# On Counting Generalized Colorings

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**Abstract.** The notion of graph polynomials definable in Monadic Second Order Logic, MSOL, was introduced in [Mak04]. It was shown that the Tutte polynomial and its generalization, as well as the matching polynomial, the cover polynomial and the various interlace polynomials fall into this category.

In this paper we present a framework of graph polynomials based on counting functions of generalized colorings. We show that this class encompasses the examples of graph polynomials from the literature. Furthermore, we extend the definition of graph polynomials definable in MSOL to allow definability in full second order, SOL. Finally, we show that the SOL-definable graph polynomials extended with a combinatorial counting function are exactly the counting functions of generalized colorings definable in SOL.

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# 1 Introduction

## 1.1 Graph invariants and graph polynomials

A *graph invariant* is a function from the class of (finite) graphs $\mathcal{G}$ into some domain $\mathcal{D}$ such that isomorphic graphs have the same picture. Usually such invariants are meant to be uniformly defined in some formalism. If $\mathcal{D}$ is the two-element boolean algebra we speak of *graph properties*. Examples are the properties of being connected, planar, Eulerian, Hamiltonian, etc. If $\mathcal{D}$ consists of the natural numbers, we speak of *numeric graph invariants*. Examples are the number of connected components, the size of the largest clique or independent set, the diameter, the chromatic number, etc. But $\mathcal{D}$ could also be a polynomial ring $\mathbb{Z}[\bar{X}]$ over $\mathbb{Z}$ with a set of indeterminates $\bar{X}$. Here examples are the characteristic polynomial, the chromatic polynomial, the Tutte polynomial, etc.

There are many graph invariants discussed in the literature, which are polynomials in $\mathbb{Z}[\bar{X}]$, but there are hardly any papers discussing classes of graph polynomials as an object of study in its generality. An outline of such a study was presented in [Mak06]. In [Mak04] the second author has introduced the **MSOL**-definable and the **SOL**-definable graph polynomials, the class of graph polynomials where the range of summation is definable in (monadic) second order logic. He has verified that all the examples of graph polynomials discussed in the literature, with the exception of the weighted graph polynomial of [NW99], are actually **SOL**-polynomials over some expansions (by adding order relations) of the graph, cf. also [Mak06]. In some cases this is straight forward, but in some cases it follows from intricate theorems.

A proper $k$-vertex-colorings of a graph $G = (V, E)$ with colors from a set $\{0, \ldots, k-1\} = [k]$ is a function from $f : V \to [k]$ such that no two distinct vertices connected by an edge have the same value. A simple case of generalized colorings are the $\phi$-colorings, $k$-vertex-colorings definable by a first order formula $\phi(F)$ over graphs with an additional function symbol $F$, and we allow all functions $f$ which are interpretations of $F$ satisfying $\phi(F)$. It will become clearer later that to define a $\phi$-coloring, the formula has to be subject to certain semantic restrictions such as invariance under permutation of the colors, the existence of a bound on the colors used, and independence of the colors not used. The general case arises by expanding the graph, allowing several color sets, and replacing functions by relations. The associated counting function $\chi_\phi(G, k)$ counts the number of generalized colorings satisfying $\phi$ as a function of $k$.

Our first result is

**Proposition A** *Let $\bar{k} = (k_1, \ldots k_\alpha)$ be the cardinalities of the various color sets. For $\phi$ subject to the conditions above, the counting function $\chi_\phi(G, \bar{k})$ is a polynomial in $\bar{k}$.*

The purpose of this paper is to present a framework, which we call *counting functions of generalized colorings*, for defining graph invariants.

In particular we shall compare the counting functions of generalized colorings with the **SOL**-definable polynomials. A special case of these, the **MSOL**-

definable polynomials were first introduced in [Mak04]. **SOL**-definable polynomials define invariants of finite first order $\tau$-structures for arbitrary vocabularies (similarity types). In contrast to counting functions of generalized colorings, they are polynomials by their very definition. The definability condition refers to summation over definable sets of relations and products over definable sets of elements of the underlying structure.

In order to relate the counting functions of generalized colorings to the **SOL**-definable polynomials, we allow additional combinatorial functions as monomials. We call the corresponding polynomials *extended* **SOL**-*polynomials*.

### 1.2 Main result

Our main results here are:

**Theorem B** *Every extended* **SOL**-*definable polynomial is a counting function of a generalized coloring of graphs definable in* **SOL**.

**Theorem C** *Every counting function of a generalized coloring of ordered graphs definable in* **SOL** *is an extended* **SOL**-*definable polynomial.*

### 1.3 Outline of the paper

We assume the reader is familiar with the basics of graph theory as, say, presented in [Die96,Bol99]. We also assume the reader is familiar the basics of finite model theory as, say, presented in [EFT94,EF95,Lib04]

Section 2 is a prelude to our general discussion, in which we discuss the chromatic polynomial and explain how it fits into the various frameworks. In Section 3 we introduce our notion of counting functions of generalized colorings definable in **SOL**. We prove they are polynomials in the number of colors and show examples of graph polynomials from the literature which fall under this class of graph polynomials. In Section 4 we give precise definition of extended **SOL**-definable polynomials. In Appendix A we give the full details skipped in Section 4, and in Appendix B we give the full proofs of both Theorem B and Theorem C.

An earlier version of some of the material of this paper was posted as [MZ06].

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## 2 Prelude: The chromatic polynomial

Before we introduce our general definitions, we discuss the oldest graph polynomial studied in the literature, the classical *chromatic polynomial* $\chi_G(k)$. It has a very rich literature. For an excellent and exhaustive monograph, cf. [DKT05].

We denote by $\mathcal{G}$ the set of graphs of the form $G = ([n], E)$. A $k$-vertex-coloring of $G$ is a function $f : [n] \to [k]$ such that whenever $(u, v) \in E$ then $f(u) \neq f(v)$. $\chi_G(k)$ denotes the number of $k$-vertex-colorings of $G$. $\chi_G(k)$ defines, for each graph, a function $\chi_G(\lambda) : \mathcal{G} \to \mathbb{N}$ which turns out to be a polynomial in $\lambda$.

We note that $\chi_G(\lambda)$ really denotes a family of polynomials indexed by graphs from $\mathcal{G}$. This family is furthermore uniformly defined based on some of the properties of the graph $G$. We are interested in various formalisms in which such uniform definitions can be given. We isolate the following themes:

(i) A *recursive* definition of $\chi_G(k)$ (using an order on the vertices or edges).
(ii) A uniform *explicit* definition of $\chi_G(k)$ over the graph using a second order logic formalism. In [Cou] it is called a *static* definition of the polynomial.
(iii) We associate with each $k \in \mathbb{N}$ a two-sorted structure $\mathcal{G}_k = \langle G, [k] \rangle$ and interpret $\chi_G(k)$ as counting the number of expansions $\langle \mathcal{G}_k, F \rangle$ satisfying some first order formula $\phi(F)$.

In [CGM07] the relationship between recursive and explicit definitions is studied. There, a framework is provided which allows to show that every recursive definition of a graph polynomial also allows an explicit definition. The converse is open but seems not to be true. Here we are interested in the relationship between explicit definition, and counting expansions.

