From a Zoo to a Zoology: Descriptive Complexity for Graph Polynomials

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Abstract. We outline a general theory of graph polynomials which covers all the examples we found in the vast literature, in particular, the chromatic polynomial, various generalizations of the Tutte polynomial, matching polynomials, interlace polynomials, and the cover polynomial of digraphs. We introduce the class of (hyper)graph polynomials definable in second order logic, and outline a research program for their classification in terms of definability and complexity considerations, and various notions of reducibilities.

1 Introduction

During the last ten years I have studied questions of computability of graph polynomials, summarized in [36, 38, 34, 41, 39, 40, 37, 8]. I found uncharted territory with plenty of amazing theorems, surprising results, and the more I got into it, the more I was perplexed. I feel that we do not have a comprehensive understanding of graph polynomials, although about particular polynomials, such as the characteristic polynomial, the chromatic polynomial, the matching polynomials and the Tutte polynomial there is more known than what could be told in several books. It is noteworthy that many authors speak in their papers of *the* graph polynomial, suggesting that theirs is the one and only one worth studying. It is also noteworthy, that very few authors who study a particular graph polynomial P, have more than this particular polynomial and possibly some immediate relatives of P, in mind.

In this paper I try to sketch a research program of how a general theory of graph polynomials could be developed. The collection of graph polynomials I have gathered from the literature looks like a zoo[1]. There are prominent animals like the elephant, the giraffe, the gorilla, and there are exotic animals defying classification, like the lamprey (petromyzon marinus, not really a fish) or platypus (ornithorhynchus anatinus, not really a water bird, not really a mammal). Some animals look different, but are related, like the elephant and the rock hyrax (procavia capensis); some look alike, but are not related, like the hedgehog (erinaceus europus) and the echidna (tachyglossus aculeatus). *Zoology* is the science

[1] It was T. Zaslavsky who suggested the titel "From a zoo to a zoology" for this research program.

of comparing and classifying animals. *Graphpolynomology* would be the art of comparing and classifying graph polynomials.

2 Graph polynomials

Let $\mathcal{G}$ be the class of graphs $G = (V, E)$ without loops and multiple edges. Let $\mathcal{R}$ be a ring and $\bar{X}$ be a (not necessarily finite) set of indeterminates. A *graph polynomial* is a function

$$p : \mathcal{G} \to \mathcal{R}[\bar{X}]$$

such that for isomorphic graphs $G_1 \simeq G_2$ we have $p(G_1) = p(G_2)$. If we consider labeled graphs, the notion of isomorphism has to be correspondingly modified. If $p(G)$ takes only values 0 or 1 in $\mathcal{R}$ we speak of *graph properties*.

There are plenty of graph polynomials which have been discussed in the literature, although no systematic treatment on graph polynomials in general is available[2]. To put our results into perspective we discuss briefly four classical graph polynomials, the *chromatic polynomial* $\chi(G, \lambda)$, the *characteristic polynomial* $P(G, \lambda)$, the *acyclic generating matching polynomials* $m(G, \lambda)$ *and* $g(G, \lambda)$ and the *Tutte polynomial* $T(G, X, Y)$. For historic reasons we also discuss briefly the very first polynomial introduced into graph theory, the *edge difference polynomial*. We also add to our discussion two more recent examples, the two *interlace polynomials*, and the *cover polynomial* defined on digraphs.

The edge-difference polynomial. The historically first polynomial in graph theory was introduced by J.J. Sylvester in 1878, [51] and further studied by J. Peterson in 1891. It is the multivariate polynomial depending on the ordering of the vertices $V = \{v_1, v_2, \ldots, v_n\}$ and defined as

$$P_G(X_1, X_2, \ldots, X_n) = \sum_{\substack{i<j \\ (v_i, v_j) \in E}} (X_i - X_j)$$

This polynomial is not a graph invariant, but it was used as a tool in studying regularity and colorability questions of graphs. In particular, N. Alon and M. Tarsi [3] observed that it can be used to study list colorings. For a survey, cf. Z. Tuza [55]. In our context, however, the edge-difference polynomial does not play a prominent rôle.

