A Most General Edge Elimination Polynomial

Ilia Averbouch*, Benny Godlin*, and J.A. Makowsky⋆

Faculty of Computer Science
Israel Institute of Technology
Haifa, Israel
{ailia,bgodlin,janos}@cs.technion.ac.il

Abstract. We look for graph polynomials which satisfy recurrence relations on three kinds of edge elimination: edge deletion, edge contraction and edge extraction, i.e., deletion of edges together with their end points. Like in the case of deletion and contraction only (J.G. Oxley and D.J.A. Welsh 1979), it turns out that there is a most general polynomial satisfying such recurrence relations, which we call $\xi(G, x, y, z)$. We show that the new polynomial simultaneously generalizes the Tutte polynomial, the matching polynomial, and the recent generalization of the chromatic polynomial proposed by K.Dohmen, A.Pönitz and P.Tittman (2003), including also the independent set polynomial of I. Gutman and F. Harary, (1983) and the vertex-cover polynomial of F.M. Dong, M.D. Hendy, K.T. Teo and C.H.C. Little (2002). We give three definitions of the new polynomial: first, the most general recursive definition, second, an explicit one, using a set expansion formula, and finally, a partition function, using counting of weighted graph homomorphisms. We prove the equivalence of the three definitions. Finally, we discuss the complexity of computing $\xi(G, x, y, z)$.

1 Introduction

There are several well-studied graph polynomials, among them the chromatic polynomial, [Big93, GR01, DKT05], different versions of the Tutte polynomial, [Bol99, BR99, Sok05], and of the matching polynomial, [HL72, LP86, GR01], which are known to satisfy certain linear recurrence relations with respect to *deletion* of an edge, *contraction* of an edge, or deletion of an edge together with its endpoints, which we call *extraction* of an edge. The generalization of the chromatic polynomial, which was introduced by K.Dohmen, A.Pönitz and P.Tittman in [DPT03], happens to satisfy such recurrence relation as well. The question that arises is, what is the most general graph polynomial that satisfies similar linear recurrence relation.

In this paper[1] all the graphs are unlabeled ; multiple edges and self loops are allowed. We denote by $G = (V, E)$ the graph with vertex set V and edge set E.

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Our investigation is motivated by the approach of J.G. Oxley and D.J.A. Welsh, [OW79], in defining a most general graph polynomial in five variables which satisfies a recurrence relation based on edge deletion and edge contraction, for which they show, that up to a simple prefactor, the resulting polynomial is the Tutte polynomial. Here we shall look for a most general polynomial which satisfies a recurrence relation based on three edge elimination operations: edge deletion, edge contraction and edge extraction.

1.1 Recursive Definitions of Graph Polynomials

Edge Elimination. We define three basic edge elimination operations on multigraphs:

- *Deletion.* We denote by G_{-e} the graph obtained from G by simply removing the edge e.
- *Contraction.* We denote by $G_{/e}$ the graph obtained from G by unifying the endpoints of e. Note that this operation can cause production of multiple edges and self loops.
- *Extraction.* We denote by $G_{\dagger e}$ the graph induced by $V \setminus \{u, v\}$ provided $e = \{u, v\}$. Extraction removes also all the edges adjacent to e.

Additionally, we require the polynomial to be *multiplicative* for disjoint unions, i.e., if $G_1 \oplus G_2$ denotes disjoint union of two graphs, then the polynomial $P(G_1 \oplus G_2) = P(G_1) \cdot P(G_2)$. This is justified by the fact that the polynomials occurring in the literature are usually multiplicative. The initial conditions are defined for an *empty set* (graph without vertices, usually, $P(\emptyset) = 1$) and for a single point $P(E_1)$. With respect to these operations, we recall the known recursive definitions of graph polynomials:

The Matching Polynomial. There are different versions of the matching polynomial discussed in the literature, for example *matching generating polynomial* $g(G, \lambda) = \sum_{i=0}^{n} a_i \lambda^i$ and *matching defect polynomial* $\mu(G, \lambda) = \sum_{i=0}^{n} (-1)^i a_i \lambda^{n-2i}$, where $n = |V|$ and a_i is the number of i-matchings in G. We shall use the bivariate version that incorporates the both above:

$$M(G, x, y) = \sum_{i=0}^{n} a_i x^{n-2i} y^i \tag{1}$$

The recursive definition of The matching polynomial satisfies the initial conditions $M(E_1) = x$ and $M(\emptyset) = 1$, and the recurrence relations

$$\begin{aligned} M(G) &= M(G_{-e}) + y \cdot M(G_{\dagger e}) \\ M(G_1 \oplus G_2) &= M(G_1) \cdot M(G_2) \end{aligned} \tag{2}$$

