An extension of the bivariate chromatic polynomial

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K.Dohmen, A.Pönitz and P.Tittman (2003), introduced a bivariate generalization of the chromatic polynomial $P(G, x, y)$ which subsumes also the independent set polynomial of I. Gutman and F. Harary, (1983) and the vertex-cover polynomial of F.M. Dong, M.D. Hendy, K.T. Teo and C.H.C. Little (2002). We first show that $P(G, x, y)$ has a recursive definition with respect to three kinds of edge elimination: edge deletion, edge contraction, and and edge extraction, i.e. deletion of an edge together with its end points. Like in the case of deletion and contraction only (J.G. Oxley and D.J.A. Welsh 1979) it turns out that there is a most general, or as they call it, a universal polynomial satisfying such a recurrence relations with respect to the three kinds of edge elimations, which we call $\xi(G, x, y, z)$. We show that the new polynomial simultaneously generalizes, $P(G, x, y)$, as well as the Tutte polynomial and the matching polynomial, We also give an explicit definition of $\xi(G, x, y, z)$ using a subset expansion formula. We also show that $\xi(G, x, y, z)$ can be viewed as a partition function, using counting of weighted graph homomorphisms. Furthermore, we expand this result to edge-labeled graphs as was done for the Tutte polynomial by T. Zaslavsky (1992) and by B. Bollobas and O. Riordan (1999). The edge labeled polynomial $\xi_{lab}(G, x, y, z, \bar{t})$ also generalizes the chain polynomial of R.C. Read and E.G. Whitehead Jr. (1999). Finally, we discuss the complexity of computing $\xi(G, x, y, z)$.

1. Introduction

There are several well-studied graph polynomials, among them the chromatic polynomial, [5, 20, 12], different versions of the Tutte polynomial, [6, 7, 32], and of the matching polynomial, [21, 24, 20], which are known to satisfy certain linear recurrence relations with respect to *deletion* of an edge, *contraction* of an edge, or deletion of an edge together with its endpoints, which we call *extraction* of an edge. The generalization of the chromatic polynomial, which was introduced by K.Dohmen, A.Pönitz and P.Tittman in [13],

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happens to satisfy such recurrence relation as well. The question that arises is, what is the most general graph polynomial that satisfies similar linear recurrence relation.

Our investigation is motivated by the approach of J.G. Oxley and D.J.A. Welsh, [30], who define a universal graph polynomial in five variables $U(G; x, y, \alpha, \sigma, \tau)$ which satisfies a recurrence relation based on edge deletion and edge contraction, for which they show, that up to a simple prefactor, the resulting polynomial is the Tutte polynomial. The existence of such a polynomial is called the *Universality Property* of the Tutte polynomial. Here we shall look for a universal polynomial with respect to a recurrence relation based on three edge elimination operations: edge deletion, edge contraction and edge extraction.

In this paper all the graphs are unlabeled unless it is explicitly mentioned; multiple edges and self loops are allowed. We denote by $G = (V, E)$ the graph with vertex set V and edge set E.

A *graph invariant* is a function from the class of (finite) graphs $\mathcal{G}$ into some domain $\mathcal{D}$ such that isomorphic graphs have the same picture. A *graph polynomial* is a graph invariant, which has a polynomial ring $\mathbb{Z}[\bar{X}]$, or, more generally, any commutative ring $\mathcal{R}[\bar{X}]$, over some (not necessarily finite) set of indeterminates $\bar{X}$, as its domain. Formally, the graph polynomial p is defined as $p : \mathcal{G} \mapsto \mathcal{R}[\bar{X}]$.

1.1. Recursive definition of graph polynomials

Edge elimination. We define three basic edge elimination operations on multigraphs:

- *Deletion.* We denote by G_{-e} the graph obtained from G by simply removing the edge e.
- *Contraction.* We denote by $G_{/e}$ the graph obtained from G by unifying the endpoints of e. Note that this operation can cause production of multiple edges and self loops.
- *Extraction.* We denote by $G_{\dagger e}$ the graph induced by $V \setminus \{u, v\}$ provided $e = \{u, v\}$. Note that this operation removes also all the edges adjacent to e.

With respect to these three operations, together with *disjoint union* of two graphs $G_1 \oplus G_2$ and with *initial conditions* defined for an *empty set* $\emptyset$ (graph without vertices) and for a single point E_1, we recall the known recursive definitions of graph polynomials:

The matching polynomial. There are different versions of the matching polynomial discussed in the literature, for example *matching generating polynomial* $g(G, \lambda) = \sum_{i=0}^{n} a_i \lambda^i$ and *matching defect polynomial* $\mu(G, \lambda) = \sum_{i=0}^{n} (-1)^i a_i \lambda^{n-2i}$, where $n = |V|$ and a_i is the number of i-matchings in G. We shall use the bivariate version that incorporates the both above:

$$M(G, x, y) = \sum_{i=0}^{n} a_i x^{n-2i} y^i \tag{1.1}$$

The recursive definition of the matching satisfies the initial conditions $M(E_1) = x$ and $M(\emptyset) = 1$, and the recurrence relations

$$\begin{aligned} M(G) &= M(G_{-e}) + y \cdot M(G_{\dagger e}) \\ M(G_1 \oplus G_2) &= M(G_1) \cdot M(G_2) \end{aligned} \tag{1.2}$$

The Tutte polynomial. We recall the definition of classical two-variable Tutte polynomial (cf. for example B.Bollobás [6]):

Definition 1. Let $G = (V, E)$ be a (multi-)graph. Let $A \subseteq E$ be a subset of edges. We denote by $k(A)$ the number of connected components in the spanning subgraph (V, A). Then two-variable Tutte polynomial is defined as follows

$$T(G, x, y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)} (y-1)^{|A|+k(A)-|V|} \tag{1.3}$$

The Tutte polynomial satisfies the initial conditions $T(E_1) = 1$ $T(\emptyset) = 1$ and has linear recurrence relation with respect to the operations above:

$$\begin{aligned} T(G, x, y) &= \begin{cases} x \cdot T(G_{-e}, x, y) & \textit{if } e \textit{ is a bridge,} \\ y \cdot T(G_{-e}, x, y) & \textit{if } e \textit{ is a loop,} \\ T(G_{-e}, x, y) + T(G_{/e}, x, y) & \textit{otherwise} \end{cases} \\ T(G_1 \oplus G_2, x, y) &= T(G_1, x, y) \cdot T(G_2, x, y) \end{aligned} \tag{1.4}$$

However, we shall use in this paper the version of the Tutte polynomial used by A.Sokal [32], known as the (bivariate) *partition function* of the Pott's model:

$$Z(G, q, v) = \sum_{A \subseteq E} q^{k(A)} v^{|A|} \tag{1.5}$$

The partition function of the Pott's model is co-reducible to the Tutte polynomial via

$$T(G, x, y) = (x-1)^{-k(E)} (y-1)^{-|V|} Z(G, (x-1)(y-1), y-1). \tag{1.6}$$

It satisfies the initial conditions $Z(E_1) = q$ and $Z(\emptyset) = 1$, and satisfies a recurrence relation which does not distinguish whether the edge e is a loop, a bridge, or none of the two:

$$\begin{aligned} Z(G, q, v) &= v \cdot Z(G_{/e}, q, v) + Z(G_{-e}, q, v) \\ Z(G_1 \oplus G_2, q, v) &= Z(G_1, q, v) \cdot Z(G_2, q, v) \end{aligned} \tag{1.7}$$

The bivariate chromatic polynomial. K.Dohmen, A.Pönitz and P.Tittman in [13] introduced a polynomial $P(G, x, y)$ that extends the classical chromatic polynomial by splitting the available x colors into y colors for proper and $x - y$ colors for arbitrary colorings.

