:

- (i) $ind_k(G)$ the number of independent sets of size k .
- (ii) $ind(G, X) = \sum_i ind(G, i) \cdot X^i$, the independent set polynomial.
- (iii) The chromatic polynomial $\chi(G, \lambda)$.
- (iv) Any graph polynomial from the literature, like matching polynomials, Tutte polynomial, interlace polynomial, cover polynomial of directed graphs, etc.

Coding many graph parameters into a graph polynomial

A particular graph polynomial is considered

interesting

if it encodes many useful graph parameters.

The characteristic polynomial

Let $G = (V(G), E(G))$ be a graph.

The characteristic polynomial $P(G, X)$ is defined as the characteristic polynomial (in the sense of linear algebra) of the adjacency matrix A_G of G defined as

$$\det(X \cdot \mathbf{1} - A_G) = \sum_{i=0}^n c_i(G) \cdot X^{n-i}.$$

It is well known that

- (i) $n = |V(G)|$
- (ii) $-c_2(G) = |E(G)|$
- (iii) $-c_3(G)$ equals twice the number of triangles of G .
- (iv) The second largest zero $\lambda_2(G)$ of $P(G; X)$ gives a lower bound to the conductivity of G

The chromatic polynomial

We define $\chi(G, k)$ to be the number of proper k -colorings of a graph G .

- For $k \in \{0, 1, 2\}$ it can be computed in polynomial time (exercise).
- For $k = 3$ it is $\#\mathbf{P}$ -complete even for bipartite graphs (Linial 1986).
- $\chi(G, \lambda)$ is a polynomial in λ (Birkhoff 1912).
- $\chi(G, -1)$ is the number of acyclic orientations of G (Stanley 1973).

The Tutte polynomial

The Tutte polynomial of G is defined as

$$T(G; X, Y) = \sum_{F \subseteq E(G)} (X - 1)^{r\langle E \rangle - r\langle F \rangle} (Y - 1)^{n\langle F \rangle}$$

where $k\langle F \rangle$ is the number of connected components of the **spanning subgraph defined by F** ,

$r\langle F \rangle = |V| - k\langle F \rangle$ is its rank

and $n\langle F \rangle = |F| - |V| + k\langle F \rangle$ is its nullity.

The Tutte polynomial **subsumes** the chromatic polynomial.

$$\chi(G, X) = (-1)^{r(G)} \cdot X^{k(G)} \cdot T(G(1 - X, 0))$$

Evaluations of the Tutte polynomial

See D. Welsh, Complexity: Knots, colourings and counting, Cambridge, 1993, and M. Korn and I. Pak, Tilings of rectangles with T-tetrominoes, TCS 319, 2004

- (i) $T(G; 1, 1)$ is the number of spanning trees of G ,
- (ii) $T(G; 1, 2)$ is the number of connected spanning subgraphs of G ,
- (iii) $T(G; 2, 1)$ is the number of spanning forests of G ,
- (iv) $T(G; 2, 2)$ is the number of spanning subgraphs of G ,
- (v) $T(G; 1 - k, 0)$ is the number of proper k -vertex colorings of G ,
- (vi) $T(G; 2, 0)$ is the number of acyclic orientations of G ,
- (vii) $T(G; 0, -2)$ is the number of Eulerian orientations of G .
- (viii) $2 \cdot T(\text{Grid}_{4x,4y}; 3, 3)$ is the number of tilings of the $(4x \times 4y)$ - grid graph with T-tetrominoes

The cover polynomial

Chung and Graham, 1995 and D'Antona and Munarini, 2000

Let $D = (V, E)$ be a directed graph.

$C \subseteq E$ is a **path-cycle cover of G** if C is a subgraph with maximal in-degree ≤ 1 and maximal out-degree ≤ 1 and C is a vertex disjoint decomposition of E with $p(C)$ paths and $c(C)$ cycles.

The **(factorial) cover polynomial** is the polynomial

$$C(D, x, y) = \sum_C (x)_{p(C)} \cdot y^{c(C)}$$

The **(geometric) cover polynomial** is the polynomial

$$C_{geom}(D, x, y) = \sum_C (x)^{p(C)} \cdot y^{c(C)}$$

Here $(x)_n = x \cdot (x - 1) \cdot \dots \cdot (x - n + 1)$ is the falling factorial.