The Tutte Polynomial. We recall the definition of classical two-variable Tutte polynomial (cf. for example B.Bollobás [Bol99]):

Definition 1. Let $G = (V, E)$ be a (multi-)graph. Let $A \subseteq E$ be a subset of edges. We denote by $k(A)$ the number of connected components in the spanning subgraph (V, A). Then two-variable Tutte polynomial is defined as follows

$$T(G, x, y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)} (y-1)^{|A|+k(A)-|V|} \tag{3}$$

The Tutte polynomial satisfies the initial conditions $T(E_1) = 1$ $T(\emptyset) = 1$ and has linear recurrence relation with respect to the operations above:

$$T(G, x, y) = \begin{cases} x \cdot T(G_{/e}, x, y) & if\ e\ is\ a\ bridge, \\ y \cdot T(G_{-e}, x, y) & if\ e\ is\ a\ loop, \\ T(G_{/e}, x, y) + T(G_{-e}, x, y) & otherwise \end{cases}$$
$$T(G_1 \oplus G_2, x, y) = T(G_1, x, y) \cdot T(G_2, x, y) \tag{4}$$

However, we shall use in this paper the version of the Tutte polynomial used by A.Sokal [Sok05], known as the (bivariate) *partition function* of the Pott's model:

$$Z(G, q, v) = \sum_{A \subseteq E} q^{k(A)} v^{|A|} \tag{5}$$

The partition function of the Pott's model is co-reducible to the Tutte polynomial via

$$T(G, x, y) = (x-1)^{-k(E)} (y-1)^{-|V|} Z(G, (x-1)(y-1), y-1). \tag{6}$$

It satisfies the initial conditions $Z(E_1) = q$ and $Z(\emptyset) = 1$, and satisfies a recurrence relation which does not distinguish whether the edge e is a loop, a bridge, or none of the two:

$$Z(G, q, v) = v \cdot Z(G_{/e}, q, v) + Z(G_{-e}, q, v)$$
$$Z(G_1 \oplus G_2, q, v) = Z(G_1, q, v) \cdot Z(G_2, q, v) \tag{7}$$

The Bivariate Chromatic Polynomial. K.Dohmen, A.Pönitz and P.Tittman in [DPT03] introduced a polynomial $P(G, x, y)$ as follows: there is two disjoint sets of colors Y and Z, and a generalized proper coloring of a graph $G = (V, E)$ is a map $\phi : V \mapsto (Y \sqcup Z)$ such that for all $\{u, v\} \in E$, if $\phi(u) \in Y$ and $\phi(v) \in Y$, then $\phi(u) \neq \phi(v)$ (The set Y is called therefore "proper colors"). For two positive integers $x > y$, the value of the polynomial is the number of generalized proper colorings by x colors, y of them are proper. To make this definition meaningful for graphs with multiple edges we require that a vertex with a self-loop can be colored only by a color in $X \setminus Y$ and that a multiple edge does not affect colorings.

Proposition 1. *The polynomial $P(G, x, y)$ satisfies the initial conditions $P(E_1)$ $= x$ and $P(\emptyset) = 1$, and the following recurrence relation:*

$$P(G, x, y) = P(G_{-e}, x, y) - P(G_{/e}, x, y) + (x-y) \cdot P(G_{\dagger e}, x, y)$$
$$P(G_1 \oplus G_2, x, y) = P(G_1, x, y) \cdot P(G_2, x, y) \tag{8}$$

Proof. Let $G = (V, E)$ be a graph, and $P(G, x, y)$ be the number of generalized colorings defined above. Let $v \in V$ be any vertex. We denote by $P^v(G, x, y)$ the number of generalized colorings of G, when v is colored by an "improper" color, i.e. $\phi(v) \in X \setminus Y$. Observing that a vertex v can have any color in $X \setminus Y$, and the coloring of $V - \{v\}$ does not depend on the color of v, we get:

Lemma 2. *$P^v(G, x, y) = (x-y) \cdot P(G_{-v}, x, y)$, where G_{-v} denotes the subgraph of G induced by $V \setminus \{v\}$.*