### 2.1 Recursive definition.

The first proof that $\chi_G(\lambda)$ is a polynomial used the observation that $\chi_G(\lambda)$ has a recursive definition using the order of the edges, which can be taken as the order induced by the lexical ordering on $[n]^2$. However, the object defined does not depend on the particular order of the edges. For details, cf. [Big93,Bol99]. The essence of the proof is as follows:

For $e = (v_1, v_2)$, we put $G - e = (V, E')$ with $E' = E - \{e\}$, and $G/e = (V^*, E^*)$ $V^* = V - \{v_2\}$ and $E^* = (E \cap (V^*)^2) \cup \{(v_1, v); (v_2, v) \in E\}$. The operation passing from $G$ to $G - e$ is called *edge removal*, and the operation passing from $G$ to $G/e$ is called *edge contraction.*

**Lemma 1.** *Let $e, f$ be two edges of $G$. Then we have $(G - e) - f = (G - f) - e$, $(G/e) - f = (G - f)/e$, $(G - e)/f = (G/f) - e$ and $(G/e)/f = (G/f)/e$.*

Let $E_n = ([n], \emptyset)$. We have $\chi_{E_n}(\lambda) = \lambda^n$. Furthermore, for any edge $e \in E$ we have $\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G/e}(\lambda)$. Let $E = (e_0, e_1, \ldots, e_m)$ be the enumeration of the edges in this lexicographic order. Using the order on the edges, this allows us to compute $\chi_G(\lambda)$. It also turns out, using Lemma 1, that the result is *independent* of the ordering of the edges.

### 2.2 Explicit descriptions.

There are other proofs that $\chi_G(\lambda)$ is a polynomial.

*Proof.* We first observe that any coloring uses at most $n$ of the $\lambda$ colors. For any $m \leq n$, let $c(m)$ be the number of colorings, with a fixed set of $m$ colors, which are vertex colorings and use all $m$ of the colors. Then, given $\lambda$ colors, the number of vertex colorings that use exactly $m$ of the $\lambda$ colors is the product of $c(m)$ and the binomial coefficient $\binom{\lambda}{m}$. So

$$\chi_G(\lambda) = \sum_{m \leq n} \binom{\lambda}{m} c(m)$$

The right side here is a polynomial in $\lambda$, because each of the binomial coefficients is. We also use that for $\lambda \leq m$ we have $\binom{\lambda}{m} = 0$.

If both the set of colors and the set of vertices are initial segments of the natural numbers with their order, we can also rewrite this in the following way:

$$\chi_G(\lambda) = \sum_{A:init(A,V)} \sum_{f:ontocol(f,A)} \binom{\lambda}{card(A)} \quad \text{(chrom-1)}$$

where $init(A, V)$ says that $A$ is an initial segment of $V$, and $ontocol(f, A)$ says that $f$ is a vertex coloring using all the colors of $A$.

Equation chrom-1 is an example of a explicit definition of the chromatic polynomial.

In [DKT05, Theorem 1.4.1] another explicit description of $\chi_G(\lambda)$ is given: Let $a(G, m)$ be the number of partitions of $V$ into $m$ independent sets, and let

$$(\lambda)_m = \lambda \cdot (\lambda - 1) \cdot \ldots \cdot (\lambda - m + 1)$$

be the *falling factorial*. Then $\chi_G(\lambda) = \sum_m a(G, m) \cdot (\lambda)_m$. This again be written as

$$\chi_G(\lambda) = \sum_{P:indpart(P,A_P,V)} (\lambda_{card(A_P)}) \quad \text{(chrom-2)}$$

where $indpart(P, A_P, V)$ says that $P$ is an equivalence relation on $V$ and $A_P$ consists of the first elements (with respect to the order on $V = [n]$) of each equivalence class.

A third explicit description for $\chi_G(\lambda)$ is given in [DKT05, Theorem 2.2.1]. It can be obtained from a two-variable polynomial $Z_G(\lambda, V)$ defined by

$$Z_G(\lambda, V) = \sum_{S:S\subseteq E} \left( \prod_{v:fcomp(v,S)} \lambda \cdot \prod_{e:e\in S} V \right) = \sum_{S:S\subseteq E} \left( \lambda^{k(S)} \cdot \prod_{e:e\in S} V \right)$$

where $fcomp(v, S)$ is the property "$v$ is the first vertex in the order of $V$ of some connected component of the spanning subgraph $< S : V >$ on $V$ induced by $S$", and $k(S)$ is the number of connected components of $< S : V >$. Now we have

$$\chi_G(\lambda) = Z_G(\lambda, -1) \quad \text{(chrom-3)}$$

The three explicit descriptions of the chromatic polynomial chrom-1, chrom-2, chrom-3 have several properties in common:

(i) They satisfy the same recursive definition, because they define the same polynomial.
(ii) They are of the form $\sum_k A_k(G)P_k(\lambda)$ where $P_k(\lambda)$ is a polynomial in $\lambda$ with integer coefficients of degree $k$.
(iii) The coefficients $A_k(G)$ are positive and have a combinatorial interpretation.
(iv) The coefficients can be alternatively obtained by collecting the terms $P_k(\lambda)$ of a summation over certain relations definable in second order logic over the graph with an order on the vertices and interpreting $k$ as the cardinality of such a relation.
(v) Although the order on the vertices is used in the explicit description of the polynomial, the polynomial is *invariant under permutations of the ordering*.

There are also significant differences.

(i) In chrom-1 it is important that the set of colors and the set of vertices are initial segments of the natural numbers with their natural order. The summation involves *one unary relation* and *one unary function*.
(ii) In chrom-2 the summation involves a *binary relation* on the vertices which is not a subset of the edge relation, but of its complement. The order relation is only needed to identify equivalence classes.
(iii) In chrom-3 we actually use a two-variable polynomial and then substitute for one variable $-1$. The summation involves a *binary relation* on vertices which is a subset of the edge relation. It can be also viewed as a *unary relation* on the set of edges. The order relation is only needed to identify connected components.

### 2.3 Counting expansions.

Let $\mathbf{F}$ be a unary function symbol and let $\phi(\mathbf{F}, \mathbf{E})$ be the formula which says that $F$ is a proper $k$-vertex coloring for the edge relation $E$. Then $\chi_G(k) = \chi_\phi(G, k)$ is the number of interpretations $F$ of $\mathbf{F}$ in $\langle [n], [k], E, f \rangle$ which satisfy $\phi(\mathbf{F}, \mathbf{E})$. We note that

(i) a coloring is invariant under permutations of the colors,
(ii) the number of colors is bounded by the size of $V$, and
(iii) the property of being a coloring is independent of the colors not used.

This is readily generalized to other formulas $\psi(F, E)$ satisfying similar properties, and will be the starting point for our notion of generalized coloring.

## 3 Generalized Chromatic Polynomials

### 3.1 Generalized colorings

Let $\mathcal{M}$ be a $\tau$-structure with universe $M$. We say a two-sorted structure $\langle \mathcal{M}, [k], R \rangle$ for the vocabulary $\tau_R$ is a generalized coloring of $\mathcal{M}$ with $k$ colors. The set $[k]$ will be referred to as the color set. We denote relation symbols by bold-face letters, and their interpretation by the corresponding roman-face letter.

