ENCOUNTERS WITH A. MOSTOWSKI

J.A. MAKOWSKY

Abstract. We trace our encounters with Prof. A. Mostowski, in person and in his papers. It turns out that he and his papers left a continuous mark on my own work, from its very beginning, and till today.

1. The Zürich Logic Seminar

I do not quite remember when I first heard the name of A. Mostowski, but it was in the early stages of my undergraduate studies at the mathematics and physics department of the ETH (Swiss Federal Institute of Technology) in Zürich, around 1967-1968.

This is how I got into Mathematical Logic at an early stage of my studies. When still at the Gymnasium, I spent much of my time studying philosophy. Plato, Leibniz, Kant, Cassirer, the Neokantians, and the Vienna Circle attracted my special interest. I also attended university seminars on Kant and Chomsky, before my matriculation exam (Matura). Originally I wanted to study mathematics and philosophy, but following good advise, I finally registered in 1967 to study mathematics and physics, leaving philosophy to my free time.

So it was only natural that in my first semester I participated in a seminar of the general studies department dedicated to Wittgenstein's *Tractatus Logico-Philosophicus*, organized by E. Specker[1] and G. Huber[2]. Wittgenstein was just slowly being rediscovered in German

Date: January, 1, 2007.

[1]E. Specker, born 1920, Professor of Mathematics and Mathematical Logic at ETH, Zürich, 1955-1987. Ernst Specker has made decisive contributions towards shaping directions in topology, algebra, mathematical logic, combinatorics and algorithms over the last 60 years. His collected works are published as [JLSS90].

[2]G. Huber, born 1923, was a disciple of H. Barth and K. Jaspers. 1956-1990 he held the chair of philosophy at ETH. His interest were in metaphysics, epistemology, ethics, history of philosophy (especially ancient Greek philosophy), contemporary social environmental problems, and politics of science. His magnum opus "Eidos und Existenz" was published in 1995.

Language Academia[3], and the seminar turned out to be a major event. E. Specker was also the lecturer in my first course on algebra, and he impressed me by his approach to mathematics, by his non-conventional personality, and his civil courage in the politically charged time of the Cold War and the international student revolts. In my second semester I took E. Specker's course in Mathematical Logic, and also started to attend the Logic Seminar. The Logic Seminar had a long tradition, originally started by P. Bernays and F. Gonseth in 1936. After the war E. Specker, J.R. Büchi, C. Böhm and G. Müller were regulars, and W. Craig, R. Sikorski and H. Wang were among its guests. In 1955 E. Specker joined as newly appointed professor, and in 1966 H. Läuchli.

The seminar was always held on Monday between 5pm and 7pm. When I studied between 1967 and 1974 it was known as the Läuchli-Specker Seminar, but Prof. P. Bernays, who was born in 1887, still attended and commented the sessions. The seminar was also an undergraduate seminar, where students could get credit, but most of the participants were regulars, undergraduates, graduates and postgraduates, including other logicians from Zürich (H. Bachmann, E. Engeler, P. Finsler, R. Fittler and B. Scarpellini) and other occasional guests. The seminar continued its activities after Specker's retirement under H. Läuchli. After Läuchli's premature death in 1997, it continued again under E. Specker, who participated uninterruptedly till the official end of the seminar in 2002. From 1973 on, E. Specker and V. Strassen also organized a joint seminar of the University of Zürich and ETH on Logic and Algorithmics[4]. Each semester the seminars were dedicated to a topic. During my studies we had seminars on such diverse topics as many-valued logics, decidable und undecidable theories

[3]Notably through Ingeborg Bachmann's essay [Bac53] and articles published in the "Kursbuch", a then newly founded highly influential periodical published by H.M. Enzensberger [Enz67]. Both Ingeborg Bachmann and H.M. Enzensberger are outstanding literary figures of German language literature of the post-war period, and both are among its most distinguished poets. Bachmann's poetry is available in English [Bac06]. For more information, cf. `http://en.wikipedia.org/wiki/Ingeborg_Bachmann`. and `http://en.wikipedia.org/wiki/Hans_Magnus_Enzensberger`. Enzensberger wrote in 1962 a poem on Gödel's incompleteness theorem, which also serves as the text of H.W. Henze's second violin concerto, [Mak81]. For Wittgenstein's influence on poetry, cf. [Per96].

[4]Among the participants of these seminars in the years 1970-1974 there were quite a few who are or were still active researchers more than twenty years later: W. Baur, C. Christen (1943-1994), W. Deuber (1942-1999), M. Fürer, J. von zur Gathen, J. Heintz, M. Sieveking, and E. Zachos.

after [ELTT65, TMR53], model theory after [Sho67, Chapter 5], categoricity after [Mor65], the spectrum problem, Hilbert's 10th problem, finite versions of the axiom of choice after [Mos45], non-standard analysis, intuitionism, and complexity theory, cf. [SS76]. In these seminars I was particularly impressed by the omnipresence of Polish logicians. The names and works of A. Ehrenfeucht, K. Kuratowski, J. Łoś, J. Łukasiewicz, A. Mostowski, H. Rasiowa, C. Ryll-Nardzewski, R. Sikorski, R. Suszko, and A. Tarski became all familiar to me in this period. In E. Specker's work on set theory [Spe57], and later also in H. Läuchli's early papers [Läu62], A. Mostowski's [Mos39, Mos45] plays an important role. E. Specker met A. Mostowski several times at conferences, definitely in Warsaw 1959 and in Jerusalem 1966. They held each other in very high esteem.

So it happened later that I did most of my Ph.D. work in Warsaw while being immatriculated in Zürich, much like A. Mostowski did most of his Ph.D. work in Zürich, while being immatriculated in Warsaw.

2. Personal encounters

I first met A. Mostowski personally in October 1971, I saw him last in May 1973, and all our personal encounters took place in Warsaw. As described in the previous section I knew of A. Mostowski first from the literature. In 1970 I had already read the famous monograph on set theory by K. Kuratowski and A. Mostowski [KM67], I was familiar with most of Mostowski's work in model theory, but I had been most impressed by his booklet *Thirty years of Foundational studies*, [Mos66].

In the summer of 1971 I attended the ASL Logic Colloquium in Cambridge and presented my results on the finite axiomatizability of categorical theories which constituted my M.Sc. thesis. It was there that I met first S. Shelah, although I had corresponded with him before. I also met W. Marek, P. Zbierski[5] and M. Srebrny. W. Marek suggested I should try to visit Warsaw University and spend some time there. Upon my return I discovered the existence of a Swiss-Polish student exchange program and applied, together with my friend D. Giorgetta, for a grant to spend a year in Warsaw. The Swiss office in charge of the program informed us that only one of us was awarded the grant for mathematics, and it was D. Giorgetta[6]. But D. Giorgetta discovered

[5] 1944-2002, another dear colleague who left us prematurely.

[6] Donato Giorgetta finished his Ph.D. under E. Specker in Mathematical Logic, and became later an actuarian for ZURICH Insurance. As a result of his stay in Warsaw he also became W. Marek's brother-in-law. Two more grants were awarded the same year: For studies in Slavic languages to D. Weiss, now professor at the University of Zürich. Among his many scientific activities he was also leader of a

upon his arrival, that the Polish side had selected both of us for a stay with the Logic Group in Warsaw and immediately wrote me a letter about this discovery. I traveled to Warsaw to see for myself what was going on, and with Prof. Mostowski's and Prof. H. Rasiowa's help the Swiss authorities were persuaded to change their decision. Officially, I was to be a guest to H. Rasiowa's group with A. Wasilewska as my formal host. It was then, in October 1971, that I first met A. Mostowski personally.