Let $e = \{u, v\} \in E$ be any edge of G, which is not a self-loop and not a multiple edge. Consider the number of colorings of G_{-e}. Any such coloring is either a coloring of G, or a coloring of $G_{/e}$, when the vertex $u = v$, which is produced by the contraction, is colored by a proper color. Together with Proposition 2, that raises:

$$P(G, x, y) = P(G_{-e}, x, y) - P(G_{/e}, x, y) + (x - y) \cdot P(G_{\dagger e}, x, y) \tag{9}$$

One can easily check that this equation is satisfied also for loops and multiple edges. Together with the fact that a singleton can be colored by any color, and the fact that the number of colorings is multiplicative, this completes the proof.

1.2 A Most General Edge Elimination Polynomial

Recursive Definition. Inspired by the characterization of the Tutte polynomial given in [OW79], see also [Bol99], Theorem 2 of Chapter 10, we look for the most general linear recurrence relation[2], which can be obtained on unlabeled graphs by introducing new variables, and which does not distinguish between local properties of the edge e which is to be eliminated[3]. To assure that the polynomial so defined is unique, we have to prove that its definition is not dependent on the order in which the edges are removed.

We start with the initial conditions $\xi(E_1) = x$ and $\xi(\emptyset) = 1$, and recurrence relation

$$\begin{aligned} \xi(G) &= w \cdot \xi(G_{-e}) + y \cdot \xi(G_{/e}) + z \cdot \xi(G_{\dagger e}) \\ \xi(G_1 \oplus G_2) &= \xi(G_1) \cdot \xi(G_2) \end{aligned} \tag{10}$$

We prove:

Theorem 3. *The recurrence relation (10) defines for each graph G a unique polynomial $\xi(G)$ if and only if one of the following conditions are satisfied:*

$$z = 0 \tag{11}$$

$$w = 1 \tag{12}$$

[2] The first paper to study general conditions under which linear recurrence relations define a graph invariant is D.N.Yetter [Yet90].

[3] It is conceivable that recurrence relations with various case distinctions depending on local properties of e and more variables give other "most general" polynomials. This is the reason why we speak of "a most general" edge elimination polynomial in the title of the paper.

Under condition (12), which allows a more general graph polynomial to be obtained, the recurrence relation (10) is restricted to

$$\begin{aligned}
\xi(G,x,y,z) &= \xi(G_{-e},x,y,z) + y\cdot\xi(G_{/e},x,y,z) + z\cdot\xi(G_{\dagger e},x,y,z)\\
\xi(G_1\oplus G_2,x,y,z) &= \xi(G_1,x,y,z)\cdot\xi(G_2,x,y,z)\\
\xi(E_1,x,y,z) &= x;\\
\xi(\emptyset,x,y,z) &= 1; \qquad (13)
\end{aligned}$$

Remark 4. *From this theorem one sees immediately that $\xi(G,x,y,z)$ gives, by choosing appropriate values for the variables and simple prefactors, the partition function of the Pott's model, the bivariate matching polynomial and the bivariate chromatic polynomial with all their respective substitution instances, including the classical chromatic polynomial, the Tutte polynomial and the independent set polynomial, [DHTL02, GH83]. The latter two polynomials are already substitution instances of the bivariate chromatic polynomial $P(G,x,y)$ of [DPT03]. The following table summarizes these observations.*

Pott's model	$Z(G,q,v)=\xi(G,q,v,0)$
Bivariate Tutte polynomial	$T(G,x,y)=(x-1)^{-k(E)}\cdot(y-1)^{-\lvert V\rvert}\cdot$ $\xi\left(G,(x-1)(y-1),(y-1),0\right)$
Bivariate matching polynomial	$M(G,x,y)=\xi(G,x,0,y)$
Bivariate chromatic polynomial	$P(G,x,y)=\xi(G,x,-1,x-y)$

Explicit definition. We now give an explicit form of the polynomial $\xi(G,x,y,z)$ using 3-partition expansion[4]:

Theorem 5. *Let $G=(V,E)$ be a (multi)graph. Then the edge elimination polynomial $\xi(G,x,y,z)$ can be calculated as*

$$\xi(G,x,y,z)=\sum_{(A\sqcup B)\subseteq E} x^{k(A\sqcup B)-k_{cov}(B)}\cdot y^{|A|+|B|-k_{cov}(B)}\cdot z^{k_{cov}(B)} \qquad (14)$$

where by abuse of notation we use $(A\sqcup B)\subseteq E$ for summation over subsets $A,B\subseteq E$, such that the subsets of vertices $V(A)$ and $V(B)$, covered by respective subset of edges, are disjoint: $V(A)\cap V(B)=\emptyset$; $k(A)$ denotes the number of spanning connected components in (V,A), and $k_{cov}(B)$ denotes the number of covered connected components, i.e. the connected components of $(V(B),B)$.

