Array Codes for Functional PIR and Batch Codes

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Array Codes for Functional PIR and Batch Codes

Research Thesis

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Abstract

A functional PIR array code is a coding scheme which encodes some $s$ information bits into a $t \times m$ array such that every linear combination of the $s$ information bits has $k$ mutually disjoint recovering sets. Every recovering set consists of some of the array’s columns while it is allowed to read at most $\ell$ encoded bits from every column in order to receive the requested linear combination of the information bits. Functional batch array codes impose a stronger property where every multiset request of $k$ linear combinations has $k$ mutually disjoint recovering sets. Locality functional array codes demand that the size of every recovering set is restrained to be at most $r$.

Formally, assume $s$ bits are encoded to be stored in a $t \times m$ array, where each column corresponds to a server that stores the encoded bits. It is required in PIR codes that every information bit has $k$ mutually disjoint recovering sets while it is allowed to read at most $\ell$ bits from every column. An array code with these parameters and properties is defined as an $(s, k, m, t, \ell)$ PIR array code. Furthermore, it will be called an $(s, k, m, t, \ell)$ batch array code if every multiset request of $k$ information bits has $k$ mutually disjoint recovering sets. In case the requests are not only of information bits but any linear combination of them, we receive an $(s, k, m, t, \ell)$ functional PIR array code, if the same linear combination is requested $k$ times or $(s, k, m, t, \ell)$ functional batch array code for a multiset request of $k$ linear combinations. In case where every request of a linear combination has $k$ recovering sets, where each is of size at most $r$, we receive an $(s, k, m, t, r)$ locality functional array code.

The main figure of merit when studying these codes is to optimize the number of columns, i.e., servers, given the values of $s, k, t, \ell, r$. Thus, the smallest $m$ such that an $(s, k, m, t, \ell)$ PIR, batch, functional PIR, functional batch code exists, is denoted by $P_{t,\ell}(s, k), B_{t,\ell}(s, k), FP_{t,\ell}(s, k), FB_{t,\ell}(s, k)$, respectively. Furthermore, we denote by $D(s, k, t, r)$ the smallest $m$ such that an $(s, k, m, t, r)$ locality func-
tional array code exists.

The results we achieve in this work can be divided into two main parts. In the first part, we initiate the study of functional PIR and batch codes in the array setup and we present lower bounds on the smallest number of columns for these families of codes. In addition, we show several general array code constructions and results for \( k = 1, 2 \). Also, we present three specific constructions of array codes that some of them give us optimal or nearly optimal codes.

In the second part, we define and study locality functional array codes. We show connections between the problem of finding the minimal number of columns for locality functional array codes and some problems in subspaces. In addition we show how covering codes are used to get lower bounds and constructions for locality functional array codes.
Abbreviations and Notations

\[ [n] \quad \text{—} \quad [1, \ldots, n] \]
\[ \Sigma \quad \text{—} \quad \mathbb{F}_2 \]
\[ s \quad \text{—} \quad \text{Number of information bits} \]
\[ k \quad \text{—} \quad \text{Number of recovering sets} \]
\[ m \quad \text{—} \quad \text{Number of columns in array code} \]
\[ t \quad \text{—} \quad \text{Number of rows in array code} \]
\[ \mathbb{F}_q \quad \text{—} \quad \text{Finite field of size } q \]
\[ \mathbb{F}^n_q \quad \text{—} \quad \text{Vector space of dimension } n \text{ over } \mathbb{F}_q \]
Chapter 1

Introduction

1.1 Problem Definition

Private information retrieval (PIR) codes and batch codes are families of codes which have several applications such as PIR protocols [2, 8, 13, 17, 36, 39], erasure codes in distributed storage systems [26, 27, 32], one-step majority-logic decoding [22, 24], load balancing in storage, cryptographic protocols [20], switch codes [5, 9, 35], and more. They have been recently generalized to functional PIR and functional batch codes [42]. In this work we study these families of codes when they are used as array codes.

The setup of storing information in array codes works as follows. Assume $s$ bits are encoded to be stored in a $t \times m$ array, where each column corresponds to a server that stores the encoded bits. The encoded bits should satisfy several properties which depend upon whether the resulting code is a PIR, batch, functional PIR, or functional batch codes. Given a design parameter $k$ of the code, it is required in PIR codes that every information bit has $k$ mutually disjoint recovering sets. Here, a recovering set is a set of columns, i.e., servers, in which given the encoded bits in the columns of the recovering set it is possible to recover the information bit. In case it is possible to read only a portion of the encoded bits in every column, we denote this parameter by $\ell$. An array code with these parameters and properties is defined as an $(s, k, m, t, \ell)$ PIR array code. Furthermore, it will be called an $(s, k, m, t, \ell)$ batch array code if every multiset request of $k$ information bits has $k$ mutually disjoint recovering sets. In case the requests are not only of information bits but any linear combination of them, we receive an $(s, k, m, t, \ell)$ functional PIR array code, if the same linear combination is requested $k$ times or $(s, k, m, t, \ell)$ functional batch array code for a multiset request of $k$ linear combinations. Yet another family of codes that will be studied in this work will be referred by locality.
functional array codes. Here we assume that \( \ell = t \) and an \((s,k,m,t,r)\) locality functional array code guarantees that every linear combination \( v \) of the information bits has \( k \) mutually disjoint recovering sets, where each is of size at most \( r \).

The main figure of merit when studying these families of codes is to optimize the number of columns, i.e., servers, given the values of \( s, k, t, \ell \). Thus, the smallest \( m \) such that an \((s,k,m,t,\ell)\) PIR, batch, functional PIR, functional batch code exists, is denoted by \( P_{t,\ell}(s,k) \), \( B_{t,\ell}(s,k) \), \( FP_{t,\ell}(s,k) \), \( FB_{t,\ell}(s,k) \), respectively. Studying the value of \( P_{t,\ell}(s,k) \) has been initiated in \([17]\) and since then several more results have appeared; see e.g. \([3,4,7,41]\). Note that the first work \([20]\) which studied batch codes defined them in their array codes setup and only later on they were studied in their one-dimensional case, also known as primitive batch codes; see e.g. \([1,23,28,34,40]\). Functional PIR and batch codes have been recently studied in \([42]\) but only for vectors, that is, \( t = \ell = 1 \). Thus, this work initiates the study of functional PIR and batch codes in the array setup.

The motivation to study functional PIR and batch codes originates from the observation that in many cases and protocols, such as PIR, the user is not necessarily interested in one of the information bits, but rather, some linear combination of them. Furthermore, functional batch codes are closely related to the family of random I/O (RIO) codes, introduced by Sharon and Alrod \([30]\), which are used to improve the random input/output performance of flash memories. A variant of RIO codes, called parallel RIO codes, was introduced in \([37]\), and linear codes of this family of codes have been studied in \([38]\). It was then shown in \([42]\) that in fact linear parallel RIO codes are equivalent to functional batch codes.

### 1.2 Our Work

The results we achieve in this work can be divided into two main parts. In the first part, we initiate the study of functional PIR and batch codes in the array setup and we present lower bounds on the smallest number of columns for these families of codes. In addition, we show general array code constructions such as a construction based on the Gadget Lemma, where we get that \( FP_{p,t,\ell}(p \cdot s,k) \leq FB_{p,t,\ell}(p \cdot s,k) \leq FB_{t,\ell}(s,p \cdot k) \). Also, results based on covering codes, for instance, \( FP_{t,\ell}(s,k) \leq FB_{t,\ell}(s,k) \leq k \cdot \left\lfloor \frac{t}{g[t,\ell]} \right\rfloor \), where \( g[t,\ell] \) denotes the smallest dimension of a linear binary covering code with length \( t \) and covering radius \( \ell \). Furthermore, results for \( k = 1,2 \) such as \( FP_{t,1}(s,1) = \left\lceil \frac{s}{t} \right\rceil \), \( FB_{2,2}(s,2) \leq 7 \cdot \left\lceil \frac{s}{8} \right\rceil \), and more other results can be found in the work.

In addition, we present three specific constructions of array codes that some of them give us optimal or nearly optimal codes. The first one, is a construc-
tion given in [17, Th.20], where it was proved in [7, Th.10] that this construction gives a PIR array code. We study how it can be used also as batch and functional PIR array codes for specific parameters and get that $B_{2,2}(6,15) = 25$ and $21 \leq FP_{2,2}(6,11) \leq 25$. The second construction is a generalization of an example given in [17] of a PIR code. We study how it can be used as PIR and batch array codes and get that for any $r \geq 3$, $P_{r^2-r+1,r-1}(r^2 + r, r) = B_{r^2-r+1,r-1}(r^2 + r, r) = r + 1$. In the third construction we have $s$ information bits and the code stores each two nonzero disjoint linear combinations of total size at most $s$ in each column. We get that $24 \leq FP_{2,2}(4,14) \leq 25$ and $88 \leq FP_{2,2}(5,48) \leq 90$.

In the second part, we define and study locality functional array codes. We show connections between the problem of finding the minimal number of columns for locality functional array codes and different problems in subspaces such as spreads, covering designs, $\lambda$-fold partitions, and covering Grassmannian codes. We show several results based on these problems in subspaces like $D(s,1,t,1) = \left\lceil \frac{2^{t-1}}{2^{s-1}} \right\rceil$, $D(s,\left\lceil \frac{s-t-1}{t} \right\rceil , t, 1) \leq \left\lceil \frac{s}{t} \right\rceil$, and $D(s,\left\lceil \frac{2^{t-1}}{2^{s-1}} \right\rceil + 1, t, r) \leq \frac{2^{t-1}}{2^{s-1}}$, where $s = rt$, and more.

In addition we show how covering codes are used to get lower bounds and constructions for locality functional array codes. The main result in this part is that for any positive integer $w$ such that $t|w$, $\left\lceil \frac{h[w,s,r]_q}{2^t - 1} \right\rceil \leq D(ws,1,t,r) \leq \frac{(2^w - 1)h[s,r]_q}{2^t - 1}$, where $h[s,r]_q$ is the smallest length of a linear covering code over $\mathbb{F}_q$ with covering radius $r$ and redundancy $s$.

### 1.3 Outline of the Thesis

The rest of the thesis is organized as follows. In Chapter 2 we formally define the codes studied in the work, discuss some of the previous related work, and list several basic properties. In Chapter 3 we show lower bounds on the number of servers for functional PIR and batch array codes. Chapter 4 lists several code constructions which are based on the Gadget Lemma, covering codes, and several more results for $k = 1, 2$. Chapter 5 presents three constructions of array codes and in Section 6 the rates of these codes are studied. Section 7 studies locality functional array codes. Lastly, Section 8 concludes and summarizes the results of this work.
Chapter 2

Background and Previous Results

In this chapter we formally define the codes we study in the thesis, and preview some of the previous results that are related to our work.

2.1 Definitions and Preliminaries

This work is focused on five families of codes, namely private information retrieval (PIR) codes that were defined recently in [17], batch codes that were first studied by Ishai et al. in [20], their extension to functional PIR codes and functional batch codes that was investigated in [42], and locality functional codes. In these five families of codes, $s$ information bits are encoded to $m$ bits. While for PIR codes it is required that every information bit has $k$ mutually disjoint recovering sets, batch codes impose this property for every multiset request of $k$ bits. Similarly, for functional PIR codes it is required that every linear combination of the information bits has $k$ mutually disjoint recovering sets, and functional batch codes impose this property for every multiset request of $k$ linear combination of the bits. Lastly, similar to functional PIR codes, for locality functional codes it is required that the size of every recovering set is limited to be at most $r$. While this description of the codes corresponds to the case of one-dimensional codewords, the goal of this work is to study their extension as array codes, which is defined as follows. The set $[n]$ denotes the set of integers $\{1, 2, \ldots, n\}$ and $\Sigma = \mathbb{F}_2$.

We start with the formal definition of the first four families of codes that will be studied in the work, while we defer the definition of locality functional array codes to Chapter 7.

Definition 2.1.
1. An \((s, k, m, t, \ell)\) **PIR array code** over \(\Sigma\) is defined by an encoding map \(E : \Sigma^s \rightarrow (\Sigma^t)^m\) that encodes \(s\) information bits \(x_1, \ldots, x_s\) into a \(t \times m\) array and a decoding function \(D\) that satisfies the following property. For any \(i \in [s]\) there is a partition of the columns into \(k\) recovering sets \(S_1, \ldots, S_k \subseteq [m]\) such that \(x_i\) can be recovered by reading at most \(\ell\) bits from each column in \(S_j, j \in [k]\).

2. An \((s, k, m, t, \ell)\) **batch array code** over \(\Sigma\) is defined by an encoding map \(E : \Sigma^s \rightarrow (\Sigma^t)^m\) that encodes \(s\) information bits \(x_1, \ldots, x_s\) into a \(t \times m\) array and a decoding function \(D\) that satisfies the following property. For any multiset request of \(k\) bits \(i_1, \ldots, i_k \in [s]\) there is a partition of the columns into \(k\) recovering sets \(S_1, \ldots, S_k \subseteq [m]\) such that \(x_{i_j}, j \in [k]\) can be recovered by reading at most \(\ell\) bits from each column in \(S_j\).

3. An \((s, k, m, t, \ell)\) **functional PIR array code** over \(\Sigma\) is defined by an encoding map \(E : \Sigma^s \rightarrow (\Sigma^t)^m\) that encodes \(s\) information bits \(x_1, \ldots, x_s\) into a \(t \times m\) array and a decoding function \(D\) that satisfies the following property. For any request of a linear combination \(v\) of the information bits, there is a partition of the columns into \(k\) recovering sets \(S_1, \ldots, S_k \subseteq [m]\) such that \(v\) can be recovered by reading at most \(\ell\) bits from each column in \(S_j, j \in [k]\).

4. An \((s, k, m, t, \ell)\) **functional batch array code** over \(\Sigma\) is defined by an encoding map \(E : \Sigma^s \rightarrow (\Sigma^t)^m\) that encodes \(s\) information bits \(x_1, \ldots, x_s\) into a \(t \times m\) array and a decoding function \(D\) that satisfies the following property. For any multiset request of \(k\) linear combinations \(v_1, \ldots, v_k\) of the information bits, there is a partition of the columns into \(k\) recovering sets \(S_1, \ldots, S_k \subseteq [m]\) such that \(v_{i_j}, j \in [k]\) can be recovered by reading at most \(\ell\) bits from each column in \(S_j\).