Proposition 1. *The polynomial $P(G, x, y)$ satisfies the initial conditions $P(E_1) = x$ and $P(\emptyset) = 1$, and the following recurrence relation:*

$$\begin{aligned} P(G, x, y) &= P(G_{-e}, x, y) - P(G_{/e}, x, y) + (x-y) \cdot P(G_{\dagger e}, x, y) \\ P(G_1 \oplus G_2, x, y) &= P(G_1, x, y) \cdot P(G_2, x, y) \end{aligned} \tag{1.8}$$

The proof is given in Section 2.

1.2. A universal edge elimination polynomial

Recursive definition. Inspired by the characterization of the Tutte polynomial given in [30], see also [6], Theorem 2 of Chapter 10, we look for the most general linear recurrence relation[1], which can be obtained on unlabeled graphs by introducing new variables, and which does not distinguish between local properties of the edge e which is to be eliminated[2]. To assure that the polynomial so defined is unique, we have to prove that its definition is not dependent on the order in which the edges are removed.

We start with the initial conditions $\xi(E_1) = x$ and $\xi(\emptyset) = 1$, and recurrence relation

$$\begin{aligned} \xi(G) &= w \cdot \xi(G_{-e}) + y \cdot \xi(G_{/e}) + z \cdot \xi(G_{\dagger e}) \\ \xi(G_1 \oplus G_2) &= \xi(G_1) \cdot \xi(G_2) \end{aligned} \tag{1.9}$$

We prove:

Theorem 2 (Universality Property). *The recurrence relation (1.9) defines for each graph G a unique polynomial $\xi(G)$ if and only if one of the following conditions are satisfied:*

$$z = 0 \tag{1.10}$$

$$w = 1 \tag{1.11}$$

Under condition (1.11), which allows a more general graph polynomial to be obtained, the recurrence relation (1.9) is restricted to

$$\begin{aligned} \xi(G, x, y, z) &= \xi(G_{-e}, x, y, z) + y \cdot \xi(G_{/e}, x, y, z) + z \cdot \xi(G_{\dagger e}, x, y, z) \\ \xi(G_1 \oplus G_2, x, y, z) &= \xi(G_1, x, y, z) \cdot \xi(G_2, x, y, z) \\ \xi(E_1, x, y, z) &= x; \\ \xi(\emptyset, x, y, z) &= 1; \end{aligned} \tag{1.12}$$

Remark 3. *From this theorem one sees immediately that $\xi(G, x, y, z)$ gives, by choosing appropriate values for the variables and simple prefactors, the partition function of the Pott's model, the bivariate matching polynomial and the bivariate chromatic polynomial with all their respective substitution instances, including the classical chromatic polynomial, the Tutte polynomial and the independent set polynomial, [10, 18]. The latter two polynomials are already substitution instances of the bivariate chromatic polynomial $P(G, x, y)$ of [13]. Table 1 summarizes these observations.*

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[1] The first paper to study general conditions under which linear recurrence relations define a graph invariant is D.N.Yetter [35]

[2] It is conceivable that recurrence relations with various case distinctions depending on local properties of e and more variables give other universal polynomials.

Table 1. Some substitution instances of $\xi(G, x, y, z)$

Pott's model	$Z(G, q, v) = \xi(G, q, v, 0)$
Bivariate Tutte polynomial	$T(G, x, y) = (x-1)^{-k(E)} \cdot (y-1)^{-\lvert V\rvert} \cdot$ $\xi\left(G, (x-1)(y-1), (y-1), 0\right)$
Bivariate matching polynomial	$M(G, x, y) = \xi(G, x, 0, y)$
Bivariate chromatic polynomial	$P(G, x, y) = \xi(G, x, -1, x-y)$

Subset expansion of $\xi(G, x, y, z)$. We now give an explicit form of the polynomial $\xi(G, x, y, z)$ using 3-partition edge expansion[3]:

Theorem 4. *Let $G = (V, E)$ be a (multi)graph. Then the edge elimination polynomial $\xi(G, x, y, z)$ can be calculated as*

$$\xi(G, x, y, z) = \sum_{(A \sqcup B) \subseteq E} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)} \tag{1.13}$$

where by abuse of notation we use $(A \sqcup B) \subseteq E$ for summation over subsets $A, B \subseteq E$, such that the subsets of vertices $V(A)$ and $V(B)$, covered by respective subset of edges, are disjoint: $V(A) \cap V(B) = \emptyset$; $k(A)$ denotes the number of spanning connected components in (V, A), and $k_{cov}(B)$ denotes the number of covered connected components, i.e. the connected components of the graph $(V(B), B)$.

Remark 5. *From Theorem 4 one can see that $\xi(G, x, y, z)$ is a polynomial definable in Monadic Second Order Logic, with quantification over sets of edges ($MSOL_2$), where an order over vertices is to be used for stating "number of connected sets", but the final result is order-independent. We shall not use logic in the sequel of the paper. For details the reader is referred to [26].*

Counting weighted graph homomorphisms. Another way to define graph polynomials is counting weighted graph homomorphisms. Let the weighted graph $H = (V_H, E_H)$ be defined as follows:

- H is a clique of size $|H| = x + 2p$ with all loops.
- All the vertices of H are separated into three parts, denoted as V^A, V^B and V^I, which induce respectively cliques K_x^A, K_p^B and K_p^I, such that $|V^A| = x$, $|V^B| = p$ and $|V^I| = p$.
- The weight function $w = (\alpha, \beta)$: $\alpha : V_H \mapsto \mathbb{R}$ and $\beta : E_H \mapsto \mathbb{R}$

$$\alpha(v) = \begin{cases} 1 & if\ v \in (V^A \cup V^B) \\ -1 & otherwise \end{cases}$$

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[3] A more precise name would be a "Pair of disjoint subsets expansion". We chose the name 3-partition expansion, as any two disjoint subsets induce a partition into three sets.

$$\beta(u,v) = \begin{cases} y+1 & if\ (u=v) \wedge (u,v \in (V^A \cup V^B)) \\ 1 & otherwise \end{cases}$$

Theorem 6. *Let $Z_H(G)$ be a homomorphism function of a graph $G = (V,E)$ into a weighted graph H above.*

$$Z_H(G) = \sum_{\substack{h\,:\,V \mapsto V_H \\ homomorphism}} \prod_{v \in V} \alpha(h(v)) \prod_{(u,v)\in E} \beta(h(u),h(v))$$

Then, for all nonnegative integers x and p and all $y \in \mathbb{R}$, we have

$$\xi(G,x,y,p\cdot y) = Z_H(G)$$

The proof is given in Section 5.