**Definition 1 (Coloring Property).** *A class $\mathcal{P}$ of generalized colorings*

$$\mathcal{P} = \{\langle \mathcal{M}, [k], R\rangle \mid R \subseteq M^s \times [k]\}$$

*of $\tau_R$ structures is a coloring property if it satisfies the following conditions:*

**Isomorphism property** *$\mathcal{P}$ is closed under $\tau_R$-isomorphisms.*
*In particular, $\mathcal{P}$ is closed under permutations of the color set $[k]$.*

**Extension property** *For every $\mathcal{M}$, $k, k'$ and $R$, if $k' \geq k$ and $\langle \mathcal{M}, [k], R\rangle \in \mathcal{P}$ then $\langle \mathcal{M}, [k'], R\rangle \in \mathcal{P}$.*
*In other words, the extension property requires that increasing the number of colors of a generalized coloring does not affect whether it belongs to the property.*

*We refer to the relation symbol $\mathbf{R}$ and its interpretations $R$ as* coloring predicates*. For fixed $k$, a specific interpretation is called a $k$-$\mathcal{P}$-coloring.*

**Definition 2 (Bounded coloring properties).**

*(i) A coloring property is* bounded*, if for every $\mathcal{M}$ there is a number $N_M$ such that for all $k \in \mathbb{N}$ the set of colors $\{x \in [k] : \exists \bar{y} \in M^m R(\bar{y}, x)\}$ has size at most $N_M$.*
*(ii) A coloring property is* range bounded*, if its range is bounded in the following sense: There is a number $d \in \mathbb{N}$ such that for every $\mathcal{M}$ and $\bar{y} \in M^m$ the set $\{x \in [k] : R(\bar{y}, x)\}$ has at most $d$ elements.*

*Clearly, if a coloring property is range bounded, it is also bounded.*

**Proposition 1 (Special case of Proposition A).** *Let $\mathcal{P}$ be a bounded coloring property. For every $\mathcal{M}$ the number $\chi_{\mathcal{P}}(\mathcal{M}, k)$ is a polynomial in $k$ of the form*

$$\sum_{j=0}^{N_m} c_{\mathcal{P}}(\mathcal{M}, j)\binom{k}{j}$$

*where $c_{\mathcal{P}}(\mathcal{M}, j)$ is the number of generalized $k$-$\mathcal{P}$-colorings $R$ with a fixed set of $j$ colors. If $\mathcal{P}$ is range-bounded then $N_m \leq d \cdot |M|^m$.*

*Proof.* We first observe that any generalized coloring $R$ uses at most $N_M$ of the $k$ colors, if it is bounded. If $R$ is range-bounded then $N_m \leq d \cdot |M|^m$. For any $j \leq N_M$, let $c_{\mathcal{P}}(\mathcal{M}, j)$ be the number of colorings, with a fixed set of $j$ colors, which are generalized vertex colorings and use all $j$ of the colors. We use the properties of the coloring property. So any permutation of the set of colors used is also a coloring. Therefore, given $k$ colors, the number of vertex colorings that use exactly $j$ of the $k$ colors is the product of $c_{\mathcal{P}}(\mathcal{M}, j)$ and the binomial coefficient $\binom{k}{j}$. So

$$\chi_{\mathcal{P}}(\mathcal{M}, k) = \sum_{j \leq N_M} c_{\mathcal{P}}(\mathcal{M}, j)\binom{k}{j}$$

The right side here is a polynomial in $k$, because each of the binomial coefficients is. We also use that for $k < j$ we have $\binom{k}{j} = 0$.

We note that bounded properties which are not range-bounded lack in uniformity. Therefore, for a range-bounded property $\mathcal{P}$ we call $\chi_{\mathcal{P}}(\mathcal{M}, k)$ a *generalized chromatic polynomial*.

*Remark 1.* The restriction to coloring properties in Theorem 1 is essential. Let $\chi_{onto}(G, k)$ be the number of functions $f : V(G) \to [k]$ which is onto. Clearly, this is not a polynomial in $k$ as for $k >\mid V(G) \mid$ it always vanishes, so it should be constantly 0.

### 3.2 Definability of generalized chromatic polynomials

We assume the reader is familiar with First and Second Order Logic, denoted by **FOL** and **SOL**, as defined in, for example, [EF95]. The formulas of $\mathbf{SOL}(\tau)$ are defined like the ones of **FOL**, with the addition that we allow countably many variables for $n$-ary relation symbols $U_{n,\alpha}$ for $\alpha \in \mathbb{N}$, for each $n \in \mathbb{N}$, and quantification over these. Monadic second order logic $\mathbf{MSOL}(\tau)$ is the restriction of $SOL(\tau)$ to unary relation variables and quantification over these.

**Definition 3.** *A generalized chromatic polynomial for $\tau$-structures with coloring predicate $R$ is definable in $\mathbf{SOL}(\tau_R)$, respectively in $\mathbf{MSOL}(\tau_R)$, if it is of the form $\chi_\phi(\mathcal{M}, \bar{\lambda})$, where $\phi \in \mathbf{SOL}(\tau_R)$, respectively in $\mathbf{MSOL}(\tau_R)$, and defines a range-bounded coloring property.*

**Definition 4 (Coloring formula).** *A first order (or second order) formula $\phi(\mathbf{R})$ is a* coloring formula*, if the class of its models, which are of the form $\langle \mathcal{M}, [k], R \rangle$, is a coloring property. If a coloring property $\mathcal{P}$ is definable by a coloring formula $\phi$ then we denote the number of generalized $k$-$\mathcal{P}$-colorings on $R$ by $\chi_{\phi(R)}(\mathcal{M}, k)$.*

The coloring property of proper vertex colorings from example 3.3 is definable by a **FOL** formula. We will discuss coloring formulas definable in second order logic **SOL** in more detail in Subsection 3.2.

### 3.3 Examples and applications of Proposition 1

Let $\mathbf{F}$ be a unary function symbol which serves as the coloring predicate. A (not necessarily proper) vertex coloring of a graph $G = (V, E)$ is a map $F : V \to [k]$ for some $k$.

(i) A vertex coloring $F$ is *proper*, if it satisfies $\forall u, v(\mathbf{E}(u, v) \to \mathbf{F}(u) \neq \mathbf{F}(v))$. Clearly, this does define a coloring property.
(ii) If we require that a vertex coloring $F$ uses all the colors, then this is not a coloring property. It violates the extension property in Definition 1.
(iii) A vertex coloring is *pseudo-complete*, if it satisfies

$$\forall x, y \exists u, v(\mathbf{E}(u, v) \wedge \mathbf{F}(u) = x \wedge \mathbf{F}(v) = y).$$

For the same reason as above this is not a coloring property.

**Connected colorings.** A vertex coloring is *connected* if each monochromatic set induces a connected subgraph. We denote by $\chi_{cc}(G,k)$ the number of connected colorings of $G$ with $k$ colors. Clearly, coloring all connected components with the same color gives a connected coloring. However, it is not clear how difficult it is to count such colorings.

*Conjecture 1.* For all $k \in \mathbb{N}$ with $k \geq 2$, computing $\chi_{cc}(G,k)$ is $\sharp\mathbf{P}$ hard.

**Harmonious colorings.** Harmonious colorings are counterparts to complete colorings. A vertex coloring is a *harmonious coloring* if it is a proper coloring and every pair of colors appears at most once on an edge. In this case, the extension property holds and the number of harmonious colorings $\chi_{harm}(G,k)$ is a polynomial in $k$. Harmonious colorings are treated in [HK83], as well as [Edw97] and [EM95].