Evaluations of the cover polynomial

- (i) $C(D, 0, 1)$ is the number of cycle covers of D , which is the permanent of the adjacency matrix of D .
- (ii) $C(D, 0, -1)$ is the determinant of the adjacency matrix of D .
- (iii) $C(D, 1, 0)$ is the number of hamiltonian paths of D .
- (iv) $C(D, x, 1)$ is the *factorial rook polynomial* of D .

Definability and Complexity

What I find intriguing?

- If one finds (in the literature) or defines (a new) a graph polynomial, how can one guess evaluations which are **combinatorially meaningful**?
- How are the **computational difficulties** of evaluating graph polynomials at different evaluation points related?

Note:

The characteristic polynomial is polynomial time computable at all evaluation points.

All other examples have many evaluation points which are $\#P$ -hard.

Evaluations, coefficients and zeros of graph polynomials

How could one prove that a graph parameter f is

not coded

in a given graph polynomial from an infinite class of graph polynomials \mathcal{P} as

- an evaluation?
- a coefficient?
- a zero?

We will use method of **logic** and **linear algebra** !

Numeric graph invariants aka graph parameters

Evaluations of graph polynomials are graph parameters,
but there are many more.

We look through our **favorite** monographs on graph theory:

- R. Diestel, Graph Theory, Springer 1996
- B. Bollobás, Modern Graph Theory, Springer 1998

Numeric graph invariants (graph parameters)

We denote by $G = (V(G), E(G))$ a graph, and by \mathcal{G} and \mathcal{G}_{simple} the class of finite (simple) graphs, respectively.

A **numeric graph invariant** or **graph parameter** is a function

$$f : \mathcal{G} \rightarrow \mathbb{R}$$

which is invariant under graph isomorphism.

- (i) Cardinalities: $|V(G)|$, $|E(G)|$
- (ii) Counting configurations:
 - $k(G)$ the number of connected components,
 - $m_k(G)$ the number of k -matchings
- (iii) Size of configurations:
 - $\omega(G)$ the clique number
 - $\chi(G)$ the chromatic number
- (iv) Evaluations of graph polynomials:
 - $\chi(G, \lambda)$, the chromatic polynomial, at $\lambda = r$ for any $r \in \mathbb{R}$.
 - $T(G, X, Y)$, the Tutte polynomial, at $X = x$ and $Y = y$ with $(x, y) \in \mathbb{R}^2$.

Multiplicative graph parameters

Let $G_1 \sqcup G_2$ denote the disjoint union of two graphs.

f is **multiplicative** if $f(G_1 \sqcup G_2) = f(G_1) \cdot f(G_2)$.

- (i) $|V(G)|, |E(G)|, k(G)$ are not multiplicative
- (ii) $\chi(G)$ and $\omega(G)$ are not multiplicative
- (iii) The number of perfect matchings $pm(G)$ is multiplicative and so is the generating matching polynomial $\sum_k m_k(G)X^k$.
Note that $m_k(G)$ is not multiplicative.
- (iv) The graph polynomials $\chi(G, \lambda)$ and $T(G, X, Y)$ are multiplicative.

Additive graph parameters

Let $G_1 \sqcup G_2$ denote the disjoint union of two graphs.

f is **additive** if $f(G_1 \sqcup G_2) = f(G_1) + f(G_2)$.

(i) $|V(G)|, |E(G)|$ are additive.

(ii) $k(G)$ are additive

Let $b(G)$ be the number of 2-connected components of G .
 $b(G)$ is additive.

(iii) $\chi(G)$ and $\omega(G)$ are not additive

(iv) If f is additive and $r \in \mathbb{R}$, then r^f is multiplicative.

Maximizing and minimizing graph parameters

Let $G_1 \sqcup G_2$ denote the disjoint union of two graphs.

f is **maximizing** if $f(G_1 \sqcup G_2) = \max\{f(G_1), f(G_2)\}$.

f is **minimizing** if $f(G_1 \sqcup G_2) = \min\{f(G_1), f(G_2)\}$.

- (i) The various chromatic numbers $\chi(G)$, $\chi_e(G)$, $\chi_t(G)$ are maximizing.
- (ii) The size of the maximal clique $\omega(G)$ and the maximal degree $\Delta(G)$ are maximizing.
- (iii) The tree-width $tw(G)$ and the clique-width $cw(G)$ of a graph are maximizing.
- (iv) The minimum degree $\delta(G)$, the girth $g(G)$ are minimizing.