The chromatic polynomial . Let $\chi(G, \lambda)$ denote the number of proper vertex colorings of G with at most λ colors. G. Birkhoff, [7], observed in 1912 that $\chi(G, \lambda)$ is, for a fixed graph G, a polynomial in λ, which is now called the *chromatic polynomial of* G. The chromatic polynomial is the oldest graph polynomial to appear in the literature, which is a graph invariant. Since then a substantial

[2] I have found over 250 entries in MathSciNet querying "graph polynomial" or "polynomial of a graph" in the review text.

body of knowledge about the chromatic polynomial of graphs and its applications has been accumulated. The recent book by F.M. Dong, K.M. Koh and K.L. Teo [18] gives an excellent and extensive survey. One of the surprising facts is a theorem of R.P. Stanley, [49], which states that $\chi(G,-1)$ is the number of acyclic orientations of G.

The Tutte polynomial. Interesting generalizations of the chromatic polynomial were introduced by H. Whitney in 1932 and W.T. Tutte in 1947. The most prominent among them is now called the *Tutte polynomial* $T(G,X,Y)$ which is a two variable polynomial from which the chromatic polynomial can be obtained via a simple substitution and multiplication with a prefactor. The exact relationship is given by

$$\chi(G,X) = (-1)^{r(G)} X^{k(G)} T(G; 1-X, 0)$$

Here $k(K)$ is the number of connected components of G and $r(G) = \mid V \mid - k(G)$.

For a modern exposition the reader is referred to [10, chapter X], [24] or [58]. Tutte's own account on how he got involved with his polynomial is very enjoyable, [54]. For this extended abstract we do not need a full definition of the Tutte polynomial. We only note that it is a polynomial in two variables related to the rank generating function of matroids. But we should note that in the years after 1980 the Tutte polynomial found important interpretations in statistical mechanics and quantum field theory, knot theory, and biology, cf. [58] and [48]. F. Jaeger in [29] showed that the Jones polynomial of knot theory on alternating knots is just an instance of the Tutte polynomial of the knot diagram viewed as a graph. L. Kauffman in [31] introduced first a generalization of the Tutte polynomial which gives the Jones polynomial for arbitrary knots. Different approaches to multivariate versions of the Tutte polynomial are discussed in [11, 48].

Other univariate graph polynomials were introduced after 1955, often first motivated by problems from chemistry and physics. The two most prominent are the characteristic and the matching polynomial (which comes in two versions).

The characteristic polynomial of a graph G, denoted by $P(G,\lambda)$ is the characteristic polynomial of the adjacency matrix M_G of the graph G, $P(G,\lambda) = \det(\lambda \cdot \mathbf{1} - M_G)$ and is completely determined by the eigenvalues of M_G, which are all real, as the matrix is symmetric.

The matching polynomials. The *acyclic polynomial of G* is the polynomial $m(G,\lambda) = \sum_k (-1)^k \cdot m_k(G) \cdot \lambda^{n-2k}$, where the coefficients $m_k(G)$ count k-matchings. A chemical point of view of these polynomials is given in [17] and [53], where algorithmic aspects are also touched. A close relative of the acyclic polynomial is the *matching generating polynomial of a graph G* defined as $g(G,\lambda) = \sum_k m_k(G)\lambda^k$, where $m(G,\lambda) = \lambda^n g(G,(-\lambda^{-2}))$. An excellent survey on these two matching polynomials may be found in [25, Chapter 1] and [35, Chapter 8.5]. We shall refer to both as *matching polynomials*. Somewhat surprisingly we have $m(G,\lambda) = P(G,\lambda)$ if and only if G is a forest.

The interlace polynomials. Two of the more interesting recent graph polynomials were introduced by R. Arratia, B. Bollobás and G. Sorkin in [5, 6]. They are called *interlace polynomials* and there is a univariate and a two-variable version. M. Aigner and H. van der Holst [2] studied these polynomials from a matrix point of view and derived various combinatorial interpretations.

The cover polynomial of directed graphs. An interesting recent graph polynomial on directed graphs is the *cover polynomial* introduced by F.R.K. Chung and R.L. Graham [14], and independently in the context of rook polynomials, by Gessel, [23]. In [14] it is presented as an attempt to create a Tutte-like polynomial for directed graphs, and is closely related to the chromatic polynomial. There is also related work by R.P. Stanley [50] and T. Chow [13], and very recently, by P. Pitteloud [46].