PIR codes are similar in their definition to locally repairable codes (LRC) with availability \([26, 27]\), however PIR codes do not impose any constraint on the size of the recovering sets as done for LRCs. In fact, these codes have more in common with one-step majority-logic decodable codes that were studied a while ago by Massey \([24]\) and later by Lin and others \([22]\) for applications of fast decoding. The main difference is that one-step majority-logic decodable codes require that each bit (both information and redundancy) will have multiple recovering sets.

We refer to each column as a **bucket** and to each entry in a bucket as a **cell**. Furthermore, it is said that a cell stores a **singleton** if one of the information bits is stored in the cell. In the rest of the thesis we will refer to every linear combination of the information bits as a binary vector of length \(s\), which indicates the information bits in this linear combination. Our goal is to fix the values of \(s, k, t\)
and $\ell$ and then seek to optimize the value of $m$. In particular, we will have that $t$ and $\ell$ are fixed, where $t \geq \ell$, and then study the growth of $m$ as a function of $s$ and $k$. Hence, we denote by $P_{t,\ell}(s, k), B_{t,\ell}(s, k), FP_{t,\ell}(s, k), FB_{t,\ell}(s, k)$ the smallest $m$ such that an $(s, k, m, t, \ell)$ PIR, batch, functional PIR, functional batch code exists, respectively. In case $\ell = t = 1$ we will simply remove them from these notations.

### 2.2 Previous Work and Basic Results

A Private Information Retrieval (PIR) protocol allows a user to retrieve a data item from a database, in such a way that the servers storing the data will get no information about which data item was retrieved. The problem was introduced in [8]. The protocol to achieve this goal assumes that the servers don’t collude. It is also assumed that the database is error-free and synchronized all the time. For a set of $k$ servers, the goal is to design a $k$-server PIR protocol, in which the efficiency of the PIR is measured by the total number of bits transmitted by all parties involved. This model is called an information-theoretic PIR.

The classical model of PIR assumes that each server stores a copy of an $s$-bit database, so the storage overhead, namely the ratio between the total number of bits stored by all servers and the size of the database, is $k$. However, recent work combines PIR protocols with techniques from distributed storage (where each server stores only some of the database) to reduce the storage overhead. This approach was first considered in [29], and several papers have developed this direction further such as the breakthrough approach presented by Fazeli, Vardy, and Yaakobi [17] which shows that $m$ servers (for some $m > k$) may emulate a $k$-server PIR protocol.

The following upper and lower bounds on the number of buckets for PIR array codes have been shown in [4] and are stated in the following theorem.

**Theorem 2.1.**

1. $P_{t, t}(s, k) \geq \frac{2ks}{s+t}$, [4, Th. 3].

2. For any integer $t \geq 2$ and any integer $s > t$, $P_{t, t}(s, k) \geq \frac{k(s-2t+1)}{(2s-2t+1)t+(s-t)^2}$, [4, Th. 4].

3. For any integer $t \geq 2$ and any integer $s > 2t$, $P_{t, t}(s, k) \geq \frac{2ks(s+1)}{(s-t)^2+3st-t^2+2t}$, [41, Th. 16].

4. For any integer $t \geq 2$ and any integer $t < s \leq 2t$, $P_{t, t}(s, k) \leq \frac{k(s-2t+1)}{(2s-2t+1)t+(s-t)^2}$, [4, Th. 6].
For every symbol has a large number of disjoint recovering sets.

One of the simplest ways to construct array PIR and batch codes uses the Gad-
gadget Lemma, which was first proved in [20].

**Lemma 2.3. (The Gadget Lemma)** Let C be an (s, k, m, 1) batch code, then for any positive integer t there exists an (ts, k, m, t) batch array code C’ (denoted also by t · C).

Note that for any two integers \( t \geq 2 \) and \( s > t \), the bound in Theorem 2.1[2] improves upon the bound in Theorem 2.1[1]. This is verified by showing that \( k \cdot s \cdot (2s - 2t^2 + 1) + (s - t)^2 \geq 0 \) by basic algebraic manipulations. However the lower bound in Theorem 2.1[1] holds for all values of \( s \), while the one in Theorem 2.1[2] only for \( s > t \). Also, in [41] it was shown that for any two integers \( t \geq 2 \) and \( s > 2t \), the bound in Theorem 2.1[3] is stronger than the bound in Theorem 2.1[2].

The result in Theorem 2.1[4] is achieved by Construction 1 in [4]. The authors presented another construction which is not reported here due to its length. For the exact details please refer to [4, Construction 4 and Th.8]. This construction was then improved in [41] and in [7]. Several more constructions of PIR array codes have also been presented in [7, 41].

In [1] the authors presented how to construct both PIR and batch codes from multiplicity codes. In [21], multiplicity codes were used to construct locally decodable codes (LDC) in order to retrieve the value of a single symbol from the information symbols with high probability, given that at most a fixed fraction of the codeword’s symbol has errors. Since PIR and batch codes are not concerned with errors, the recovering procedure can be modified so that each information symbol has a large number of disjoint recovering sets.

The following theorem summarizes some of the known basic previous results, as well as several new ones. The proofs are rather simple and are thus omitted.

**Theorem 2.2.** For every s, k, t, \( \ell \), a positive integers:

1. \( P_{t, \ell}(s, 1) = B_{t, \ell}(s, 1) = \lceil s/\ell \rceil \).
2. \( FP_{t, \ell}(s, k_1 + k_2) \leq FP_{t, \ell}(s, k_1) + FP_{t, \ell}(s, k_2) \) (also for P, B, and FB).
3. \( FP_{t, \ell}(s, a \cdot k) \leq a \cdot FP_{t, \ell}(s, k) \) (also for P, B, and FB).
4. \( FP_{t, \ell}(s_1 + s_2, k) \leq FP_{t, \ell}(s_1, k) + FP_{t, \ell}(s_2, k) \) (also for P, B, and FB).
5. \( FP_{t, \ell}(a \cdot s, k) \leq a \cdot FP_{t, \ell}(s, k) \) (also for P, B, and FB).
6. \( FP_{t, \ell}(s, k) \leq a \cdot FP_{a \cdot t, \ell}(s, k) \) (also for P, B, and FB).

One of the simplest ways to construct array PIR and batch codes uses the Gadget Lemma, which was first proved in [20].
It is easily verified that the Gadget Lemma holds also for PIR codes and therefore
\[ P_{t,\ell}(s, k) \leq P_{t,1}(s, k) \leq P(\lceil s/t \rceil, k) \] and
\[ B_{t,\ell}(s, k) \leq B_{t,1}(s, k) \leq B(\lceil s/t \rceil, k). \]
However, unfortunately, the Gadget Lemma does not hold in general for functional PIR and batch codes. Even a weaker variation of the Gadget Lemma, where \( \ell = t \), does not hold in general for functional PIR and batch codes either. Assume by contradiction that if there is an \((s, k, m, 1, 1)\) functional PIR code \( C \), then for any positive integer \( t \) there exists a \((ts, k, m, t, t)\) functional PIR array code. Then, this will imply that \( FP_{t,1}(ts, k) \leq FP(s, k) \). However, it is known that \( FP(2, 2) = 3 \) by the simple parity code. Thus, under this assumption it would hold that \( FP_{2,2}(4, 2) \leq FP(2, 2) = 3 \). But, according to a lower bound on functional PIR array codes, which will be shown in Theorem 3.5, it holds that
\[ FP_{2,2}(4, 2) \geq \frac{22}{15} \cdot \frac{15}{3} > 3, \]
which is a contradiction.
Chapter 3

Lower Bounds on Array Codes

In this section we present several lower bounds on functional PIR and batch array codes. Let \( \{ \binom{a}{b} \} \) be the Stirling number of the second kind, which calculates the number of partitions of a set of \( a \) elements into \( b \) nonempty subsets. It is well known that \( \{ \binom{a}{b} \} = \frac{1}{b!} \sum_{i=0}^{b} (-1)^{b-i} \binom{b}{i} i^a \).

**Theorem 3.1.** For all \( s, k, t \) and \( \ell \) positive integers \( F_{B_{t,\ell}}(s, k) \geq m^* \), where \( m^* \) is the smallest positive integer such that

\[
\sum_{i=k}^{m^*} \binom{m^*}{i} \cdot \binom{t}{\ell}^i \geq \binom{2s + k - 2}{k}.
\]

**Proof** Let \( C \) be an optimal \((s, k, m^*, t, \ell)\) functional batch array code. Since there are \( s \) information bits, there are \( (2^s - 1) \) possible linear combination requests and there are \( \binom{2^s + k - 2}{k} \) possible multiset requests of length \( k \). For each multiset request of \( k \) linear combinations \( v_1, \ldots, v_k \) of the information bits, there is a partition of the buckets of the code \( C \) into \( k \) recovering sets \( S_1, \ldots, S_k \subseteq [m^*] \) such that \( v_j, j \in [k] \) can be recovered by reading at most \( \ell \) bits from each column in \( S_j \).

In each bucket there are \( t \) cells where at most \( \ell \) cells from them can be read. Thus, there are \( \sum_{j=1}^{\ell} \binom{t}{j} \) nonzero linear combinations that can be obtained from one bucket. For any positive integer \( n \), there are \( \left( \sum_{j=1}^{\ell} \binom{t}{j} \right)^n \) nonzero linear combinations that can be obtained from \( n \) buckets while using all the \( n \) buckets.

In order to satisfy a multiset request, the buckets must be divided into \( k \) disjoint recovering sets such that each set can satisfy one requested linear combination. There are

\[
\sum_{i=k}^{m^*} \binom{m^*}{i} \cdot \binom{t}{\ell}^i \cdot \binom{t}{j}^i \].
possibilities to divide at most $m^*$ buckets into $k$ nonempty disjoint sets. Each subset of the buckets of size at least $k$ can be divided into $k$ nonempty sets. Thus, we take the sum over all the subsets of the buckets of size at least $k$, where for each such subset we count the number of possibilities to divide it into $k$ nonempty subsets using Stirling number of the second kind. From each subset of size $p$ where $k \leq p \leq m^*$, there exist $(\sum_{j=1}^{\ell} \binom{t}{j})^p$ linear combinations. Therefore, for a given partition of $i$, $k \leq i \leq m^*$ buckets into $k$ subsets such that the sizes of the subsets are $p_1, p_2, \ldots, p_k$ where $\sum_{j=1}^{k} p_j = i$, the number of different $k$-sets of linear combinations such that each linear combination taken from one subset is

$$\prod_{p \in \{p_1, p_2, \ldots, p_k\}} \left(\sum_{j=1}^{\ell} \binom{t}{j}\right)^p = \left(\sum_{j=1}^{\ell} \binom{t}{j}\right)^i.$$ 

In order to satisfy each multiset request by a set of $k$ linear combinations such that each linear combination satisfies one requested linear combination. It must hold that the number of different $k$-sets of linear combinations such that each linear combination taken from one subset of the buckets, for all partitions of the $m^*$ buckets into $k$ nonempty disjoint subsets, is larger than the number of multiset requests. Thus,

$$\sum_{i=k}^{m^*} \binom{m^*}{i} \cdot \left\{i\right\} \cdot \left(\sum_{j=1}^{\ell} \binom{t}{j}\right)^i \geq \binom{2^s + k - 2}{k}. \quad (3.1)$$

A similar lower bound can be obtained for functional PIR array codes. While in functional batch array codes there exist $\binom{2^s + k - 2}{k}$ possible multiset requests, in functional PIR array codes there exist $2^s - 1$ possible requests.

**Corollary 3.2.** For all $s, k, t$ and $\ell$ positive integers $FP_{t,\ell}(s, k) \geq m^*$, where $m^*$ is the smallest positive integer such that

$$\sum_{i=k}^{m^*} \binom{m^*}{i} \cdot \left\{i\right\} \cdot \left(\sum_{j=1}^{\ell} \binom{t}{j}\right)^i \geq 2^s - 1. \quad (3.2)$$

Another combinatorial bound for functional PIR array codes is shown in the following theorem.

**Theorem 3.3.** For all $s, k, t$ and $\ell$ positive integers $FP_{t,\ell}(s, k) \geq m^*$, where $m^*$ is
the smallest positive integer such that

\[
\sum_{i=1}^{m^* - k + 1} \binom{m^*}{i} \cdot \left( \sum_{j=1}^{\ell} \binom{i}{j} \right)^i \geq k \cdot (2^s - 1).
\]

**Proof** Let \( C \) be an optimal \((s, k, m^*, t, \ell)\) functional PIR array code. Since there are \( s \) information bits, there are \((2^s - 1)\) possible requests. The code \( C \) must satisfy each request \( k \) times by \( k \) linear combinations from \( k \) disjoint recovering sets. In other words, for each request there are \( k \) nonempty disjoint recovering sets, such that each set has a linear combination equal to the request. Each recovering set must be of size at most \( m^* - k + 1 \), in order to have other \( k - 1 \) nonempty recovering sets.

In each bucket there are \( t \) cells where at most \( \ell \) cells from them can be read. Thus, there are \( \sum_{i=1}^{\ell} \binom{i}{j} \) nonzero linear combinations that can be obtained from one bucket and \((\sum_{i=1}^{\ell} \binom{i}{j})^n\) from \( n \) buckets, for any positive integer \( n \), while using all the \( n \) buckets. We are interested in counting the different linear combinations that can be obtained from at most \( m^* - k + 1 \) buckets. Thus, there are

\[
\sum_{i=1}^{m^* - k + 1} \binom{m^*}{i} \cdot \left( \sum_{j=1}^{\ell} \binom{i}{j} \right)^i
\]

such linear combinations. It must hold that the number of different linear combinations that can be got from at most \( m^* - k + 1 \) buckets is larger than \( k \) times the number of the possible requests. Thus,\n
\[
\sum_{i=1}^{m^* - k + 1} \binom{m^*}{i} \cdot \left( \sum_{j=1}^{\ell} \binom{i}{j} \right)^i \geq k \cdot (2^s - 1). \tag{3.3}
\]

The following corollary is derived from Theorem 3.3.