The girth is the minimum length of a cycle in G .

The connection matrix of a graph parameter

Connection matrix $M(f, 0)$.

Let G_i be an enumeration of all finite graphs (up to isomorphism).

The **connection matrix** $M(f, 0) = m_{i,j}(f, 0)$ is defined by

$$m_{i,j}(f, 0) = f(G_i \sqcup G_j)$$

The **rank** of $M(f, 0)$ is denoted by $r(f, 0)$.

Examples: Check with $|V(G)|$ and $2^{|V(G)|}$.

Computing $r(f, 0)$

Proposition:

- (i) If f is multiplicative, $r(f, 0) = 1$.
- (ii) If f is additive, $r(f, 0) = 2$.
- (iii) If f is maximizing or minimizing, $r(f, 0)$ is infinite.
- (iv) For the average degree $d(G)$ of a graph, $r(d, 0)$ is infinite.

Proof: The first three statements are easy.

For $f = d(G)$ we have

$$M(d, 0) = 2 \frac{|E_1| + |E_2|}{|V_1| + |V_2|}.$$

This contains, for graphs with a fixed number e of edges, the Cauchy matrix $(\frac{2e}{i+j})$, hence $r(d, 0)$ is infinite. □.

Characterizing multiplicative graph parameters

M. Freedman, L. Lovász and A. Schrijver, 2007

Theorem: ([FLS] Proposition 2.1.)

Assume $f(G) \neq 0$ for some graph G .

f is multiplicative iff $M(f, 0)$ has rank 1 and is positive semi-definite.

Recall: A finite square matrix M over an ordered field is **positive semi-definite** if for all vectors \bar{x} we have $\bar{x}M\bar{x}^{tr} \geq 0$. An infinite matrix is positive semi-definite, if every finite principal submatrix is positive semi-definite.

Definability of graph polynomials
in Monadic Second Order Logic MSOL

Simple MSOL-definable graph polynomials

The graph polynomial $ind(G, X) = \sum_i ind(G, i) \cdot X^i$, can be written also as

$$ind(G, X) = \sum_{I \subseteq V(G)} \prod_{v \in I} X$$

where I ranges over all independent sets of G .

To be an independent set is definable by a formula of Monadic Second Order Logic (MSOL) $\phi(I)$.

A **simple MSOL-definable graph polynomial** $p(G, X)$ is a polynomial of the form

$$p(G, X) = \sum_{A \subseteq V(G): \phi(A)} \prod_{v \in A} X$$

where A ranges over all subsets of $V(G)$ satisfying $\phi(A)$ and $\phi(A)$ is a MSOL-formula.

General MSOL-definable graph polynomials

For the general case

- One allows several indeterminates X_1, \dots, X_t .
- One gives an inductive definition.
- One allows an ordering of the vertices.
- One requires the definition to be **invariant under the ordering**, i.e., different orderings still give the same polynomial.
- This also allows to define the modular counting quantifiers $C_{m,q}$ "there are, modulo q exactly m elements..."

The general case includes the Tutte polynomial, the cover polynomial, and **virtually all graph polynomials from the literature**.

A graph polynomial which is **not MSOL-definable**

A proper coloring of a graph is **harmonious** if every pair of colors appears at most once on an edge.

Let $\chi_{harm}(G)$ be the **smallest** k such that G has a harmonious coloring with k colors.

Let $\chi_{harm}(G, k)$ denote the number of harmonious colorings of G with k colors.

Theorem: (Kotek, Makowsky, Zilber, 2008)

- (i) $\chi_{harm}(G, k)$ is a polynomial in k
- (ii) Assuming $\mathbf{P} \neq \mathbf{NP}$, $\chi_{harm}(G, k)$ is **not an MSOL-definable** polynomial in the most general sense.

A graph polynomial which is not MSOL-definable - Proofs

(i):

Verify the extension property, that adding unused colors preserves harmonicity. Then it follows from Makowsky and Zilber (2006) [MZ].

(ii):

It was shown by Edwards and McDiarmid (1995) that computing $\chi_{\text{harm}}(G)$ is **NP**-hard even on trees. It was shown by Courcelle, Makowsky, Rotics (2000) that for all $t \in \mathbb{N}$ evaluating MSOL-definable polynomials on rational points is in **P** for graphs of tree-width at most t . □

Finiteness of $r(f, 0)$

Finite Rank Theorem:([GKM])

Let $p(G, \bar{X})$ be a general MSOL-definable graph polynomial in t indeterminates, and $\bar{x} \in \mathbb{R}^t$.