I spent the period from April 1972 till July 1972 as a Swiss-Polish exchange student during the preparation of my Ph.D. thesis in Warsaw. I was accompanied by my first wife. Prof. Mostowski took personal care of us. He took us himself to register at the police at the Mostowski Palace. I remember the astonished face of the clerk at the information desk, when A. Mostowski introduced himself. "Are you, is it possible, Panie Professorze, are you a relative of the original owner of this palace?" "No, no, I am not, just a coincidence of names", he replied, and inquired where we had to register. Only recently I learned that this was not quite true. A. Mostowski was a remote relative of Tadeusz Mostowski, after whom the palace is named[7] .

At that time I was working in model theory and tried to settle the question asked by M. Morley in [Mor65], whether there are finitely axiomatizable complete theories which were categorical in uncountable powers. I was working hard. I have been working on this before, but now had no new results. I had made some progress on this before, and the results were the content of my M.Sc. thesis [Mak71]. I just had submitted two papers with these results to Polish journals, [Mak72,

research project "Towards a History of Verbal Propaganda in Soviet Union and Socialist Poland", and for art studies to Roman Signer, by now a leading avant-garde artist. Roman Signer is well known for his explosions, his moving everyday objects and motorized vehicles. He is regarded as a representative of an expanded concept of sculpture and work and has meanwhile become a role model for a younger generation of artists.

[7]The building, at present the seat of Warsaw Metropolitan Police, is dated from 18th century. In 1762 Jan Hilzen, Minsk Governor, purchased the area on which the Palace is situated today and after demolition of a hitherto existing building he constructed a new mansion. In 1795 Tadeusz Mostowski, Hilzen's grandson, inherited the residence. In 1822 he ceded it to the Government of Polish Kingdom. In 1823 the Palace was reconstructed and adapted as a seat of Federal Commission of Internal Affairs and the Police. A well-known architect, Antoni Corazzi, designed the new shape of the Palace. The classical, monumental facade of the Palace is characterized by a beautiful break with bas-reliefs and four-column Corinthian portico. The interior hides a real architectural pearl, so called "White Ballroom", embellished with wonderful architectural profiles and ornamentation.

Mak74b]. During my first stay in Warsaw I also studied generalized quantifiers, and finally decided to change topic. I read a lot, discussed a lot of mathematics with A. Mostowski whom I saw weekly, with W. Marek, and attended the logic seminar. The seminar was held in Polish, but as there were also other foreign guests, the speakers often chose to write English on the blackboard. I studied the beginnings of abstract model theory and read everything I could get my hands on about the model theory of extensions of First Order Logic. I had become an expert in the field, but still had no serious new results. A. Mostowski's wife, Maria Mostowska, then the director of the excellent library of the Mathematical Institute of the Polish Academy of Sciences, was always very helpful and knowledgeable when I tried to dig in the hidden treasures of this library.

In this period I also came to the conclusion that my marriage reached the end of its course. When I confessed to Professor Mostowski[8] in June that my work was stalling due to a marriage crisis, he asked politely and with empathy whether it was serious. I tried to explain a bit, but after my first attempts he just remarked that he was married since before the War, and in these days there were much bigger crises and none of them did lead to the dissolution of his marriage. Neither did these crises hinder his mathematical work. Mathematics, it seemed, was his anchor and his rescue island. Mathematics was the only rock solid secure thing on earth. I remember A. Mostowski once coming into the room I shared with W. Marek, J. Onyszkiewicz and P. Zbierski, telling us calmly about an incredible incident which just had occurred. During an advanced exam a student known for his excellence had failed when asked to sketch a proof of the Fundamental Theorem of Algebra. We suggested that the student may have been nervous and scared and as a consequence may have had a blackout. "Scared ?, blackout ?", A. Mostowski asked, "even if I were woken up in the middle of the night from deep sleep, and I were asked at gun point to prove the Fundamental Theorem of Algebra, I would just go ahead and prove it". It seemed that his maxim was "I prove therefore I am".

When my marriage crisis reached its peak in July, I asked Professor Mostowski for permission to interrupt my stay in Warsaw. He was very understanding. He also told me of the planned Logic Year to be held at the newly founded Banach Center, and suggested I should be one of its invited guests and lecture about abstract model theory.

---

[8]When I used to speak to him alone, we spoke German and I addressed him as "Herr Mostowski", as he insisted that I use this more casual form.

In the following months my reading and my conversations with A. Mostowski and W. Marek bore fruit. I finally managed to prove some theorems in abstract model theory and got encouragement and suggestions from G. Kreisel, whom I had the pleasure to meet first in Zürich in the autumn of 1972. When I returned to Warsaw in February 1973 I had already most of the results which would later constitute my Ph.D. thesis, cf. [Mak73, Mak75].

My stay at the Logic Year at the Banach Center was one of the most formative periods of my scientific life. I lectured on abstract model theory for two months. Many scientific friendships were formed there and last till today. In the days where the Cold War was still the order of the day, the Logic Year provided a rare opportunity where established and young scientists from the Eastern Block and the West could dedicate their time to do mathematics and socialize without the interference of world politics. Both the research conditions and the social life were excellent and left a mark on all the participants. All were extremely happy about this. So it seemed at least. We later learned that A. Mostowski was reprimanded by the political authorities in Poland for having created a too liberal atmosphere. Be that as it may, A. Mostowski and H. Rasiowa were excellent hosts, and in spite of his various duties and obligations, A. Mostowski attended my lectures and continued to show interest for my work and discuss it with me. During that period I also had a very intensive correspondence with G. Kreisel in Stanford, and in April 1973 Kreisel sent a telegram offering me a visiting assistant professor position in Stanford starting in October 1973. I left Warsaw in June 1973. I never thought that this was the last time I would see A. Mostowski. He had given me a lot as a human being and as scientist. He incorporated the old ideal of a humble but proud person serving science and respecting people as they are. He was always ready to encourage curiosity and true talent. He was suspicious of showmanship and preferred to let mathematics shine in its own light. His understatement concerning his own work reaches a peak in his booklet *Thirty years of Foundational studies*, [Mos66], where according to the index of authors he refers only five times to his own work, among which once for pointing out an error in one of his papers.

In the following sections I will discuss those of A. Mostowski's papers in more detail which had a direct influence on my own work.

3. Categoricity in power

A first-order theory T is categorical if all of its models are isomorphic. Because of the compactness theorem for First Order Logic this can only be the case if the model unique up to isomorphism is finite. For infinite models, J. Łoś, and independently R. Vaught, [Vau57], introduced the notion of categoricity in power: A first-order theory T is categorical in an infinite cardinal κ if all of its models of cardinality κ are isomorphic. R. Vaught uses the notion to prove decidability of various theories. His method is now known under the heading of Vaught's Test. In [Łoś54] J. Łoś gives examples for this newly defined notion and states some unsolved problems concerning it.