**Corollary 3.4.** \( FP_{t,\ell}(s, k) \geq \left\lceil \frac{\log_2(k(2^s - 1) + 1)}{\log_2(\sum_{i=0}^{\ell} \binom{i}{j})} \right\rceil \), for all \( s, k, t \) and \( \ell \) positive integers.

**Proof** The proof of Theorem 3.3 can be modified by using a weaker constraint, that the size of each subset is at most \( m \). Thus, it must hold that \( \sum_{i=1}^{m} \binom{m}{i} \cdot \left( \sum_{j=1}^{\ell} \binom{i}{j} \right)^i \geq k \cdot (2^s - 1) \). From the equality \( \sum_{i=0}^{m} \binom{m}{i} \cdot x^i = (x + 1)^m \), we
get that,
\[
\sum_{i=1}^{m} \binom{m}{i} \cdot \left( \sum_{j=1}^{\ell} \binom{t}{j} \right)^{i} = \left( 1 + \sum_{j=1}^{\ell} \binom{t}{j} \right)^{m} - 1
\]
\[
= \left( \sum_{j=0}^{\ell} \binom{t}{j} \right)^{m} - 1 \geq k \cdot (2^s - 1).
\]

Therefore, a lower bound over the minimal number of buckets, is
\[
FP_{t, \ell}(s, k) \geq \left\lceil \log_2 \left( \frac{k(2^{s-1} - 1)}{(2^s - 1) + \sum_{j=0}^{\ell} \binom{t}{j}} \right) \right\rceil.
\]

Lastly in this section we show a different lower bound for functional PIR array codes, which is motivated by the corresponding lower bound for PIR array codes from [4, Th. 3].

**Theorem 3.5.** For any \( s, k, t \) and \( \ell \) positive integers, \( FP_{t, \ell}(s, k) \geq 2^{k(2^{s-1} - 1)} \cdot \frac{2k^{2^s - 1}}{(2^s - 1) + \sum_{j=0}^{\ell} \binom{t}{j}} \).

**Proof** Suppose there exists an \( (s, k, m, t, \ell) \) functional PIR array code. There are \( 2^s - 1 \) possible linear combination requests which are denoted by \( u_i \) for \( 1 \leq i \leq 2^s - 1 \). For \( i \in [2^s - 1] \), we define by \( \alpha_i \) to be the number of recovering sets of size 1 of the \( i \)-th linear combination request \( u_i \).

Since it is possible to read at most \( \ell \) bits from each bucket, every bucket can satisfy at most \( \sum_{i=1}^{\ell} \binom{t}{i} \) linear combinations. Thus, the number of recovering sets of size 1 is \( m \cdot \sum_{i=1}^{\ell} \binom{t}{i} \), and \( \sum_{j=1}^{2^s - 1} \alpha_j \leq m \cdot \sum_{i=1}^{\ell} \binom{t}{i} \). Hence, there exists \( q \in [2^s - 1] \) such that \( \alpha_q \leq \frac{m \cdot \sum_{i=1}^{\ell} \binom{t}{i}}{2^s - 1} \), so out of its \( k \) disjoint recovering sets of \( u_q \), at most \( \alpha_q \) of them are of size 1, and the size of each of the remaining \( k - \alpha_q \) subsets is at least 2. Hence,

\[
m \geq \alpha_q + 2(k - \alpha_q) = 2k - \alpha_q \geq 2k - \frac{m \cdot \sum_{i=1}^{\ell} \binom{t}{i}}{2^s - 1},
\]

and therefore \( m(1 + \frac{\sum_{i=1}^{\ell} \binom{t}{i}}{(2^s - 1)}) \geq 2k \), which implies that \( FP_{t, \ell}(s, k) \geq \frac{2k(2^s - 1)}{(2^s - 1) + \sum_{j=0}^{\ell} \binom{t}{j}} \).

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Chapter 4

General Constructions of Array Codes

In this section we present several constructions of array codes for functional PIR and batch codes.

4.1 Basic Constructions

Even though the Gadget Lemma cannot be extended in general for functional PIR and batch codes, here we show a variation of it that will hold. For any positive integer $i$, $0^i$ denotes the zero vector of length $i$, and for any two vectors $v$ and $u$, the vector $vu$ is defined to be the concatenation of $u$ after $v$.

**Lemma 4.1.** For any positive integer $p$, if there exists an $(s, p \cdot k, m, t, \ell)$ functional batch array code, then there exists an $(p \cdot s, k, m, p \cdot t, \ell)$ functional batch array code. Therefore,

$$FP_{p \cdot t, \ell}(s, k) \leq FB_{p \cdot t, \ell}(p \cdot s, k) \leq FB_{t, \ell}(s, p \cdot k),$$

and in particular, $FP_{t, 1}(s, k) \leq FB_{t, 1}(s, k) \leq FB(\lceil \frac{s}{t} \rceil, t \cdot k)$.

**Proof** Let $C$ be an $(s, p \cdot k, m, t, \ell)$ functional batch array code with encoding function $E$ and decoding function $D$. We construct an $(p \cdot s, k, m, p \cdot t, \ell)$ functional batch array code $C'$ by using the code $C$. Let $S = \{x_{i,j} : 1 \leq i \leq p, 1 \leq j \leq s\}$ be the set of $p \cdot s$ information bits. The $p \cdot s$ information bits can be partitioned into $p$ parts, each of size $s$, such that part $i, i \in [p]$ is $S_i = \{x_{i,j} : 1 \leq j \leq s\}$. The code $C'$ will be represented by a $pt \times m$ array $A$, that contains $p$ subarrays $A_1, A_2, \ldots, A_p$ each of dimension $t \times m$. In the encoding function of the code $C'$,
the $i$-th subarray $A_i$ stores the encoded bits of the set $S_i$ by applying the encoding function $E$ of the code $C$ over the information bits in the set $S_i$.

Let $R = \{v_1, v_2, \ldots, v_k\}$ be a multiset request of size $k$ of the $p \cdot s$ information bits, where $v_i, i \in [k]$ is a binary vector of length $ps$ that represents the $i$-th request. For each $i \in [k]$, denote $v_i = (v_i^1, v_i^2, \ldots, v_i^p)$ where $v_i^j, j \in [p]$ is a vector of length $s$ that represents the linear combination of the bits in $S_i$. Let $R^* = \{v_i^j : 1 \leq i \leq k, 1 \leq j \leq p\}$ be a multiset request of size $pk$, that has $pk$ vectors of length $s$ each. By using the decoding function $D$ of the code $C$ with the request $R^*$ we get $pk$ recovering sets. For each $i \in [k]$ and $j \in [p]$, let $B_i^j = \{(h_{i,1}, a_{i,1}), (h_{i,2}, a_{i,2}), \ldots, (h_{i,s}, a_{i,s})\}$ be a recovering set for $v_i^j$ of size $a_i$, where for each $g \in [a_i]$, $(h_{i,g}, a_{i,g})$ is a pair of a bucket $h_{i,g}$ with a vector $a_{i,g}$ of length $t$ that indicates the cells which are read from the bucket $h_{i,g}$. For each $B_i^j$ and $f \in [p]$, let $B_{i,f} = \{(h_{i,1}, 0^{(f-1)}u_{i,1}0^{(p-f)}), \ldots, (h_{i,s}, 0^{(f-1)}u_{i,s}0^{(p-f)})\}$ be a recovering set for $v_i^j$, that reads the cells of subarray $A_f$. For each $i \in [k]$, to satisfy the request $v_i$, the union $\bigcup_{f=1}^{p} B_{i,f}$ is taken, since for each $f \in [p]$ the subset $B_{i,f}$ can satisfy the request $v_i^j$.

For each $f_1, f_2 \in [p], i_1, i_2 \in [k]$ and $j_1, j_2 \in [p]$, $B_{i_1,f_1}$ and $B_{i_2,f_2}$ have disjoint subsets of buckets if $i_1 \neq i_2$ or $j_1 \neq j_2$, because $B_{i_1}^j$ and $B_{i_2}^j$ have disjoint subsets of buckets if $i_1 \neq i_2$ or $j_1 \neq j_2$. Thus, for any $i \neq j \in [k], \bigcup_{f=1}^{p} B_{i,f}$ and $\bigcup_{f=1}^{p} B_{j,f}$ have disjoint subsets of buckets.

It remains to show that we read at most $\ell$ cells from each bucket. For any $v_i, i \in [k]$ it is clear that if the recovering set $B^j_{i,f_1}$ was used then $f_1 = j$, which implies that the recovering sets $B^j_{i,f_2}$ for each $f_2 \neq f_1$ was not used. Thus, the recovering sets that were used to satisfy $v_i$ have disjoint subsets of buckets. Thus, each bucket can appear in at most one of these recovering sets, and it is known that each one of these subsets uses at most $\ell$ cells from each bucket from the properties of the code $C$.

The last claim in the lemma holds by setting $p = t$ and $t = 1$.

Another general construction is stated in the next theorem.

**Theorem 4.2.** For any positive integers, $s, k, t, t_0$, and $\ell$, $FB_{t,\ell}(s, k) \leq m + m_0$, where $m = FB_{t_0,\ell}(s, k)$ and $m_0 = FB_{t_0}(m \cdot t_0, k)$.

**Proof** Let $C_1, C_2$ be an $(s, k, m, t + t_0, \ell), (m \cdot t_0, k, m_0, t, \ell)$ functional batch array code, respectively. We construct an $(s, k, m + m_0, t, \ell)$ functional batch array code $C$ by using the codes $C_1, C_2$. First, the $s$ information bits are encoded using
the encoder function of the code $C_1$ to get a $(t + t_0) \times m$ array $A$. Then, the $t_0 \cdot m$ bits in the last $t_0$ rows of $A$ are encoded into a $t \times m_0$ array $B$ using the encoder function of the code $C_2$. The code $C$ will be represented by a $t \times (m + m_0)$ array, where the first $m$ buckets (columns) will be the first $t$ rows of the array $A$ and the last $m_0$ buckets will be the array $B$.

Let $R = \{v_1, \ldots, v_k\}$ be a multiset request of size $k$, where $v_i, i \in [k]$ is a binary vector of length $s$ that represents the $i$-th request. Denote by $\{E_1, \ldots, E_k\}$ the $k$ recovering sets that are obtained by using the decoding function of the code $C_1$ with the request $R$. For each $i \in [k]$, assume that $|E_i| = p_i$ and denote $E_i = \{(h_{i,1}, u_{i,1}), \ldots, (h_{i,p_i}, u_{i,p_i})\}$ where for each $j \in [p_i]$, $(h_{i,j}, u_{i,j})$ is a pair of a bucket $h_{i,j}$ with a vector $u_{i,j}$ of length $t + t_0$ that indicates the cells which are read from the bucket $h_{i,j}$. For each $i \in [k]$ and $j \in [p_i]$, let $u'_{i,j}$ be the vector with the last $t_0$ entries of $u_{i,j}$ and let $R'_{i,j}$ be the sum of the bits in the cells that indicated by $u'_{i,j}$.

Let $R' = \{\sum_{j=1}^{p_1} R'_{1,j}, \ldots, \sum_{j=1}^{p_k} R'_{k,j}\}$ be a multiset request of size $k$. Denote by $\{F_1, \ldots, F_k\}$ the $k$ recovering sets that are obtained by using the decoding function of the code $C_2$ with the multiset request $R'$. To satisfy $v_i$, the code $C$ can use the recovering set $F_i \cup E'_i$, where $E'_i = \{(h_{i,1}', u_{i,1}''), \ldots, (h_{i,k}', u_{i,k}'')\}$ where for each $j \in [k], u''_{i,j}$ is the vector with the first $t$ entries of $u_{i,j}$.

It remains to show that at most $\ell$ cells are read from each bucket. Each $v_i, i \in [k]$ has a recovering set $F_i \cup E'_i$, where the recovering set $F_i$ of $C_2$ uses at most $\ell$ cells from each bucket from the property of the code $C_2$. Also, the recovering set $E_i$ of $C_1$ uses at most $\ell$ cells from each bucket from the property of the code $C_1$. Thus, $E'_i$ also uses at most $\ell$ cells.

Note that a similar statement can hold for functional PIR array code, where for any positive integers $s, k, t, t_0$, and $\ell$, $FP_{t,\ell}(s,k) \leq m + m_0$, where $m = FP_{t+t_0,\ell}(s,k)$ and $m_0 = FB_{t,\ell}(m \cdot t_0,k)$.

### 4.2 Constructions based upon Covering Codes

In this section it is shown how covering codes are used to construct array codes. Denote by $d_H(x, y)$ the Hamming distance between two vectors $x, y$, and denote by $w_H(x)$ the Hamming weight of $x$. Also define $\langle x, y \rangle$ as the inner product of the two vectors $x, y$. Next we remind the definition of covering codes [10].

**Definition 4.1.** Let $n \geq 1, R \geq 0$ be integers. A code $C \subseteq \mathbb{F}_q^n$ is called an $R$-covering code if for every word $y \in \mathbb{F}_q^n$ there is a codeword $x \in C$ such that $d_H(x, y) \leq R$. The notation $[n, k, R]_q$ denotes a linear code over $\mathbb{F}_q$ of length $n$.
$n$, dimension $k$, and covering radius $R$. The value $g[n, R]_q$ denotes the smallest dimension of a linear code over $\mathbb{F}_q$ with length $n$ and covering radius $R$. The value $h[s, R]_q$ is the smallest length of a linear code over $\mathbb{F}_q$ with covering radius $R$ and redundancy $s$. In case $q = 2$ we will remove it from these notations.

The following property is well known for linear covering codes; see e.g. [10, Th. 2.1.9].

**Property 4.3.** For an $[n, k, R]$ linear covering code with some parity check matrix $H$, every syndrome vector $s \in \Sigma^{n-k}$ can be represented as the sum of at most $R$ columns of $H$.

The connection between linear codes and functional batch array codes is established in the next theorem.