Remark 7. *A general characterization of graph parameters which can be obtained from homomorphism functions by choosing appropriate weights is given in [15]. This characterization requires that the weights α are positive reals. However, in Theorem 6, we use negative values for α.*

1.3. Comparison with the weighted graph polynomial

The weighted graph polynomial $U(G,\bar{x},y)$ introduced by S.D.Noble and D.J.A.Welsh in [28] is defined for a graph $G = (V,E)$ as

$$U(G,\bar{x},y) = \sum_{A \subseteq E} y^{|A|-r(A)} \prod_{i=1}^{|V|} x_i^{s(i,A)}$$

where $s(i,A)$ denotes the number of connected components of size i in the spanning subgraph (V,A), and $r(A) = |V| - k(A)$ is the rank of (V,A).

The main difference between $U(G,\bar{x},y)$ and $\xi(G,x,y,z)$ is the number of variables, which grows in the case of U and is fixed in the case of ξ. Furthermore, in the definition of $s(i,A)$ the numeric value of the index of the variable x_i is used. This has as a consequence that one cannot freely rename the variables of U. In ξ, as well as in all graph polynomials definable in $MSOL_2$ in an order invariant way, the variables can be renamed. This allows one to show that $U(G,\bar{x},y)$ is not an $MSOL_2$-definable polynomial.

$U(G,\bar{x},y)$ also gives the Tutte polynomial and the matching polynomial as its substitution instances. One can see that the polynomial $U(G,\bar{x},y)$ distinguishes between graphs for which $\xi(G,x,y,z)$ gives the same value. As an example we look at the trees shown on Fig. 1. We do not know whether $\xi(G,x,y,z)$ can be obtained as a substitution instance of $U(G,\bar{x},y)$.

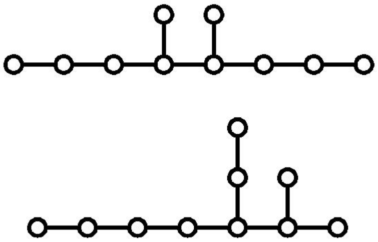

Fig. 1: Non-isomorphic trees having the same $\xi(G,x,y,z)$.

1.4. A labeled version of ξ

For edge-labeled[4] graphs we define the labeled version of our polynomial: Let $G = (V,E,c)$ be an edge-labeled multigraph s.t. $c : E \mapsto \Lambda$, where Λ is a set of labels, without any algebraic structure defined over it, and suppose that for each $\lambda \in \Lambda$ three elements (weights) w_λ, y_λ and z_λ of the domain ring are chosen. Then using the same approach as for the unlabeled version, we define a linear recurrence relation:

$$\begin{aligned} \xi_{lab}(G) &= w_{c(e)} \cdot \xi_{lab}(G_{-e}) + y_{c(e)} \cdot \xi_{lab}(G_{/e}) + z_{c(e)} \cdot \xi_{lab}(G_{\dagger e}) \\ \xi_{lab}(G_1 \oplus G_2) &= \xi_{lab}(G_1) \cdot \xi_{lab}(G_2) \\ \xi_{lab}(E_1) &= x; \\ \xi_{lab}(\emptyset) &= 1; \end{aligned} \tag{1.14}$$

Note that we do introduce labels on edges but not on vertices, as this would make the definition of the edge contraction unclear, unless we had defined an algebraic structure over Λ. For example, if Λ were a ring, we could define the label of the vertex produced by a contraction of an edge $\{u,v\}$ to be the sum of the labels of u and v. In that case, we would get a generalization of the weighted graph polynomial for labeled graphs $W(G,\bar{x},y)$, also introduced by S.D.Noble and D.J.A.Welsh in [28]. However, this polynomial is "too strong", in sense that it has the same well-definability problems as $U(G,\bar{x},y)$ discussed in Section 1.3. We prove then

Theorem 8. *Every one of the conditions*

$$\forall e \in E\left(z_{c(e)} = 0\right) \tag{1.15}$$

$$\forall e \in E\left(w_{c(e)} = 1\right) \wedge \forall e_1, e_2 \in E\left(y_{c(e_1)} z_{c(e_2)} = y_{c(e_2)} z_{c(e_1)}\right) \tag{1.16}$$

is sufficient for the recurrence relation (1.14) defining a unique graph polynomial that does not depend on the order of graph deconstruction.

Remark 9. *The conditions in Theorem 8 are not necessary. To see this, we look at a graph with two connected components and use condition (1.15) for edges in the first component and condition (1.16) for edges in the second component.*

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[4] In [7] they speak of edge-colorings rather than edge-labelings. As we also discuss chromatic polynomials we prefer our terminology as it avoids confusions.

Under the uniqueness condition (1.16), which allows a more general graph polynomial to be obtained, the recurrence relation (1.14) is restricted to

$$\begin{aligned} \xi_{lab}(G) &= \xi_{lab}(G_{-e}) + y \cdot t_{c(e)} \cdot \xi_{lab}(G_{/e}) + z \cdot t_{c(e)} \cdot \xi_{lab}(G_{\dagger e}) \\ \xi_{lab}(G_1 \oplus G_2) &= \xi_{lab}(G_1) \cdot \xi_{lab}(G_2) \\ \xi_{lab}(E_1) &= x; \\ \xi_{lab}(\emptyset) &= 1; \end{aligned} \tag{1.17}$$

where

$$\begin{aligned} y_{c(e)} &= y \cdot t_{c(e)} \\ z_{c(e)} &= z \cdot t_{c(e)} \end{aligned} \tag{1.18}$$

x, y and z are unlabeled variables, and $\bar{t}$ is the unique solution of (1.18). Like the unlabeled case, we also introduce an explicit form as a subset expansion:

Theorem 10. *The expression*

$$\xi_{lab}(G, x, y, z, \bar{t}) = \sum_{(A \sqcup B) \subseteq E} x^{k(A \sqcup B)} \left(\prod_{e \in A \sqcup B} (y t_{c(e)}) \right) \left(\frac{z}{xy} \right)^{k_{cov}(B)} \tag{1.19}$$

defines the same graph polynomial as the recurrence relation (1.17).

Note that the degree of x and y in the denominator does never exceed the degree of the respective variable in the nominator.