Then the rank $r(p(G, \bar{x}), 0)$ is finite.

Freedman, Lovász and Schrijver introduced various connection matrices $M(f, \otimes_k)$ for k -labeled graphs and various operations, which they used to characterize graph parameters arising from partition functions and coloring models.

The theorem above holds also for the **several variations** of connection matrices for labeled graphs and relational structures.

The theorem only depends on the fact that the connection matrix $M(f, \otimes)$ is defined for an **MSOL-smooth** operation \otimes in the sense of [M].

Proof: We use the **bilinear** version of the **Feferman-Vaught Theorem** for graph polynomials from [M] to estimate the $r(f, k)$.

The estimates are very bad, and just suffice to establish finiteness of $r(f, k)$, but they grow with multiple exponentials in k .

Applications of the Finite Rank Theorem, I

Corollary:[GKM] The following numeric graph invariants f have $r(f, k) < \infty$:

- (i) The number of acyclic orientations
- (ii) The number of eulerian orientations

Proof: They are both instances of the Tutte polynomial.

Applications of the Finite Rank Theorem, II

Corollary:([GKM])

$\omega(G)$ **is not an instance** of any MSOL-definable graph polynomial, but **is the degree** of some MSOL-definable graph polynomial,

Proof: $\omega(G_1 \sqcup G_2) = \max\{\omega(G_1), \omega(G_2)\}$. So $r(\omega, 0) = \infty$.

$\omega(G)$ can be obtained as **degree** of the graph polynomial

$$\text{clique}(G, X) = \sum_i \text{clique}_i(G) X^i = \sum_{C \subseteq V} \prod_{v \in C} X$$

where $\text{clique}_i(G)$ is the number of cliques of size i , and C varies over all cliques of G .

Clearly, $\text{clique}(G, X)$ is a simple MSOL-definable graph polynomial. □

Applications of the Finite Rank Theorem, III

Corollary:([GKM])

- (i) If f satisfies $f(G_1 \sqcup G_2) = \max\{f(G_1), f(G_2)\}$, then f is **not an instance** of an MSOL-definable graph polynomial.
- (ii) If f satisfies $f(G_1 \sqcup G_2) = \min\{f(G_1), f(G_2)\}$, then f is **not an instance** of an MSOL-definable graph polynomial.

Applications of the Finite Rank Theorem, IV

Let $d(G)$ denote the **average degree** of G . We have

$$d(G) = \frac{1}{|V(G)|} \cdot \sum_{v \in V(G)} d_G(v),$$

where $d_G(v)$ denotes the degree of a vertex v of G .

Corollary:([GKM])

$d(G)$ is **not an instance** of an MSOL-definable graph polynomial.

Proof: For $f = d(G)$ we have

$$M(d, 0) = 2 \frac{|E_1| + |E_2|}{|V_1| + |V_2|}.$$

This contains, for graphs with a fixed number e of edges, the Cauchy matrix

$$\left(\frac{2e}{i+j} \right),$$

hence $r(d, 0)$ is infinite. □

Specific coefficients of graph polynomials

Let $p(G, \bar{X})$ be an *MSOL*-definable graph polynomial with values in $\mathbb{R}[\bar{X}]$ with m indeterminates X_1, \dots, X_m , and let

$$X_1^{\alpha_1} \cdot X_2^{\alpha_2} \cdot \dots \cdot X_m^{\alpha_m}$$

be a **specific monomial** of $p(G, \bar{X})$ with coefficient $c_\alpha(G)$, where $\alpha = (\alpha_1, \dots, \alpha_m)$.

Theorem:[GKM] Then there is an invariantly *MSOL*-definable graph polynomial $p_\alpha(G, \bar{X})$ such that $c_\alpha(G)$ is an evaluation of $p_\alpha(G, \bar{X})$.

Remark: The theorem remains valid for monomials of the form

$$X_1^{n_1(G)-\alpha_1} \cdot X_2^{n_2(G)-\alpha_2} \cdot \dots \cdot X_m^{n_m(G)-\alpha_m},$$

where $n_i(G) = |V(G)|$ or $n_i(G) = |E(G)|$.

