The Complexity of Multivariate Matching Polynomials

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February 28, 2007

Abstract

We study various versions of the univariate and multivariate matching and rook polynomials. We show that there is most general multivariate matching polynomial, which is, up the some simple substitutions and multiplication with a prefactor, the original multivariate matching polynomial introduced by C. Heilmann and E. Lieb. We follow here a line of investigation which was very successfully pursued over the years by, among others, W. Tutte, B. Bollobas and O. Riordan, and A. Sokal in studying the chromatic and the Tutte polynomial.

We show here that evaluating these polynomials over the reals is $\sharp\mathbf{P}$-hard for all points in $\mathbb{R}^k$ but possibly for an exception set which is semi-algebraic and of dimension strictly less than k. This result is analoguous to the characterization due to F. Jaeger, D. Vertigan and D. Welsh (1990) of the points where the Tutte polynomial is hard to evaluate. Our proof, however, builds mainly on the work by M. Dyer and C. Greenhill (2000).

1 Introduction

In this paper we study generalizations of the matching and rook polynomials and their complexity. The matching polynomial was originally introduced in [5] as a multivariate polynomial. Some of its general properties, in particular the so called *half-plane property*, were studied recently in [2]. We follow here a line of investigation which was very successfully pursued over the years by, among others, W. Tutte [15], B. Bollobas and O. Riordan [1], and A. Sokal [14], in studying the chromatic and the Tutte polynomial. The paper is part of a general research program on graph polynomials as described in [11, 12].

*Partially supported by a grant of the Graduate School of the Technion–Israel Institute of Technology

†Partially supported by a Grant of the Fund for Promotion of Research of the Technion–Israel Institute of Technology

Recurrence relations. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Let $g(G, x)$ be the *matching defect polynomial* defined as $g(G, x) = \sum_{k=0}^{\frac{n}{2}} m_k x^k$, where m_k is a number of k-matchings in G. We denote by G_{-e} the graph where the edge $e = (u, v)$ is deleted from G, and by G_{-u-v} the graph where the vertices u, v are deleted together with all the edges connected to u and v. We denote by E_n the graph of n isolated vertices, and K_n the clique on n vertices. The matching polynomial satisfies the following recurrence relations:

$$\begin{aligned} g(G, x) &= g(G_{-e}, x) + x \cdot g(G_{-u-v}, x) \\ g(G_1 \sqcup G_2, x) &= g(G_1, x) \cdot g(G_2, x) \\ g(E_1, x) &= 1; \\ g(\emptyset, x) &= 1; \end{aligned} \tag{1}$$

We shall study graph polynomials $P(G, \bar{x}, \bar{y}, \bar{z})$ which are well defined using the following general recurrence relations:

$$\begin{aligned} P(G) &= z_e P(G_{-e}) + y_e \cdot P(G_{-u-v}) \\ P(G_1 \sqcup G_2) &= P(G_1) \cdot P(G_2) \\ P(E_1) &= x_v; \\ P(\emptyset) &= 1; \end{aligned} \tag{2}$$

where z_e, y_e are indeterminates representing weighted edges and x_v are indeterminates representing weighted vertices. Let $V(M) \subseteq V$ denote the subset of vertices incident with edges of a matching M. The polynomial

$$U_0(G, \bar{x}, \bar{y}, \bar{z}) = \prod_{e \in E(G)} z_e \cdot \prod_{v \in V(G)} x_v$$

satisfies the recurrence relations (2) with $y_e = 0$ for all $e \in E(G)$, but is *trivial* in the sense that it does only depend on the sets $V(G)$ and $E(G)$ but does not reflect which vertices are connected by which edges.

We shall prove

Theorem 1. *The polynomial*

$$U(G) = \sum_{\substack{M \subseteq E, \\ M \text{ is a matching}}} \left[\left(\prod_{e \in M} y_e \right) \left(\prod_{v \in V \setminus V(M)} x_v \right) \right] \tag{3}$$

is the most general non-trivial polynomial which is well defined and satisfies the recurrence relations (2). Furthermore, the univariate matching defect *or* acyclic *polynomial and the* rook polynomial*, and the original multivariate matching polynomial, can be obtained from it by simple substitutions and multiplication with a prefactor.*

The number of indeterminates of $U(G)$ depends on the graph G. If instead of indexing the indeterminates by edges and vertices, we we allow only a fixed number of weights of the edges and vertices, say k_e and k_v respectively, we get a polynomial in $k_e + k_v$ many variables. The corresponding polynomial is denoted by $U_{k_e,k_v}(G)$ and is the most general polynomial for these recurrence relations for edge and vertex colored graphs.