Remark 11. *The* labeled Sokal polynomial *[32], Zaslavsky's* normal function of the colored matroid *[36], Heilmann and Lieb's* labeled matching polynomial *[21], and the chain polynomial [31, 33] are substitution instances of $\xi_{lab}(G, x, y, z, \bar{t})$ up to a simple prefactor.*

The remainder of the paper is organized as follows: in Section 2 we prove the recurrence relation of the generalized chromatic polynomial. In Section 3 we establish the most general linear recurrence relation with respect to the three edge elimination operations, restrict it to be multiplicative and order-invariant for some specific graphs, and then prove the order-invariance of the resulting function in general. In Sections 4 and 5 we prove Theorem 4 and Theorem 6, and show that the subset expansion of Theorem 4 and and the partition function of Theorem 6 are indeed the same polynomial. The section 6 expands our results to the edge-labeled graphs. The section 7 contains examples of known graph polynomials which can be obtained as substitution instances of the edge elimination polynomial. Finally, in Section 8 we deal with the complexity of its computation.

2. The recursive definition of the generalized chromatic polynomial

Recall the definition given by K.Dohmen, A.Pönitz and P.Tittman in [13]: There are two disjoint sets of colors Y and Z; a generalized coloring of a graph $G = (V, E)$ is a map $\phi : V \mapsto (Y \sqcup Z)$ such that for all $\{u, v\} \in E$, if $\phi(u) \in Y$ and $\phi(v) \in Y$, then $\phi(u) \neq \phi(v)$ (The set Y is called therefore "proper colors"). For two positive integers $x > y$, the value of the polynomial is the number of generalized colorings by x colors, y of them are proper. To make this definition meaningful for graphs with multiple edges we require that a vertex with a self-loop can be colored only by a color in $X \setminus Y$ and that a multiple edge does not affect colorings. Let $G = (V, E)$ be a graph, and $P(G, x, y)$ be the number of generalized colorings defined above. Let $v \in V$ be any vertex. We denote by $P^v(G, x, y)$ the number of generalized colorings of G, when v is not colored by a proper color, i.e. $\phi(v) \in X \setminus Y$.

Lemma 12. *$P^v(G, x, y) = (x - y) \cdot P(G_{-v}, x, y)$, where G_{-v} denotes the subgraph of G induced by $V \setminus \{v\}$.*

Proof. By inspection: the vertex v can have any color in $X \setminus Y$, and the coloring of the remainder does not depend on it. □

Let $e = \{u, v\} \in E$ be any edge of G, which is not a self-loop and not a multiple edge. Consider the number of colorings of G_{-e}. Any such coloring is either a coloring of G, or a coloring of $G_{/e}$, when the vertex $u = v$, which is produced by the contraction, is colored by a proper color. Together with Lemma 12, that raises:

$$P(G, x, y) = P(G_{-e}, x, y) - P(G_{/e}, x, y) + (x - y) \cdot P(G_{\dagger e}, x, y) \tag{2.1}$$

One can easily check that this equation is satisfied also for loops and multiple edges. Together with the fact that a singleton can be colored by any color, and the fact that the number of colorings is multiplicative, this proves Proposition 1 in section 1.1.

3. The most general recurrence relation

We are looking for the most general *graph invariant* $\xi(G)$ which satisfies some linear recurrence relation with respect to edge deletion, edge contraction and edge extraction operations, and can be obtained by introducing new variables in this recurrence relation. Note that if the initial conditions of the recurrence relation, $\xi(\emptyset)$ and $\xi(E_1)$ are polynomials, then by induction on the number of edges, the resulting graph invariant is a graph polynomial.

Additionally, we require this graph invariant to be *multiplicative* for disjoint unions, i.e., if $G_1 \oplus G_2$ denotes disjoint union of two graphs, then the polynomial $\xi(G_1 \oplus G_2) = \xi(G_1) \cdot P(G_2)$. This is justified by the fact that the graph polynomials occurring in the literature are usually multiplicative.

From this consideration alone we obtain the initial condition and the product rule:

$$\xi(G_1 \oplus G_2) = \xi(G_1) \cdot \xi(G_2) \tag{3.1}$$

$$\xi(\emptyset) = 1; \tag{3.2}$$

Indeed, the disjoint union with an empty set gives the same graph, so the resulting function should also remain the same.

At this stage, we formulate the edge elimination rule introducing a new variable wherever we can. We set

$$\begin{aligned} \xi(G,x,y,z,t) &= t\cdot\xi(G_{-e}) + y\cdot\xi(G_{/e}) + z\cdot\xi(G_{\dagger e}) \qquad (3.3)\\ \xi(E_1,x,y,z,t) &= x; \end{aligned}$$

If we compute now this recursively defined function $\xi(G,x,y,z,t)$ using some arbitrary order of graph decomposition steps (which are elimination of edges using (3.3) and disjoint union using (3.1)), we will get a polynomial, which may depend on the used order. Recall that we are looking for an order-independent graph invariant. We shall first use the graph shown on Fig. 2 to obtain the condition necessary for our recursive definition being order-independent, and then prove that this condition is also sufficient. Let G be a graph as presented on Fig. 2. H_1 and H_2 are disjoint subgraphs of G, connected by edges e_1 and e_2.

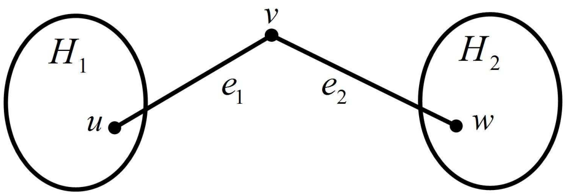

Fig. 2: Testing order invariance of edge removal.

Since we are looking for a graph invariant, we must obtain the same result by applying the edge elimination rule first on the edge e_1 and then on the edge e_2, as in case when we apply the edge elimination rule first on the edge e_2 and then on the edge e_1.

$$\begin{aligned} \xi(G) &= t\cdot\xi(G_{-e_1}) + y\cdot\xi(G_{/e_1}) + z\cdot\xi(G_{\dagger e_1}) = \\ &= t\cdot\xi(H_1)\cdot[x\cdot t\cdot\xi(H_2) + y\cdot\xi(H_2) + z\cdot\xi(H_{2-w})] + \\ &\quad y\cdot\left[t\cdot\xi(H_1)\xi(H_2) + y\cdot\xi(G_{/e_1/e_2}) + z\cdot\xi(H_{1-u})\xi(H_{2-w})\right] + \\ &\quad z\cdot\xi(H_{1-u})\xi(H_2) \qquad (3.4) \end{aligned}$$

On the other hand,

$$\begin{aligned} \xi(G) &= t\cdot\xi(G_{-e_2}) + y\cdot\xi(G_{/e_2}) + z\cdot\xi(G_{\dagger e_2}) = \\ &= t\cdot\xi(H_2)\cdot[x\cdot t\cdot\xi(H_1) + y\cdot\xi(H_1) + z\cdot\xi(H_{1-u})] + \\ &\quad y\cdot\left[t\cdot\xi(H_1)\xi(H_2) + y\cdot\xi(G_{/e_1/e_2}) + z\cdot\xi(H_{1-u})\xi(H_{2-w})\right] + \\ &\quad z\cdot\xi(H_{2-w})\xi(H_1) \qquad (3.5) \end{aligned}$$