Remark 6. *From Theorem 5 one can see that $\xi(G,x,y,z)$ is a polynomial definable in Monadic Second Order Logic, with quantification over sets of edges ($MSOL_2$), where an order over vertices is to be used for stating "number of connected sets", but the final result is order-independent. We shall not use logic in the sequel of the paper. For details the reader is referred to [Mak05].*

[4] A more precise name would be "Pair of two disjoint subsets expansion". We chose the name 3-partition expansion, as any two disjoint subsets induce a partition into three sets.

Counting Weighted Graph Homomorphisms. Another common way to define graph polynomials is counting weighted graph homomorphisms. Let the weighted graph $H = (V_H, E_H)$ be defined as follows:

- H is obtained by joining 3 cliques with all loops: $H = K_x^A \bowtie K_p^B \bowtie K_p^I$, such that K_x^A has x vertices, K_p^B and K_p^I have p vertices each.
- We denote by V^A the vertices of K^A, by V^B the vertices of K^B and by V^I the vertices of K^I
- The weight function $w = (\alpha, \beta)$: $\alpha : V_H \mapsto \mathbb{R}$ and $\beta : E_H \mapsto \mathbb{R}$
- $\alpha(v) = \begin{cases} 1 & if\ v \in (V^A \cup V^B) \\ -1 & otherwise \end{cases}$
- $\beta(u,v) = \begin{cases} y+1 & if\ (u = v) \wedge (u, v \in (V^A \cup V^B)) \\ 1 & otherwise \end{cases}$

In [AGM08] the following is shown:

Theorem 7. *Let $Z_H(G)$ be a homomorphism function of a graph $G = (V,E)$ into a weighted graph H above.*

$$Z_H(G) = \sum_{\substack{h\,:\,V \mapsto V_H \\ homomorphism}} \prod_{v \in V} \alpha(h(v)) \prod_{(u,v) \in E} \beta(h(u), h(v))$$

Then, for all nonnegative integers x and p and all $y \in \mathbb{R}$, we have

$$\xi(G, x, y, p \cdot y) = Z_H(G)$$

Remark 8. *A general characterization of graph parameters which can be obtained from homomorphism functions by choosing appropriate weights is given in [FLS07]. This characterization requires that the weights α are positive reals. However, in Theorem 7, we use negative values for α.*

2 The Most General Recurrence Relation

We are looking for the most general linear recurrence relation with respect to edge deletion, edge contraction and edge extraction operation that can be obtained by introducing new variables. Recall that we are interested in a *graph invariant*, e.i. the resulting function should not depend on the order of graph deconstruction. Moreover, this invariant should be a multiplicative graph polynomial.

From this consideration alone we obtain the initial condition $\xi(\emptyset) = 1$; and the product rule:

$$\xi(G_1 \oplus G_2) = \xi(G_1) \cdot \xi(G_2)$$

Indeed, the disjoint union with an empty set gives the same graph, so the resulting function should also remain the same.

We now formulate the edge elimination rule introducing a new variable wherever we can. We set

$$\begin{aligned}&\xi(G,x,y,z,t) = t\cdot\xi(G_{-e},x,y,z,t) + y\cdot\xi(G_{/e},x,y,z,t) + z\cdot\xi(G_{\dagger e},x,y,z,t)\\&\xi(E_1,x,y,z,t) = x;\end{aligned} \tag{15}$$

Let G be a graph as presented on Fig. 1. Note that the subgraphs H_1, H_{1-u}, H_2 and H_{2-w} can be different and have (in general) different ξ. Since we are