Complexity. Next we want to study the complexity of the evaluation of the various matching polynomials. Again we follow, inspired by [7], the example of the chromatic and the Tutte polynomials.

The complexity of evaluation of the chromatic polynomial $\chi(G, n)$ is known to be polynomial for $n \in \{0, 1, 2\}$, and $\sharp\mathbf{P}$-hard for every $n \in \mathbb{N} \setminus \{0, 1, 2\}$, [16]. For $\lambda_1, \lambda_2 \in \mathbb{R} - \mathbb{N}$, the evaluation of $\chi(G, \lambda_1)$ and $\chi(G, \lambda_2)$ can be shown to be reducible to each other in polynomial time, using a polynomial time algebraic reduction (AP-reduction), which allows the use of λ_1, λ_2 at unit cost, [9]. In [7] this was generalized for the classical Tutte polynomial in two variables.

In the case of the matching polynomials the proof method of [9] does not work, and even the methods described in [7] do not suffice. Instead we reduce the problem to counting graph homomorphisms, inspired by to the methods described in [3, 6], but with a few additional twists. Note that [3] uses some ideas developed already in [7].

For the matching generating polynomial we get:

Theorem 2. *The evaluation of $g(G, x)$ is $\sharp\mathbf{P}$-hard for $x \in \mathbb{N} \setminus \{0\}$. Furthermore, for $x_0 \in \mathbb{R} \setminus \{0, 1\}$, evaluation of $g(G, 1)$ is AP-reducible to $g(G, x_0)$.*

Using Theorem 2 we can formulate similar theorems for the univariate acyclic and rook polynomials. For the multivariate versions we prove:

Theorem 3. *For any finite k_e and k_v, the weighted matching polynomial*

$$U_{k_e,k_v}(G, \overline{x}^{(k_e)}, \overline{y}^{(k_v)})$$

is $\sharp\mathbf{P}$-hard to evaluate for all points in $\mathbb{R}^{k_e+k_v}$ except for its k_e-ary subspace defined by $\overline{y} = 0$.

Outline of the paper. The rest of the paper is organized as follows: Section 2 contains the basic definition used in the paper. In Section 3 we discuss the most general matching polynomial in a weighted and in non-weighted versions and prove Theorem 1. In Section 4 we first analyze the complexity of $g(G, x)$, and then extend this to analyze the complexity of bivariate and multivariate matching polynomials. Finally, we prove Theorem 3.

Acknowledgments We would like to thank B. Courcelle and M. Bläser for valuable suggestions and discussions.

2 Matching polynomials

Matching polynomial. Let $G(V, E)$ be a simple, undirected, loop-free graph with $|V| = n$ vertices. A *k-matching* in a graph G is a subset $M \subseteq E$ of k edges, no two of which have a vertex in common. We denote by $m_k(G)$ the number of k-matchings of a graph G, when we define $m_0(G) = 0$ by convention. Of course, for $k > \frac{n}{2}$ always $m_k(G) = 0$. We denote by $V(M)$ the set of the vertices, which participate in the matching. We denote by a *perfect matching* of a graph the matching, which covers all the vertices: $V(M) = V$. Trivially, this is $\frac{n}{2}$-matching, and it can exist only if n

is even. Historically, the first type of matching polynomial (known in literature as *acyclic* or *matching defect* polynomial) was defined as:

$$\mu(G,x) = \sum_{k=0}^{\frac{n}{2}} (-1)^k m_k(G) x^{n-2k}$$

At least, two closely related polynomials are studied in the literature: the *matching generating* polynomial, and the *rook* polynomial [10, Chapter 8.5]

The matching generating polynomial is defined by:

$$g(G,x) = \sum_{k=0}^{\frac{n}{2}} m_k(G) x^k$$

The rook polynomial was introduced by J.Riordan [13] and is discussed in detail in [4, Chapter 1].

Let G be a spanning subgraph of the complete bipartite graph $K_{n,n}$. We define the rook polynomial $\rho(G,x)$ by

$$\rho(G,x) = \sum_{k=0}^{n} m_k(G) x^{n-k}.$$

Note that the number of vertices in G is now $2n$.