The first papers relating to these questions all come from the Polish school. In 1956 A. Ehrenfeucht and A. Mostowski [EM56] introduced the notion of indiscernibles. These ideas were further developed by A. Ehrenfeucht, [Ehr57], where he shows (using the general continuum hypothesis) that a theory categorical in an uncountable successor cardinal cannot define a connected antisymmetric relation of any finite arity. In 1959 C. Ryll-Nardzewski [RN59], gave a characterization of countable theories categorical in $\aleph_0$. This result was also independently found by E. Engeler [Eng59] and L. Svenonius [Sve57].

In 1969 in the Zürich Logic Seminar we have read the fundamental paper by M. Morley on categoricity in power [Mor65]. This is probably the most influential paper in model theory written so far. It builds on technical work of the Polish schools in Warsaw (Mostowski), Wrocław (Ryll-Nardzewski) and Berkeley (Tarski, Ehrenfeucht, Vaught). In it, M. Morley, proves a truly deep theorem:

Theorem 1 (Morley, 1965)**.** *A countable first-order theory is categorical in one uncountable power iff it is categorical in all uncountable powers.*

In proving this, the beginnings of a rich structure theory for models of first-order theories is laid. M. Morley also formulated extremely fruitful *open problems* emanating from his main results. The continuation of this line of research has become an important chapter in the history of logic[9]. Model theory from 1965 till today has been strongly influenced by Morley's paper, with S. Shelah its most dominant figure. But at the core of Morley's work we have his precursors, A. Ehrenfeucht and A. Mostowski.

[9]We have no space tell this fascinating story. The reader may consult the excellent book by W. Hodges [Hod93].

I also tried my luck with one of these problems, the question of finite axiomatizability of complete κ-categorical theories. At first my modest attempts looked promising, [Mak72, Mak74b]. An almost strongly minimal theory is a simple case of a theory categorical in all uncountable powers. I managed to show[10]:

Theorem 2 (Makowsky, 1971). *There are no complete finitely axiomatizable $\aleph_0$-categorical strongly minimal theories.*

A theory categorical in all uncountable powers is superstable, but the converse is not true. Using an idea of H. Läuchli, my supervisor, I managed to show:

Theorem 3 (Makowsky, 1971). *There is a complete finitely axiomatizable superstable theory.*

H. Läuchli also had suggested a sufficient group-theoretic hypothesis which would enable one to construct a complete finitely axiomatizable theory categorical in all uncountable powers. The group-theoretic hypothesis is still an open problem in combinatorial group theory, cf. [Mak74b], but it did not attract the attention of the group theorists, in spite of my various efforts to popularize the problem among combinatorial group theorists. After 1972 I was stuck with these results, but so were all the many others who tried. Only after 1980 the questions were completely answered. M. Peretyat'kin [Per80] showed

Theorem 4 (Peretyat'kin, 1980). *There are complete finitely axiomatizable theories categorical in all uncountable powers.*

B. Zilber [Zil93] and G. Cherlin, L. Harrington and A. Lachlan [CHL80] showed

Theorem 5 (Zilber, 1980). *There are no complete finitely axiomatizable theories categorical in all infinite powers.*

Very recently I wrote a paper with B. Zilber, [MZ0x], relating categorical theories to graph polynomials. In it, indiscernibles again play a crucial role. So we are back to A. Ehrenfeucht and A. Mostowski, with [EM56].

4. Generalized quantifiers and infinitary logics

Inspired by A. Mostowski's lovely booklet [Mos66], I already got interested in generalized quantifiers and extensions of First Order Logic

[10]R. Vaught told me, when I presented my result in the Berkeley Logic Seminar in 1973, that he also proved it much earlier, but he had not published his proof.

during my undergraduate studies. I also read [Hen61], where L. Henkin introduced partially ordered quantifier prefixes and infinitary formulas, and presented his ideas at the Symposium on Foundations of Mathematics held in Warsaw in the Fall of 1959. It was W. Marek who pointed out to me P. Lindström's work [Lin66, Lin69]. P. Lindström proved in these papers:

Theorem 6 (Lindström, 1966)**.** *First Order Logic is the only model theoretic logic which both has the compactness and the Löwenheim-Skolem-Tarski property.*

The exact framework for this theorem is a bit complicated. For an elaborate statement of Lindström's Theorems in their proper setting, the reader may consult the introductory chapters of [BF85] by J. Barwise, H.-D. Ebbinghaus and J. Flum [Bar85, Ebb85, Flu85].

The origins of this theorem go back to A. Mostowski. In [Mos57] a generalization of quantifiers is proposed, and a few structural characterizations are given for quantifiers on pure sets, the cardinality quantifiers. In the paper, A. Mostowski also states several open problems, mostly asking whether classical theorems for First Order Logic have their corresponding counterparts for First Order Logic augmented with a generalized quantifier. This paper triggered extensive research into various generalized quantifiers, and in the following years an abundance of generalized quantifiers appeared in the literature, establishing completeness and compactness theorems, or exhibiting counterexamples. A. Mostowski approached the question of Craig's interpolation theorem for such extensions, [Mos68]. However, no general theory evolved till the work of P. Lindström.

I once asked P. Lindström how he had found his celebrated theorem. He answered that he was looking for a non-trivial application of the Ehrenfeucht-Fraïssé games, cf. [Ehr61]. What he really proved was

Theorem 7. *Given two countable structures $\mathcal{A}, \mathcal{B}$ over a finite vocabulary which are elementarily equivalent, then there exists a countable structure $\mathcal{C}$ which is an elementary extension of both $\mathcal{A}$ and $\mathcal{B}$.*

In his proof he used the ingenious encoding of the back-and-forth property of Ehrenfeucht-Fraïss'e games within the two-sorted pair $[\mathcal{A}, \mathcal{B}]$ and applied the compactness and the Löwenheim-Skolem-Tarski property. He noted further that if the vocabularies of the two structures are made disjoint, the same argument could be used to show that no generalized quantifier, as defined in Mostowski's paper, could be added to the logic. Further analyzing his argument he realized that an even more general definition of generalized quantifier would allow the same trick. So he worked out his definition of abstract logics.

As already suggested by A. Mostowski in [Mos57, Mos68], it is a natural question to ask whether there were analogous characterizations of logics involving the Craig interpolation or the Beth definability theorem. Besides generalized quantifiers one can also introduce the logic $\mathcal{L}_{\kappa,\lambda}$ with disjunctions and conjunctions of size $< \kappa$ and strings of standard quantifiers of length $< \lambda$. E. Engeler in his thesis was the first to study such infinitary first-order languages, [Eng58]. C. Karp proved a completeness theorem for the language $\mathcal{L}_{\omega_1,\omega}$, [Kar64]. E. Lopez-Escobar, [LE65] proved the Craig interpolation theorem for this logic. D. Scott, [Sco65], studied the model theory of $\mathcal{L}_{\omega_1,\omega}$ further, showing, among other results, the following theorem:

Theorem 8 (Scott, 1965)**.** *For every countable structure $\mathcal{A}$ with finite vocabulary there is a sentence $\sigma(\mathcal{A})$ in $\mathcal{L}_{\omega_1,\omega}$ which has, up to isomorphisms, only $\mathcal{A}$ as a model.*

The sentence $\sigma(\mathcal{A})$ is today called the *Scott sentence* of $\mathcal{A}$. J. Barwise, in his thesis [Bar67], showed

Theorem 9 (Barwise, 1967)**.** *There are infinitely many sublogics of $\mathcal{L}_{\omega_1,\omega}$ which satisfy the Craig interpolation theorem.*

On the other hand, A. Mostowski, [Mos68], was the first to show how to prove that certain logics do not satisfy the Craig interpolation theorem. In 1970 H. Friedman rediscovered Lindström Theorems and discussed in several widely circulated manuscripts, [Fri71a, Fri71b], the status of the Craig's interpolation theorem and Beth's definability theorem in various logics, and finally published his [Fri73].