Solving the above two equations, we get:

$$tz\cdot\xi(H_1)\xi(H_{2-w}) + z\cdot\xi(H_{1-u})\xi(H_2) = tz\cdot\xi(H_{1-u})\xi(H_2) + z\cdot\xi(H_1)\xi(H_{2-w})$$

Hence, we have the following necessary condition:

$$z = 0 \ \ or \tag{3.6}$$

$$t = 1 \ \ or \tag{3.7}$$

$$\xi(H_1)\xi(H_{2-w}) = \xi(H_{1-u})\xi(H_2) \ \ for\ any\ H_1\ and\ H_2 \tag{3.8}$$

In case of (3.8), if H_1 is a singleton, we get $x \cdot \xi(H_{2-w}) = 1 \cdot \xi(H_2)$ for any $w \in V(H_2)$, which leads to $\xi(G) = x^{|V(G)|}$.
In the case (3.6) (which also includes the trivial polynomial above), the resulting function is a substitution instance of the Pott's model:

$$\xi(G, x, y, 0, t) = t^{|E|} \cdot Z(G, x, \frac{y}{t}) \tag{3.9}$$

Since the partition function of the Pott's model can be also obtained when $t = 1$, the latter case is considered more general.That brings us back to the recurrence relation (1.12). To complete the proof of Theorem 2, we need now to show that any two steps of the graph decomposition using (1.12) are interchangeable. This includes two parts,

- Edge elimination and disjoint union.
- Decomposition of a graph by elimination of any two edges in different order;

The proof of the first part is simple. Let G be a disjoint union of two graphs: $G = H_1 \oplus H_2$. Without loss of generality, assume that the edge e, which is being eliminated, is in $E(H_1)$. Then use the linearity of our recurrence relation to show that

$$\begin{aligned} \xi(G) &= \big(\xi(H_{1-e}) + y \cdot \xi(H_{1/e}) + z \cdot \xi(H_{1\dagger e})\big) \cdot \xi(H_2) = \\ &= \xi(H_{1-e}) \cdot \xi(H_2) + y \cdot \xi(H_{1/e}) \cdot \xi(H_2) + z \cdot \xi(H_{1\dagger e}) \cdot \xi(H_2) \end{aligned}$$

The second part requires analyzing of three possible cases (Fig. 3):

- Case 1: Two the edges have no common vertices (graphs G_1, G_2, G_3);
- Case 2: Two the edges have one common vertex, and at least one exclusive vertex (graphs G_4, G_5);
- Case 3: Two the edges have no exclusive vertices (graphs G_6, G_7).

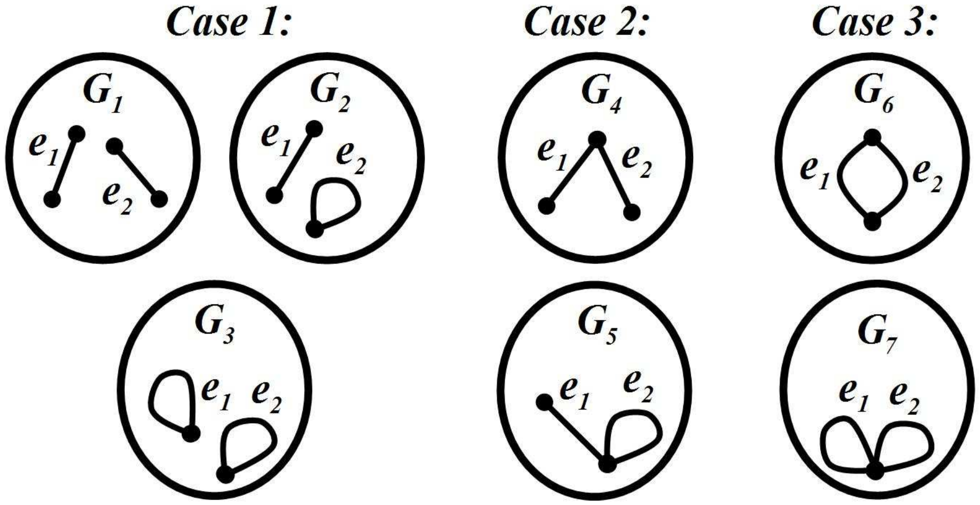

Fig. 3: Different cases to check order invariance of edge removal.

In the first case, the edge elimination operations are independent and commutative, e.g. $G_{-e1/e2} \cong G_{/e2-e1}$ and $G_{\dagger e1/e2} \cong G_{/e2\dagger e1}$. Thus, if we first eliminate e_1 and then e_2, we have:

$$\begin{aligned}
\xi(G) &= \xi(G_{-e_1}) + y\xi(G_{/e_1}) + z\xi(G_{\dagger e_1}) = \\
&= \xi(G_{-e_1-e_2}) + y\xi(G_{-e_1/e_2}) + z\xi(G_{-e_1\dagger e_2}) + \\
&+ \; y\xi(G_{/e_1-e_2}) + y^2\xi(G_{/e_1/e_2}) + yz\xi(G_{/e_1\dagger e_2}) + \\
&+ \; z\xi(G_{\dagger e_1-e_2}) + yz\xi(G_{\dagger e_1/e_2}) + z^2\xi(G_{\dagger e_1\dagger e_2})
\end{aligned}$$

On the other hand, if we first eliminate e_2 and then e_1, we have:

$$\begin{aligned}
\xi(G) &= \xi(G_{-e_2}) + y\xi(G_{/e_2}) + z\xi(G_{\dagger e_2}) = \\
&= \xi(G_{-e_2-e_1}) + y\xi(G_{-e_2/e_1}) + z\xi(G_{-e_2\dagger e_1}) + \\
&+ \; y\xi(G_{/e_2-e_1}) + y^2\xi(G_{/e_2/e_1}) + yz\xi(G_{/e_2\dagger e_1}) + \\
&+ \; z\xi(G_{\dagger e_2-e_1}) + yz\xi(G_{\dagger e_2/e_1}) + z^2\xi(G_{\dagger e_2\dagger e_1})
\end{aligned}$$

It is simple to see that two the expressions are equal.

The second case is slightly more confusing, since the edge extraction operation is not commutative with others: indeed, if we have extracted for example the edge e_1 in G_4, there is no more e_2 to eliminate, and vice versa. The other operations, deletion and contraction, are still commutative with each other. Therefore we should consider only the 3 sequences of non-commutative edge elimination operations listed below:

- Extraction of the first edge eliminates also the second;
- Contraction of the first edge and then extraction of the second gives a graph with the two edges extracted;
- Deletion of the first edge and then extraction of the second is equivalent to simply extraction of the second edge.