*mcc*($t$)**-colorings.** We say a vertex coloring $f : V \to [k]$ has small monochromatic connected components if $f^{-1}(a)$ induces a graph with each of its connected components of size at most $t$. For a fixed $t$, this is a coloring property. Related graph invariants were introduced in [LMST07] and in [ADOV03].

**Proposition 2.** *$\chi_{cc}(G,k)$, $\chi_{harm}(G,k)$ and $\chi_{mcc(t)}(G,k)$ are generalized chromatic polynomials.*

**Complete colorings.** A vertex coloring is *complete*, if it is both proper and pseudo-complete. Complete colorings are studied in the context of the *achromatic number of a graph* $G$ which is the largest number $k$ such that $G$ has a complete coloring with $k$ colors. The achromatic number of $G$ and the number of complete colorings is a function of $G$ but not of $k$. In other words, to be a complete coloring is not a coloring property in our sense. Let $\chi_{complete}(G,k)$ denote the number of complete colorings of $G$ with $k$ colors. Using the same argument as in Remark 1 we see that $\chi_{complete}(G,k)$ ultimately vanishes for large enough $k$, and therefore is not a polynomial in $k$.

The achromatic number was introduced in [HHR67]. For a survey of recent work, cf. [HM97].

### 3.4 Generalized multi-colorings

To construct also graph polynomials in several variables, we extend the definition to deal with several color-sets, and also call them generalized chromatic polynomials.

Let $\mathcal{M}$ be a $\tau$-structure with universe $M$. We say an $(\alpha+1)$-sorted structure $\langle \mathcal{M}, [k_1], \ldots, [k_\alpha], R\rangle$ for the vocabulary $\tau_{\alpha,R}$ with $R \subset M^m \times [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$ is a *generalized coloring* of $\mathcal{M}$ for colors $\bar{k}^\alpha = (k_1, \ldots, k_\alpha)$. By abuse of notation, $m_i = 0$ is taken to mean the color-set $k_i$ is not used in $R$.

**Definition 5 (Multi-color Coloring Property).** *A class of generalized multi-colorings $\mathcal{P}$ is a coloring property if it satisfies the following conditions:*

**Isomorphism property** *: $\mathcal{P}$ is closed under $\tau_{\alpha,R}$-isomorphisms.*
**Extension property** *: For every $\mathcal{M}$, $k_1 \leq k'_1, \ldots, k_\alpha \leq k'_\alpha$, and $R$, if $\langle \mathcal{M}, [k_1], \ldots, [k_\alpha], R \rangle \in \mathcal{P}$ then $\langle \mathcal{M}, [k'_1], \ldots, [k'_\alpha], R \rangle \in \mathcal{P}$.*
**Non-occurrence property** *: Assume $R \subset M^m \times [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$ with $m_i = 0$, and $\langle \mathcal{M}, [k_1], \ldots, [k_\alpha], R \rangle \in \mathcal{P}$, then for every $k'_i \in \mathbb{N}$, $\langle \mathcal{M}, [k_1], \ldots, [k'_i], \ldots, [k_\alpha], R \rangle \in \mathcal{P}$.*

*The extension property and the non-occurrence property require that increasing the number of colors respectively adding unused color-sets does not affect whether the generalized coloring belongs to $\mathcal{P}$.*

*We denote by $\chi_{\mathcal{P}}(\mathcal{M}, k_1, \ldots, k_\alpha)$ the number of generalized $\bar{k}^\alpha - \mathcal{P}$-multi-coloring $R$ on $\mathcal{M}$. If $\mathcal{P}$ is definable by some formula $\phi(\mathbf{R})$ we also write $\chi_{\phi(R)}(\mathcal{M}, k_1, \ldots, k_\alpha)$.*

The notions of *bounded* and *range-bounded multi-coloring properties* are defined as for the coloring properties with respect to each color-set.

**Proposition 3 (Special case of Proposition A).** *Let $\mathcal{P}$ be a bounded multi-coloring property with bound $N$. In the case of range bounded multi-colorings $N \leq d\cdot \mid M \mid^m$. For every $\mathcal{M}$ the number $\chi_{\mathcal{P}}(\mathcal{M}, k_1, \ldots, k_\alpha)$ is a polynomial in $k_1, \ldots, k_\alpha$ of the form*

$$\sum_{\bar{j}^\alpha \leq [N]^\alpha} c_{\phi(R)}(\mathcal{M}, \bar{j}^\alpha) \prod_{1 \leq \beta \leq \alpha} \binom{k_\beta}{j_\beta}$$

*where $c_{\phi(R)}(\mathcal{M}, \bar{j}^\alpha)$ is the number of generalized $\bar{k}^\alpha - \phi$-colorings $R$ with fixed sets of $j_\beta$ colors respectively.*

*Proof.* Similar to the one variable case.

*Example 1.* Recall in example 3.3 we denoted by $\chi_{mcc(t)}(G, k)$ the number of vertex colorings for which no color induces a graph with a connected component larger than $t$. Let $\chi_{mcc}(G, k, t) = \chi_{mcc(t)}(G, k)$ be the counting function of generalized multi-colorings satisfying the above condition, where $t$ is considered a color-set. We note for every $t \geq |V|$, every vertex coloring has only connected components of size no more then $t$. So, if $\chi_{mcc}(G, k, t)$ were a polynomial in $t$ then by interpolation $\chi_{mcc}(G, k, t)$ would be the number of vertex colorings, so the function $\chi_{mcc}(G, k, t)$ is must not be a polynomial in $t$. The set of such vertex colorings does not satisfy the non-occurrence condition. This example shows the motivation for requiring this condition of coloring properties.

### 3.5 The most general case

We shall also allow several simultaneous coloring predicates $R_1, \ldots, R_s$. The notion of coloring properties for this situation extends naturally. We shall call multi-coloring properties and multi-coloring simply also coloring properties and colorings, if the situation is clear from the context. The notion of definability of multi-colorings with several coloring predicates is analogous to the simple case.

### 3.6 Closure properties

**Proposition 4 (Sums and products).** *The sum and product of two generalized chromatic polynomials* $\chi_{\phi(\mathbf{R})}(G, \lambda)$ *and* $\chi_{\psi(\mathbf{R})}(G, \lambda)$ *is again a generalized chromatic polynomial.*

*Proof.* For the sum we take $chi_{\theta_1}(G, \lambda)$ with

$$\begin{aligned}\theta_1(R, R', U) =& ((U = \emptyset) \wedge \phi(R) \wedge (R' = \emptyset)) \vee \\ & ((U = M) \wedge (R = \emptyset) \wedge \psi(R')).\end{aligned}$$

For the product we take $\chi_{\theta_2}(G, \lambda)$ with $\theta_2(\mathbf{R}, \mathbf{R}') = (\phi(\mathbf{R}) \wedge \psi(\mathbf{R}'))$.

### 3.7 More examples of generalized chromatic polynomials

We now show how many graph polynomials can be viewed as generalized chromatic polynomials.

**Combinatorial polynomials.** The following combinatorial polynomials can be thought of as generalized chromatic polynomials:

(i) For the polynomial $\lambda^n$ we take all maps $[n] \to [k]$ for $\lambda = k$. So $\lambda^n = \chi_{true(f)}$ where $true(f)$ is $\forall v(f(v) = f(v))$. It is a first order definable range-bounded coloring property.