In the light of this situation I proved, [Mak73]:

Theorem 10 (Makowsky, 1973)**.** *$\mathcal{L}_{\omega_1,\omega}$ is the smallest model-theoretic logic which satisfies the Craig interpolation theorem and contains all the Scott sentences of countable structures.*

Actually, the theorem holds for a property strictly weaker than Craig interpolation, the so called Δ-interpolation, first discussed in an abstract framework by S. Feferman, [Fef74, Fef75]. Both these papers appeared in the special issue of Fundamenta Mathematicae dedicated to A. Mostowski's 60th birthday[11]. My own Ph.D. thesis [Mak74a, MSS76, Mak75] deals with Δ-interpolation, and I owe a lot to A. Mostowski and S. Feferman, for their interest and encouragement which I received during its preparation.

---

[11]My paper [Mak74b] also appears in this issue.

By 1975 the subject of abstract model theory was well established and attracted many researchers, and I continued working in the field in collaboration with S. Shelah till 1985, [MS79, MS81, MS83]. The monumental book [BF85] was written when the field most flourished. It contains essentially all the results obtained in this field till 1984. Then the interest in the subject faded, as it became clear that many theorems depend on set-theoretic assumptions, such as the existence of large cardinals, cf. [Mak85]. However, the concepts developed in abstract model theory found new life in finite model theory, a discipline developed in theoretical computer science, under the name of *Descriptive Complexity*. The books [EF95, Imm99, Lib04] bear witness to this revival. But this is another story.

5. The spectrum problem

Descriptive complexity has its origin in the spectrum problem. A. Mostowski was among the first to study this problem as well.

We denote by $Spec(\phi)$ the set of natural numbers n such that ϕ has a model of size n. A first-order spectrum is a set of natural numbers S such that there is a first-order sentence ϕ_S over some finite vocabulary such that $S = Spec(\phi_S)$. In 1952 H. Scholz, [Sch52], posed the following problem.

Scholz's Question: Characterize the sets of natural numbers which are first-order spectra.

This problem inaugurated a new column of *Problems* to be published in the Journal of Symbolic Logic and edited by L. Henkin. Other questions published in the same issue were authored by G. Kreisel and L. Henkin. They deal with a question about interpretations of non-finitist proofs dealing with recursive ordinals and the no-counter-example interpretation (Kreisel), the provability of formulas asserting the provability or independence of provability assertions (Henkin), and the question whether the ordering principle is equivalent to the axiom of choice (Henkin)[12]. All in all 9 problems were published, the last in 1956.

The context in which Scholz's question question was formulated is given by the various completeness and incompleteness results for First Order Logic which were the main concern of logicians of the period. An easy consequence of Gödel's classical completeness theorem of 1929 states that validity of first-order sentences in all (finite and infinite) structures is recursively enumerable, whereas Church's and Turing's classical theorems state that it is not recursive. In contrast to this, it was shown in 1950 by B. Trakhtenbrot [Tra50], that validity of first-order sentences in all finite structures (f-validity) is not recursively enumerable, and hence satisfiability of first-order sentences in some finite structure (f-satisfiability) is not decidable, although it is recursively enumerable. Thus, what H. Scholz was really asking is whether one could prove anything meaningful about f-satisfiability besides its undecidability.

The first to publish a paper in response to H. Scholz's problem was G. Asser [Ass55]. He did give a rather intricate necessary and sufficient condition for an arithmetical function $f_S(n)$ to be the characteristic function of a spectrum S. The condition shows that such a function is elementary in the sense of Kalmar. In his thesis written under the supervision of A. Mostowski in 1950, A. Grzegorczyk defined a hierarchy of low complexity recursive functions, today known as the Grzegorczyk Hierarchy. The thesis was published as [Grz53]. As Kalmar's elementary functions correspond to the third level $\mathcal{E}^3$ of the Grzegorczyk Hierarchy, we have

Theorem 11 (Asser, 1955)**.** *If f_S is the characteristic function of a first-order spectrum S, then $f_S \in \mathcal{E}^3$.*

Furthermore, G. Asser showed that not every characteristic function $f \in \mathcal{E}^3$ is the characteristic function of a first-order spectrum. Asser also noted that his characterization did not establish whether the complement of a spectrum is a spectrum.

[12]Incidentally, A. Mostowski had published a paper answering this question in 1939, [Mos39]. The paper apppeared in Fundamenta Mathematicae, but was overlooked and is not referenced in the Mathematical Reviews. But it was reviewed in the Journal of Symbolic Logic the same year, JSL 4 (1939) pp. 129-130 by A.A. Bennett. This is not surprising, as the Nazis effectively stopped the functioning of Fundamenta Mathematicae in 1939, and its circulation was made almost impossible till the end of World War II. In the first number of Fundamenta Mathematicae after the War the victims of the War and the Nazi atrocities are listed.

Asser's Question: Is the complement of a first-order spectrum a first-order spectrum?

About the same time, A. Mostowski [Mos56] also considered the problem. He showed

Theorem 12 (Mostowski, 1956)**.** *All sets of natural numbers, whose characteristic functions are in the second level of the Grzegorczyk Hierarchy $\mathcal{E}^2$, are first-order spectra.*

Further important work on the spectrum problem is contained in J. Bennett's thesis, [Ben62], which was never published and contains a wealth of interesting results. Scholz's original question was finally answered in 1972, after twenty years, when N. Jones and A. Selman related first-order spectra to non-deterministic time bounded Turing machines. Their result was first published in a conference version in 1972, [JS72], and the journal version appeared in 1974, [JS74]. Let $\mathbf{NE} = \bigcup_{c\geq 1} \mathbf{NTIME}(2^{c\cdot n})$ the family of sets of natural numbers recognizable in exponential time by non-deterministic Turing machines. **coNE** denotes the family of sets of natural numbers S such that $\mathbb{N} - S \in \mathbf{NE}$. N. Jones and A. Selman showed

Theorem 13 (Jones and Selman, 1972)**.** *A set $S \subseteq \mathbb{N}$ is a first-order spectrum if and only if $S \in \mathbf{NE}$.*

Corollary 14. *First order spectra are closed under complementation if and only if if and only if $\mathbf{NE} = \mathbf{coNE}$.*

This shows why Asser's question could not be answered easily. The question whether $\mathbf{NE} = \mathbf{coNE}$ is an outstanding open problem of complexity theory, cf. [GJ79, Joh90]. It is related in spirit to the more famous question of whether $\mathbf{NP} = \mathbf{coNP}$, and for both it is widely believed that the answer is negative. However, a positive answer to $\mathbf{NE} = \mathbf{coNE}$ would have less dramatic consequences.