For the polynomials $\mu(G,x)$, $g(G,x)$ and $\rho(G,x)$ we have the following relations from [4, Chapter 1].

Proposition 4.

$$\mu(G,x) = x^n g(G,(-x^{-2})) = \rho(G,x^2)$$

Finally the multivariate matching polynomial introduced in [5] is defined as

$$M(G,\bar{x},\bar{y}) = \sum_{\substack{M \subseteq E, \\ M \text{ is a matching}}} \prod_{e=(u,v)\in M} y_e x_u x_v \tag{4}$$

3 The most general matching polynomial

We start from the recurrence relation satisfied by the matching defect polynomial. Let $G = (V,E)$ be a simple undirected graph without loops. Let $e = \{u,v\} \in E$ be an edge. Let $G_{-e} = (V, E \setminus \{e\})$ denote the graph with removed edge e and G_{-u-v} denote the induced subgraph of G without vertices u and v. Let $G_1 \sqcup G_2$ denote the disjoint union of two graphs G_1 and G_2. Let E_1 denote a single vertex, and $\emptyset$ denote the graph without vertices. We have seen that the matching generating polynomial satisfies the recurrence relations (1). Similar recurrence relations hold for the matching defect and the rook polynomials.

We are looking for a general recursion scheme common to all the matching polynomials and ask under what conditions this defines a unique most general polynomial. By introducing new coefficients and weights on edges and vertices of G we look at the following recursion scheme:

Empty graph: $P(\emptyset) = 1$

Singleton graph: $P(E_1^v) = x_v$

Disjoint union: $P(G_1 \sqcup G_2) = t \cdot P(G_1) \cdot P(G_2)$

Edge reduction: $P(G) = z_e \cdot P(G_{-e}) + y_e \cdot P(G_{-u-v})$

For the disjoint union and the empty graph there are no weights to be taken into account. We could also introduce an indeterminate for the empty graph, but the general results will be essentially (up to substitution) the same. If we have a fixed ordering $<_{E(G)}$ of the edges of the graph G this recursion scheme defines a unique polynomial in the indeterminates $\bar{x}, \bar{y}, \bar{z}$ and t. We would like the polynomial to be independent of the ordering on the edges. Such an order independence property is called in the literature also *confluence property* or *Church-Rosser property.*

Lemma 5. *The recursion scheme has the confluence property iff*

(i) $t = 1$, *and*

(ii) $z_e = 1$ *for all* $e \in E(G)$ *or* $y_e = 0$ *for all* $e \in E(G)$.

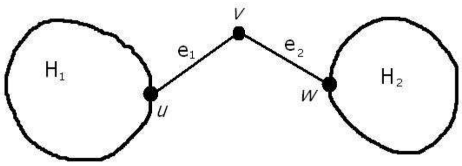

Fig. 1: Applying confluence restriction.

Proof. The conditions are necessary: For $t = 1$ we just apply the recursion for the disjoint union with an empty set. The case of $z_e = 1$, or $y_e = 0$ is evident from Fig.1.
To see that the conditions are sufficient it is enough to see that the order of applying the recursion at any two steps is interchangeable. □

In the case $y_e = 0$ we obtain the trivial polynomial U_0. In the sequel we are interested in the non-trivial polynomials.

Now we rewrite the recurrence relation using Lemma 5:

$$\begin{aligned} P(G, \overline{x}, \overline{y}) &= P(G_{-e}, \overline{x}, \overline{y}) + y_e \cdot P(G_{-u-v}, x, y) \\ P(G_1 \sqcup G_2, \overline{x}, \overline{y}) &= P(G_1, \overline{x}, \overline{y}) \cdot P(G_2, \overline{x}, \overline{y}) \\ P(E_1^v, \overline{x}, \overline{y}) &= x_v; \\ P(\emptyset, \overline{x}, \overline{y}) &= 1; \end{aligned} \tag{5}$$

Additionally, by multiple application of the edge reduction rule, we can obtain the vertex reduction rule. Let $u_1, u_2, ...u_d$ be the neighbors of v, and $e_i = u_i, v$. Then,

$$P(G, \bar{x}, \bar{y}) = x_v \cdot P(G, \bar{x}, \bar{y}) + \sum_{i=1}^{d} y_{e_i} P(G_{-v-u_i}, \bar{x}, \bar{y}) \tag{6}$$