**Theorem 4.4.** Let $C$ be a $[t, t-s, \ell]$ linear covering code. Then, there exists an $(s, 1, 1, t, \ell)$ functional batch array code. In particular, $FB_{t,\ell}(t - g[t, \ell], 1) = 1$.

**Proof** Let $x = (x_1, \ldots, x_s)$ the vector of dimension $1 \times s$ with the $s$ information bits, and let $H$ be a parity check matrix of the code $C$, with dimension $s \times t$. We construct an $(s, 1, 1, t, \ell)$ functional batch array code $C'$ by taking each entry of the vector $c = (xH)^T$ as a cell in the code. The dimension of $c$ is $t \times 1$, and thus, we get one bucket with $t$ cells where each cell has a linear combination of the $s$ information bits.

Let $u \in \Sigma^t$ be a request which represents the linear combination $\langle u, x \rangle$ of the $s$ information bits. From Property 4.3, we know that there exists a vector $y \in \Sigma^t$ such that $y \cdot H^T = u$, where $w = w_H(y) \leq \ell$. Let $A = \{i : i \in [t], y_i = 1\}$, where $y_i$ is the entry number $i$ of $y$. Thus, $\langle u, x \rangle = u \cdot x^T = y \cdot H^T \cdot x^T = y \cdot c = \sum_{i \in A} c_i$, where $c_i$ is the entry number $i$ of $c$. Therefore, to satisfy the request $\langle u, x \rangle$ we should read $|A| = w \leq \ell$ cells from the code $C'$.

Recall that $g[t, \ell]$ is the smallest dimension of a linear code with length $t$ and covering radius $\ell$. Thus, there exists a $[t, g[t, \ell], \ell]$ linear covering code. We get that there exists a $(t - g[t, \ell], 1, 1, t, \ell)$ functional batch array code, which implies that $FB_{t,\ell}(t - g[t, \ell], 1) = 1$.

Theorem 4.4 holds also for functional PIR array code and thus the following results are derived.

**Corollary 4.5.** Let $s, k, t$ and $\ell$ be positive integers. Then,

1. $FP_{t,\ell}(s, k) \leq FB_{t,\ell}(s, k) \leq k \cdot \left\lceil \frac{s}{t - g[t, \ell]} \right\rceil$. 

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2. \( FP_{t+0,\ell}(s,k) \leq FP_{t,t}(s,k) \), where \( t_0 = g[t + t_0, \ell] \). Also works for FB.

3. \( FP_{t,\ell}(s,k) \leq FB_{t,\ell}(s,k) \leq k \cdot \left( \left\lceil \frac{s}{\alpha} \right\rceil + 1 \right) \), where \( \left\lceil \frac{s}{\alpha} \right\rceil \leq t - g[t, \ell] \), and \( \alpha = (t + 1) - g[(t + 1), \ell] \).

The third claim of Corollary 4.5 is derived from Theorem 4.4 and Theorem 4.2.

### 4.3 The Cases of \( k = 1, 2 \)

Even though the cases of \( k = 1, 2 \) are the most trivial ones when the codewords are vectors, they are apparently not easily solved for array codes. In this section we summarize some of our findings on these important and interesting cases.

**Theorem 4.6.** For each \( s, t, \ell \) positive integers:

1. \( FP_{t,\ell}(s,1) \geq \left\lceil \frac{s}{\log_2(\binom{s}{\ell})} \right\rceil \).

2. \( FP_{t,\ell}(s,1) = \left\lceil \frac{s}{\ell} \right\rceil \).

3. \( FP_{t,\ell}(\left\lfloor \log_2(t+1) \right\rfloor,1) = 1 \) and \( \left\lceil \frac{s}{\log_2(t+1)} \right\rceil \leq FP_{t,1}(s,1) \leq \left\lceil \frac{s}{\log_2(t+1)} \right\rceil \).

4. \( FP_{t,\alpha}(s,1) \leq \left\lceil \frac{s}{t-g[t,\alpha+1]} \right\rceil \), where \( 0 < \alpha < 1 \).

5. \( FP_{t,1/2}(s,1) = \frac{t}{2} + 1 \), where \( t \) is even, \( \frac{t}{2} \) is integer, and \( \frac{t}{2} \leq t - 1 \).

**Proof**

1. From corollary 3.4

2. The lower bound over \( FP_{t,\ell}(s,1) \) is obtained by using the lower bound from the first claim of this theorem, \( FP_{t,\ell}(s,1) \geq \left\lceil \frac{s}{\log_2(\binom{s}{\ell})} \right\rceil = \left\lceil \frac{s}{\ell} \right\rceil \). The upper bound can be verified by showing that there exists an \( (s,1,\left\lfloor \frac{s}{\ell} \right\rfloor, t, t) \) functional PIR array code. There are \( t \) cells in each bucket. Then, in order to write all the \( s \) information bits there is a need to \( \left\lfloor \frac{s}{\ell} \right\rfloor \) buckets. Each request is a linear combination of the \( s \) information bits. Thus, each request can be satisfied by reading the information bits which included in the request. It was shown that \( FP_{t,\ell}(s,1) \geq \left\lceil \frac{s}{\ell} \right\rceil \) and there exists an \( (s,1,m,t,t) \) functional PIR array code. Therefore, \( FP_{t,\ell}(s,1) = \left\lfloor \frac{s}{\ell} \right\rfloor \).

3. A \( ([\log_2(t+1)],1,1,t,1) \) functional PIR array code \( C \) can be obtained by writing all the \( 2^{[\log_2(t+1)]-1} \leq t \) linear combinations of the information bits in at most \( t \) cells of one bucket. Each request is a linear combination.
of the information bits, and hence, for each request there exists a cell in the bucket that satisfies it. Thus, the appropriate cell can satisfy the request. The minimum number of buckets is $1$. Thus, $FP_{t,1}(s,t) = 1$. The lower bound over $FP_{t,1}(s,t) = 1$ is derived from the first claim of this theorem. Thus $FP_{t,1}(s,t) \geq \lceil \frac{s}{\log_2(t+1)} \rceil = \lceil \frac{s}{\log_2(t+1)} \rceil$. The upper bound is shown by using Theorem 2.2(5), $FP_{t,1}(s,t) \leq \lceil \frac{s}{\log_2(t+1)} \rceil \cdot \lceil \frac{s}{\log_2(t+1)} \rceil$. 4. From Corollary 4.5(1).

5. The lower bound over $FP_{t,t/2}(s,t) = 1$ can be found using the lower bound from the first claim of this theorem.

For the upper bound, from Corollary 4.5(3) we get that $FP_{t,t/2}(s,t) \leq \lceil \frac{s}{t} \rceil + 1$. Since $g[t + 1, t/2] = t - g[t, t/2]$ in order to use Corollary 4.5(3). Since $s/t \leq t - 1$, it is derived that $FP_{t,t/2} = \frac{s}{t} + 1$. Example 1. In this example we demonstrate the construction of a $(12, 4, 4, 2, 2)$ functional PIR array code according to Theorem 4.6(5). The construction is given
in Table 4.1. It can be verified that $FP_{4,2}(12, 1) = 4$. Note that in this example and in the rest of the thesis the notation $x_{i_1}x_{i_2}\cdots x_{i_h}$ is a shorthand to the summation $x_{i_1} + x_{i_2} + \cdots + x_{i_h}$.

An improvement for the case of $\ell = 1$ is proved in the following theorem.

**Theorem 4.7.** For any positive integers $s_1, s_2,$ and $t$,

$$FP_{t,1}(s_1 + s_2, 1) \leq \left\lceil \frac{s_1}{\log_2(t + 1)} \right\rceil + 1,$$

where $2^{s_2} - 1 \leq \left(\left\lceil \frac{s_1}{\log_2(t + 1)} \right\rceil + 1\right)(t - (2^\left\lceil \log_2(t + 1) \right\rceil - 1))$.

**Proof** A construction of an $(s_1 + s_2, 1, m, t, 1)$ functional PIR array code for $m = \left\lceil \frac{s_1}{\log_2(t + 1)} \right\rceil + 1$ is presented. The first $s_1$ information bits are divided into $m - 1$ parts, where $h_i, i \in [m - 1]$ is the size of part $i$, and $h_i \leq \left\lceil \log_2(t + 1) \right\rceil$. Then, all the linear combinations of part $i \in [m - 1]$ are written in the $i$-th bucket, so in each of the first $m - 1$ buckets there are at least $t - (2^\left\lceil \log_2(t + 1) \right\rceil - 1)$ empty cells. In the last bucket, the parity of each of the first $2^\left\lceil \log_2(t + 1) \right\rceil - 1$ rows is stored. Since $2^{s_2} - 1 \leq m \cdot (t - (2^\left\lceil \log_2(t + 1) \right\rceil - 1))$, each of the $2^{s_2} - 1$ linear combinations of the $s_2$ bits can be written in the empty cells of the $m$ buckets.

Let $v = (v_1, \ldots, v_m)$ be a request such that for any $i \in [m - 1]$ the length of $v_i$ is $h_i$, the length of $v_m$ is $s_2$, and for simplicity assume that they are all nonzero. The linear combination $v_m$ is satisfied by the cell where it is stored and assume it is in the $j$-th bucket, where $j < m$. Assume that the cell in the $j$-th bucket where the linear combination $v_j$ is stored is in row $r$. We read from each bucket $b \in [m - 1]$, where $b \neq j$ the cell with the linear combination represented by $v_b + u_b$, where $u_b$ is the vector that represents the cell in bucket $b$ in row $r$, but if $v_b + u_b = 0$ do not read from bucket $b$. Also, we read the cell in row $r$ from the last bucket. Then, the obtained linear combination is the combination that is represented by

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Table 4.2: \((15, 1, 7, 4, 1)\) functional PIR array code

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<td>(x_7)</td>
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<td>(x_{11})</td>
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<tr>
<td>(x_2)</td>
<td>(x_4)</td>
<td>(x_6)</td>
<td>(x_8)</td>
<td>(x_{10})</td>
<td>(x_{12})</td>
<td>(x_2 x_4 x_6 x_8 x_{10} x_{12})</td>
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<td>(x_9 x_{10})</td>
<td>(x_{11} x_{12})</td>
<td>(x_1 \cdots x_{12})</td>
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Table 4.3: \((8, 2, 7, 2, 2)\) functional PIR array code

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<td>(x_3 x_4)</td>
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<td>(x_8)</td>
<td>(x_7 x_8)</td>
<td>(x_3 x_4 x_7 x_8)</td>
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</tbody>
</table>

\((v_1, \ldots, v_{m-1})\), because \(\sum_{1 \leq b \leq m, b \neq j} u_b = v_j\) and for each \(b \in [m - 1]\) where \(b \neq j\) we read the linear combination that is represented by \(v_b + u_b\) from bucket \(b\).

For any \(t, s_1, s_2\) where \(s = s_1 + s_2\) and \(s_2 \geq \lceil \log_2(t + 1) \rceil\), the upper bound in Theorem 4.7 improves upon the one in Theorem 4.6(3) since \(\lceil \frac{s}{\lceil \log_2(t+1) \rceil} \rceil \geq \lceil \frac{s}{\lceil \log_2(t+1) \rceil} \rceil + 1\).

**Example 2.** In this example the construction of a \((15, 1, 7, 4, 1)\) functional PIR array code is demonstrated based on Theorem 4.7. It can be verified that the parameters \(t = 4, s_1 = 12\) and \(s_2 = 3\) satisfy the constraints of Theorem 4.7. The construction is given in Table 4.2. The first \(s_1 = 12\) information bits are partitioned into 6 parts, each part of size 2. All the nonzero linear combinations of part \(i, i \in [6]\) are written in the \(i\)-th bucket with one cell remains empty. The sum of each of the first 3 rows is written. Now, there are still 7 empty cells, which are used to store all the nonzero linear combinations of the last \(s_2 = 3\) bits in the empty cells. It can be concluded that \(FP_{4,1}(15, 1) \leq 7\), and from Theorem 4.6(3) we get that \(FP_{4,1}(15, 1) \geq 7\). Thus, \(FP_{4,1}(15, 1) = 7\).

Lastly, we report on several results for \(k = 2\).

**Theorem 4.8.** \(6 \leq FB_{2,2}(8, 2) \leq 7\).
Proof The lower bound is obtained from Theorem 3.1. The upper bound is verified using the construction which appears in Table 4.3, i.e., the construction gives an (8, 2, 7, 2, 2) functional batch array code. There are 8 information bits, 7 buckets, each one with 2 cells, and we show that this code can satisfy each multiset request of size 2. Let $S_1 = \{x_1, x_2, x_3, x_4\}$ be a set of the first 4 information bits and $S_2 = \{x_5, x_6, x_7, x_8\}$ be a set of the last 4 information bits. Let $R = \{v_1, v_2\}$ be a multiset request of size 2, where $v_1$ and $v_2$ are vectors of size 8. For each $i \in [2], v_i = (v_i^1, v_i^2)$ where $v_i^j, j \in [2]$ is a vector of length 4 that represents a linear combination of the bits in $S_j$. The possible linear combinations of $S_1$ are divided into four different types in the following way.

1. The first type $T_1$ includes the vectors that can be satisfied by using only one bucket from the buckets $1 – 3$.

2. The second type $T_2$ includes any vector $u$ that satisfies the following constraint. The vectors $u + (1, 1, 0, 0)$ and $u + (0, 0, 1, 1)$ can be satisfied by one bucket from buckets $1 – 3$. (The vector $1,1,0,0$ represents the linear combination $x_1 + x_2$.)

3. The third type $T_3$ includes any vector $u$ that satisfies the following constraint. The vectors $u + (1, 1, 1, 1)$ and $u + (1, 1, 0, 0)$ can be satisfied by one bucket from the buckets $1 – 3$.

4. The fourth type $T_4$ includes any vector $u$ that satisfies the following constraint. The vectors $u + (1, 1, 1, 1)$ and $u + (0, 0, 1, 1)$ can be satisfied by one bucket from the buckets $1 – 3$.

These four types are disjoint and their union covers all the nonzero linear combinations of $S_1$. From the symmetry of the first four information bits and the last four bits, the linear combinations of $S_2$ are divided in the same way. It is possible to see that every two buckets from buckets $1 – 3$ can satisfy each possible linear combination of the first four bits. In the same way, every two buckets from buckets $4 – 6$ can satisfy each possible linear combination of the last four bits. Also, the last bucket can satisfy each vector $(u, u)$, where $u \in \{(1, 1, 0, 0), (0, 0, 1, 1), (1, 1, 1, 1)\}$.