(ii) Similarly, for $\lambda_{(n)} = \lambda \cdot (\lambda - 1) \cdot \ldots \cdot (\lambda - n + 1)$ we take all injective maps, which is easily expressed by a first order formula which defines a range-bounded coloring property.

(iii) Finally, for $\binom{\lambda}{n}$ we take the ranges of injective maps. This is a range-bounded coloring property of a second order formula $\phi(\mathbf{P})$ which says that $P \subseteq [k]$ is the range of an injective map $f : [n] \to [k]$.

**Connected components.** We denote by $k(G)$ the number of connected components of $G$. The polynomial $\lambda^{k(G)}$ can be written as $\chi_{\phi_{connected}}(G, \lambda)$ with $\phi_{connected}(f)$ the formula

$$((u, v) \in E \to f(u) = f(v)).$$

**Matching polynomial.** Let $G = (V, E)$ be a graph. A subset $M \subseteq E$ is a matching if no two edges in $E$ have a common vertex. The matching polynomial of $G$ is given by

$$g(G, \lambda) = \sum_j \mu(G, j)\lambda^j$$

where $\mu(G, j)$ is the number of of matchings of size $j$.

We look at the structure $G_k$ and at pairs $(M, F)$ with $M \subset E$ and $F : E \to [k]$ such that $M$ is a matching and the domain of $F$ is $M$, which can be expressed by a formula $match(M, F)$. We have

$$\chi_{match(M,F)}(G, k) = \sum_j \mu(G, j)k^j = g(G, k)$$

which shows that it is a generalized chromatic polynomial.

There are two close relatives to the matching polynomial, cf. [God93].

(i) The *acyclic polynomial*

$$m(G,k) = \sum_j (-1)^j \mu(G,j) k^{n-2j} = k^n g(G, -k^{-2})$$

(ii) The *rook polynomial*, which is defined for bipartite graphs only:

$$r(G,k) = \sum_j \mu(G,j) k^{n-j} = k^n g(G, -x^{-1}).$$

The rook polynomial does not look like a generalized chromatic polynomial, but it is a substitution instance of $g(G,k)$. Similarly, the acyclic polynomial is a product of $k^n$ with a substitution instance of $g(G,k)$.

**Tutte polynomial.** We use the Tutte polynomial in the following form:

$$Z(G,q,v) = \sum_{A \subseteq E} q^{k(A)} v^{|A|}$$

where $conn(A)$ is the number of connected components of the spanning subgraph $(V,A)$. This form of the Tutte polynomial is discussed in [Sok05]. For this purpose we look at the three-sorted structure

$$G_{k,l} = \langle V, [k], [l], E \rangle$$

and at the triples $(A, F_1, F_2)$ with $A \subseteq E$, $F_1 : V \to [k]$ whose interpretation depends on $A$, and $F_2 : A \to [l]$ such that, simultaneously, for $(u,v) \in A \to F_1(u) = F_1(v)$. This is expressed in the formula $Tutte(A, F_1, F_2)$. Now we have

$$\chi_{Tutte(A,F_1,F_2)}(G,k,l) = \sum_{A \subseteq E} k^{conn(A)} l^{|A|}$$

which is the evaluation of $Z(G,q,v)$ for $q = k, v = l$.

# 4 SOL-definable graph polynomials

## 4.1 SOL($\tau$)-polynomials

We are now ready to introduce the **SOL**-definable polynomials. Let $\mathcal{R}$ be a commutative semi-ring, which contains the semi-ring of the integers $\mathbb{N}$. For our discussion $\mathcal{R} = \mathbb{N}$ or $\mathcal{R} = \mathbb{Z}$ suffices, but the definitions generalize. Our polynomials have a fixed finite set of variables (indeterminates, if we distinguish them from the variables of **SOL**), $\mathbf{X}$.

**Definition 6 (SOL-monomials).** *Let $\mathcal{M}$ be a $\tau$-structure. We first define the* **SOL**-*definable* $\mathcal{M}$-monomials. *inductively.*

*(i) Elements of $\mathbb{N}$ are* **SOL**-*definable $\mathcal{M}$-monomials.*
*(ii) Elements of* **X** *are* **SOL**-*definable $\mathcal{M}$-monomials.*
*(iii) Finite products of monomials are* **SOL**-*definable $\mathcal{M}$-monomials.*
*(iv) Let $\phi(\bar{a})$ be a $\tau \cup \{\bar{a}\}$-formula in* **SOL***, where $\bar{a} = (a_1, \ldots, a_m)$ is a finite sequence of constant symbols not in $\tau$. Let $t$ be a $\mathcal{M}$-monomial. Then*

$$\prod_{\bar{a}:\langle \mathcal{M}, \bar{a}\rangle \models \phi(\bar{a})} t$$

*is a* **SOL**-*definable $\mathcal{M}$-monomial.*

*The polynomial $t$ may depend on relation or function symbols occurring in $\phi$.*

We note that the degree of a $\mathcal{M}$-monomial is polynomially bounded by the cardinality of $\mathcal{M}$.

**Definition 7 (SOL-polynomials).** *The $\mathcal{M}$-polynomials definable in* **SOL** *are defined inductively:*

*(i) $\mathcal{M}$-monomials are* **SOL**-*definable $\mathcal{M}$-polynomials.*
*(ii) Let $\phi(\bar{a})$ be a $\tau \cup \{\bar{a}\}$-formula in* **SOL** *where $\bar{a} = (a_1, \ldots, a_m)$ is a finite sequence of constant symbols not in $\tau$. Let $t$ be a $\mathcal{M}$-polynomial. Then*

$$\sum_{\bar{a}:\langle \mathcal{M}, \bar{a}\rangle \models \phi(\bar{a})} t$$

*is a* **SOL**-*definable $\mathcal{M}$-polynomial.*
*(iii) Let $\phi(\bar{R})$ be a $\tau \cup \{\bar{R}\}$-formula in* **SOL** *where $\bar{R} = (R_1, \ldots, R_m)$ is a finite sequence of relation symbols not in $\tau$. Let $t$ be a $\mathcal{M}$-polynomial definable in* **SOL**. *Then*

$$\sum_{\bar{R}:\langle \mathcal{M}, \bar{R}\rangle \models \phi(\bar{R})} t$$

*is a* **SOL**-*definable $\mathcal{M}$-polynomial.*

*The polynomial $t$ may depend on relation or function symbols occurring in $\phi$.*

An $\mathcal{M}$-polynomial $p_{\mathcal{M}}(\mathbf{X})$ is an expression with parameter $\mathcal{M}$. The family of polynomials, which we obtain from this expression by letting $\mathcal{M}$ vary over all $\tau$-structures, is called, by abuse of terminology, a $\mathbf{SOL}(\tau)$-polynomial.

Among the **SOL**-definable polynomials we find most of the known graph polynomials from the literature.

### 4.2 Properties of SOL-definable polynomials

The proofs of the following properties are in Appendix A.

**Proposition 5.** *(i) If we write an* **SOL***-definable polynomial as a sum of monomials, then the coefficients of the monomials are in* $\mathbb{N}$.