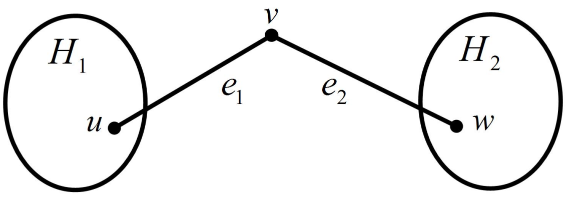

Fig. 1. Testing order invariance of edge removal

looking for a graph invariant, we must obtain the same result by applying the edge elimination rule first on the edge e_1 and then on the edge e_2, as in case when we apply the edge elimination rule first on the edge e_2 and then on the edge e_1.

$$\begin{aligned}\xi(G) &= t\cdot\xi(G_{-e_1}) + y\cdot\xi(G_{/e_1}) + z\cdot\xi(G_{\dagger e_1}) =\\&= t\cdot\xi(H_1)\cdot[x\cdot t\cdot\xi(H_2) + y\cdot\xi(H_2) + z\cdot\xi(H_{2-w})] +\\&\quad y\cdot\left[t\cdot\xi(H_1)\xi(H_2) + y\cdot\xi(G_{/e_1/e_2}) + z\cdot\xi(H_{1-u})\xi(H_{2-w})\right] +\\&\quad z\cdot\xi(H_{1-u})\xi(H_2)\end{aligned} \tag{16}$$

On the other hand,

$$\begin{aligned}\xi(G) &= t\cdot\xi(G_{-e_2}) + y\cdot\xi(G_{/e_2}) + z\cdot\xi(G_{\dagger e_2}) =\\&= t\cdot\xi(H_2)\cdot[x\cdot t\cdot\xi(H_1) + y\cdot\xi(H_1) + z\cdot\xi(H_{1-u})] +\\&\quad y\cdot\left[t\cdot\xi(H_1)\xi(H_2) + y\cdot\xi(G_{/e_1/e_2}) + z\cdot\xi(H_{1-u})\xi(H_{2-w})\right] +\\&\quad z\cdot\xi(H_{2-w})\xi(H_1)\end{aligned} \tag{17}$$

Solving the above two equations, we get:

$$tz\cdot\xi(H_1)\xi(H_{2-w}) + z\cdot\xi(H_{1-u})\xi(H_2) = tz\cdot\xi(H_{1-u})\xi(H_2) + z\cdot\xi(H_1)\xi(H_{2-w})$$

Hence, we have the following necessary condition:

$$z = 0 \;\; or \tag{18}$$

$$t = 1 \;\; or \tag{19}$$

$$\xi(H_1)\xi(H_{2-w}) = \xi(H_{1-u})\xi(H_2) \;\; for\ any\ H_1\ and\ H_2 \tag{20}$$

Under condition (20) we get the polynomial $\xi(G) = x^{|V(G)|}$ which we also get under the assumption $z = 0$ or $t = 1$. In case of $z = 0$ it can be easily seen that the resulting function is a substitution instance of the Pott's model:

$$\xi(G,x,y,0,t) = t^{|E|} \cdot Z(G,x,\frac{y}{t}) \tag{21}$$

Since the partition function of the Pott's model can be also obtained when $t = 1$, the latter case is considered more general. That brings us back to the recurrence relation (13). To complete the proof of Theorem 3, we need now to show that any two steps of the graph decomposition using (13) are interchangeable. This involves two parts,

- Edge elimination and disjoint union, and
- Decomposition of a graph by elimination of any two edges in different order.

The proof of the first part is simple. Let G be a disjoint union of two graphs: $G = H_1 \oplus H_2$. Without loss of generality, assume that the edge e, which is being eliminated, is in $E(H_1)$. Then use the linearity of our recurrence relation to show that

$$\begin{aligned}\xi(G) &= \big(\xi(H_{1-e}) + y\cdot\xi(H_{1/e}) + z\cdot\xi(H_{1\dagger e})\big)\cdot\xi(H_2) = \\ &= \xi(H_{1-e})\cdot\xi(H_2) + y\cdot\xi(H_{1/e})\cdot\xi(H_2) + z\cdot\xi(H_{1\dagger e})\cdot\xi(H_2)\end{aligned}$$

The second part requires analyzing of three possible cases:

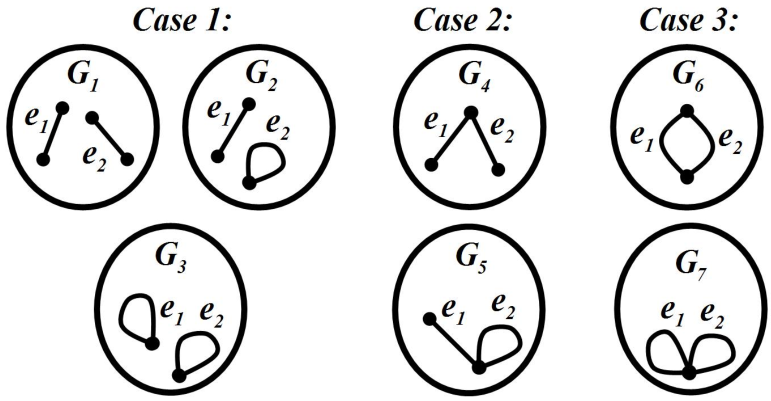

Fig. 2. Different cases to check order invariance of edge removal

- Case 1: Two the edges have no common vertices (graphs G_1, G_2, G_3);
- Case 2: Two the edges have one common vertex, and at least one exclusive vertex (graphs G_4, G_5);
- Case 3: Two the edges have no exclusive vertices (graphs G_6, G_7).

The detailed analysis of these cases is straightforward.

3 The Explicit Form of $\xi(G)$

We need to show that the expression (14) satisfies the *initial conditions* of (13), that it is *multiplicative* and that it also satisfies the *edge elimination rule* of (13). Then by induction on the number of edges in G the theorem holds.

The first fact is trivial; the second one can be easily checked by reader. Indeed, the summation over subsets of edges of $G(V, E) = G_1(V_1, E_1) \oplus G_2(V_2, E_2)$ can be regarded as a summation over the subsets of E_1, and then independently over the subsets of E_2. Therefore, we just need to prove that

Lemma 9. *The explicit expression given by (14) satisfies the edge elimination rule of (13).*

Proof. Let $G = (V, E)$ be the (multi)graph of interest. Let $N(G)$ be defined as

$$N(G, x, y, z) = \sum_{(A \sqcup B) \subseteq E} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A|+|B|-k_{cov}(B)} \cdot z^{k_{cov}(B)} \tag{22}$$

where $k(A)$ denotes the number of connected components in (V, A), and $k_{cov}(B)$ denotes the number of the connected components of $(V(B), B)$, where $V(B) \subseteq V$ are the vertices covered by the edges of B. Let e be the edge we have chosen to reduce. Any particular choice of A and B can be regarded as a vertex-disjoint edge coloring in 2 colors A and B, when part of the edges remains uncolored. We divide all the coloring into three disjoint cases:

- Case 1: e is uncolored;
- Case 2: e is colored by B, and it is the only edge of a colored connected component;
- Case 3: All the rest. That means, e is colored by A, or e is colored by B but it is not the only edge of a colored connected component.

In case 1, we just sum over colorings of G_{-e}:

$$N_1(G) = \sum_{(A \sqcup B) \models \; Case\ 1} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A|+|B|-k_{cov}(B)} \cdot z^{k_{cov}(B)} = N(G_{-e})$$

In case 2, the edge e is a connected component of $(V(B), B)$. Therefore, if we analyze now $N(G_{\dagger e})$, we will get

- The number of edges colored by A is the same;
- The number of edges colored by B is reduced by one;
- The total number of colored connected components is reduced by one;
- The number of covered connected components colored B is reduced by one;

This gives us

$$N_2(G) = \sum_{(A \sqcup B) \models \; Case\ 2} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A|+|B|-k_{cov}(B)} \cdot z^{k_{cov}(B)} = z \cdot N(G_{\dagger e})$$

And finally, in case 3, e is a part of a bigger colored connected component, or it is alone a connected component colored by A. In this case, we analyze the colorings of $G_{/e}$:

– Either $|A|$ or $|B|$ is reduced by 1, the other remained the same;
– The total number of colored connected components remained the same;
– The number of covered connected components colored B remained the same.

According to the above,

$$N_3(G) = \sum_{(A \sqcup B) \models \ Case\ 3} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A|+|B|-k_{cov}(B)} \cdot z^{k_{cov}(B)} = y \cdot N(G_{/e})$$

which together with $N(G) = N_1(G) + N_2(G) + N_3(G)$ completes the proof.