If one of the vectors $\{v_1^1, v_2^1\}$ is included in $T_1$ (assume it is $v_1^1$) and one of the vectors $\{v_1^2, v_2^2\}$ is included in $T_2$ (assume it is $v_2^2$), then these two vectors can be satisfied by one bucket from $1 – 3$ and one bucket from $4 – 6$. Then the remaining two buckets of $1 – 3$ can satisfy $v_1^2$ and the remaining two buckets of $4 – 6$ can satisfy $v_2^2$. Therefore, in this case the request $R$ is satisfied by disjoint sets.

If there exist $2 \leq q_1, q_2 \leq 4$ where $v_1^1 \in T_{q_1}$ and $v_2^2 \in T_{q_2}$. Then, there exists a vector $u'$ where $v_1^1 + u'$ can be satisfied by one bucket from buckets $1 – 3$ and...
\( v_1^2 + u' \) can be satisfied by one bucket from buckets 4 – 6. Thus, the code can satisfy \( v_1^2 \) and \( v_2^2 \), that consist the request \( v_1 \), by one bucket from 1 – 3, one bucket from 4 – 6, and the last bucket, which satisfies the request \( (u', u') \) for each possible \( u' \). Then, the remaining two buckets of 1 – 3 can satisfy \( v_1^2 \) and the remaining two buckets of 4 – 6 can satisfy \( v_2^2 \). Similarly, if there exist \( 2 \leq q_1, q_2 \leq 4 \) where \( v_1^2 \in T_{q_1} \) and \( v_2^2 \in T_{q_2} \), the code can satisfy the requests \( v_1 \) and \( v_2 \) by disjoint sets.

The last case is when \( \{v_1^2, v_1^3\} \subseteq T_1 \) and \( \{v_2^2, v_2^3\} \subseteq T_2 \), where \( 2 \leq q \leq 4 \) (or \( \{v_1^2, v_2^2\} \subseteq T_3 \) and \( \{v_1^3, v_2^3\} \subseteq T_4 \)). In the beginning we satisfy \( v_1^1 \) by one bucket from 1 – 3. Then, take a vector \( u'' \), such that \( v_2^2 + u'' \) can be satisfied by one bucket, denote it by \( b_1 \). The vector \( v_1^1 + u'' \) can be satisfied by the remaining two buckets from 1 – 3, denote them by \( b_2, b_3 \). Then, the request \( R_2 = \{v_3^1, v_2^3\} \) can be satisfied by \( \{b_1, b_2, b_3, 7\} \) (where 7 is the last bucket). Lastly, the request \( v_1^1 \) can be satisfied by the remaining two buckets from 4 – 6. Thus, we can conclude that there exists 2 recovering sets for each possible request, and hence, \( FB_{2,2}(8, 2) \leq 7 \).

The result in Theorem 4.8 can be generalized to different values of \( s \).

**Corollary 4.9.** \( \log_7(2^{s-1}(2^s - 1)) \leq FB_{2,2}(s, 2) \leq 7 \cdot \left\lceil \frac{s}{8} \right\rceil \).

**Proof** The upper bound is derived from Theorem 4.8 and Theorem 2.2(5). The lower bound is obtained from Theorem 3.1 where \( FB_{2,2}(s, 2) \geq m \) where \( m \) is the smallest positive integer such that \( \sum_{i=2}^{m} \binom{m}{i} \cdot \left( \sum_{j=1}^{n} \binom{n}{j} \right)^i \geq \left( \frac{2}{3} \right)^2 \). It is known that \( \frac{1}{2} \cdot 2^{i-1} = 1 \). Thus, \( \sum_{i=2}^{m} \binom{m}{i} \cdot (2^{i-1} - 1) \cdot 3^i \geq 2^{s-1} \cdot (2^s - 1) \).

For each \( i \geq 2 \), \( (2^{i-1} - 1) \cdot 3^i \leq 6^i \). Hence, it must hold that \( \sum_{i=0}^{m} \binom{m}{i} \cdot 6^i \geq \sum_{i=2}^{m} \binom{m}{i} \cdot 6^i \geq 2^{s-1} \cdot (2^s - 1) \). From the equality \( \sum_{i=0}^{m} \binom{m}{i} \cdot x^i = (x + 1)^m \), we get that \( \sum_{i=0}^{m} \binom{m}{i} \cdot 6^i = 7^m \geq 2^{s-1} \cdot (2^s - 1) \). Thus, \( FB_{2,2}(s, 2) \geq m \geq \log_7(2^{s-1} \cdot (2^s - 1)) \).

According to Corollary 4.9, we get that for \( s \) large enough \( \log_7(2^{s-1} \cdot (2^s - 1)) = \log_7(2^{s-1}) + \log_7(2^s - 1) \approx (s - 1) \cdot \log_7(2) + 2s \cdot \log_7(2) = (2s - 1) \cdot \log_7(2) \approx 0.71s \). In addition, the result in Theorem 4.8 can be modified to different value of \( t \).

**Corollary 4.10.** \( 6 \leq FB_{3,1}(8, 2) \leq 7 \).

**Proof** The lower bound is obtained from Theorem 3.1. The upper bound is verified by Theorem 4.5(2), where \( FB_{3,1}(8, 2) \leq FB_{2,2}(8, 2) \leq 7 \).
Chapter 5

Specific Constructions of Array Codes

In this section we discuss three constructions of array codes.

5.1 Construction A

We start with a construction given in [17, Th.20], where it was proved in [7, Th.10] that this construction gives a PIR array code for any integer $t \geq 2$. We study how it can be used also as batch and functional PIR array codes for $t = 2$. First, the construction for the general case is presented.

**Construction I.** Let $t \geq 2$ be a fixed integer. The number of information bits is $s = t(t+1)$, the number of cells in each bucket (the number of the rows) is $t$. The number of buckets is $m = m' + m''$, where $m' = \binom{t(t+1)}{t}$, and $m'' = \binom{t(t+1)}{t+1}/t$. In the first $m'$ buckets all the tuples of $t$ bits out of the $t(t+1)$ information bits are stored, which needs $\binom{t(t+1)}{t}$ buckets. In the last $m''$ buckets we store all possible summations of $t+1$ bits, such that each one of the $t(t+1)$ bits appears in exactly one summation in every bucket (in each summation there are $t+1$ bits and there are $t$ rows). There are $\binom{t(t+1)}{t+1}$ such summations and since there are $t$ rows then $t$ summations can be stored in each bucket, so the number of buckets of this part is $m'' = \binom{t(t+1)}{t+1}/t$.

For any integer $t \geq 2$ denote the code that is obtained from Construction I by $C^A_t$. Construction I for the case of $t = 2$ is demonstrated in Table 5.1.

Now we want to show that the code $C^A_2$ is a $(6, 15, 25, 2, 2)$ batch array code, by using several properties which are proved in the following three lemmas. For
where $|F|$ can satisfy the bit requests of $x$. Case 1: If $k = 0$, then none of the bits $x_3, x_4, x_5, x_6$ is requested and the property clearly holds.

Case 2: If $k_4 = 0$, then it necessarily holds that $k_3 \leq 5$. Assume by contradiction that $k_3 > 5$. Then, it holds that $k_1 \geq k_2 > 5$, and hence, $k = k_1 + k_2 + k_3 > 15$, which is a contradiction. Thus $k_3 \leq 5$ and the code can use $k_3$ buckets from $\mathcal{F}_3$.

Case 3: If $k_5 = 0$, then it necessarily holds that $k_4 \leq k_3 \leq 4$. Assume by contradiction that $k_4 > 4$. Then, it holds that $k_1 \geq k_2 \geq k_3 > 4$, and hence, $k = k_1 + k_2 + k_3 + k_4 > 15$, which is a contradiction. Assume by contradiction that $k_3 > 4$, when $k_4 \geq 1$. Then, it holds that $k_1 \geq k_2 > 4$, and hence, $k = k_1 + k_2 + k_3 + k_4 > 15$, which is a contradiction. Thus $k_3 \leq 4$ and the code $C_4^3$ can satisfy the bit requests of $x_3$ by taking $k_3$ buckets from $\mathcal{F}_3$. Then the code $C_2^4$ can satisfy the bit requests of $x_4$ by taking $k_4 \leq 4$ buckets from $\mathcal{F}_4 \setminus (\mathcal{F}_4 \cap \mathcal{F}_3)$, where $|\mathcal{F}_4 \setminus (\mathcal{F}_4 \cap \mathcal{F}_3)| = 4$.

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Lemma 5.1. For any multiset request $(k_1, \ldots, k_6)$ of size $k = 15$, the code $C_2^4$ can satisfy all the requests of bits $x_3, x_4, x_5, x_6$ by using only the first 15 buckets.

Proof The proof is divided into the following cases according to number of different information bits that appear in the request.

Case 1: If $k_3 = 0$, then none of the bits $x_3, x_4, x_5, x_6$ is requested and the property clearly holds.

Case 2: If $k_4 = 0$, then it necessarily holds that $k_3 \leq 5$. Assume by contradiction that $k_3 > 5$. Then, it holds that $k_1 > k_2 > 5$, and hence, $k = k_1 + k_2 + k_3 > 15$, which is a contradiction. Thus $k_3 \leq 5$ and the code can use $k_3$ buckets from $\mathcal{F}_3$.

Case 3: If $k_5 = 0$, then it necessarily holds that $k_4 \leq k_3 \leq 4$. Assume by contradiction that $k_4 > 4$. Then, it holds that $k_1 \geq k_2 \geq k_3 > 4$, and hence, $k = k_1 + k_2 + k_3 + k_4 > 15$, which is a contradiction. Assume by contradiction that $k_3 > 4$, when $k_4 \geq 1$. Then, it holds that $k_1 \geq k_2 > 4$, and hence, $k = k_1 + k_2 + k_3 + k_4 > 15$, which is a contradiction. Thus $k_3 \leq 4$ and the code $C_4^3$ can satisfy the bit requests of $x_3$ by taking $k_3$ buckets from $\mathcal{F}_3$. Then the code $C_2^4$ can satisfy the bit requests of $x_4$ by taking $k_4 \leq 4$ buckets from $\mathcal{F}_4 \setminus (\mathcal{F}_4 \cap \mathcal{F}_3)$, where $|\mathcal{F}_4 \setminus (\mathcal{F}_4 \cap \mathcal{F}_3)| = 4$. 

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Case 4: If \( k_6 = 0 \), then it necessarily holds that \( k_5 \leq k_4 \leq 3 \) and \( k_3 \leq 4 \). Assume by contradiction that \( k_5 > 3 \). Then, it holds that \( k_1 \geq k_2 \geq k_3 \geq k_4 \geq k_5 > 3 \), and hence, \( k = k_1 + k_2 + k_3 + k_4 + k_5 > 15 \), which is a contradiction. Assume by contradiction that \( k_4 > 3 \), when \( k_5 + k_4 \geq 2 \). Then, it holds that \( k_1 \geq k_2 \geq k_3 > 4 \), and hence, \( k = k_1 + k_2 + k_3 + k_4 + k_5 > 15 \), which is a contradiction. Thus, \( k_3 \leq 4 \) and the code \( C_2^4 \) can satisfy the bit requests of \( x_3 \) by taking \( k_3 \) buckets from \( F_3 \). Also, \( k_4 \leq 3 \), then the code \( C_2^4 \) can satisfy the bit requests of \( x_4 \) by taking \( k_4 \) buckets from \( F_4 \setminus (F_4 \cap F_3) \). Lastly, the code \( C_2^4 \) can satisfy the bit requests of \( x_5 \) by taking \( k_5 \leq 3 \) buckets from \( F_5 \setminus ((F_5 \cap F_4) \cup (F_5 \cap F_3)) \), where \( |F_5 \setminus ((F_5 \cap F_4) \cup (F_5 \cap F_3))| = 3 \).

Case 5: If \( k_6 > 0 \), then it necessarily holds that \( k_6 \leq k_5 \leq 2 \), \( k_4 \leq 3 \) and \( k_3 \leq 4 \). Assume by contradiction that \( k_6 > 2 \). Then, it holds that \( k_1 \geq k_2 \geq k_3 \geq k_4 \geq k_5 > 2 \), and hence, \( k = \sum_{i=1}^{6} k_i > 15 \), which is a contradiction. Assume by contradiction that \( k_5 > 2 \) when \( k_6 \geq 1 \). Then, it holds that \( k_1 \geq k_2 \geq k_3 \geq k_4 \geq 2 \), and hence, \( k = \sum_{i=1}^{6} k_i > 15 \), which is a contradiction. Assume by contradiction that \( k_4 > 3 \) when \( k_6 + k_5 \geq 2 \). Then, it holds that \( k_1 \geq k_2 \geq k_3 > 3 \), and hence, \( k = \sum_{i=1}^{6} k_i > 15 \), which is a contradiction. Assume by contradiction that \( k_3 > 4 \) when \( k_6 + k_5 + k_4 > 3 \). Then, it holds that \( k_1 \geq k_2 > 4 \), and hence, \( k = \sum_{i=1}^{6} k_i > 15 \), which is a contradiction. Thus, \( 1 \leq k_3 \leq 4 \) and the code \( C_2^4 \) can satisfy the bit requests of \( x_3 \) by taking \( k_3 \) buckets from \( F_3 \). Then the code \( C_2^4 \) can satisfy the bit requests of \( x_4 \) by taking \( k_4 \leq 3 \) buckets from \( F_4 \setminus (F_4 \cap F_3) \). Then the code \( C_2^4 \) can satisfy the bit requests of \( x_5 \) by taking \( k_5 \leq 2 \) buckets from \( F_5 \setminus ((F_5 \cap F_4) \cup (F_5 \cap F_3)) \). Lastly, the code \( C_2^4 \) can satisfy the bit requests of \( x_6 \) by taking \( k_6 \leq 2 \) buckets from \( F_6 \setminus ((F_6 \cap F_5) \cup (F_6 \cap F_4) \cup (F_6 \cap F_3)) \), where \( |F_6 \setminus ((F_6 \cap F_5) \cup (F_6 \cap F_4) \cup (F_6 \cap F_3))| = 2 \).