The most general matching polynomial - proof of Theorem 1 We need to show that the following explicit definition provides the same sentence as the recursive one (5).

$$U(G,\bar{x},\bar{y}) = \sum_{\substack{M\subseteq E,\\ M \text{ is a matching}}} \left[\left(\prod_{e\in M} y_e \right) \left(\prod_{v\in V\setminus V(M)} x_v \right) \right] \tag{7}$$

Here: $G(V,E)$ is a simple loop-free graph with both vertices and edges labeled; every vertex $v \in V$ has corresponding variable x_v; every edge $e \in E$ has corresponding variable y_e; $V(M) \subseteq V$ is a subset of vertices touched by edges of matching M; The summation is over all the possible matchings, including the empty one.

Proof. We use induction on the number of vertices $|V|$:
Base: for graphs of size $n \leq 2$ we have:

- $U(\emptyset,\bar{x},\bar{y}) = 1$
- $U(E_1^v,\bar{x},\bar{y}) = x_v$;
- $U(E_2^{uv},\bar{x},\bar{y}) = U(E_1^u,\bar{x},\bar{y}) \cdot U(E_1^v,\bar{x},\bar{y}) = x_u x_v$
- $U(K_2^{uv},\bar{x},\bar{y}) = U(G_{-e},\bar{x},\bar{y}) + y_e \cdot U(\emptyset,\bar{x},\bar{y}) = x_u x_v + y_e$

Here E_1^v and E_2^{uv} denotes the graph with isolated vertices u, v respectively. In all those cases we have $P(G,\bar{x},\bar{y}) = U(G,\bar{x},\bar{y})$.
Closure: we assume the equation (7) holds for all the graphs with at most n vertices, and proof that it holds also for the graphs with $n+1$ vertices. Let G be a graph with $n+1$ vertices and v one of its vertices, with neighbors $u_1, u_2, ..., u_d$. Then we apply the vertex reduction rule (6) and the inductive assumptions and obtain:

$$\begin{aligned} P(G,\bar{x},\bar{y}) &= x_v \cdot \sum_{\substack{M\subseteq E(G_{-v}),\\ M \text{ is a matching}}} \left[\left(\prod_{e\in M} y_e \right) \left(\prod_{v\in V\setminus V(M)} x_v \right) \right] + \\ &+ \sum_{i=1}^{d} y_{e_i} \cdot \sum_{\substack{M\subseteq E(G_{-v-u_i}),\\ M \text{ is a matching}}} \left[\left(\prod_{e\in M} y_e \right) \left(\prod_{v\in V\setminus V(M)} x_v \right) \right] \end{aligned}$$

The first part summarizes over all the matchings in G, which do not include v, and the second - over all the matchings including v. Hence, we have $P(G,\bar{x},bary) = U(G,\bar{x},bary)$. □

The most general unlabeled version of matching polynomial can be obtained by restriction of number of vertex and edge weights to 1:

$$U_{1,1}(G,x,y) = \sum_{\substack{M\subseteq E,\\ M \text{ is a matching}}} \left[\left(\prod_{e\in M} y \right) \left(\prod_{v\in V\setminus V(M)} x \right) \right] \tag{8}$$

The Heilmann-Lieb polynomial 4 is now obtained from $U(G,\bar{x},\bar{y})$ by the following.

Proposition 6. $M(G,\bar{x},\bar{y}) = \left(\prod_{v\in V(G)} x_v\right) \cdot U(G,\bar{x}',\bar{y})$ *with* $x_v{}' = \frac{1}{x_v}$.

4 Complexity

Graph homomorphisms. Let C be a set of k colors, where k is a constant. Let $H = (C, E_H)$ be a graph with vertex set C. Given a graph $G = (V, E)$ with vertex set V, a map $X : V \mapsto C$ is called an homomorphism from G to H, or an H-coloring of G if for all $v_1, v_2 \in V$, $\{v_1, v_2\} \in E \to \{X(v_1), X(v_2)\} \in E_H$. Let $\Omega_H(G)$ denote the set of all H-coloring of G. The problem of counting homomorphisms is to determine $|\Omega_H(G)|$ given a graph G for some fixed H. We refer to this problem from now on as $\sharp\mathbf{H}$.