The characterization of spectra using non-deterministic complexity classes was independently found also by C. Christen and R. Fagin. Claude Christen[13]'s thesis, [Chr74] remains unpublished, and only a small part was published in German [Chr76]. Christen discovered all his results independently, and only in the late stage of his work his attention was drawn to Bennett's work [Ben62] and the paper of Jones and Selman [JS72]. It turned out that most of his independently found

[13]Claude Christen, born 1943, joined the faculty of CS at the University of Montreal in 1976 and died there, a full professor, prematurely, on April 10, 1994.

results were already in print or published by R. Fagin after completion of C. Christen's thesis.

Ronald Fagin's thesis is a treasure of results dealing also with generalized spectra, which are models of existential second order sentences. For a published account of his work, cf. [Fag74, Fag75].

R. Fagin's main result is:

Theorem 15 (Fagin, 1975)**.** *A class of finite structures K is recognizable by a polynomial time non-deterministic Turing machine if and only if it is definable by an existential sentence in Second Order Logic.*

This theorem is rightly seen as the beginning of a new discipline: Descriptive Complexity Theory, [EF95, Imm99, Lib04]. A very personal account of its evolution and of Fagin's work in finite model theory is given in [Fag90]. Its main purpose is to characterize low level complexity classes in terms of logical definability. Scholz's question and the early answers of G. Asser's and A. Mostowski's papers pioneered this line of investigations. So did the papers by by J.R. Büchi, B. Trakhtenbrot and C.C. Elgot, [Büc60, Elg61, Tra61]. They showed

Theorem 16 (Büchi, Elgot, Trakhtenbrot, 1960)**.** *The regular languages (sets of finite words) are exactly those languages which are definable in Monadic Second Order Logic.*

The corresponding problem for more complex sets of integers, initiated by S. Kleene, [Kle43], was also investigated by A. Mostowski already in a series of papers starting in 1947, [Mos47].

And where is my encounter with A. Mostowski in all this? It is twofold. C. Christen did his work while I was studying in Zürich and we became good friends. Both he and I learned about the spectrum problem in the Zürich Logic Seminar, where also A. Mostowski's paper was discussed. C. Christen finally presented his work in the Specker-Strassen seminar, which I also attended. It took another thirty years till I returned to the spectrum problem. In [DFL97] the following was shown:

Theorem 17 (Dirand, Fagin and Loescher, 1997)**.** *Let $S \subset \mathbb{N}$. S is ultimately periodic iff there is a first-order sentence ϕ_S with one unary function symbol and a finite number of unary relation symbols such that $S = Spec(\phi_S)$.*

Y. Gurevich and S. Shelah have generalized this to Monadic Second Order Logic, [GS03]. In [FM04], E. Fischer and I generalized this further to arbitrary vocabularies, provided the class of models is of some bounded width.

6. The Feferman-Vaught Theorem

One theorem which accompanies my mathematical work throughout, is the so called Feferman-Vaught theorem which tells us how to reduce the first-order properties of a generalized sum or product of structures to the first-order properties of its summands, respectively factors. The theorems actually extends, in the case of generalized sums, to Monadic Second Order Logic. The theorem, and especially its proof, has its origin in A. Mostowski's [Mos52]. This theorem has far reaching applications in establishing the decidability of first-order and monadic second-order theories, in automata theory, and in algorithmics. The latter two applications where thoroughly discussed in [Mak04]. Some passages of this section are taken from [Mak04].

I first explain the theorem, and for this we need some technical notation. For a vocabulary τ, $FOL(\tau)$ denotes the set of τ-formulas in First Order Logic. $SOL(\tau)$ and $MSOL(\tau)$ denote the set of τ-formulas in Second Order and Monadic Second Order Logic, respectively. A *sentence* is a formula without free variables. For a class of τ-structures K, $Th_{FOL}(K)$ is the set of sentences of $FOL(\tau)$ that are true in all $\mathfrak{A} \in K$. We write $Th_{FOL}(\mathfrak{A})$ for $K = \{\mathfrak{A}\}$. Similarly, $Th_{SOL}(K)$ and $Th_{MSOL}(K)$ denote the corresponding sets of sentences for SOL and $MSOL$. For a set of sentences $\Sigma \subseteq SOL(\tau)$ we denote by $Mod(\Sigma)$ the class of τ-structures which are models of Σ.

The Feferman-Vaught Theorem stands at the beginnings of model theory. W. Hodges, in his delightful book, [Hod93], very carefully traces the history of early model theoretic developments. Most of the references in the sequel are taken from it.

A. Tarski published four short abstracts on model theory in 1949 in the Bulletin of the American Mathematical Society [Tar49c, Tar49d, Tar49b, Tar49a]. He also had sent his manuscript of *Contribution to the theory of models, I* to E.W. Beth for publication as [Tar54]. Seemingly inspired by these, E.W. Beth published two papers on model theory [Bet53, Bet54]. In [Bet54] he, and independently R. Fraïssé in [Fra55], showed, among other things, that

Theorem 18 (Beth 1954, Fraïssé 1955)**.**
Let $\mathfrak{A}, \mathfrak{B}$ be linear orders, $\mathfrak{C} = \mathfrak{A} \sqcup_{<} \mathfrak{B}$ their ordered disjoint union. Then $Th_{FOL}(\mathfrak{C})$ is uniquely determined by $Th_{FOL}(\mathfrak{A})$ and $Th_{FOL}(\mathfrak{B})$, and can be computed from $Th_{FOL}(\mathfrak{A})$ and $Th_{FOL}(\mathfrak{B})$.

In the early fifties A. Tarski had many young researchers gathered in Berkeley, among them A.Ehrenfeucht, S. Feferman, R. Fraïssé and

R. Vaught. One of the many questions studied in this early period of logic, and the one which interests us here is the following:

Question: Let $\mathfrak{A}, \mathfrak{B}$ be τ-structures, $\mathfrak{A} \times \mathfrak{B}$ the cartesian product and $\mathfrak{A} \sqcup \mathfrak{B}$ the disjoint union. Assume we are given $Th_{FOL}(\mathfrak{A})$ and $Th_{FOL}(\mathfrak{B})$.
What can we say about $Th_{FOL}(\mathfrak{A} \times \mathfrak{B})$ and $Th_{FOL}(\mathfrak{A} \sqcup \mathfrak{B})$?
What happens in the case of infinite sums and products?

This question triggered many landmark papers and also led to the study of ultraproducts.

The Feferman-Vaught Theorem evolved as follows. In [Mos52] Mostowski proves, among other things[14] the analogue of Theorem 18 for products

Theorem 19 (Mostowski 1952).
Let $\mathfrak{A}, \mathfrak{B}$ relational structures or algebras, $\mathfrak{C} = \mathfrak{A} \times \mathfrak{B}$ their cartesian product. Then $Th_{FOL}(\mathfrak{C})$ is uniquely determined by, and can be computed effectively from, $Th_{FOL}(\mathfrak{A})$ and $Th_{FOL}(\mathfrak{B})$.

Finally, S. Feferman and R. Vaught answered the question in the outermost generality, [FV59].

S. Feferman recalls[15], A. Tarski did not really appreciate the answer given. A special case of this answer reads as follows.

Theorem 20 (Feferman and Vaught, 1959).
Let $\langle \mathfrak{A}_i, i \in I \rangle$ be structures of the same similarity type. Then the theory of the infinite cartesian product $Th_{FOL}(\prod_{i \in I} \mathfrak{A}_i)$ and the theory of the disjoint union $Th_{FOL}(\bigsqcup_{i \in I} \mathfrak{A}_i)$ are uniquely determined by the theories of $\langle Th_{FOL}(\mathfrak{A}_i), i \in I \rangle$.