Thus, if we first eliminate e_1 and then e_2, we have:

$$\begin{aligned}
\xi(G) &= \xi(G_{-e_1}) + y\xi(G_{/e_1}) + z\xi(G_{\dagger e_1}) = \\
&= \xi(G_{-e_1-e_2}) + y\xi(G_{-e_1/e_2}) + z\xi(G_{\dagger e_2}) + \\
&+ \; y\xi(G_{/e_1-e_2}) + y^2\xi(G_{/e_1/e_2}) + yz\xi(G_{\dagger(e_1 \text{ and } e_2)}) + z\xi(G_{\dagger e_1})
\end{aligned}$$

On the other hand, if we first eliminate e_2 and then e_1, we have:

$$\begin{aligned}
\xi(G) &= \xi(G_{-e_2}) + y\xi(G_{/e_2}) + z\xi(G_{\dagger e_2}) = \\
&= \xi(G_{-e_2-e_1}) + y\xi(G_{-e_2/e_1}) + z\xi(G_{\dagger e_1}) + \\
&+ \; y\xi(G_{/e_2-e_1}) + y^2\xi(G_{/e_2/e_1}) + yz\xi(G_{\dagger(e_1 \text{ and } e_2)}) + z\xi(G_{\dagger e_2})
\end{aligned}$$

It is simple to see that two the expressions are equal.

In the third case, the edge elimination steps are symmetric with respect to the order among e_1 and e_2. This completes the proof of Theorem 2.

4. The subset expansion of the polynomial $\xi(G,x,y,z)$

In this section we prove Theorem 4. In order to do so, we need to show that

- The expression (1.13) satisfies the initial conditions of (1.12);
- The expression (1.13) is multiplicative;
- The expression (1.13) satisfies the edge elimination rule of (1.12).

Then by induction on the number of edges in G the theorem holds. The first fact is trivial; the second one can be easily checked by reader. Indeed, the summation over subsets of edges of $G(V,E) = G_1(V_1,E_1) \oplus G_2(V_2,E_2)$ can be regarded as a summation over the subsets of E_1, multiplied by an independent summation over the subsets of E_2. Therefore, we just need to prove that

Lemma 13. *The subset expansion given by (1.13) satisfies the edge elimination rule of (1.12).*

Proof. Let $G = (V,E)$ be the (multi)graph of interest. Let $N(G)$ be defined as

$$N(G,x,y,z) = \sum_{(A\sqcup B)\subseteq E} x^{k(A\sqcup B)-k_{cov}(B)} \cdot y^{|A|+|B|-k_{cov}(B)} \cdot z^{k_{cov}(B)} \tag{4.1}$$

where $k(A)$ denotes the number of connected components in (V,A), and $k_{cov}(B)$ denotes the number of the connected components of $(V(B),B)$, where $V(B) \subseteq V$ are the vertices covered by the edges of B. Note that the subsets of vertices $V(A)$ and $V(B)$ covered respectively by the edges of A and the edges of B are disjoint: $V(A)\cap V(B) = \emptyset$. Let e be the edge we have chosen to reduce. Any particular choice of A and B can be regarded as a vertex-disjoint edge coloring in 2 colors A and B, when part of the edges remains uncolored. We divide all the coloring into three disjoint cases:

- Case 1: e is uncolored;
- Case 2: e is colored by B, and it is the only edge of a colored connected component;
- Case 3: All the rest. That means, e is colored by A, or e is colored by B but it is not the only edge of a colored connected component.

In case 1, we just sum over colorings of G_{-e}:

$$N_1(G) = \sum_{(A\sqcup B)\models\ Case\ 1} x^{k(A\sqcup B)-k_{cov}(B)} \cdot y^{|A|+|B|-k_{cov}(B)} \cdot z^{k_{cov}(B)} = N(G_{-e}) \tag{4.2}$$

In case 2, the edge e is a connected component of $(V(B),B)$. Therefore, if we analyze now $N(G_{\dagger e})$, we will get

- The number of edges colored by A is the same;
- The number of edges colored by B is reduced by one;
- The total number of colored connected components is reduced by one;
- The number of covered connected components colored B is reduced by one;

This gives us

$$N_2(G) = \sum_{(A\sqcup B)\models\ Case\ 2} x^{k(A\sqcup B)-k_{cov}(B)} \cdot y^{|A|+|B|-k_{cov}(B)} \cdot z^{k_{cov}(B)} = z\cdot N(G_{\dagger e}) \tag{4.3}$$

And finally, in case 3, e is a part of a bigger colored connected component, or it is alone a connected component colored by A. In this case, we analyze the colorings of $G_{/e}$:

- Either $|A|$ or $|B|$ is reduced by 1, the other remained the same;
- The total number of colored connected components remained the same;
- The number of covered connected components colored B remained the same.

According to the above,

$$N_3(G) = \sum_{(A \sqcup B) \models \; Case \; 3} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A|+|B|-k_{cov}(B)} \cdot z^{k_{cov}(B)} = y \cdot N(G_{/e}) \quad (4.4)$$

which together with $N(G) = N_1(G) + N_2(G) + N_3(G)$ completes the proof. □

5. Definition of $\xi(G, x, y, z)$ as a Partition Function

In this section we prove Theorem 6. Let h be any mapping $h : V \mapsto V_H$. It is an homomorphism from G into H. Let us define:

$$E_A(h) = \left\{ (u, v) \in E : h(u) = h(v) \in V^A \right\}$$

$$E_B(h) = \left\{ (u, v) \in E : h(u) = h(v) \in V^B \right\}$$

Since $E_A(h)$ and $E_B(h)$ are the only edges that are mapped into the edges of H with a weight different from 1, we can write:

$$Z_H(G) = \sum_{h:V \mapsto V_H} \prod_{v \in V} \alpha(h(v)) \prod_{(u,v) \in (E_A(h) \cup E_B(H))} (y+1) \quad (5.1)$$

We use the Newton binomial to expand the second product of (5.1):

$$\begin{aligned} Z_H(G) &= \sum_{h:V \mapsto V_H} \prod_{v \in V} \alpha(h(v)) \left(\sum_{\substack{A \subseteq E_A(h) \\ B \subseteq E_B(h)}} \prod_{(u,v) \in (A \cup B)} y \right) = \\ &= \sum_{h:V \mapsto V_H} \prod_{v \in V} \alpha(h(v)) \left(\sum_{\substack{A \subseteq E_A(h) \\ B \subseteq E_B(h)}} y^{|A|+|B|} \right) \quad (5.2) \end{aligned}$$

It is clear that the edge subsets A and B are vertex-disjoint. Now, for any mapping $h : V \mapsto V_H$ and any vertex-disjoint edge subsets $A, B \subseteq E$, let an auxiliary function $\varphi(A, B, h)$ be as follows:

$$\varphi(A, B, h) = \begin{cases} 1 & if \quad (A \subseteq E_A(h)) \wedge (B \subseteq B_A(h)) \\ 0 & otherwise \end{cases}$$