Weighted graph homomorphisms. M. Dyer and C. Greenhill [3] provide a proof that counting graph homomorphisms is $\sharp\mathbf{P}$-complete for every H containing connected component, which is not clique with all the loops, and not full bipartite without loops. They introduce a symmetric matrix A of weights of edges of H, such that $A_{ij} = 0$ iff $\{i, j\} \notin E_H$, and diagonal matrix D of positive weights of vertices of H. Then, they define the weight $w_{A,D}(X)$ of some particular H-coloring X, as

$$w_{A,D}(X) = \prod_{\{v,w\} \in E} A_{X(v)X(w)} \prod_{v \in V} D_{X(v)X(v)}$$

The problem of counting weighted graph homomorphisms is, given a graph G, to determine for some fixed A and D the value of $Z_{A,D}(G)$ defined by

$$Z_{A,D}(G) = \sum_{X \in \Omega(G)} w_{A,D}(X)$$

This problem is referred from now on as $EVAL(A, D)$. For the case of matrix D restricted to be a unit matrix, the problem is referred as $EVAL(A)$.

Complexity of evaluation of $g(G, x)$. We use some ideas from [3] concerning the complexity of counting H-colorings of graphs. Our new key observation is that by allowing indeterminates in the matrices A and D we can interpret $Z_{A,D}(G)$ as polynomials in these indeterminates.

The matching polynomial $g(G, x)$ is obtained by observing that a matching of a graph is an independent set in the corresponding line graph.

Proposition 7. *The number of independent sets in graph G is equal to the number of H-colorings of the same graph G, when H consists of 2 vertices v_1 and v_2, and 2 edges (v_1, v_2) and (v_2, v_2).*

Proof. We define a bijection between independent sets and H-colorings as follows: given an independent set $U \subseteq V$, the coloring is $X(v) = v_1$ iff $v \in U$ and $X(v) = v_2$ otherwise. It is easy to see that every independent set gives a rise to a legal H-coloring and vice-versa. ☐

Proposition 8. *Let $G(V, E)$ be a graph and let $G'(E, E')$ be its line graph.*
Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ *and* $D = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$.
Then for any $\lambda \neq 0$:

$$Z_{A,D}(G') = g(G, \lambda)$$

Proof. We define a bijection between H-colorings of $G'(E,E')$ with weight λ^k and k-matchings of $G(V,E)$ as follows: given k-matching $M \subseteq E$, the coloring is $X(e) = v_1$ iff $e \in M$ and $X(e) = v_2$ otherwise. It is easy to see that every k-matching gives a rise to a legal H-coloring with weight λ^k and vice-versa. □

We need the following from [3, Theorem 3.2]:

Theorem 9 (Dyer and Greenhill). *Let A be a symmetric matrix with no two linearly dependent columns. There is a polynomial-time reduction from $EVAL(A)$ to $EVAL(A,D)$, for any diagonal matrix D of positive vertex weights.*

Now we restrict A and D to our very specific matrices mentioned in the proposition 8. On the other hand we allow arbitrary real values for λ.

Proposition 10. *Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$. Then for any $\lambda \neq 0$, there is a polynomial-time reduction from $EVAL(A)$ to $EVAL(A,D)$.*

Proof. The proof is identical to the proof of [3, Theorem 3.2] with the following observations and changes:

(i) Lemma 3.8 holds also for negative λ, if computations are over the complex numbers, rather than the reals.

(ii) Actually, for our case we do not need [3, Theorem 3.1], which only holds for the reals and deals with the case where the matrix ADA is singular. In our case we have $ADA = \begin{pmatrix} 1 & 1 \\ 1 & \lambda+1 \end{pmatrix}$ which is singular iff $\lambda = 0$.

(iii) [3, Lemma 3.4] holds also over the complex numbers.

□

Proof of Theorem 2.
We use in our proof the following well known result by L. Valiant [16]:

Proposition 11. *Counting independent sets in graphs is #P-complete, even when restricted to line graphs.*

Proof. Assume there is a polynomial algorithm for evaluating of the generating matching polynomial $g(G,\lambda)$ on any graph G at some $\lambda = \lambda_0 \neq 0$.

By Propositions 10 and 8, there is a polynomial algorithm for $EVAL(A)$ on any line graph[1], which, according to Proposition 7, gives a rise to a polynomial algorithm for counting independent sets in any line graph, in contradiction with Valiant's results. □

[1]A graph G can be reconstructed from its line graph G' in polynomial time, cf. for instance P. Lehot [8].

5 Complexity of bivariate and multivariate matching polynomials

Proof of Theorem 3. We first deal with the bivariate case.