*(ii) (Normal Form, Lemma 2) Let* $\Psi(\mathcal{M})$ *be an* **SOL***-monomial viewed as a polynomial. Then* $\Psi(\mathcal{M})$ *is a product of a finite number* $s$ *of terms of the form* $\prod_{\bar{a}:\langle \mathcal{M},\bar{a}\rangle \models \phi_i} t_i$, *where* $i \in [s]$, $t_i \in \mathbb{N} \cup \mathbf{X}$ *and* $\phi_i \in \mathbf{SOL}$.

*(iii) (Proposition 6) The product of two* $\mathbf{SOL}(\tau)$*-polynomials is again a* $\mathbf{SOL}(\tau)$*-polynomial.*

*(iv) (Proposition 7) The sum of two* $\mathbf{SOL}(\tau)$*-polynomials is again a* $\mathbf{SOL}(\tau)$*-polynomial.*

*(v) (Proposition 8) Let* $P : Str(\tau) \to \mathbb{N}[\bar{X}]$ *be of the form*

$$P(\mathcal{M}, \bar{X}) = \sum_{\bar{R}:\langle \mathcal{M},\bar{R}\rangle \models \chi_R} \prod_{\bar{b}:\langle \mathcal{M},\bar{R},\bar{b}\rangle \models \psi} \sum_{\bar{a}:\langle \mathcal{M},\bar{R},\bar{a},\bar{b}\rangle \models \phi} \Phi(\langle \mathcal{M}, R, \bar{a}, \bar{b}\rangle, \bar{X}),$$

*where* $\Phi(\mathcal{A}, \bar{X})$ *is a* **SOL***-definable monomial, then* $P(\mathcal{M}, \bar{X})$ *is a* **SOL***-definable polynomial.*

### 4.3 Combinatorial polynomials

As for the generalized chromatic polynomials, it is note-worthy to see which combinatorial polynomials are **SOL**-definable polynomials. The following are all **SOL**-definable generalized chromatic polynomials. We denote by $card_{\mathcal{M},\bar{v}}(\varphi(\bar{v}))$ the number of $\bar{v}$'s that satisfy $\varphi$.

**Cardinality, I:** The cardinality of a definable set $card_{\mathcal{M},\bar{v}}(\varphi(\bar{v})) = \sum_{v \in \varphi(v)} 1$ is an evaluation of a **SOL**-definable polynomial.

**Cardinality, II:** The cardinality as the exponent in a monomial $X^{card_{\mathcal{M},\bar{v}}(\varphi(\bar{v}))} = \prod_{v:\varphi(v)} X$ is an **SOL**-definable polynomial.

**Cardinality, III:** Exponentiation of cardinalities $card_{\mathcal{M},\bar{v}}(\varphi(\bar{v}))^{card_{\mathcal{M},\bar{v}}(\psi(\bar{v}))} = \prod_{v:\psi(v)} \sum_{u:\varphi(u)} 1$ is equivalent to an evaluation of a **SOL**-definable polynomial by proposition 8.

**Factorials:** The factorial of the cardinality of a definable set $card_{\mathcal{M},\bar{v}}(\varphi(\bar{v}))! = \sum_{\pi:\varphi(v)\to\varphi(v)} 1$ is an evaluation of a **SOL**-definable polynomial.

**Binomial coefficients:** The binomial coefficient $\binom{X}{card_{\mathcal{M},\bar{v}}(\varphi(\bar{v}))} = \frac{(X)_{|\varphi|}}{|\varphi|!}$ is not an evaluation of a **SOL**-definable polynomial. It contains terms involving division, which contradicts Proposition 5

**Falling factorial:** The falling factorial $(X)_{card_{\mathcal{M},\bar{v}}(\varphi(\bar{v}))} = \binom{X}{|\varphi|} \cdot |\varphi|!$ is not a **SOL**-definable polynomial, because it contains negative terms, which contradicts Proposition 5 However, if the underlying structure has a linear order, then it is an evaluation of an **SOL**-definable polynomial.

In the next sub-section we will show how the **SOL**-definable polynomials are related to the **SOL**-definable generalized chromatic polynomials via the addition of $\binom{X}{card_{\mathcal{M},\bar{v}}(\varphi(\bar{v}))}$. This example shows the motivation for this addition, as currently the **SOL**-definable polynomials are not expressible enough to include basic combinatorial functions.

### 4.4 Extended SOL($\tau$)-polynomials

Motivated by the discussion in Subsection 4.3 we define now the *extended* **SOL**-*polynomials.*

**Definition 8 (Extended SOL-polynomials).**

*(i) For every $\phi(\bar{v}) \in \mathbf{SOL}(\tau)$ we define the cardinality of the set defined by $\phi$:*

$$card_{\mathcal{M},\bar{v}}(\phi(\bar{v})) = \mid \{\bar{a} \in M^m : \langle \mathcal{M}, \bar{a} \rangle \models \phi(\bar{a})\} \mid .$$

*(ii) The* extended $\mathbf{SOL}(\tau)$-monomials *are defined inductively as:*

*(ii.a)* $\mathbf{SOL}(\tau)$*-monomials are extended* $\mathbf{SOL}(\tau)$*-monomials.*

*(ii.b) For every $\phi(\bar{v}) \in \mathbf{SOL}(\tau)$ and for every $X \in \mathbf{X}$,*

$$\binom{X}{card_{\mathcal{M},\bar{v}}(\phi(\bar{v}))}$$

*is an extended* $\mathbf{SOL}(\tau)$*-monomial.*

*(ii.c) Finite product of extended* $\mathbf{SOL}(\tau)$*-monomials are extended* $\mathbf{SOL}(\tau)$*-monomial.*

*Note that $\binom{X}{card_{\mathcal{M},\bar{v}}}$ may not occur within the scope of $\prod$.*

*(iii) The extended* $\mathbf{SOL}(\tau)$*-polynomials are defined as in definition 7 with respect to extended* $\mathbf{SOL}(\tau)$*-monomials.*

*(iv) Similarly, we define also* extended **MSOL**-polynomials.

We note that all of the combinatorial functions from sub-section 4.3 are expressible by extended **SOL**-definable polynomials.

## 5 Proof of Theorems B and C

To prove Theorem B we proceed by induction, proving it first for **SOL**-monomials, and then we use the normal form for **SOL**-polynomials (Proposition 5). To prove Theorem C we code the color sets inside the graph. Using that the coloring is bounded, we can code the color sets in a fixed Cartesian product $M^{d \cdot m}$ of the domain of the structure $\mathcal{M}$. The details are spelled out in Appendix B.

## 6 Conclusions

Starting with the classical chromatic polynomial we have introduced generalized multi-colorings. We have shown that the corresponding counting functions are always polynomials, which we called generalized chromatic polynomials.

We have then shown that the class of generalized chromatic polynomials is very rich and covers virtually all examples of graph polynomials which have been studied in the literature.

We presented the class of **SOL**-definable graph polynomials which were introduced in [Mak04], and extended them by allowing generalized binomial coefficients as monomials. We have then shown that the class of extended **SOL**-definable graph polynomials coincides with the class of **SOL**-definable generalized chromatic polynomials. This, along with the extensive scope of the class, suggests that the frameworks presented in this paper are natural to the study of graph polynomials.

Theorems B and C can also be used to analyze the complexity of evaluations of **SOL**-definable polynomials at integer points. They fit nicely into the framework developed by S. Toda in his unpublished thesis and in [TW92].