This can be used to treat the coefficient of $X^{|V(G)|-3}$ of the characteristic polynomial.

Graph polynomials which are not MSOL-definable

without the assumption $\mathbf{P} \neq \mathbf{NP}$

Let $c : E \rightarrow [k]$ be an edge-coloring. c is **path-rainbow** if between any two vertices $u, v \in V$ there is a path where all the edges have different colors.

We denote by $\chi_{rainbow}(G, k)$ the number of path-rainbow colorings of G with k colors.

Theorem:(T. Kotek and J.A.M.)

- (i) $\chi_{rainbow}(G, k)$ is a polynomial in k .
- (ii) $\chi_{rainbow}(G, k)$ is not MSOL-definable (but SOL-definable).

Proof: A more sophisticated use of connection matrices.

The same works also for **harmonious colorings**.

Complexity of evaluations

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Exceptional points and uniform reductions,
CSR'08, pp. 86-98

The complexity of the chromatic polynomial, I

Theorem:

- $\chi(G, 3)$ is $\#\mathbf{P}$ -complete (Valiant 1979).
- $\chi(G, -1)$ is $\#\mathbf{P}$ -complete (Linial 1986).

Question: What is the complexity of computing $\chi(G, \lambda)$ for $\lambda = \lambda_0 \in \mathbb{Q}$ or even for $\lambda = \lambda_0 \in \mathbb{C}$?

The complexity of the chromatic polynomial, II

Let $G_1 \bowtie G_2$ denote the join of two graphs.

We observe that

$$\chi(G \bowtie K_n, \lambda) = (\lambda)^n \cdot \chi(G, \lambda - n) \quad (\star)$$

Hence we get

(i) $\chi(G \bowtie K_1, 4) = 4 \cdot \chi(G, 3)$

(ii) $\chi(G \bowtie K_n, 3 + n) = (n + 3)^n \cdot \chi(G, 3)$ hence
for $n \in \mathbb{N}$ with $n \geq 3$ it is $\#\mathbf{P}$ -complete.

The complexity of the chromatic polynomial, III

If we have an oracle for some $q \in \mathbb{Q} - \mathbb{N}$ which allows us to compute $\chi(G, q)$ we can compute $\chi(G, q')$ for any $q' \in \mathbb{Q}$ as follows:

Algorithm $A(q, q', |V(G)|)$:

- (i) Given G the degree of $\chi(G, q)$ is at most $n = |V(G)|$.
- (ii) Use the oracle and (\star) to compute $n + 1$ values of $\chi(G, \lambda)$.
- (iii) Using Lagrange interpolation we can compute $\chi(G, q')$ in polynomial time.

We note that this algorithm is purely algebraic and works for all graphs G , $q \in (F) - \mathbb{N}$ and $q' \in F$ for any field F extending \mathbb{Q} .

The complexity of the chromatic polynomial, IV

We summarize the situation for the chromatic polynomial as follows:

- (i) We have an **exception set** $C = \mathbb{N}$ which is a countable union of semi-algebraic sets of dimension 0 in the field \mathbb{C} .
- (ii) We have a numeric graph invariant $f(G) = |G|$ which is **FP**.
- (iii) We have **one algebraic** algorithm $A(q, q', f(G))$ which runs in polynomial time in q, q' and $f(G)$ which calls the oracle $\chi(-, q')$.
 q, q' are in any finite dimensional algebraic extension field F of \mathbb{Q} .
- (iv) The algorithm $A(q, q', f(G))$ reduces **uniformly**,
for any $q \in F - C$, the evaluation of $\chi(G, q)$ into the evaluation of $\chi(G, 3)$.

The nature of the algorithm A , I

In the case of $\chi(-, q)$ and $\chi(-, q')$

- The input of A is $f(G) \in F$,
in this case the degree of the $\chi(G, \lambda)$
- The output of A is a rational function $A(q, q', f(G)) \in F(x_0, x_1, \dots, x_{f(G)+2})$.
the Lagrange interpolation for $f(G) + 1$ points for q, q'
- The final result of the reduction is obtained by evaluating this rational function at

$$x_0 = \chi(G, q'), \quad x_1 = \chi(G \bowtie K_1, q'), \quad \dots, \quad x_n = \chi(G \bowtie K_n, q')$$

$$x_{n+1} = q, \quad x_{n+2} = q'$$

A suitable model of computation for A is

the unit-cost model *BSS*
advocated by L. Blum, M. Shub and S. Smale.