We can rewrite (5.2) as follows:

$$\begin{aligned} Z_H(G) &= \sum_{h:V \mapsto V_H} \prod_{v \in V} \alpha(h(v)) \sum_{A \sqcup B} \varphi(A, B, h) \cdot y^{|A|+|B|} = \\ &= \sum_{A \sqcup B} y^{|A|+|B|} \cdot \sum_{h:V \mapsto V_H} \varphi(A, B, h) \cdot \prod_{v \in V} \alpha(h(v)) \quad (5.3) \end{aligned}$$

Here, $\varphi(A,B,h)$ is a constraint over the mapping: to fit the constraint, every connected components of (V_A, A) should be mapped into a single vertex of V^A, and every connected component of (V_B, B) should be mapped into a single vertex of V^B. The rest of the vertices $I(A,B) = V \setminus (V_A \cup V_B)$, can be mapped each one to any of the vertices of H. Let us denote by $I' \subseteq I(A,B)$ the subset of $I(A,B)$ mapped to V^I. Then:

$$\begin{aligned}
Z_H(G) &= \sum_{A \sqcup B} y^{|A|+|B|} x^{k_{cov}(A)} p^{k_{cov}(B)} \sum_{I' \subseteq I(A,B)} (-p)^{|I'|} (x+p)^{|I(A,B)|-|I'|} = \\
&= \sum_{A \sqcup B} y^{|A|+|B|} x^{k_{cov}(A)} p^{k_{cov}(B)} (x+p-p)^{|I(A,B)|} = \\
&= \sum_{(A \sqcup B) \subseteq E} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A|+|B|-k_{cov}(B)} \cdot (y \cdot p)^{k_{cov}(B)} = \\
&= \xi(G, x, y, y \cdot p) \qquad (5.4)
\end{aligned}$$

where we use

$$k(A \sqcup B) - k_{cov}(B) = k_{cov}(A) + |I(A,B)|.$$

This completes the proof.

6. The edge elimination polynomial of a labeled graph

To obtain the edge-labeled version of our polynomial, we use the same approach as in Section 3: we are looking for a multiplicative graph invariant satisfying linear recurrence relation with respect to the edge elimination operations. We start with

$$\begin{aligned}
\xi_{lab}(G_1 \oplus G_2) &= \xi_{lab}(G_1) \cdot \xi_{lab}(G_2) \\
\xi_{lab}(\emptyset) &= 1; \qquad (6.1)
\end{aligned}$$

and define an edge elimination rule introducing a new variable wherever we can (now every variable has index e for the edge which is currently being eliminated[5]

$$\begin{aligned}
\xi_{lab}(G) &= w_e \cdot \xi(G_{-e}) + y_e \cdot \xi_{lab}(G_{/e}) + z_e \cdot \xi_{lab}(G_{\dagger e}) \\
\xi_{lab}(E_1) &= x;
\end{aligned} \qquad (6.2)$$

The same considerations as in Section 3, using the same graph (Fig. 2), we get that the recursion (6.2) defines a unique graph polynomial when either $z_e = 0$ or $w_e = 1$ and $y_{e_1} z_{e_2} = y_{e_2} z_{e_1}$. We expand this restriction to any two edges of the graph:

- $z_e = 0$ for *every* edge e, or
- $w_e = 1$, $y_e = y \cdot t_e$ and $z_e = z \cdot t_e$ for *every* edge e (here y and z do not depend on e).

In the first case we obtain an instance of the labeled Sokal polynomial:

$$\xi_{lab}(G, \bar{w}, x, \bar{y}, \bar{0}) = \left(\prod_{e \in E} w_e \right) \cdot Z(G, q, \bar{v}) \qquad (6.3)$$

---

[5] We do not use an index for vertices, because the vertex set of the graph is being changed during decomposition.

where $q = x$ and $v_e = \frac{y_e}{w_e}$. Since the Sokal polynomial can be also obtained when $\bar{w} = \bar{1}$, the latter case is considered more general. That brings us to the recurrence relation (1.17). We have now to prove two propositions:

Proposition 14. *The recurrence relation (1.17) is order-invariant.*

Proposition 15. *The formula (1.19) defines the same polynomial as the recurrence relation (1.17).*

Both the proofs are similar to the respective unlabeled version and left to the reader.

7. Application to some known graph polynomials

In this section we present different known graph polynomials as substitution instances of $\xi(G)$ and $\xi_{lab}(G)$. Two issues should be addressed here:

Zero coefficients. When some of the arguments x, y or z of our polynomial are zero, we generally get 0 in all the summands that contain this variable in some positive power, and the undefined expression 0^0 in all the summands that contain it in power 0. However, being a polynomial, our function is continuous, and thus we can use the fact that for any nonnegative integer k,

$$x^k|_{x=0} = \lim_{x \to 0} x^k = \begin{cases} 1 & if\ k = 0 \\ 0 & otherwise \end{cases}$$

Hence, if in our substitution some variable equals 0, the value of the resulting polynomial is still well-defined.

Multiple edges and loops. Some of the graph polynomials are defined only for simple and loop-free graphs. However, their definition can be easily generalized to multigraphs, such that the equalities below hold in case of a simple input graph.

Labeled versions of the Tutte polynomial

Proposition 16. *The Sokal polynomial (in both unlabeled and labeled versions) can be obtained by*

$$Z(G, q, v) = \xi(G, q, v, 0)$$

$$Z_{lab}(G, q, \bar{v}) = \xi_{lab}\,(G, q, 1, 0, \bar{v})$$

in particular, the chromatic polynomial can be obtained by

$$\chi(G, \lambda) = Z(G, \lambda, -1) = \xi(G, \lambda, -1, 0)$$

Proof. By inspection of summands with $B = \emptyset$. All the other summands are eliminated by $z = 0$. □

By a simple substitution of variables, we get the following three corollaries:

Corollary 17. *The classical Tutte polynomial can be obtained by*

$$T(G,x,y) = (x-1)^{-k(E)} \cdot (y-1)^{-|V|} \cdot \xi\left(G,(x-1)(y-1),(y-1),0\right)$$

Recall that $r(S) = |V| - k(S)$ is the rank of the spanning subgraph with edge set S. The Zaslavsky's normal function of the colored matroid [36], applied to a graph $G = (V,E)$ with edge coloring function $c : E \mapsto \Lambda$, is defined by

$$R(G,c) = \sum_{S\subseteq E} \left(\prod_{e\in S} x_{c(e)}\right)\left(\prod_{e\notin S} y_{c(e)}\right)(x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)}.$$

Corollary 18. *The Zaslavsky's normal function of the edge-colored graph can be obtained by*

$$R(G,c) = (x')^{-k(E)} \cdot (y')^{-|V|} \cdot \left(\prod_{e\in E} y_{c(e)}\right) \cdot \xi_{lab}(G,x'y',y',0,\bar{t})$$

where $t_{c(e)} = \frac{x_{c(e)}}{y_{c(e)}}$ *and* $x' = x-1, y' = y-1$.