A. Mostowski's proof already contains all the ingredients of the proof via reduction sequences, as used later in [FV59].

Another version also allows for the index structure to vary.

Theorem 21 (Feferman and Vaught, 1959).
Let $\mathfrak{A}$ be structure and I an index set. Then the theory of the infinite cartesian product $Th_{FOL}(\prod_{i \in I} \mathfrak{A})$ and the theory of the disjoint union $Th_{FOL}(\bigsqcup_{i \in I} \mathfrak{A})$ are uniquely determined by $Th_{FOL}(\mathfrak{A})$ and $Th_{MSOL}(I)$.

[14] In his own words: "The paper deals with the notion of direct product in the theory of decision problems. (...) [It discusses] a theory of which the primitive notions are representable as powers of certain base-relations and [reduces] all the problems concerning this theory (in particular the decision problem) to problems concerning the theory of the base relations".

[15] Personal communication, December 2000.

By combining Theorems 20 and 21 with transductions and interpretations, similar results can be stated for a wide variety of *generalized products*. In the original paper [FV59] the transductions or interpretations are hidden in an unfortunate lengthy definition of generalized products.

In a sequence of papers by A. Ehrenfeucht, H. Läuchli, S.Shelah and Y. Gurevich, cf. [Ehr61, LL66, She75, Gur79, Gur85] it emerged that Theorems 20 and 21 remain true for $MSOL$ rather than FOL in the case of the sum $Th_{MSOL}(\bigsqcup_{i \in I} \mathfrak{A}_i)$ and the multiple disjoint union $Th_{MSOL}(\bigsqcup_{i \in I} \mathfrak{A})$ but not for products.

All these extensions of A. Mostowski's approach from 1952 had far-reaching applications. The Feferman-Vaught Theorem played an important role in, or can be used to simplify, the proofs of decidability of various theories. This includes the first-order theories of Abelian groups, linear orders, and the monadic second order theories of various classes of orders and trees. There is no space here to discuss all this.

More recently, I used, together with B. Courcelle and U. Rotics the Feferman-Vaught Theorem to simplify and extend results in algorithmic graph theory, knot theory, and the computational complexity of graph polynomials, [CMR98, CMR01, Mak04].

7. Epilogue and conclusions

I have met A. Mostowski regularly only for a relatively short time in his last years. His last student, with whom I became friends, was K. Apt, who worked on infinitistic rules of proof, cf. [Apt73], a topic initiated by A. Mostowski in [Mos61]. K. Apt later worked in program semantics, where some of these ideas seemed to be on the back of his mind, [AO83]. At this time I also worked on related problems, [GFMdR85]. These papers have to do with proving termination of nondeterministic programs under various assumptions of fair execution. Termination of programs is not provable in general in the framework of first order logic, as it depends heavily on the existence of suitable well-orderings. The proof rules used in these papers reflect upon this. Is it a coincidence, that we both got involved with such questions, or is it yet another example of A. Mostowski's influence? Be that, as it may.

My own published work started in the model theory of first order theories, and then moved on to abstract model theory. In both these areas A. Mostowski's papers played an important role. From 1978 on I tried my hand in various questions in the foundations of Computer

Science, mostly in Database Theory, Program Semantics and Verification. Here, A. Mostowski's influence is rather indirect. He did not live to see the importance of Logic for Computer Science. It was left to H. Rasiowa and her students to play an important role these aspects of Logic. However, from 1995 on I returned to Logic, dealing mostly with applications of Logic to Combinatorics and Algorithmics. In these areas I found myself rereading papers by A. Mostowski and E. Specker, and finding there sources of inspiration.

I had several seminal encounters with great teachers. I only list here those related to mathematical logic. E. Specker was the first, in 1967. His lessons on mathematics and life still have an impact on me now. I owe a lot to S. Feferman, G. Kreisel, A. Lachlan, W. Marek, and A. Mostowski who directly influenced my early work. I owe a lot to my co-authors, especially S. Shelah, J. Stavi and B. Courcelle, and to my own graduate students. In this paper I tried to show, what scientific impact A. Mostowski and his papers had on my own work. I think the evidence speaks for itself.

Acknowledgements. I am indebted to W. Marek for his encouragement, insistence, patience and editorial care during the writing of this memorial essay.

References

[AO83] K.R. Apt and E.-R. Olderog. Proof rules and transformations dealing with fairness. *Sci. Comput. Programming*, 3:65–100, 1983.

[Apt73] K.R. Apt. Infinitistic rules of proof and their semantics. *Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys.*, 21:879–886, 1973.

[Ass55] G. Asser. Das Repräsentationsproblem im Prädikatenkalkül der ersten Stufe mit Identität. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 1:252–263, 1955.

[Bac53] I. Bachmann. Ludwig Wittgenstein - Zu einem Kapitel der jüngsten Philosophiegeschichte. *Frankfurter Hefte*, 8.7, 1953.

[Bac06] I. Bachmann. *Darkness Spoken. The collected poems. English and German. Translated by P. Filkins.* Zephyr Press, 2006.

[Bar67] K.J. Barwise. *Infinitary logic and admissible sets.* PhD thesis, Department of Philosophy, Stanford University, Stanford, CA, USA, 1967.

[Bar85] K.J. Barwise. Model-theoretic logics: Background and aims. In *Model-Theoretic Logics*, Perspectives in Mathematical Logic, chapter 1. Springer Verlag, 1985.

[Ben62] James H. Bennett. *On spectra.* PhD thesis, Princeton University, 1962.

[Bet53] E. W. Beth. On Padoa's method in the theory of definitions. *Indagationes Mathematicae*, 15:330–339, 1953.

[Bet54] E. W. Beth. Observations métamathématiques sur les structures simplement ordonneés. In *Applications scientifiques de la logique* *mathématique*, volume 5 of *Collection de Logique Mathématique, Serie A*, pages 29–35. Paris and Louvain, 1954.

[BF85] J. Barwise and S. Feferman, editors. *Model-Theoretic Logics*. Perspectives in Mathematical Logic. Springer Verlag, 1985.

[Büc60] J.R. Büchi. Weak second–order arithmetic and finite automata. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 6:66–92, 1960.

[CHL80] G. Cherlin, L. Harrington, and A. Lachlan. $\aleph_0$-categorical $\aleph_0$-stable structures. *Annals of Pure and Applied Logic*, 18:227–270, 1980.

[Chr74] C.-A. Christen. *Spektren und Klassen elementarer Funktionen*. PhD thesis, Department of Mathematics, ETH-Zurich, Switzerland, 1974.

[Chr76] C.A. Christen. Spektralproblem und Komplexitätstherie. In E. Specker and V. Strassen, editors, *Komplexität von Entscheidungsproblemen*, volume 43 of *Lecture Notes in Computer Science*, pages 102–126. Springer, 1976.

[CMR98] B. Courcelle, J.A. Makowsky, and U. Rotics. Linear time solvable optimization problems on graph of bounded clique width, extended abstract. In J. Hromkovic and O. Sykora, editors, *Graph Theoretic Concepts in Computer Science, 24th International Workshop, WG'98*, volume 1517 of *Lecture Notes in Computer Science*, pages 1–16. Springer Verlag, 1998.