A zoo of graph polynomials Without giving all the necessary references, we list a few of the many graph polynomials we found in the literature. There are variations of matching polynomials, like the rook polynomials, cf. [47]. There are polynomials counting the number of (induced) subgraphs of a certain kind. Let $\mathcal{H}$ be a graph property and put $ind_{\mathcal{H}}(G, k)$ be number of induced subgraphs of size k having property $\mathcal{H}$ in a given graph G. Then we can look at the polynomial

$$gen_{\mathcal{H}}(G, \lambda) = \sum_k ind_{\mathcal{H}}(G, k)\lambda^k$$

For $\mathcal{H}$ consisting of all the K_n's (cliques), E_n's (isolated points), C_n's (cycles), P_n's (paths) the corresponding polynomials have been studied. Instead of graph properties one can also use subsets of graphs with desirable properties such as vertex covers, coverings with subgraphs of special type etc. Some of these have been studied in a very general context as Farrell polynomials, cf. [22, 39]. There are interlace polynomials [5], Go-polynomials [21], Penrose polynomials [1], and many more. It is worth searching for all these at `scholar.google.com`.

3 Recursive definitions

One of the outstanding features of the more prominent graph polynomials are recursive definitions with respect to some order independent way of deconstructing the input graph. The main paradigm stems from the chromatic polynomial and the Tutte polynomial. We first note that for the chromatic polynomial we have

$$\chi(G) = \chi(G - e) - \chi(G/e)$$

where e is an edge and $G - e$ and G/e denotes the deletion respectively contraction of the edge e. Furthermore, for the disjoint union we have $\chi(G_1 \sqcup G_2) = \chi(G_1) \cdot \chi(G_2)$, for the graph consisting of n isolated verices E_n $\chi(E_n) = \lambda^n$. One easily verifies that $\chi(G - e - f) = \chi(G - f - e)$, $\chi(G/e/f) = \chi(G/f/e)$, $\chi(G - e/f) = \chi(G/f - e)$ and $\chi(G/e - f) = \chi(G - f/e)$, which is a kind of *Church-Rosser property* or *confluence property*. From this we get a recursive definition of $\chi(G)$ by choosing any order of the edges. Similarly, for the Tutte polynomial we have

$$T(G, X, Y) = \begin{cases} X \cdot T(G/e, X, Y) & \text{if } e \text{ is a bridge} \\ Y \cdot T(G - e, X, Y) & \text{if } e \text{ is a loop} \\ T(G/e, X, Y) + T(G - e, X, Y) & else \end{cases}$$

together with multiplicativity for disjoint unions and $T(E_n, X, Y) = 1$. Again one can verify the Church-Rosser property, and gets a recursive defintion for the Tutte polynomial. In [11] this recursive defintion was used as the starting point for the defintion of the *colored Tutte polynomial*.

In [5, 6] similar but more complicated recursive definitions are given for the various interlace polynomials. Here the recursion also involves a *pivot operation* G^{ab} on a graph G and two vertices a, b. In [14] such recursive definitions are given for the cover polynomial of directed graphs. Even for the matching polynomial one can give such rules: for the acyclic polynomial we have $m(E_n) = \lambda^n$, multiplicativity for disjoint unions and, for an edge $e = (u, v)$

$$m(G, \lambda) = m(G - e) - m(G - u - v, \lambda)$$

It is a curious fact that the literature does not explore this aspect of the matching polynomial further, and does not even note the Church-Rosser property, although it is easily verified.

The recursive definition of a graph polynomial gives an easy but slow way of computing the graph polynomial. As the recursion unwinds a number of subtasks exponential in the size of the graph has to be computed. But the nature of the recursion usually gives deeper insights. Furthermore, the various Tutte and interlace polynomials can be proven to be, in a certain sense, the most general graph polynomials satisfying their specific recursion scheme. Similar characterizations very recently shown for generalizations of the cover and the matching polynomials in [15].

Although some particular recursion schemes based on the behaviour of the graph polynomial under deletion of vertices or edges, contractions of edges, pivoting, etc. are well studied, no general theory has emerged so far, and it remains an interesting challenge to develop a satisfactory framework for recursion schemes for graph invariants.