4 Computational Complexity of $\xi(G)$

In this section we consider the complexity of computation of $\xi(G)$. In general, this polynomial is $\sharp\mathbf{P}$-hard to compute, as every instance stated in the Remark 4 is $\sharp\mathbf{P}$-hard. Moreover, C. Hoffmann proves in [Hof08] that at every point $(x, y, z) \in \mathbb{Q}$, with $x \neq 0$, $z \neq -xy$, $(x, z) \notin \{(1, 0), (2, 0)\}$, $y \notin \{-2, -1, 0\}$, evaluating $\xi(G, x, y, z)$ is $\sharp\mathbf{P}$-hard.

Recall that, according to Remark 6, the formula (14) can be used to give an order invariant definition in Monadic Second Order Logic, with quantification over sets of edges, and an auxiliary order. Hence, from the general theorem from [Mak05, Mak04], we immediately get that $\xi(G)$ is polynomial time computable on graphs of tree-width at most k where the exponent of the run time is independent of k. The drawback of the general method of [Mak05, Mak04] lies in the huge hidden constants, which make it practically unusable. However, an explicit dynamic algorithm for computing the polynomial $\xi(G)$ on graphs of bounded tree-width, given the tree decomposition of the graph, where the constants are simply exponential in k, can be constructed along the same ideas as presented in [Tra06, FMR08, MRAG06].

5 Conclusions and Open Questions

We have introduced a new graph polynomial $\xi(G, x, y, z)$ from which the Tutte polynomial, the matching polynomial and the bivariate chromatic polynomial can be obtained by simple evaluations. We have given three equivalent definitions of this polynomial, and we have shown that it is the most general graph polynomial satisfying a recurrence relation (without case distinctions) with respect to three edge elimination operations.

There are still some challenging open questions.

Recursive Definitions with Case Distinctions. Contrary to the approach in [OW79], we have avoided case distinctions in the recurrence relation. This was justified because it still gives the Tutte polynomials as special cases. Alternatively we could have introduced a polynomial in more variables which does incorporate a case distinction with respect to some local properties of an edge, such as being a bridge or a loop, or we could have allowed the deletion of single vertices, and distinguish between cases where they are isolated with or without loops, etc.

Question 1. Does one get essentially stronger polynomials if one allows also deletion of single vertices and takes into account case distinctions?

Distinctive Power. We know that the polynomial $\xi(G)$ has at least the same distinctive power as the Tutte polynomial and the bivariate chromatic polynomial together, but more than every one of them individually. Indeed, since $T(G, x, y)$ and $P(G, x, y)$ are both substitution instances of $\xi(G)$, if $\xi(G)$ coincides for two graphs, so do $T(G, x, y)$ and $P(G, x, y)$. On the other hand, we do not know whether $\xi(G)$ has more distinctive power.

Question 2. Are there two graphs G_1, G_2 such that for all x, y we have $T(G_1, x, y) = T(G_2, x, y)$ and $P(G_1, x, y) = P(G_2, x, y)$, but such that for some x, y, z $\xi(G, x, y, z) \neq \xi(G_2, x, y, z)$?

Complexity on Graph Classes of Bounded Clique-Width. We have noted that for graphs of tree-width at most k computing the edge reduction polynomial $\xi(G)$ is fixed parameter tractable (FPT) in the sense of [DF99, FG06]. Another graph parameter, introduced in [CO00] and discussed there is the clique-width. It was stated as an open problem whether the Tutte polynomial is fixed parameter tractable for graphs of clique-width at most k, [GHN05, MRAG06]. Very recently, F. Fomin, P. Golovach, D. Lokshtanov and S. Saurabh [FGLS08] showed that computing the chromatic number of graphs of clique-width at most k is $W[1]$-hard, and therefore not fixed parameter tractable. It follows from this that it is also true for evaluating the Tutte polynomial and our polynomial $\xi(G)$. Further more, this shows that the results on the complexity of evaluating the chromatic polynomial in [MRAG06] are optimal.