Generalized chromatic polynomials first occurred in model theory, in Zilber's study of $\omega$-stable $\aleph_0$-categorical theories [Zil84a,Zil84b,Zil93]. The connection to our present work is described in [MZ06].

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# A SOL-definable graph polynomials

## A.1 Properties of SOL-definable polynomials

**Lemma 2 (Normal Form).** *Let $\Psi(\mathcal{M})$ be a* **SOL**−*monomial. Then $\Psi(\mathcal{M})$ is a product of a finite number $s$ of terms of the form $\prod_{\bar{a}:\langle\mathcal{M},\bar{a}\rangle\models\phi_i} t_i$, where $i \in [s]$, $t_i \in \mathbb{N} \cup \mathbf{X}$ and $\phi_i \in SOL$.*

*Proof. By induction:*

*(i) Let $i \in \mathbb{N}$ and $\Psi(\mathcal{M}) = i$. Then $\Psi(\mathcal{M}) = \prod_{\bar{a}:\bar{a}\subseteq[n]^0} i$.*
*(ii) Let $\Psi(\mathcal{M})$ be a finite product of monomials, then clearly it is of the desired form.*
*(iii) Let $X \in \mathbf{X}$ and $\Psi(\mathcal{M}) = X$. Then $\Phi(\mathcal{M}) = \prod_{\bar{a}:\bar{a}\subseteq[n]^0} X$.*
*(iv) Let $\phi(\mathcal{M},\bar{a})$ be a SOL formula. Let $t(\bar{b})$ be a $\mathcal{M}$−monomial. Let $\Psi(\mathcal{M}) = \prod_{\bar{b}:\langle\mathcal{M},\bar{b}\rangle\models\phi_b} t(\bar{b})$. Then by the induction hypothesis,*

$$\Psi(\mathcal{M}) = \prod_{\bar{b}:\langle\mathcal{M},\bar{b}\rangle\models\phi_b}\left(\prod_{\bar{a}_1:\langle\mathcal{M},\bar{b},\bar{a}_1\rangle\models\phi_1} t_1 \cdots \prod_{\bar{a}_s:\langle\mathcal{M},\bar{b},\bar{a}_s\rangle\models\phi_s} t_s\right) =$$
$$= \prod_{\bar{b}:\langle\mathcal{M},\bar{b}\rangle\models\phi_b}\ \prod_{\bar{a}_1:\langle\mathcal{M},\bar{b},\bar{a}_1\rangle\models\phi_1} t_1 \cdots \prod_{\bar{b}:\langle\mathcal{M},\bar{b}\rangle\models\phi_b}\ \prod_{\bar{a}_s:\langle\mathcal{M},\bar{b},\bar{a}_s\rangle\models\phi_s} t_s.$$

*We note*

$$\prod_{\bar{b}:\langle\mathcal{M},\bar{b}\rangle\models\phi_b}\ \prod_{\bar{a}_i:\langle\mathcal{M},\bar{b},\bar{a}_i\rangle\models\phi_i} t_i = \prod_{\bar{a}_i,\bar{b}:\langle\mathcal{M},\bar{a}_i,\bar{b}\rangle\models\phi_b\wedge\phi_i} t_i,$$

*so $\Psi(\mathcal{M})$ is of the desired form.*

**Proposition 6.** *The product of two* **SOL**$(\tau)$*-polynomials is again a* **SOL**$(\tau)$*-polynomial.*

*Proof.* Let $P_1(\mathcal{M})$ and $P_2(\mathcal{M})$ be **SOL**−definable polynomials. Every **SOL**−definable polynomial is of the form

$$\sum_{R_1:\phi_1(R_1)} \cdots \sum_{R_s:\phi_s(R_s)} \Phi(\mathcal{M},\bar{R}),$$

where $\Phi(\mathcal{M},\bar{R})$ is a monomial. Without loss of generality, we may assume $P_1$ and $P_2$ have the same number of sums (by adding sums of the form $\sum_{U:U=\emptyset}$).
**Base:** $P_1(\mathcal{M}) \cdot P_2(\mathcal{M})$ is a $SOL$−polynomial.
**Step:** For $i \in \{1,2\}$, let $P_i(\mathcal{M}) = \sum_{R_i:\phi_i(R_i)} \Phi_i(\langle\mathcal{M},R_i\rangle)$. Then $P_1(\mathcal{M}) \cdot P_2(\mathcal{M}) = \sum_{R_1,R_2:\phi_1(R_1)\wedge\phi_2(R_2)} \Phi_1(\langle\mathcal{M},R_1\rangle)\cdot\Phi_2(\langle\mathcal{M},R_2\rangle)$. By the induction hypothesis, this is a $SOL$−polynomial.

**Proposition 7.** *The sum of two* **SOL**$(\tau)$*-polynomials is again a* **SOL**$(\tau)$*-polynomial.*

*Proof.* Let $P_1(\mathcal{M})$ and $P_2(\mathcal{M})$ be as in the previous proposition.
**Base:** Let $P_1(\mathcal{M})$ and $P_2(\mathcal{M})$ be either:

(i) $P_i(\mathcal{M}) = \sum_{R_i:\phi(R_i)} \Phi_i(\mathcal{M}, R_i)$, where $\Phi_i$ is a monomial, or
(ii) $P_i(\mathcal{M}) = \Phi_i(\mathcal{M})$, where $\Phi_i$ is a monomial.

We may assume w.l.o.g that $P_i$ is of the first form, since it holds that $\Phi_i(\mathcal{M}) = \sum_{S:S=\emptyset} \Phi_i(\mathcal{M})$. We set

$$P(\mathcal{M}) = \sum_{S:S\in\{\emptyset,M\}} \sum_{R_1,R_2:\theta_{ch}(R_1,R_2)} \left(P'_1(\mathcal{M}, S) \cdot P'_2(\mathcal{M}, S)\right),$$

where

$$\begin{aligned}\theta_{ch}(R_1, R_2) = {} & (\phi_1(R_1) \wedge (S = \emptyset) \wedge (R_2 = \emptyset)) \vee \\ & (\phi_2(R_2) \wedge (S = M) \wedge (R_1 = \emptyset)),\end{aligned}$$

and $P'_1(\mathcal{M}, S)$ respectively $P'_2(\mathcal{M}, S)$ are the corresponding $P_i(\mathcal{M})$ with every formula $\phi$ replaced with $\phi \wedge (S = \emptyset)$ respectively $\phi \wedge (S = M)$. Then $P(\mathcal{M}) = P_1(\mathcal{M}) + P_2(\mathcal{M})$.
**Step:** For $i \in \{1, 2\}$, let $P_i(\mathcal{M}) = \sum_{R_i:\phi_i(R_i)} \Phi_i(\mathcal{M}, R_i)$ such that $\Phi_i(\mathcal{M}, R_i) = \sum_{R'_i:\psi_i(R'_i)} \Theta_i(\mathcal{M}, R_i, R'_i)$.
Let $\Phi_i^{S,A}(\mathcal{M}, R_i, S) = \sum_{R'_i:\psi_i(R'_i)\wedge(S=A)} \Theta_i(\mathcal{M}, R_i, R'_i)$. We note

$$\Phi_i^{S,A}(\mathcal{M}, R_i, S) = \begin{cases} \Phi_i(\mathcal{M}, R_i) & S = A \\ 0 & S \neq A \end{cases}.$$

Set

$$P(\mathcal{M}) = \sum_{S:S\in\{\emptyset,M\}} \sum_{R_1,R_2:\theta_{ch}} \left(\Phi_1^{S,\emptyset}(\mathcal{M}, R_1, S) + \Phi_2^{S,M}(\mathcal{M}, R_2, S)\right).$$

By the induction hypothesis, $\Phi_1^{S,\emptyset} + \Phi_2^{S,M}$ is a **SOL**−polynomial. So, the claim holds.