4 Complexity

It is natural to ask how difficult it is to compute the various graph polynomials. The characteristic polynomial is computable in polynomial time using classical algorithms for the determinant of a matrix. Computing the chromatic polynomial is $\sharp\mathbf{P}$-hard due to its connection to counting colorings. This also makes computing the Tutte polynomial $\sharp\mathbf{P}$-hard. The same is true for the acyclic polynomial due to its connection to counting matchings, cf. [57]. Furthermore, F. Jaeger, D. Vertigan and D. Welsh, [30], have characterized completely the points (a, b) in the complex plane $\mathbb{C}^2$, where evaluating the Tutte polynomial $T(G, a, b)$ is difficult for arbitrary graphs. J. Oxley and D. Welsh [45] also noted that the Tutte polynomial for series–parallel graphs, which are graphs of tree-width at most 2, can be computed in polynomial time. This was extended to arbitrary fixed tree-width k independently by A. Andrejak [4] and S. Noble [43], and therefore also holds for the chromatic polynomial. Actually, they showed that computing the Tutte polynomial is fixed parameter tractable **FPT** on graph classes of tree-width at most k. In other words, it is computable in time $f(k)n^d$, where d is independent of k and n is the size of the input. For an extensive discussion of the complexity class **FPT**, cf. [19].

5 Enter logic

Already B. Courcelle in [16] observed that graph properties definable in Monadic Second Order Logic (MSOL) are in **FPT** on graph classes of tree-width at most k, cf. also [19]. This approach was extended to graph polynomials by the author in [37]. The fact that the Tutte polynomial is in **FPT** also follows from [37], which also covers the acyclic and the matching polynomial and a wide range of other graph polynomials where summations are restricted to families of subsets of edges which are definable in MSOL. Without going into the more delicate details, such polynomials are in a polynomial ring $\mathcal{R}[\bar{X}]$ and are of the form

$$g(G, \bar{X}) = \sum_{A:\phi(A)} \prod_{v:v \in A} t(v)$$

where A is a unary relation variable, $\phi(A)$ is an MSOL-definable property of the graph, and $t(v)$ is a term in $\mathcal{R}[\bar{X}]$ which may depend uniformly on v. We speak here of MSOL-polynomials.

In the same paper [37], it is shown that, in combination with the work of P. Seymour and S. Oum [44], graph polynomials, where summations are restricted to families of subsets of vertices which are MSOL-definable, are in **FPT** for graph classes of clique-width at most k. However, this method does not apply to the chromatic polynomial, the Tutte polynomial and the matching polynomials.

6 The need for a general framework

Although there is a large *zoo of graph polynomials*, there is no *general zoology.* We offer here an outline of what such a zoology could look like. Our general framework is somewhat inspired by [26], but both the scope and the emphasis are quite different. Initial work in this direction may be found in [41].

The literature on Turing complexity or algebraic complexity does not provide a natural framework to develop a complexity theory of graph polynomials. In particular there is no agreed upon notion of *efficient reducibility between graph* *polynomials*. The existing framworks do allow the formulation of hardness results by reductions to $\sharp\mathbf{P}$-hard instances which are easily recognizable as polynomial time computable in an intuitive sense. But in the existings frameworks no *hardest graph polynomial* could be identified.

6.1 SOL-polynomials as a complexity class

We propose a class of graph polynomials which covers all the examples, so far, from the literature, which has sufficient closure properties, and which guarantees that all its members are computable in exponential time in the unit cost computational model over the underlying ring $\mathcal{R}$, in the sense of the Blum-Shub-Smale model of computation, [9]. For this purpose we allow (full) Second Order Logic in the definition of the polynomials:

$$g(G,\bar{X}) = \sum_{R:\psi(R)} \prod_{\bar{v}:\bar{v}\in R} t(\bar{v})$$

R now can be a relation variable of any fixed arity, and ψ any formula of Second Order Logic (SOL). We speak then of SOL-polynomials. If $\mathcal{L}$ is a sublogic of SOL and ψ is an $\mathcal{L}$ formula, we speak of $\mathcal{L}$-polynomials.