The chain polynomial $Ch(G,\omega,\bar{u})$ was first introduced in [31] and can also be defined, cf. [33], as

$$Ch(G,\omega,\bar{u}) = \sum_{S\subseteq E}(1-\omega)^{|S|-r(S)} \prod_{e\in E-S} u_e.$$

From [33] we get

Corollary 19. *The chain polynomial can be obtained by*

$$Ch(G,\omega,\bar{u}) = \left(\prod_{e\in E} u_e\right) \cdot (1-\omega)^{-|V|} \cdot \xi_{lab}(G,1-\omega,1,0,\bar{v})$$

where $v_e = \frac{1-\omega}{u_e}$.

Matching polynomials The next two propositions deal with various forms of matching polynomials:

Proposition 20. *The generalized matching polynomial (1.1) can be obtained by*

$$M(G,x,y) = \sum_{i=0}^{n} a_i x^{n-2i} y^i = \xi(G,x,0,y)$$

In particular, the generating matching polynomial is $g(G,x) = \xi(G,1,0,x)$ *and the defect matching polynomial is* $\mu(G,x) = \xi(G,x,0,-1)$

Proposition 21. *The original Heilmann and Lieb's multivariate matching polynomial introduced in [21] can be obtained by*

$$M_{col}(G,\bar{x},\bar{y}) = \sum_{\substack{M \subseteq E,\\ M \text{ is a matching}}} \prod_{e=\{u,v\}\in M} y_e x_u x_v = \xi_{lab}(G,1,0,1,\bar{t}) \tag{7.1}$$

where $t_e = y_e x_u x_v$ for every edge $e = \{u,v\}$.

Proof. By inspection of non-zero summands of $\xi(G)$. There should be no edges in A, and every edge of B should be in different connected component, so B has to be a matching. □

Finally, the Dohmen-Pönitz-Tittman generalization of the chromatic polynomial [13] is also a substitution instance of $\xi(G)$:

Proposition 22.

$$P(G,x,y) = \xi(G,x,-1,x-y)$$

Proof. Using recursion scheme (1.1), by induction on number of edges $|E|$. □

Using results of K.Dohmen, A.Pönitz and P.Tittman [13] we can also derive that the independent set polynomial [10, 18] (which is a substitution instance of $P(G,x,y)$) is also a substitution instance of $\xi(G)$.

8. Computational complexity of $\xi(G)$

In this section we analyze the complexity of computation of $\xi(G)$ and $\xi_{lab}(G)$. In general, these polynomials are $\sharp\mathbf{P}$-hard to compute, as every instance stated in the previous section is $\sharp\mathbf{P}$-hard. Moreover, C. Hoffmann proves in [22] that at every point $(x,y,z) \in \mathbb{Q}^3$, with $x \neq 0$, $z \neq -xy$, $(x,z) \notin \{(1,0),(2,0)\}$, $y \notin \{-2,-1,0\}$, evaluating $\xi(G,x,y,z)$ is $\sharp\mathbf{P}$-hard.

Recall that, according to Remark 5, the formulas (1.13) and (1.19) can be used to give an order invariant definition in Monadic Second Order Logic, with quantification over sets of edges, and an auxiliary order.

Hence, due to a general theorem from [26, 25], we have

Proposition 23. *$\xi(G)$ and $\xi_{lab}(G)$ are polynomial time computable on graphs of tree-width at most k where the exponent of the run time is independent of k.*

Recall also from Remark 5 that the weighted graph polynomial $U(G,\bar{x},y)$ is not definable using $MSOL$, and, hence, the results of [26, 25] are not applicable. Indeed, the run time of the algorithm introduced by C.Noble in [29] for graphs of tree width at most k is polynomial, but its highest degree depends on k.

The drawback of the general method of [26, 25] lies in the huge hidden constants, which make it practically unusable. However, an explicit dynamic algorithm for computing the polynomial $\xi_{lab}(G)$ on graphs of bounded tree-width, given the tree decomposition of the graph, where the constants are simply exponential in k, can be constructed along the same ideas as presented in [34, 16].

9. Conclusions and open questions

Starting by proving a recurrence relation for $P(G, x, y)$, and inspired by the characterization of the Tutte polynomial given in [30], see also [6, Theorem 2 of Chapter 10], we have introduced the graph polynomials $\xi(G)$ and $\xi_{lab}(g)$ by showing that they are the most general polynomials satisfying a linear recurrence relation with respect to the three edge elimination operations. We have shown that the various Tutte polynomials, matching polynomials and their derivatives are, up to multiplication with simple prefactors, substitution instances of our polynomials. We have shown that our polynomials are not more difficult to compute than the various Tutte polynomials or matching polynomials, and that they are also easy to compute on graphs of tree-width at most k. Nevertheless, there are still some challenging open questions.

Recurrence with case distinctions. Contrary to the approach in [30], we have avoided case distinctions in the recurrence relation. This was justified because it still gives the Tutte polynomials as special cases. Alternatively we could have introduced a polynomial in more variables which does incorporate a case distinction with respect to some local properties of an edge, such as being a bridge or a loop, or we could have allowed the deletion of single vertices, and distinguish between cases where they are isolated with or without loops, etc.

Question 1. *Does one get essentially stronger polynomials if one allows also deletion of single vertices and takes into account case distinctions?*

Distinctive power. We know that the polynomial $\xi(G)$ has at least the same distinctive power as the Tutte polynomial and the bivariate chromatic polynomial together, but more than every one of them individually. Indeed, since $T(G, x, y)$ and $P(G, x, y)$ are both substitution instances of $\xi(G)$, if $\xi(G)$ coincides for two graphs, so do $T(G, x, y)$ and $P(G, x, y)$. On the other hand, we do not know whether $\xi(G)$ has more distinctive power.

Question 2. *Are there two graphs G_1, G_2 such that for all x, y we have $T(G_1, x, y) = T(G_2, x, y)$ and $P(G_1, x, y) = P(G_2, x, y)$, but such that for some x, y, z $\xi(G, x, y, z) \neq \xi(G_2, x, y, z)$?*

Complexity on graph classes of bounded clique-width. We have noted that for graphs of tree-width at most k computing the edge reduction polynomial $\xi(G)$ is fixed parameter tractable (FPT) in the sense of [9, 14]. Another graph parameter, introduced in [8] and discussed there is the clique-width. It was stated as an open problem whether the Tutte polynomial is fixed parameter tractable for graphs of clique-width at most k, [19, 27]. Very recently, F. Fomin, P. Golovach, D. Lokshtanov and S. Saurabh [17] showed that computing the chromatic number of graphs of clique-width at most k is $W[1]$-hard, and therefore not fixed parameter tractable. It follows from this that it is also true for evaluating the Tutte polynomial and our polynomial $\xi(G)$. Further more, this shows that the results on the complexity of evaluating the chromatic polynomial in [27] are optimal.

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