Lemma 12. *$U_{1,1}(G,x,y)$ is hard to evaluate at any point of $(x_0,y_0) \in \mathbb{R}^2$ such that $y_0 \neq 0$.*

Proof. It turns out that the most general matching polynomial in its unlabeled version, discussed in the previous section,and the matching generating polynomial, are algebraically reducible: $U_{1,1}(G,x,y) = x^n \cdot g(G, \frac{y}{x^2})$.
We consider three cases: (i) $x_0 \neq 0$; $y_0 \neq 0$, (ii) $x_0 = 0$; $y_0 \neq 0$, and (iii) $y_0 = 0$.
(iii) is easy: $U_{1,1}(G,x,0) = x^n$.
(ii) is hard to evaluate according to Valiant's result about $\sharp\mathbf{P}$-hardness of counting perfect matchings: $U_{1,1}(G,0,y) = m_{n/2}(G)y^{n/2}$, and therefore the number of perfect matchings is given by $m_{n/2} = \frac{U_{1,1}(G,0,y)}{y^{n/2}}$.
Finally for (i), if we can evaluate $U_{1,1}(G,x,y)$ at $x = x_0$ and $y = y_0$ in polynomial time, then we can also evaluate $g(G, \frac{y}{x^2})$ in polynomial time using
$g(G, \frac{y}{x^2}) = \frac{U_{1,1}(G,x,y)}{x^n}$. □

Proof of Theorem 3, continued.

Proof. Assume we have an algorithm, which evaluates $U_{k_v,k_e}(G,\overline{x},\overline{y})$ at some point $(\overline{x_0},\overline{y_0}) \in \mathbb{R}^{k_e+k_v}$.

We apply this algorithm to the graph G' obtained from G by giving all the edges and all the vertices the same weight respectively. This can be done in polynomial time. We get that

$$U_{k_v,k_e}(G',\overline{x},\overline{y}) = U_{1,1}(G,x,y)$$

$U_{1,1}(G,x,y)$ is hard to compute using Lemma 12, provied ($\overline{y_0} \neq 0$). □

We can extend this result for every function, which can be obtained from the polynomial $U_{k_v,k_e}(G,\overline{x},\overline{y})$ by pre-factoring and substitution of variables. To be more precise, let $\mathcal{G}$ denote the class of all finite graphs, and $f_1 : \mathcal{G} \times \mathbb{R}^{k_v+k_e} \mapsto \mathbb{R}$ be polynomial time computable in the size of G with unit cost for the indeterminates. A variable substitition is a function which maps the indeterminates from $\bar{x}$ into terms of $\mathbb{R}[\bar{x}]$.

Proposition 13. *Assume $s_1, s_2, \ldots, s_{k_v}$ and $s'_1, s'_2, \ldots, s'_{k_e}$ are variable substitutions, replacing variables by terms in the polynomial ring.*
Let F be a graph polynomial obtained from $U_{k_v,k_e}(G,\overline{x},\overline{y})$ by

$$F(G,\overline{x},\overline{y}) = f_1(G,\overline{x},\overline{y}) \cdot U_{k_v,k_e}(G, s_1(\overline{x},\overline{y}), \ldots, s_{k_v}(\overline{x},\overline{y}), s'_1(\overline{x},\overline{y}), \ldots, s'_{k_e}(\overline{x},\overline{y}))$$

Then, the polynomial $F(G,\overline{x},\overline{y})$ is $\sharp\mathbf{P}$-hard to evaluate at every point of (k_v+k_e)-ary space, in which $f_1 \neq 0$ for arbitrary graphs, and $s_2 \neq 0$.

Proof. Assume this is wrong. Then we can efficiently evaluate $U_{k_v,k_e}(G, s_1(\overline{x},\overline{y}),\ldots,s_{k_v}(\overline{x},\overline{y}), s'_1(\overline{x},\overline{y}),\ldots,s'_{k_e}(\overline{x},\overline{y}))$ at any point by:

$$U_{k_v,k_e}(G, s_1(\overline{x},\overline{y}),\ldots,s_{k_v}(\overline{x},\overline{y}), s'_1(\overline{x},\overline{y}),\ldots,s'_{k_e}(\overline{x},\overline{y})) = \frac{1}{f_1(G,\overline{x},\overline{y})}\cdot F(G,\overline{x},\overline{y}).$$

But this contradicts Theorem 3. □

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