7 Towards a general framework

The purpose of the general framework is to initiate a comparative study of the many graph, digraph and hypergraph polynomials which have appeared in the literature. For an extensive list of references cf. [37] and [39]. In particular, we address the following:

Universality All the polynomials we have encountered in the literature can be put into the framework of SOL-polynomials. Sometimes this is obvious. The matching polynomial can be written as

$$g(G,\lambda) = \sum_{\substack{M:M\subseteq E \\ M \text{ is a matching}}} \prod_{e:e\in M} \lambda$$

Sometimes this needs a non-trivial proof, which is the case for the interlace polynomial.

Definability. Not all SOL-definable properties of graphs are MSOL-definable, though, and sometimes it is useful to look at variations of MSOL which allow quantification over edge sets or subsets of fixed relations, which we call MSOL-2. Guarded Second Order Logic is the fragment of Second Order Logic in which the relation variables have to range over subsets of the relations specified in the vocabulary. If the edge relation of the graph is the only relation specified by the vocabulary Guarded SOL is just MSOL-2.
As graph properties can be viewed as polynomials with constant value **true** or **false** (or **1** or **0**) this gives (too) easy examples of SOL-polynomials which

are not MSOL-polynomials. The definition of the matching polynomial given above shows it is an MSOL-2-polynomial. It is hard to believe that it could be an MSOL-polynomial, but we do not know how to prove this. The interlace polynomial is an SOL-polynomial, of which we do not know whether it is an MSOL-polynomial. In [15] it shown to be definable in MSOL with a parity quantifier, but we do not know whether this can be avoided.

Recursion schemes and definability. The existence of a recursion scheme and SOL-definability both garantee that a graph polynomial is computable in exponential time. In the examples we know, every graph polynomial defined by a recursion scheme is also SOL-definable. Is this always true?
On the other hand it is unlikely that every SOL-polynomial has such a recursion scheme. How can we prove that for a given SOL-polynomial no recursion scheme exists? Can we give sufficient conditions which assure that an SOL-polynomial has a certain recursion scheme?

Comparability. Given two graph polynomials $f(G, \bar{x})$ and $g(G, \bar{x})$, we say that g is weaker than f, and write $g \preceq f$, if for any two graphs G_1, G_2 with $f(G_1, \bar{x}) = f(G_2, \bar{x})$ we also have $g(G_1, \bar{x}) = g(G_2, \bar{x})$. If $g \preceq f$ and $f \preceq g$, we say the polynomials are graph-equivalent. Comparability of graph polynomials is undecidable. This follows from the undecidablity of the consequence problem of First Order Logic if restricted to finite graphs, which was proven by M. Taitslin [52] and sharpened by I. Lavrov [32], cf. [27, Theorem 5.5.1]. We note that the two matching polynomials $m(G, \lambda)$ and $g(G, \lambda)$ are graph equivalent, but incomparable with respect to the characteristic polynomial $P(G, \lambda)$, and also with respect to the chromatic polynomial, and the Tutte polynomial. The study of this partial order among SOL-polynomials, MSOL-polynomials, or other restricted classes of graph polynomials is a natural topic of investigation. In particular, one can ask: is there a strongest SOL-polynomial, what are its additional structural properties, is it a lattice, etc.

Complexity. For a logic $\mathcal{L}$ which captures a complexity class $\mathbf{C}$ on ordered structures[3], we speak also of $\mathbf{C}$-polynomials. Interesting cases for $\mathbf{C}$ are deterministic and non-deterministic Log-Space, denoted by $\mathbf{L}$ and $\mathbf{NL}$ respectively, and determinsitic polynomial time $\mathbf{P}$. All examples in the literature actually are $\mathbf{P}$-polynomials, most actually are $\mathbf{NL}$-polynomials. For example, to see that the matching polynomial

$$g(G, \lambda) = \sum_{\substack{M: M \subseteq E \\ M \text{ is a matching}}} \prod_{e: e \in M} \lambda$$

is a $\mathbf{P}$-polynomial, it suffices to note that "$M \subseteq E$ is a matching" is a property recognizable in polynomial time.
We have seen before that the chromatic polynomial is $\sharp\mathbf{P}$-hard to compute, hence $\mathbf{P}$-polynomials are usually not computable in polynomial time. Neither are they in $\sharp\mathbf{P}$, as they can have arbitrary values in the polynomial ring. We