The uniform difficult point property for $\chi(G, \lambda)$

(i) We have shown:

For all $q \in \mathbb{Q} - \{0, 1, 2\}$ and $q' \in \mathbb{Q}$ the numeric graph invariants $\chi(-, q)$ and $\chi(-, q')$ polynomial time Turing reducible to each other.

(ii) But we have shown much more:

There is ONE **algebraic reduction scheme** for all the instances $\chi(G, q)$ to $\chi(G, q')$, where q, q' are not in \mathbb{N} .

Uniform algebraic reductions for evaluations of graph polynomials.

Let $f = \Phi(G, \bar{q})$ and $g = \Phi(G, \bar{q}')$ two evaluations of the same graph polynomial Φ . We say that f **algebraically reduces to g uniformly** in \bar{q}, \bar{q}' , and we write $f <_{UA}^P g$, if there exists

- (i) a finite set $\mathcal{A}_\Phi = \{\alpha_1, \dots, \alpha_a\}$ of size a of polynomial time computable numeric graph invariants $\alpha : Graphs \rightarrow \mathbb{Q}$, depending on Φ only;
- (ii) a polynomial time computable family $r_i : i \in \mathbb{N}$ of polynomial time computable graph transductions $r_i : Graphs \rightarrow Graphs$, depending on Φ only;
The family is polynomial time computable in Φ and i .
- (iii) a polynomial time computable function $A_\Phi : \mathbb{Q}^a \rightarrow \mathbb{Q}(x_1, x_2, \dots)$, depending on Φ only;

such that for every $G \in Graphs$ we have that

$$f(G) = A_\Phi(\alpha_1(G), \dots, \alpha_a(G)) (g(r_1(G)), \dots, g(r_{poly(G)}(G)), \bar{q}, \bar{q}')$$

The uniform difficult point property UDPP

Let $\Phi(G, \bar{x}^m)$ be a graph polynomial in m variables.

$\Phi(G, \bar{x}^m)$ has the **uniform difficult point property (DPP)** if the following holds:

There exists an **exception set** C_Φ which is a countable union of semi-algebraic sets of dimension $< m$ in the field \mathbb{C} , and for all q not in the exception set C , $\Phi(-q)$ is $\sharp\mathbf{P}$ hard.

Furthermore, for any $\bar{q}_1, \bar{q}_2 \in F^m - C_\Phi$ we have

$$\Phi(G, \bar{q}_1) <_{UA}^P \Phi(G, \bar{q}_2).$$

In other words, all the evaluations for \bar{q} not in the exception set, are of the **same difficulty and uniformly algebraically reducible** to each other.

The Tutte polynomial

The **paradigm of the DPP** was inspired by the work of Linial and Jaeger, Vertigan and Welsh.

- (i) For the classical Tutte polynomial, the **uniform DPP** was proven by Jaeger, Vertigan and Welsh in 1990.
- (ii) For the colored Tutte polynomial as defined by Bollobás and Riordan (1999), the **uniform DPP** was proven by Bläser, Dell and Makowsky in 2007.
- (iii) This also holds for the multivariate Tutte polynomial, the **Pott's model**, if restricted to a fixed finite number of variables.

More polynomials with the uniform DPP

The uniform DPP was also proven for

- (i) the **cover polynomial** $C(G, x, y)$ introduced by Chung and Graham (1995) by [Bläser Dell, 2007](#)
- (ii) the **interlace polynomial** (aka Martin polynomial) introduced by Martin (1977) and independently by Arratia, Bollobás and Sorkin (2000), by [Bläser and Hoffmann, 2007](#)
- (iii) the **matching polynomial**, by [Averbouch, Kotek and Makowsky, 2007](#)
- (iv) the **harmonious chromatic polynomial**, by [Averbouch, Kotek and Makowsky, 2007](#)

What is the pattern behind this?

In establishing the UDPP one uses the fact that in the examples the evaluations at integer points are in $\#P$.

We call such graph polynomials **counting**.

There seems to be **dichtomy property**:

- Either all the evaluations of a graph polynomial Φ are polynomial time computable, or
- Φ has the uniform difficult point property UDPP.

Conjecture: This dichtomy holds for all **counting MSOL-definable** graph polynomials.

Note that it holds for the harmonious chromatic polynomial, which is **not** MSOL-definable.

Thank you for your attention !