[CMR01] B. Courcelle, J.A. Makowsky, and U. Rotics. On the fixed parameter complexity of graph enumeration problems definable in monadic second order logic. *Discrete Applied Mathematics*, 108(1-2):23–52, 2001.

[DFL97] A. Durand, R. Fagin, and B. Loescher. Spectra with only unary function symbols. In M. Nielsen and W. Thomas, editors, *CSL'97*, volume 1414 of *Lecture Notes in Computer Science*, pages 189–202. Springer, 1997.

[Ebb85] H.D. Ebbinghaus. Extended logics: The general framework. In *Model-Theoretic Logics*, Perspectives in Mathematical Logic, chapter 2. Springer Verlag, 1985.

[EF95] H. Ebbinghaus and J. Flum. *Finite Model Theory*. Springer Verlag, 1995.

[Ehr57] A. Ehrenfeucht. On theories categorical in power. *Fund. Math.*, 44:241–248, 1957.

[Ehr61] A. Ehrenfeucht. An application of games to the completeness problem for formalized theories. *Fund. Math.*, 49:129–141, 1961.

[Elg61] C.C. Elgot. Decision problems of finite automata design and related arithmetics. *Trans. Amer. Math. Soc.*, 98:21–52, 1961.

[ELTT65] Yu. Ershov, L. Lavrov, I.A. Taĭmanov, and M.A. Taĭtslin. Elementary theories. *Russian Mathematical Surveys*, 20.4:35–105, 1965.

[EM56] A. Ehrenfeucht and A. Mostowski. Models of axiomatic theories admitting automorphisms. *Fund. Math.*, 43:50–68, 1956.

[Eng58] Erwin Engeler. *Untersuchungen zur Modelltheorie*. PhD thesis, Department of Mathematics, ETH-Zurich, Switzerland, 1958.

[Eng59] E. Engeler. A characterization of theories with isomorphic denumerable models. *Not. Amer. Math. Soc.*, 6:161, 1959. Abstract.

[Enz67] H.M. Enzensberger, editor. *Kursbuch*, volume 8. Suhrkamp, 1967.

[Fag74] R. Fagin. Generalized first-order spectra and polynomial time recognizable sets. *American Math. Society Proc.*, 7:27–41, 1974.

[Fag75] R. Fagin. Monadic generalized spectra. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 21:89–96, 1975.

[Fag90] R. Fagin. Finite model theory - a personal perspective. In S. Abiteboul and P.C. Kannelakis, editors, *ICDT'90*, volume 470 of *Lecture Notes in Computer Science*, pages 3–24. Springer Verlag, 1990.

[Fef74] S. Feferman. Two notes on abstract model theory, I: Properties invariant on the range of definable relations between structures. *Fund. Math.*, 82:153–165, 1974.

[Fef75] S. Feferman. Two notes on abstract model theory, II: Languages for which the set of valid sentences is s.i.i.d. *Fund. Math.*, 89:153–165, 1975.

[Flu85] J. Flum. Characterizing logics. In *Model-Theoretic Logics*, Perspectives in Mathematical Logic, chapter 3. Springer Verlag, 1985.

[FM04] E. Fischer and J.A. Makowsky. On spectra of sentences of monadic second order logic with counting. *Journal of Symbolic Logic*, 69(3):617–640, 2004.

[Fra55] R. Fraïssé. Sur quelques classifications des relations, basées sur les isomorphisms restraints, I: Études générale. *Publications Scientifiques de l'Université d'Alger, Serie A*, 2:15–60, 1955.

[Fri71a] H. Friedman. Back and forth, $L(q)$, $L_{\infty,\omega}(q)$ and Beth's theorem. Mimeographed notes, Stanford University, 1971.

[Fri71b] H. Friedman. The Beth and Craig theorems in infinitary languages. Mimeographed notes, Stanford University, 1971.

[Fri73] H. Friedman. Beth's theorem in cardinality logics. *Israel Journal of Mathematics*, 14:205–212, 1973.

[FV59] S. Feferman and R. Vaught. The first order properties of algebraic systems. *Fund. Math.*, 47:57–103, 1959.

[GFMdR85] O. Grumberg, N. Francez, J.A. Makowsky, and W.P. de Roever. A proof rule for fair termination of guarded commands. *Information and Control*, 66.1-2:83–102, 1985.

[GJ79] M.G. Garey and D.S. Johnson. *Computers and Intractability.* Mathematical Series. W.H. Freeman and Company, 1979.

[Grz53] A. Grzegorczyk. Some classes of recursive functions. *Rozprawy Mat.*, 4:1–46, 1953.

[GS03] Y. Gurevich and S. Shelah. Spectra of monadic second-order formulas with one unary function. In *LICS'03*, pages 291–300. IEEE, 2003.

[Gur79] Y. Gurevich. Modest theory of short chains, I. *Journal of Symbolic Logic*, 44:481–490, 1979.

[Gur85] Y. Gurevich. Monadic second order theories. In *Model-Theoretic Logics*, Perspectives in Mathematical Logic, chapter 14. Springer Verlag, 1985.

[Hen61] L. Henkin. Some remarks on infinitely long formulas. In *Infinitistic Methods*, pages 87–103. Polish Scientific Publishers, 1961.

[Hod93] W. Hodges. *Model Theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1993.

[Imm99] N. Immerman. *Descriptive Complexity*. Graduate Texts in Computer Science. Springer, 1999.

[JLSS90] G. Jäger, H. Läuchli, B. Scarpellini, and V. Strassen, editors. *E. Specker, Selecta*. Birkhäuser, 1990.

[Joh90] D.S. Johnson. A catalog of complexity classes. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume 1, chapter 2. Elsevier Science Publishers, 1990.

[JS72] N. D. Jones and Alan L. Selman. Turing machines and the spectra of first-order formulas with equality. In *STOC '72: Proceedings of the fourth annual ACM symposium on Theory of computing*, pages 157–167, New York, NY, USA, 1972. ACM Press.

[JS74] N.D. Jones and A.L. Selman. Turing machines and spectra of first order formulas. *Journal of Symbolic Logic*, 39:139–150, 1974.

[Kar64] C. Karp. *Languages with expressions of infinite length*. Studies in Logic. North Holland, 1964.

[Kle43] S. C. Kleene. Recursive predicates and quantifiers. *Trans. Amer. Math. Soc.*, 53:41–73, 1943.

[KM67] K. Kuratowski and A. Mostowski. *Set Theory*. Studies in Logic. North Holland, 1967.

[Läu62] H. Läuchli. Auswahlaxiom in der algebra. *Comment. Math. Helv.*, 37:1–18, 1962.

[LE65] E.G.K. Lopez-Escobar. An interpolation theorem for denumerably long formulas. *Fund. Math.*, 57:253–272, 1965.

[Lib04] L. Libkin. *Elements of Finite Model Theory*. Springer, 2004.

[Lin66] P. Lindström. First order predicate logic with generalized quantifiers. *Theoria*, 32:186–195, 1966.

[Lin69] P. Lindström. On extensions of elementary logic. *Theoria*, 35:1–11, 1969.

[LL66] H. Läuchli and J. Leonhard. On the elementary theory of order. *Fund. Math.*, 59:109–116, 1966.

[Łoś54] J. Łoś. On the categoricity in power of elementary deductive systems and some related problems. *Colloq. Math.*, 3:58–62, 1954.