[3] in the sense of descriptive complexity, [28, 20, 33]

propose two classes of graph polynomials as *natural complexity classes* for graph polynomials: the class of **P**-polynomials which we call $\mathbf{P} - POL$, and the class of SOL-polynomials, which we call $SOL - POL$. Clearly we have, $\mathbf{P} - SOL \subseteq SOL - POL$. Is the inclusion proper? Do these classes have complete problems with respect to some notion of reduction (see below)?
On the other hand we have seen that the characteristic polynomial $P(G, \lambda)$ is computable in polynomial time over the ring $\mathcal{R}$. Therefore it is computable in polynomial time in the unit cost model BSS over the ring $\mathcal{R}$ in the sense of Blum, Shub and Smale, in short, it is in $\mathbf{P}_{\mathcal{R}}$, [9]. Can we characterize the graph polynomials in **P**-POL which are in $\mathbf{P}_{\mathcal{R}}$?

Reducibilities. Reducibilities have now two components:

(i) Computations in the ring, performed on the polynomial, in the uniform computational model BSS, or in the non-uniform model of L. Valiant [56, 12]. Algebraic circuits (straight-line programs) or $\mathbf{P}_{\mathcal{R}}$-programs are natural choices, where $\mathcal{R}$ is the underlying ring.

(ii) Transductions of the graphs (relational structures), expressible in the logic $\mathcal{L}$ for suitably chosen $\mathcal{L}$, or computable by Turing machine transducers in the corresponding complexity class **C**.

For **P**-polynomials over $\mathcal{R}$, $\mathbf{P}_{\mathcal{R}}$ and **P**-transductions, respectively transductions definable in Fixed Point Logic, are natural choices. For details see [28, 20, 33]. In this case we speak of **P**-reducibility between two graph polynomials f, g and write $g \preceq_P f$. The comparability and reducibility relations between graph polynomials do not coincide. The chromatic polynomial $\chi(G, \lambda)$ is **P**-reducible to the Tutte polynomials, but it is not comparable to the Tutte polynomial. This can be easily seen from the formula

$$\chi(G, X) = (-1)^{r(G)} X^{k(G)} T(G; 1 - X, 0)$$

Recall that $k(K)$ is the number of connected components of G and $r(G) = | V | - k(G)$. The formula shows that the chromatic polynomial is computable in polynomial time from the Tutte polynomial, but the Tutte polynomial remains invariant under the addition of isolated vertices to the graph G, whereas the chromatic polynomial does not.
It is open whether the matching polynomials $m(G, \lambda)$ and $g(G, \lambda)$ are **P**-reducible to the Tutte polynomial.

Easy loci. The Tutte polynomial is $\sharp$**P**-hard to evaluate on all the points of the complex plane with the exception of a quasi-algebraic set of lower dimension, cf. [30]. M. Bläser and J.A. Makowsky, [8], have generalised this for the colored Tutte polynomial studied in [11]. Is a similar phenomenon also observable for arbitrary SOL-polynomials for which evaluation is $\sharp$**P**-hard at least at some point?

Graph invariants. We say a graph polynomial g is a graph invariant, if, whenever G_1 and G_2 are isomoprphic, then $g(G_1) = g(G_2)$. We say g is a *complete* graph invariant, if additionally, whenever $g(G_1) = g(G_2)$, then G_1 and G_2 are isomorphic. There are artificial graph polynomials even in one variable, which are complete graph invariants. They are artificial, because they use coding tricks, are expensive to compute, and computing other graph invariants from such a complete polynomial may be very hard. It remains open whether there are "natural" complete graph invariants, in particular, it is not obvious what we could mean by "natural".

Reduction-complete polynomials. A graph polynomial is *reduction complete* in a complexity class **C** equipped with a notion of reducibility, if every other **C**-polynomial is reducible to it. To speak about reduction-complete polynomials it may be reasonable to fix the number of variables of the polynomials under consideration. Are there any reduction-complete **P**-polynomials? Is the Tutte polynomial reduction-complete? We note that the Tutte polynomial has been shown to be the most general graph polynomial with respect to certain reduction rules (contraction and deletion of edges), cf. [10, Chapter X]. But this excludes the matching polynomial from the discussion.

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