Lemma 5.2. In the code \( C_2^4 \), for any information bit \( x_i \) and for any bucket \( b_1 \in [15] \setminus F_i \), there exists a bucket \( b_2, 16 \leq b_2 \leq 25 \) such that \( \{b_1, b_2\} \) is a recovering set of \( x_i \). In addition, the \( ([15] \setminus F_i) \) recovering sets are mutually disjoint.

Proof For any information bit \( x_i \), the buckets of \( [15] \setminus F_i \) are the buckets from the first \( m' = 15 \) buckets that does not include \( x_i \). Each bucket \( b_1 \in [15] \setminus F_i \) has two singletons \( x_{j_1}, x_{j_2} \) which are different than \( x_i \). From the construction of the code \( C_2^4 \) we know that there exist a bucket \( b_2 \) from the last 10 buckets that has the summation \( x_i + x_{j_1} + x_{j_2} \). Thus, the subset \( \{b_1, b_2\} \) is a recovering set of \( x_i \).

We want to show that for any two different buckets \( b_1', b_1'' \in [15] \setminus F_i \), the recovering sets \( \{b_1', b_2'\} \) and \( \{b_1'', b_2''\} \) of \( x_i \) are disjoint. It holds that \( \{b_1', b_2'\} \cap \{b_1'', b_2''\} = \emptyset \) because it holds that \( b_1' \neq b_1'' \) and \( b_1' \neq b_2'' \) because \( b_1' \in [15] \setminus F_i \setminus F_j \) and \( \{b_1', b_2'\} \cap \{b_1'', b_2''\} = \emptyset \).
but \( b''_2 \notin [15] \). In addition, \( \{b'_1\} \cap \{b''_1, b''_2\} = \emptyset \) because it holds that \( b'_2 \notin [15] \) but \( b''_1 \in [15] \) and \( b'_2 \neq b''_2 \) because each bucket in the last 10 buckets has exactly one summation with \( x_i \).

For any information bit \( x_i, i \in [6] \) denote by \( R^i_b \) the recovering set that uses bucket \( b \in [15] \) and can satisfy \( x_i \). For example, \( R^1_1 = \{1\} \) and \( R^1_{12} = \{12, 22\} \).

**Lemma 5.3.** For the two information bits \( x_1, x_2 \), the buckets \( \{10, 11, \ldots, 15\} \) are divided into 3 pairs, \( \mathcal{P} = \{(10, 15), (11, 14), (12, 13)\} \), such that for any pair \( (b_1, b_2) \in \mathcal{P} \), it holds that \( |R^1_{b_1} \cap R^2_{b_2}| > 0 \) and \( |R^2_{b_1} \cap R^1_{b_2}| > 0 \).

**Proof** For the first pair, \( (10, 15) \), it holds that \( R^1_{10} = \{10, 20\} \), \( R^2_{10} = \{10, 25\} \), \( R^1_{15} = \{15, 25\} \), and \( R^2_{15} = \{15, 20\} \). Then, it holds that \( |R^1_{10} \cap R^2_{15}| = |\{10, 20\} \cap \{15, 20\}| > 0 \) and \( |R^2_{10} \cap R^1_{15}| = |\{10, 25\} \cap \{15, 25\}| > 0 \). Similarly, the claim holds also for the pairs \( (11, 14) \) and \( (12, 13) \).

Now, we are ready to show that the code \( C^A_2 \) is a \( (6, 15, 25, 2, 2) \) batch array code.

**Theorem 5.4.** The code \( C^A_2 \) is a \( (6, 15, 25, 2, 2) \) batch array code. In particular, \( B_{2,2}(6,15) = 25 \).

**Proof** The lower bound is derived from Theorem 2.1. Therefore, \( B_{2,2}(6,15) \geq \frac{30-6-7}{(4^2+36-4+4)} > 24 \). The upper bound is derived from the code \( C^A_2 \). Let \( (k_1, \ldots, k_6) \) be a multiset request of size \( k = 15 \). The first step is to satisfy all the requests of bits \( x_3, x_4, x_5, x_6 \) according to Lemma 5.1 by using only the first \( m' = 15 \) buckets. Then, the remaining requests are of the bits \( x_1, x_2 \). Denote by \( \alpha_1, \alpha_2 \) the number of the remaining buckets from the first \( m' = 15 \) buckets that include \( x_1, x_2 \) as singleton, but not both of them, respectively. Then, take \( \min\{k_2, \alpha_2\} \) buckets as a recovering set of \( x_2 \) and take \( \min\{k_1, \alpha_1\} \) buckets as recovering sets of \( x_1 \). The first bucket which contains the singletons \( x_1, x_2 \) is not used yet. Denote by \( r \) the number of bit requests from the multiset request that were satisfied so far. Furthermore, denote by \( k'_1, k'_2 \) the number of remaining bit requests of \( x_1, x_2 \), respectively, where \( k'_1 = k_1 - \min\{k_1, \alpha_1\} \) and \( k'_2 = k_2 - \min\{k_2, \alpha_2\} \). After this step we still have \( 15 - r \) buckets in the first \( m'' = 15 \) buckets, including the first bucket and all the last \( m'' = 10 \) buckets. Therefore, for \( x_1 \) and \( x_2 \) there are \( 15 - r \) possible recovering sets.

The second step is to satisfy the remaining \( 15 - r \) bit requests from the multiset request. If \( k'_1 = 0 \) or \( k'_2 = 0 \), then it is possible to satisfy them by using the remaining \( k - r = 15 - r \) recovering sets of \( x_1 \) or \( x_2 \). Otherwise, \( k'_1 > 0 \) and
$k'_2 > 0$. So far we used all the buckets from the set $(\mathcal{F}_1 \cup \mathcal{F}_2) \setminus \{1\}$ which is of size 8 and another $p$ buckets from the subset \{10, 11, \ldots, 15\}. Thus, $k'_1 + k'_2 = 7 - p$.

Let $\mathcal{G} \subseteq \{10, 11, \ldots, 15\}$ be the subset of buckets from \{10, 11, \ldots, 15\} that were not used in the first step and let $p = 6 - |\mathcal{G}|$. According to Lemma 5.2 there are at least $7 - p$ remaining recovering sets for each bit of $\{x_1, x_2\}$, which are the set $\{1\}$ and the sets of $\mathcal{R}_b^i$ where $b \in \mathcal{G}$ and $i \in [2]$. According to Lemma 5.3, the buckets \{10, 11, \ldots, 15\} are divided into 3 pairs, where the $b$-th bucket is paired with the $(25 - b)$-th bucket, for $10 \leq b \leq 15$. The subset $\mathcal{G}$ is partitioned into two subsets, $\mathcal{U}_1 = \{b \in \mathcal{G} : (25 - b) \in \mathcal{G}\}$ and $\mathcal{U}_2 = \{b \in \mathcal{G} : (25 - b) \notin \mathcal{G}\}$. Let $\beta_1 = |\mathcal{U}_1|$ and $\beta_2 = |\mathcal{U}_2|$. The following cases are considered.

**Case 1:** If $p$ is even and $k'_1$ is even (or $k'_2$ is even). Since $p$ is even, it is deduced that $\beta_2$ is even as well. Assume that $k'_1$ is even, then also $(k'_1 - \beta_2)$ is even. In order to satisfy $x_1$ we can take $\min\{\beta_2, k'_1\}$ recovering sets that use $\min\{\beta_2, k'_1\}$ buckets from $\mathcal{U}_2$. We can see that $\beta_1 + \beta_2 = 6 - p$ and $k'_1 \leq 6 - p = \beta_1 + \beta_2$ then $k'_1 - \beta_2 \leq \beta_1$. If $k'_2 > \beta_2$, then we can satisfy the remaining requests of $x_1$ with $(k'_1 - \beta_2)/2$ pairs of buckets from $\mathcal{U}_1$, where for each bucket $b$ from the $(k'_1 - \beta_2)$ buckets we can take $\mathcal{R}_b^i$ as a recovering set for $x_1$. It is possible to show that each recovering set for $x_1$ that uses a bucket from $\mathcal{U}_2$ intersects with only one recovering set for $x_2$ that uses a bucket from $\mathcal{G}$. Also, each pair of recovering sets for $x_1$ that uses a pair of bucket from $\mathcal{U}_1$ intersects with only two recovering sets for $x_2$ that use buckets from $\mathcal{G}$.

Thus, from the $7 - p$ recovering sets of $x_2$ it is not possible to use only $\max\{k'_1, \beta_2 + 2 \cdot \frac{k'_1 - \beta_2}{2}\} = k'_1$ of them. Thus it is possible to use the remaining $7 - p - k'_1 = k'_2$ to satisfy the $k'_2$ requests of $x_2$. The case when $k'_1$ is odd but $k'_2$ is even can be solved similarly while changing between $x_1$ and $x_2$.

**Case 2:** If $p$ is odd and $k'_1$ is odd (or $k'_2$ is odd). Then $\beta_2$ is odd. Assume that $k'_1$ is odd, then also $(k'_1 - \beta_2)$ is even and the rest is similar to Case 1.

**Case 3:** If $p$ is even and $k'_1, k'_2$ are odd. Then start with satisfying $x_1$ with a recovering set $\{1\}$. Then we still have an even number of remaining requests of $x_1$ that must be satisfied, and the rest is similar to Case 1.

**Case 4:** If $p$ is odd and $k'_1, k'_2$ are even. Then start with satisfying $x_1$ with a recovering set $\{1\}$. Then we still have an odd number of remaining requests of $x_1$ that must be satisfied, and the rest is similar to Case 2.

Thus, we can conclude that the code can satisfy each multiset of 15 information bits, and hence, $B_{2,2}(6, 15) = 25$.

In addition it is possible to show that the code $C^A_2$ is a $(6, 11, 25, 2, 2)$ functional PIR array code.

**Theorem 5.5.** The code $C^A_2$ is a $(6, 11, 25, 2, 2)$ functional PIR array code. In particular, $21 \leq FP_{2,2}(6, 11) \leq 25$. 

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Proof The lower bound is obtained from Theorem 3.5 where $FP_{2,2}(6, 11) \geq 2^{11.63/3+63} = 21$. The upper bound can be obtained from the code $C_2^A$. Given a request $R$, a linear combination of the information bits, that the code $C_2^A$ must satisfy $k = 11$ times by disjoint recovering sets. Because of the symmetry of $x_i, i \in [6]$, it is sufficient to check requests according to their length (number of information bits). Thus, the proof is divided into the following cases according to number of information bits that appear in the request.

Case 1: If the request contains one information bit then it is the case of PIR.

Case 2: If the request contains two information bits, then assume that it is $x_1 + x_2$. Then, the recovering sets are the following $\{1\}, \{2, 6\}, \{3, 7\}, \{4, 8\}, \{5, 9\}, \{16, 11\}, \{17, 10\}, \{18, 13\}, \{19, 12\}, \{20, 25\}, \{21, 24\}, \{22, 23\}$.

Case 3: If the request contains three information bits, then assume that it is $x_1 + x_2 + x_3$. Then, the recovering sets are the following $\{16\}, \{1, 2\}, \{17, 10\}, \{18, 11\}, \{19, 12\}, \{20, 7\}, \{21, 8\}, \{22, 9\}, \{23, 5\}, \{24, 4\}, \{25, 3\}$.

Case 4: If the request contains four information bits, then assume that it is $x_1 + x_2 + x_3 + x_4$. Then, the recovering sets are the following $\{16, 2\}, \{17, 3\}, \{18, 4\}, \{19, 5\}, \{20, 25\}, \{21, 24\}, \{22, 23\}, \{10, 15\}, \{11, 14\}, \{12, 13\}, \{6, 7, 8, 9\}$.

Case 5: If the request contains five information bits, then assume that it is $x_2 + x_3 + x_4 + x_5 + x_6$. Then, the recovering sets for the request are the following $\{16, 1\}, \{17, 2\}, \{18, 3\}, \{19, 4\}, \{20, 5\}, \{21, 11\}, \{22, 12\}, \{23, 13\}, \{24, 14\}, \{25, 15\}, \{6, 7, 8, 9\}$.

Case 6: If the request contains all the information bits, that it is $x_1 + x_2 + x_3 + x_4 + x_5 + x_6$. Then, the recovering sets are the following $\{16\}, \{17\}, \{18\}, \{19\}, \{20\}, \{21\}, \{22\}, \{23\}, \{24\}, \{25\}, \{1, 10, 15\}, \{2, 8, 14\}, \{3, 9, 11\}, \{4, 7, 12\}, \{5, 6, 13\}$.

5.2 Construction B

Next we generalize an example given in [17] of a PIR code for any integer $r \geq 3$ and study how it can be used also as batch array codes. We first present the construction for the general case.

Construction II. Let $r \geq 3$ be a fixed integer, the number of information bits is $s = r(r+1)$, the number of the buckets is $m = r+1$, and the number of the cells in each bucket is $t = (r-1)r+1$. The information bits are partitioned into $r+1$ parts each of size $r$, denote by $S_i$ the part $i$ of the bits. For each $i \in [r+1]$, write the linear combination $\sum_{j \in S_i} x_j$ to bucket $i$. For each $i, i \in [r+1]$ write each
Table 5.2: Construction II for $r = 3$

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<th>3</th>
<th>4</th>
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<tbody>
<tr>
<td>$x_1x_2x_3$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
<td>$x_1$</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$x_2$</td>
<td>$x_3$</td>
<td>$x_3$</td>
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<tr>
<td>$x_5$</td>
<td>$x_4x_5x_6$</td>
<td>$x_4$</td>
<td>$x_5$</td>
<td>$x_6$</td>
</tr>
<tr>
<td>$x_7$</td>
<td>$x_7$</td>
<td>$x_5$</td>
<td>$x_6$</td>
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</tr>
<tr>
<td>$x_8$</td>
<td>$x_9$</td>
<td>$x_7x_8x_9$</td>
<td>$x_8$</td>
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</tr>
<tr>
<td>$x_{10}$</td>
<td>$x_{10}$</td>
<td>$x_{11}$</td>
<td>$x_9$</td>
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<tr>
<td>$x_{11}$</td>
<td>$x_{12}$</td>
<td>$x_{12}$</td>
<td>$x_{10x_{11}x_{12}}$</td>
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</tbody>
</table>

one of the subsets of size $r - 1$ of $S_i$ as singletons in a different bucket other than bucket $i$.