**Proposition 8.** *Let $P : Str(\tau) \to \mathbb{N}[\bar{X}]$ be of the form*

$$P(\mathcal{M}, \bar{X}) = \sum_{\bar{R}:\langle\mathcal{M},\bar{R}\rangle\models\chi_R} \prod_{\bar{b}:\langle\mathcal{M},\bar{R},\bar{b}\rangle\models\psi} \sum_{\bar{a}:\langle\mathcal{M},\bar{R},\bar{a},\bar{b}\rangle\models\phi} \Phi(\langle\mathcal{M}, R, \bar{a}, \bar{b}\rangle, \bar{X}),$$

*where $\Phi(\mathcal{A}, \bar{X})$ is a* **SOL**-*definable monomial, then $P(\mathcal{M}, \bar{X})$ is a SOL-definable polynomial.*

*Proof.* We note by expanding the product

$$\prod_{\bar{b}:\langle\mathcal{A},\bar{b}\rangle\models\psi} \sum_{\bar{a}:\langle\mathcal{A},\bar{a},\bar{b}\rangle\models\phi} \Phi(\mathcal{A}, \bar{X}) = \sum_{f:\mathcal{A}\models\vartheta} \prod_{\bar{a},\bar{b}:\langle\mathcal{A},\bar{a},\bar{b}\rangle\models\varphi} \Phi(\mathcal{A}, \bar{X}),$$

where $\vartheta$ says $f$ is a function $f : \{\bar{b} \mid \langle\mathcal{A}, \bar{b}\rangle \models \psi\} \to \{\bar{a} \mid \langle\mathcal{A}, \bar{a}, \bar{b}\rangle \models \phi\}$, and $\varphi = \left(f(\bar{b}) = \bar{a}\right) \wedge \psi \wedge \phi$. So the proposition holds.

By induction the last proposition holds for functions defined by alternating $\prod$ and $\sum$, as long as all $\sum$ within the scope of a $\prod$ iterate over elements (and not over relations).

## B Proof of Main Theorems

### B.1 Proof of Theorem B

We now turn to prove the equivalence of extended **SOL**−definable polynomials and **SOL**-definable generalized chromatic polynomials. For the proof we need a normal form lemma for **SOL**-definable polynomials.

We now turn to prove Theorem B for **SOL**-definable polynomials, first for monomials only, then for polynomials.

**Lemma 3.** *Every extended* **SOL**−*monomial* $\Psi(\mathcal{M})$ *is a* **SOL**−*definable generalized chromatic polynomial.*

*Proof. The binomial coefficients are generalized chromatic polynomials as shown in Subsection 3.7. Next, we look at a term* $\prod_{\bar{a}:\langle \mathcal{M},\bar{a}\rangle \models \phi} t$ *in the finite product that* $\Psi(\mathcal{M})$ *consists of, as guaranteed by the normal form lemma. For* $t = c \in \mathbb{N}$ *we note*

$$\prod_{\bar{a}:\langle \mathcal{M},\bar{a}\rangle \models \phi} c = |\{F \mid F : \{\bar{a} : \langle \mathcal{M},\bar{a}\rangle \models \phi\} \to [c]\}| = \chi_{\theta(R_1 \dots R_{\lceil \log c \rceil})},$$

*where* $\theta(\bar{R}) = (\bigwedge_i \forall \bar{a} \ (R_i(\bar{a}) \to \phi(\bar{a}))) \wedge \left(\forall \bar{a} \quad \phi_{<c_0}(\bar{a},\bar{R})\right)$, *and* $\phi_{<c}(\bar{a},\bar{R})$ *says the vector* $f_{R_1}(\bar{a}) \dots f_{R_{\lceil \log c \rceil}}(\bar{a})$ *is in* $[c]$, *where* $f_{R_i}$ *is the characteristic function of* $R_i$. *We note also that*

$$\prod_{\bar{a}:\langle \mathcal{M},\bar{a}\rangle \models \bar{a}} X_i = \chi_{\psi(F)},$$

*where* $\psi(F) =$ "$F : \{\bar{a} : \phi(\bar{a})\} \to [X_i]$". *The lemma follows by the closure of* **SOL**−*definable generalized chromatic polynomial under finite product.*

**Theorem B** *Every extended* **SOL**$(\tau)$*-polynomial over some* $\tau$*-structure* $\mathcal{A}$ *is a counting function of a generalized coloring definable in* **SOL**$(\tau)$ *in a suitable expansion of* $\mathcal{A}$.

*Proof.* Let $\Psi(\mathcal{M}) = \sum_{\bar{R}:\langle \mathcal{M},\bar{R}\rangle \models \phi(\bar{R})} t(\bar{R})$, where $t(\bar{R})$ is a **SOL**-polynomial, and therefore a **SOL**-definable generalized chromatic polynomial, $t(\bar{R}) = \chi_{\theta(\bar{S})}(\langle \mathcal{M},\bar{R}\rangle, \bar{k})$. Then $\Psi(\mathcal{M}) = \chi_{\theta(\bar{S},\bar{R}) \wedge \phi(\bar{R})}(\mathcal{M})$. Similarly for sums over tuples of elements. The theorem follows from the previous lemma.

### B.2 Proof of Theorem C

Now we can prove the converse of Theorem B.

**Theorem C** *Every generalized chromatic polynomial definable in* **SOL** *on ordered* $\tau$*-structures* $\mathcal{M}$ *is an extended* **SOL***-definable polynomial.*

*Proof.* From Theorem 1 we know that for every $\mathcal{M}$ the number of elements given by $\chi_{\phi(R)}(\mathcal{M}, k)$ is a polynomial in $k$ of the form

$$\sum_{j=0}^{d\cdot|M|^m} c_{\phi(R)}(\mathcal{M}, j)\binom{k}{j}$$

where $c_{\phi(R)}(\mathcal{M}, j)$ is the number of generalized $k - \phi$-colorings $R$ with a fixed set of $j$ colors. Furthermore the total number of colors used is bounded by $N = d\cdot \mid M \mid^m$. By assumption $\mathcal{M}$ has a linear order $<_M$. Therefore the lexicographic order on $M^{d\cdot m}$ is definable in **SOL**. We interpret the set of colors used inside $\mathcal{M}$ by the set $M^{d\cdot m}$.

We replace $R$ by a new relation where each occurrence of a color is replace by a $(d \cdot m)$-tuple, and call this new relation $S$. We also modify the formula $\phi$ to a formula $\psi$ by adding the requirement that all the colors used by $S$ form an initial segment in the lexicographic order. Let us denote by $I_S$ the initial segment of this lexicographic ordering of the colors used by $S$. Clearly $I_S$ is definable in **SOL** in the expanded structure.

The extended **SOL**-polynomial we look for now is

$$\sum_{S:\psi(S)} \binom{\lambda}{card(I_S)}.$$