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References

[AGM07] Averbouch, I., Godlin, B., Makowsky, J.A.: The most general edge elimination polynomial. arXiv (2007), `http://uk.arxiv.org/pdf/0712.3112.pdf`

[AGM08] Averbouch, I., Godlin, B., Makowsky, J.A.: An extension of the bivariate chromatic polynomial (submitted, 2008)

[Big93] Biggs, N.: Algebraic Graph Theory, 2nd edn. Cambridge University Press, Cambridge (1993)

[Bol99] Bollobás, B.: Modern Graph Theory. Springer, Heidelberg (1999)

[BR99] Bollobás, B., Riordan, O.: A Tutte polynomial for coloured graphs. Combinatorics, Probability and Computing 8, 45–94 (1999)

[CO00] Courcelle, B., Olariu, S.: Upper bounds to the clique–width of graphs. Discrete Applied Mathematics 101, 77–114 (2000)

[DF99] Downey, R.G., Fellows, M.F.: Parametrized Complexity. Springer, Heidelberg (1999)

[DHTL02] Dong, F.M., Hendy, M.D., Teo, K.L., Little, C.H.C.: The vertex-cover polynomial of a graph. Discrete Mathematics 250, 71–78 (2002)

[DKT05] Dong, F.M., Koh, K.M., Teo, K.L.: Chromatic Polynomials and Chromaticity of Graphs. World Scientific, Singapore (2005)

[DPT03] Dohmen, K., Pönitz, A., Tittmann, P.: A new two-variable generalization of the chromatic polynomial. Discrete Mathematics and Theoretical Computer Science 6, 69–90 (2003)

[FG06] Flum, J., Grohe, M.: Parameterized complexity theory. Springer, Heidelberg (2006)

[FGLS08] Fomin, F., Golovach, P., Lokshtanov, D., Saurabh, S.: Clique-width: On the price of generality. Department of Informatics, University of Bergen, Norway (preprint, 2008)

[FLS07] Freedman, M., Lovász, L., Schrijver, A.: Reflection positivity, rank connectivity, and homomorphisms of graphs. Journal of AMS 20, 37–51 (2007)

[FMR08] Fischer, E., Makowsky, J.A., Ravve, E.V.: Counting truth assignments of formulas of bounded tree width and clique-width. Discrete Applied Mathematics 156, 511–529 (2008)

[GH83] Gutman, I., Harary, F.: Generalizations of the matching polynomial. Utilitas Mathematicae 24, 97–106 (1983)

[GHN05] Giménez, O., Hliněný, P., Noy, M.: Computing the Tutte polynomial on graphs of bounded clique-width. In: Kratsch, D. (ed.) WG 2005. LNCS, vol. 3787, pp. 59–68. Springer, Heidelberg (2005)

[GR01] Godsil, C., Royle, G.: Algebraic Graph Theory. Graduate Texts in Mathematics. Springer, Heidelberg (2001)

[HL72] Heilmann, C.J., Lieb, E.H.: Theory of monomer-dymer systems. Comm. Math. Phys 28, 190–232 (1972)

[Hof08] Hoffmann, C.: A most general edge elimination polynomial–thickening of edges (2008) arXiv:0801.1600v1 [math.CO]

[LP86] Lovasz, L., Plummer, M.D.: Matching Theory. Annals of Discrete Mathematics, vol. 29. North-Holland, Amsterdam (1986)

[Mak04] Makowsky, J.A.: Algorithmic uses of the Feferman-Vaught theorem. Annals of Pure and Applied Logic 126(1-3), 159–213 (2004)

[Mak05] Makowsky, J.A.: Colored Tutte polynomials and Kauffman brackets on graphs of bounded tree width. Disc. Appl. Math. 145(2), 276–290 (2005)

[MRAG06] Makowsky, J.A., Rotics, U., Averbouch, I., Godlin, B.: Computing graph polynomials on graphs of bounded clique-width. In: Fomin, F.V. (ed.) WG 2006. LNCS, vol. 4271, pp. 191–204. Springer, Heidelberg (2006)

[OW79] Oxley, J.G., Welsh, D.J.A.: The Tutte polynomial and percolation. In: Bundy, J.A., Murty, U.S.R. (eds.) Graph Theory and Related Topics, pp. 329–339. Academic Press, London (1979)

[Sok05] Sokal, A.: The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In: Survey in Combinatorics, 2005. London Mathematical Society Lecture Notes, vol. 327, pp. 173–226 (2005)

[Tra06] Traldi, L.: On the colored Tutte polynomial of a graph of bounded treewidth. Discrete Applied Mathematics 154(6), 1032–1036 (2006)

[Yet90] Yetter, D.N.: On graph invariants given by linear recurrence relations. Journal of Combinatorial Theory, Series B 48(1), 6–18 (1990)