For any integer $r \geq 3$ denote the code that is obtained from Construction II by $C_r^B$. Construction II for the case of $r = 3$ is demonstrated in Table 5.2. It is possible to show that for any $r \geq 3$ the code $C_r^B$ is an $(r^2 + r, r, r + 1, r^2 - r + 1, r - 1)$ PIR array code.

**Theorem 5.6.** For any integer $r \geq 3$ the code $C_r^B$ from Construction II is an $(r^2 + r, r, r + 1, r^2 - r + 1, r - 1)$ PIR array code. In particular,

$$\frac{r \cdot (4r^2 + 3r - 1)}{4r^2 - r + 1} \leq P_{r^2-r+1,r-1}^2(r^2 + r, r) \leq r + 1.$$

**Proof** The lower bound can be obtained by using Theorem 2.1(2).

$$P_{r^2-r+1,r-1}^2(r^2 + r, r) \geq P_{r^2-r+1,r+1}^2(r^2 + r, r) \geq \frac{r \cdot (r^2 + r)(4r - 1)}{(4r - 1)(r^2 - r + 1) + (2r - 1)^2} = \frac{r(4r^3 - r^2 + 4r^2 - r)}{4r^3 - 4r^2 + 4r - r^2 + r - 1 + 4r^2 - 4r + 1} = \frac{r^2(4r^2 + 3r - 1) - r \cdot (4r^2 + 3r - 1)}{4r^2 - r + 1}.$$

The upper bound is verified by using the code $C_r^B$. There are $s = r(r + 1)$ information bits, and the number of buckets is $m = r + 1$. For each $i \in [m]$, there exists a cell with the linear combination $\sum_{q \in S_i} x_q$ and another $r(r - 1)$ cells to store
one \((r-1)\)-subset from each \(S_j\), \(j \in [r+1]\), where \(j \neq i\). Thus, the number of the rows is \(r^2 - r + 1\).

Let \(x_j\) be a request that the code \(C^B_r\) must satisfy by \(r\) disjoint recovering sets. Assume that \(x_j \in S_i, i \in [r+1]\). There are \(r - 1\) buckets which include \(x_j\) as a singleton, because \(x_j\) appears in \(r - 1\) subsets of length \(r - 1\) of part \(S_i\). Thus, each bucket of the \(r - 1\) buckets is taken as a recovering set, while reading only one cell from it. In addition, in the \(i\)-th bucket there exists a cell with \(\sum_{q \in S_j} x_q\), which includes \(x_j\). The \((r-1)\)-subset, \(S_i \setminus \{x_j\}\), is written in a bucket \(p\), which is different from bucket \(i\), and is different from the buckets that were taken so far (because \(x_j \notin S_i \setminus \{x_j\}\)). Thus, the set \(\{i, p\}\) is a recovering set of \(x_j\), and it is sufficient to read from bucket \(i\) one cell, which is \(\sum_{q \in S_i} x_q\) and to read \(r - 1\) cells with the \(r - 1\) bits of \(S_i \setminus \{x_j\}\) from bucket \(p\). Thus, there exist \(r\) disjoint recovering sets for \(x_j\), where at most \(r - 1\) cells are read from each bucket.

Next we want to show that for any integer \(r \geq 3\) the code \(C^B_r\) is an \((r^2 + r, r, r + 1, r^2 - r + 1, r - 1)\) batch array code, by using a property stated in the following lemma.

**Lemma 5.7.** For any integer \(r \geq 3\) it holds that every two buckets of the code \(C^B_r\) can form a recovering set of every bit \(x_i\) by reading at most \(r - 1\) cells from each bucket.

**Proof** Given a pair of buckets from \(C^B_r\), for simplicity we assume that they are the first two buckets. The first bucket has a cell with \(\sum_{i \in S_1} x_i\), and has exactly \(r - 1\) bits as singletons from each \(S_j, 2 \leq j \leq r + 1\). Hence, the first bucket does not include exactly one of the information bits from each \(S_j, 2 \leq j \leq r + 1\). Thus, the number of bits that do not appear as singletons in the first bucket is \(2r\). Hence, the first bucket can satisfy each information bit except to these \(2r\) bits, by reading exactly one cell.

The second bucket contains \(r - 1\) bits out of the \(r\) bits of \(S_1\) as singletons. Thus, each one of these \((r-1)\) bits from \(S_1\) can be satisfied by reading each one of them as a singleton from the second bucket. Also, the remaining bit of \(S_1\) can be satisfied by reading the \(r - 1\) singletons of \(S_1\) from the second bucket with the cell \(\sum_{i \in S_1} x_i\) in the first bucket.

The first two buckets include different \((r-1)\)-subsets of each part other than \(S_1, S_2\). Then, the information bit that does not appear as a singleton cell or as part of the cell \(\sum_{i \in S_1} x_i\) in the first bucket, definitely appears as a singleton cell or in the cell \(\sum_{i \in S_2} x_i\) in the second bucket. Then, each bit \(x_q \in S_j\) where \(3 \leq j \leq r + 1\) can be satisfied by reading it as a singleton from the second bucket. There are \(r - 1\) such bits, and thus, it remains to show that the code can satisfy the bit \(x_{q_1} \in S_2\) that
is not part of the \((r - 1)\)-subset of singletons which are stored in the first bucket. We can satisfy \(x_{q_1}\) by reading the \(r - 1\) singletons of \(S_2\) from the first bucket with the cell \(\sum_{i \in S_2} x_i\) in the second bucket. Thus, the first two buckets of the code \(C_B^r\) can form a recovering set of every bit \(x_i\). Similarly, it holds for any two buckets of the code \(C_B^r\).

Now, we are ready to show that for any integer \(r \geq 3\) the code \(C_B^r\) is \((r^2 + r, r, r + 1, r^2 - r + 1, r - 1)\) batch array code.

**Theorem 5.8.** For any integer \(r \geq 3\) the code \(C_B^r\) from Construction \(\Pi\) is an \((r^2 + r, r, r + 1, r^2 - r + 1, r - 1)\) batch array code. In particular,

\[
\frac{r \cdot (4r^2 + 3r - 1)}{4r^2 - r + 1} \leq B_{r^2 - r + 1, r - 1}(r^2 + r, r) \leq r + 1.
\]

**Proof** The lower bound is follows from the lower bound of \(P_{r^2 - r + 1, r - 1}(r^2 + r, r)\). The upper bound is achieved by using Construction \(\Pi\). Let \(R = \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}\) be a multiset request of \(r\) information bits. First, we want to show that the code \(C_B^r\) can satisfy the first \(r - 1\) bits of the request by using only \(r - 1\) buckets. From Construction \(\Pi\) it is known that each information bit \(x_i\) appears as a singleton in \(r - 1\) buckets out of the \(r + 1\) buckets. Thus, in each subset of buckets of size at least 3, there is at least one bucket that contains a cell with \(x_i\). Therefore, the first \(r - 1\) bits of the request can be read by singletons from \(r - 1\) different buckets.

After the first step, we still have 2 buckets and from Lemma 5.7 it is known that these two buckets can satisfy each \(x_{i_j}\) in particular \(x_{i_1}\).

According to Theorem 5.6 and Theorem 5.8 it can be verified that for any \(r \geq 3\), \(r < \frac{r \cdot (4r^2 + 3r - 1)}{4r^2 - r + 1} \leq B_{r^2 - r + 1, r - 1}(r^2 + r, r) \leq B_{r^2 - r + 1, r - 1}(r^2 + r, r) \leq r + 1\). Thus, we conclude that Construction \(\Pi\) gives optimal PIR and batch array codes.

### 5.3 Construction C

We now present our third construction, and study how it can be used as PIR and functional PIR array codes for specific parameters.

**Construction III.** Let \(s \geq 2\) be a fixed integer. The number of information bits is \(s\), the number of cells in each bucket (the number of the rows) is 2. We write each two nonzero disjoint linear combinations of total size at most \(s\), and hence,
Table 5.3: Construction $\text{III}$ for $s = 4$

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<tr>
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For any integer $s \geq 2$ denote the code that is obtained from Construction $\text{III}$ by $C^C_s$. Construction $\text{III}$ for the case of $s = 4$ is demonstrated in Table 5.3 and provides the following results. First, we show that the code $C^C_4$ is a $(4, 16, 25, 2, 1)$ PIR array code.

**Theorem 5.9.** The code $C^C_4$ from Construction $\text{III}$ is a $(4, 16, 25, 2, 1)$ PIR array code. In particular, $23 \leq P_{2,1}(4, 16) \leq 25$.

**Proof** The lower bound is obtained by using Theorem 2.1[2]. Thus, $P_{2,1}(4, 16) \geq P_{2,2}(4, 16) \geq \frac{16 \cdot 4}{5 \cdot 2 + 4} > 22$. The upper bound is verified using the code $C^C_4$. Let $x_i, i \in [4]$ be a request, that the code $C^C_4$ must satisfy 16 times. From the symmetry of the code, assume that $x_1 = x_1$. The following are the recovering sets of $x_1$, where from each bucket only one cell is read: $\{\{1\}, \{2\}, \{3\}, \{7\}, \{8\}, \{9\}, \{19\}, \{10, 6\}, \{11, 5\}, \{13, 4\}, \{14, 18\}, \{15, 17\}, \{16, 12\}, \{20, 23\}, \{21, 24\}, \{22, 25\}\}$.

Next, we show that the code $C^C_4$ is a $(4, 14, 25, 2, 2)$ functional PIR array code.

**Theorem 5.10.** The code $C^C_4$ from Construction $\text{III}$ is a $(4, 14, 25, 2, 2)$ functional PIR array code. In particular, $24 \leq FP_{2,2}(4, 14) \leq 25$. 

$\sum_{i=2}^{s} \binom{\binom{s}{i} \cdot \{\binom{i}{2}\}}{2}$ buckets. Then,

$$m = \sum_{i=2}^{s} \binom{s}{i} \binom{i}{2} = \sum_{i=2}^{s} \binom{s}{i} (2^{i-1} - 1) = \frac{3^s + 1}{2} - 2^s.$$
Proof The lower bound is obtained using Theorem $\ref{thm:lb}$ \(FP_{2,2}(4, 14) \geq \frac{2^{4+15}}{15+3} > 23\). The upper bound is verified using the code \(C^p_1\). Let \(R\) be a linear combination request, that the code \(C^p_1\) must satisfy 14 times. From the symmetry of the code, the proof is divided into the following cases according to the number of information bits that appear in \(R\). If the number of information bits that appear in \(R\) is \(p\) then we assume that the request is \(x_1 + x_2 + \cdots + x_p\).

Case 1: The recovering sets are the following \(\{\{1\}, \{2\}, \{3\}, \{7\}, \{8\}, \{9\}, \{19\}, \{10, 6\}, \{11, 5\}, \{13, 4\}, \{14, 18\}, \{16, 12\}, \{20, 23\}, \{21, 24\}, \{22, 25\}\}.

Case 2: The recovering sets are the following \(\{\{1\}, \{13\}, \{16\}, \{23\}, \{2, 4\}, \{3, 5\}, \{7, 10\}, \{8, 11\}, \{9, 12\}, \{14, 15\}, \{17, 18\}, \{19, 20\}, \{21, 22\}, \{24, 25\}\}.

Case 3: The recovering sets are the following \(\{\{7\}, \{10\}, \{13\}, \{22\}, \{1, 24\}, \{2, 23\}, \{3, 25\}, \{4, 17\}, \{5, 14\}, \{6, 16\}, \{8, 20\}, \{9, 21\}, \{11, 12\}, \{18, 19\}\}.

Case 4: The recovering sets are the following \(\{\{19\}, \{20\}, \{21\}, \{22\}, \{23\}, \{24\}, \{25\}, \{1, 6\}, \{2, 5\}, \{3, 4\}, \{7, 11\}, \{8, 10\}, \{9, 13\}, \{12, 16\}, \{14, 18\}, \{15, 17\}\}.

Construction \(\text{III}\) for the case of \(s = 5\) is demonstrated in Table \(\ref{tab:construction}\) and provides the following result.

**Theorem 5.11.** The code \(C^5_1\) from Construction \(\text{III}\) is a \((5, 48, 90, 2, 2)\) functional PIR array code. In particular, \(88 \leq FP_{2,2}(5, 48) \leq 90\).

Proof The lower bound is obtained using Theorem $\ref{thm:lb}$ \(FP_{2,2}(5, 48) \geq \frac{2^{48+31}}{31+3} > 87\). The upper bound is verified using the code \(C^p_5\). Let \(R\) be a linear combination request that the code \(C^p_5\) must satisfy 48 times. From the symmetry of the code, the proof is divided into the following cases according to the number of information bits that appear in \(R\). If the number of information bits that appear in \(R\) is \(p\) then we assume that the request is \(x_1 + x_2 + \cdots + x_p\).

Case 1: The recovering sets are the following \(\{\{1\}, \{2\}, \{3\}, \{4\}, \{11\}, \{12\}, \{13\}, \{14\}, \{15\}, \{16\}, \{41\}, \{42\}, \{43\}, \{44\}, \{61\}, \{17, 26\}, \{18, 32\}, \{19, 38\}, \{20, 23\}, \{21, 29\}, \{22, 35\}, \{24, 33\}, \{25, 39\}, \{27, 30\}, \{28, 36\}, \{31, 40\}, \{34, 37\}, \{45, 8\}, \{46, 9\}, \{47, 10\}, \{49, 6\}, \{50, 7\}, \{48, 51\}, \{53, 5\}, \{52, 54\}, \{55, 67\}, \{57, 72\}, \{58, 69\}, \{59, 66\}, \{64, 56\}, \{65, 60\}, \{62, 78\}, \{63, 79\}, \{71, 81\}, \{73, 82\}, \{83, 80\}, \{68, 85\}, \{70, 88\}, \{74, 86\}, \{75, 90\}, \{76, 89\}, \{77, 87\}\}.