[Mak71] J. A. Makowsky. Kategorizität und endliche axiomatisierbarkeit. Master's thesis, Department of Mathematics, ETH-Zurich, Switzerland, 1971.

[Mak72] J.A. Makowsky. Note on almost strongly minimal theories. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.*, 20:529–534, 1972.

[Mak73] J.A. Makowsky. Langages engendrés à partir des formules de Scott. *C.R. hebd. Acad. Sc. Paris*, 276:1585–1587, 1973.

[Mak74a] J. A. Makowsky. *Δ-logics and generalized quantifiers*. PhD thesis, Department of Mathematics, ETH-Zurich, Switzerland, 1974.

[Mak74b] J.A. Makowsky. On some conjectures connected with complete sentences. *Fund. Math.*, 81:193–202, 1974.

[Mak75] J.A. Makowsky. Securable quantifiers, κ-unions and admissible sets. In A. Rose et al., editor, *Logic Colloquium,93*, volume 80 of *Studies in Logic*, pages 409–428. North Holland, 1975.

[Mak81] J.A. Makowsky. Münchhausen's theorem und seine bedeutung für die musik und musikwissenschaft. In H.W.Henze, editor, *Die Zeichen,* *Neue Aspekte der musikalischen Aesthetik II*, volume 2, pages 278–307. S. Fischer, Frankfurt, 1981.

[Mak85] J.A. Makowsky. Compactness, embeddings and definability. In *Model-Theoretic Logics*, Perspectives in Mathematical Logic, chapter 18. Springer Verlag, 1985.

[Mak04] J.A. Makowsky. Algorithmic uses of the Feferman-Vaught theorem. *Annals of Pure and Applied Logic*, 126:1–3, 2004.

[Mor65] M. Morley. Categoricity in power. *Trans. Amer. Math. Soc.*, 114:514–38, 1965.

[Mos39] A. Mostowski. Über die Unabhängigkeit des Wohlordnungssatzes vom ordnungsprinzip. *Fund. Math.*, 32:201–252, 1939.

[Mos45] A. Mostowski. Axiom of choice for finite sets. *Fund. Math.*, 33:137–168, 1945.

[Mos47] A. Mostowski. On definable sets of positive integers. *Fund. Math.*, 34:81–112, 1947.

[Mos52] A. Mostowski. On direct products of theories. *Journal of Symbolic Logic*, 17:1–31, 1952.

[Mos56] A. Mostowski. Concerning a Problem of H.Scholz. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 12:210–214, 1956.

[Mos57] M. Mostowski. On a generalization of quantifiers. *Fund. Math.*, 44:12–36, 1957.

[Mos61] A. Mostowski. Formal system of analysis based on an infinitistic rule of proof. In *Infinitistic Methods*, pages 141–166. Polish Scientific Publishers, 1961.

[Mos66] A. Mostowski. *Thirty Years of Foundational Studies*. Acta Philosophica Fennica. Basil Blackwell, Oxford, 1966.

[Mos68] A. Mostowski. Craig's interpolation theorem in some extended systems of logic. In *Logic, Methodology and Philosophy of Science, III*, Studies in Logic, pages 87–103. North Holland, 1968.

[MS79] J.A. Makowsky and S. Shelah. The theorems of beth and Craig in abstract model theory,I: The abstract setting. *Trans. Amer. Math. Soc.*, 256:215–239, 1979.

[MS81] J.A. Makowsky and S. Shelah. The theorems of beth and Craig in abstract model theory,II: Compact logics. *Archiv für Mathematische Logik und Grundlagen der Mathematik*, 21:13–35, 1981.

[MS83] J.A. Makowsky and S. Shelah. Positive results in abstract model theory: A theory of compact logics. *Annals of Pure and Applied Logic*, 25(3):263–300, 1983.

[MSS76] J.A. Makowsky, S. Shelah, and J. Stavi. Δ–logics and generalized quantifiers. *Annals of Mathematical Logic*, 10:155–192, 1976.

[MZ0x] J.A. Makowsky and B. Zilber. Polynomial invariants of graphs and totally categorical theories. *XXX*, xx:xx–yy, 200x. Preliminary version avaliable at `www.cs.technion.ac.il/~janos/PUBLIC/ZILBER`.

[Per80] M.G. Peretyat'kin. An example of an ω_1-categorical complete finitely axiomatizable theory. *Algebra and Logic*, 19:202–229, 1980.

[Per96] M. Perloff, editor. *Wittgenstein's ladder*. University of Chicago Press, 1996.

[RN59] C. Ryll-Nardzewski. On the categoricity in power $\leq \aleph_0$. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.*, 7:545–548, 1959.

[Sch52] H. Scholz. Problem # 1: Ein ungelöstes Problem in der symbolischen Logik. *Journal of Symbolic Logic*, 17:160, 1952.

[Sco65] D. Scott. Logic with denumerably long formulas and finite strings of quantifiers. In *Theory of Models*, Studies in Logic, pages 329–341, 1965.

[She75] S. Shelah. The monadic theory of order. *Annals of Mathematics*, 102:379–419, 1975.

[Sho67] J. Shoenfield. *Mathematical Logic*. Addison-Wesley Series in Logic. Addison-Wesley, 1967.

[Spe57] E. Specker. Zur Axiomatik der Mengenlehre (Fundierungs- und Auswahlaxiom). *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 3:173–210, 1957.

[SS76] E. Specker and V. Strassen, editors. *Komplexität von Entscheidungsproblemen*, volume 43 of *Lecture Notes in Computer Science*. Springer Verlag, 1976.

[Sve57] L. Svenonius. $\aleph_0$-categoricity in first order predicate calculus. *Theoria*, 25:173–178, 1957.

[Tar49a] A. Tarski. Arithmetical classes and types of algebraically closed and real-closed fields. *Bulletin of the AMS*, 55:63, 1949.

[Tar49b] A. Tarski. Arithmetical classes and types of boolean algebras. *Bulletin of the AMS*, 55:64, 1949.

[Tar49c] A. Tarski. Arithmetical classes and types of mathematical systems. *Bulletin of the AMS*, 55:63, 1949.

[Tar49d] A. Tarski. Metamathematical aspects of arithmetical classes and types. *Bulletin of the AMS*, 55:63–64, 1949.

[Tar54] A. Tarski. Contribution to the theory of models, I. *Indagationes Mathematicae*, 16:572–581, 1954.

[TMR53] A. Tarski, A. Mostowski, and R. M. Robinson. *Undecidable Theories*. North-Holland Publishing Company, 1953.

[Tra50] B. Trakhtenbrot. On the impossibility of an algorithm for the decision problem in finite domains. *Doklady Akademii Nauk SSSR*, 70:569–572, 1950.

[Tra61] B. Trakhtenbrot. Finite automata and the logic of monadic predicates. *Doklady Akademii Nauk SSSR*, 140:326–329, 1961.

[Vau57] R. Vaught. Applications to the Löwenheim-Skolem-Tarski theorem to problems of completeness and decidability. *Indagationes Mathematicae*, 16:467–472, 1957.

[Zil93] B. Zilber. *Uncountably Categorical Theories*, volume 117 of *Translations of Mathematical Monographs*. American Mathematical Society, 1993.

E-mail address, J.A. Makowsky: `janos@cs.technion.ac.il`

(J.A. Makowsky) Department of Computer Science,
Technion–Israel Institute of Technology,
32000 Haifa, Israel