Case 2: The recovering sets are the following \(\{\{1\}, \{23\}, \{29\}, \{35\}, \{66\}, \{67\}, \{68\}, \{81\}, \{2, 5\}, \{3, 6\}, \{4, 7\}, \{9, 53\}, \{8, 49\}, \{10, 88\}, \{11, 20\}, (2020-38 - 2020)
Table 5.4: Construction $\mathbf{III}$ for $s = 5$

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Case 3: The recovering sets are the following \{11\}, \{17\}, \{23\}, \{33\}, \{65\},
\{66\}, \{69\}, \{72\}, \{2, 6\}, \{3, 5\}, \{10, 11\}, \{9, 12\}, \{7, 14\}, \{4, 20\},
\{13, 28\}, \{15, 22\}, \{16, 21\}, \{17, 34\}, \{18, 27\}, \{19, 40\}, \{23, 64\}, \{24, 62\},
\{25, 33\}, \{26, 61\}, \{29, 63\}, \{30, 48\}, \{31, 38\}, \{32, 44\}, \{35, 88\}, \{36, 39\},
\{37, 85\}, \{42, 90\}, \{43, 89\}, \{46, 68\}, \{47, 67\}, \{50, 71\}, \{51, 87\}, \{52, 84\},
\{54, 74\}, \{55, 70\}, \{56, 75\}, \{57, 78\}, \{58, 79\}, \{59, 73\}, \{60, 76\}, \{77, 81\},
\{82, 86\}.

Case 4: The recovering sets are the following \{41\}, \{45\}, \{49\}, \{53\}, \{65\},
\{66\}, \{69\}, \{72\}, \{1, 8\}, \{2, 6\}, \{3, 5\}, \{10, 11\}, \{9, 12\}, \{7, 14\}, \{4, 20\},
\{13, 28\}, \{15, 22\}, \{16, 21\}, \{17, 34\}, \{18, 27\}, \{19, 40\}, \{23, 64\}, \{24, 62\},
\{25, 33\}, \{26, 61\}, \{29, 63\}, \{30, 48\}, \{31, 38\}, \{32, 44\}, \{35, 88\}, \{36, 39\},
\{37, 85\}, \{42, 90\}, \{43, 89\}, \{46, 68\}, \{47, 67\}, \{50, 71\}, \{51, 87\}, \{52, 84\},
\{54, 74\}, \{55, 70\}, \{56, 75\}, \{57, 78\}, \{58, 79\}, \{59, 73\}, \{60, 76\}, \{77, 81\},
\{82, 86\}.

Case 5: The recovering sets are the following \{61\}, \{62\}, \{63\}, \{64\}, \{65\},
\{66\}, \{81\}, \{82\}, \{83\}, \{84\}, \{85\}, \{86\}, \{87\}, \{88\}, \{89\}, \{90\}, \{66, 4\}, \{67, 3\},
\{68, 2\}, \{69, 7\}, \{70, 6\}, \{71, 1\}, \{72, 9\}, \{73, 5\}, \{74, 17\}, \{75, 10\}, \{76, 8\},
\{77, 18\}, \{78, 11\}, \{79, 12\}, \{80, 13\}, \{41, 40\}, \{42, 34\}, \{43, 28\}, \{44, 19\},
\{45, 39\}, \{46, 33\}, \{47, 27\}, \{48, 14\}, \{49, 38\}, \{50, 32\}, \{51, 20\}, \{52, 15\},
\{53, 37\}, \{54, 26\}, \{55, 21\}, \{56, 16\}, \{57, 31\}, \{58, 25\}, \{59, 22\}.

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Chapter 6

Asymptotic Analysis of Array Codes

The goal of this section is to provide a figure of merit in order to compare between the different constructions of array codes. For simplicity we consider the case where $\ell = t$, that is, it is possible to read all the bits in every bucket. Under this setup, it holds that $F_{P,t}(s,k) \leq sk/t$ for all $s,k$, and $t$. This motivates us to define the following values

$$R_X(t,k) = \limsup_{s \to \infty} \frac{X_{t,t}(s,k)}{sk/t},$$

where $X \in \{P,B,FP,FB\}$. The case where $t = 1$ has been studied in several previous works. For example, for functional PIR array codes we have $R_{FP}(1,k) \geq \frac{1}{kH(1/k)}$ for any even integer $k \geq 4$ [42, Th. 13]. Also, for functional batch array codes it holds from [42, Th. 21] that $R_{FB}(1,k) \leq \frac{1}{kH(c_k)}$, where $c_1 = \frac{1}{2}$ and $c_{k+1}$ is the root of the polynomial $H(z) = H(c_k) - zH(c_k)$. For the case $k = 1$ we have $R_{FB}(1,1) = R_{FP}(1,1) = 1$ from Theorem 4.6. According to the bounds and constructions studied in the work, we can already summarize several results in the following theorems for $t = 2$ and general values.

**Theorem 6.1.**

1. $R_{FP}(2,2) \leq R_{FB}(2,2) \leq \frac{7}{8} = 0.875$, and $R_{FB}(2,2) \geq 0.71$.
2. $R_{FP}(2,11) \leq \frac{25}{33} = 0.758$.
3. $R_{FP}(2,14) \leq \frac{25}{28} = 0.893$.  

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4. $\mathcal{R}_{FP}(2,48) \leq \frac{3}{4} = 0.75$.
5. $\mathcal{R}_{P}(2,16) \leq \frac{25}{32} = 0.78125$.
6. $\mathcal{R}_{B}(2,15) \leq \frac{5}{9} = 0.556$.

Proof
1. From Theorem 4.8 we have $FB_{2,2}(s,2) \leq 7 \cdot \left\lceil \frac{s}{8} \right\rceil$. Thus, $\mathcal{R}_{FB}(2,2) = \limsup_{s \to \infty} \frac{FB_{2,2}(s,2)}{2s/2} \leq \limsup_{s \to \infty} \frac{7\lceil s/8 \rceil}{s} \leq \limsup_{s \to \infty} \frac{(7s/8) + 7}{s} = \frac{7}{8}$.

From Corollary 4.9 we have $FB_{2,2}(s,2) \geq 0.71s$. Thus, $\mathcal{R}_{FB}(2,2) = \limsup_{s \to \infty} \frac{FB_{2,2}(s,2)}{2s/2} \geq \limsup_{s \to \infty} \frac{0.71s}{s} = 0.71$.

2. From Theorem 5.5 we have $FP_{2,2}(6,11) \leq 25$. Then, it is possible to use Theorem 2.2(5) to get that $FP_{2,2}(s,11) \leq 25 \cdot \left\lceil \frac{s}{6} \right\rceil$. Thus, $\mathcal{R}_{FP}(2,11) = \limsup_{s \to \infty} \frac{FP_{2,2}(s,11)}{11s/2} \leq \limsup_{s \to \infty} \frac{25\lceil s/6 \rceil}{11s/2} \leq \limsup_{s \to \infty} \frac{(25s/6) + 25}{11s/2} = \frac{25}{32} = 0.78125$.

3. From Theorem 5.10 we have $FP_{2,2}(4,14) \leq 25$. Then, it is possible to use Theorem 2.2(5) to get that $FP_{2,2}(s,14) \leq 25 \cdot \left\lceil \frac{s}{4} \right\rceil$. Thus, $\mathcal{R}_{FP}(2,14) = \limsup_{s \to \infty} \frac{FP_{2,2}(s,14)}{14s/2} \leq \limsup_{s \to \infty} \frac{25\lceil s/4 \rceil}{14s/2} \leq \limsup_{s \to \infty} \frac{(25s/4) + 25}{14s/2} = \frac{25}{32} = 0.75$.

4. From Theorem 5.11 we have $FP_{2,2}(5,48) \leq 90$. Then, it is possible to use Theorem 2.2(5) to get that $FP_{2,2}(s,48) \leq 90 \cdot \left\lceil \frac{s}{5} \right\rceil$. Thus, $\mathcal{R}_{FP}(2,48) = \limsup_{s \to \infty} \frac{FP_{2,2}(s,48)}{48s/2} \leq \limsup_{s \to \infty} \frac{90\lceil s/5 \rceil}{48s/2} \leq \limsup_{s \to \infty} \frac{(90s/5) + 90}{48s/2} = \frac{90}{120} = \frac{3}{4} = 0.75$.

5. From Theorem 5.9 we have $P_{2,1}(4,16) \leq 25$. Then, it is possible to use Theorem 2.2(5) and get that $P_{2,1}(s,16) \leq 25 \cdot \left\lceil \frac{s}{4} \right\rceil$. Thus, $\mathcal{R}_{P}(2,16) = \limsup_{s \to \infty} \frac{P_{2,1}(s,16)}{16s/2} \leq \limsup_{s \to \infty} \frac{25\lceil s/4 \rceil}{16s/2} \leq \limsup_{s \to \infty} \frac{(25s/4) + 25}{16s/2} = \frac{25}{32} = 0.78125$.

6. From Theorem 5.4 we have $B_{2,2}(6,15) = 25$. Then, it is possible to use Theorem 2.2(5) and get that $B_{2,2}(s,15) \leq 25 \cdot \left\lceil \frac{s}{6} \right\rceil$. Thus, $\mathcal{R}_{B}(2,15) = \limsup_{s \to \infty} \frac{B_{2,2}(s,15)}{15s/2} \leq \limsup_{s \to \infty} \frac{25\lceil s/6 \rceil}{15s/2} \leq \limsup_{s \to \infty} \frac{(25s/6) + 25}{15s/2} = \frac{25}{45} = 0.556$.

Theorem 6.2.

1. For any $r \geq 3$, $\mathcal{R}_{P}(r^2 - r + 1, r) \leq \frac{(r+1)(r^2-r+1)}{r(r^2+1)}$ (also for $B$).
2. For any \( t \geq 2 \), \( \mathcal{R}_P(t, k) \leq \frac{m}{k(t+1)} \), where \( k = \binom{t(t+1)}{i} \) and \( m = k + \frac{t(t+1)}{i} \).

3. For any two integers \( t \) and \( k \), \( \mathcal{R}_F(t, k) \leq \frac{1}{kH(c_k)} \), where \( c_1 = \frac{1}{2} \) and \( c_{k+1} \) is the root of the polynomial \( H(z) = H(c_k) - zH(c_k) \).

4. For any positive integers \( t, k \) and \( a \), \( \mathcal{R}_X(t, a \cdot k) \leq \mathcal{R}_X(t, k) \), where \( X \in \{P, B, FP, FP\} \).

5. For any positive integers \( t, k \) and \( a \), \( \mathcal{R}_X(t, k) \leq \mathcal{R}_X(a \cdot t, k) \), where \( X \in \{P, B, FP, FP\} \).

**Proof**

1. From Theorem 5.6 we have for any \( r \geq 3 \), \( P_{r^2-r+1,r+1}(s^2 + r, r) \leq \frac{r+1}{r^2 - r + 1} \). Then, it is possible to use Theorem 2.2 to get that \( P_{r^2-r+1,r+1}(s, r) \leq (r+1) \cdot \left[ \frac{s}{r^2 + r} \right] \). Thus, for a given \( r \), it holds that

\[
\mathcal{R}_P(r^2 - r + 1, r) = \limsup_{s \to \infty} \frac{P_{r^2-r+1,r^2-r+1}(s, r)}{rs/(r^2 - r + 1)}
\]

\[
\leq \limsup_{s \to \infty} \frac{P_{r^2-r+1,r-1}(s, r)}{rs/(r^2 - r + 1)}
\]

\[
= \limsup_{s \to \infty} \frac{(r+1) \cdot s}{rs/(r^2 - r + 1)}
\]

\[
= \limsup_{s \to \infty} \frac{(r+1) s}{rs/(r^2 - r + 1)} = \frac{(r+1)(r^2 - r + 1)}{r(r^2 + r)}.
\]

2. From Theorem 2.1 we have for any \( t \geq 2 \) and \( p = t + 1 \), \( P_{t+1}(t(t+1), k) \leq m \), where \( k = \binom{t(t+1)}{i} \) and \( m = k + \frac{t(t+1)}{i} \). Then, it is possible to use Theorem 2.2 to get that \( P_{t+1}(s, k) \leq m \cdot \left[ \frac{s}{t(t+1)} \right] \). Thus, for a given \( t \), it holds that \( \mathcal{R}_P(t, k) = \limsup_{s \to \infty} \frac{P_{t}(s, k)}{sk/t} \leq \limsup_{s \to \infty} \frac{m}{sk/t} = m \cdot \frac{1}{k(t+1)} \).

3. From Lemma 4.1 we have \( FB(t, k) \leq FB(s, k) \leq FB([s/t], t \cdot k) \). Thus, \( \mathcal{R}_FB(t, k) = \limsup_{s \to \infty} \frac{FB(s, k)}{sk/t} \leq \limsup_{s \to \infty} \frac{FB([s/t], t \cdot k)}{sk/t} = \limsup_{s \to \infty} \frac{FB([s/t], t \cdot k)}{s/t} \cdot \frac{1}{k} \). Thus, according to [42 Th. 21], \( \mathcal{R}_FB(t, k) \leq \frac{1}{kH(c_k)} \), where \( c_1 = \frac{1}{2} \) and \( c_{k+1} \) is the root of the polynomial \( H(z) = H(c_k) - zH(c_k) \).
4. From Theorem 2.2(3) we have that for any positive integer \( a \) and any \( X \in \{P, B, FP, FP\} \), \( X_{t,t}(s, a \cdot k) \leq a \cdot X_{t,t}(s, k) \). Thus, \( R_X(t, a \cdot k) = \limsup_{s \to \infty} \frac{X_{t,t}(s,a \cdot k)}{sk/t} \leq \limsup_{s \to \infty} \frac{a \cdot X_{t,t}(s,k)}{sk/t} = \limsup_{s \to \infty} \frac{X_{t,t}(s,k)}{sk/t} = R_X(t,k) \).

5. From Theorem 2.2(6) we have that for any positive integer \( a \) and any \( X \in \{P, B, FP, FP\} \), \( a \cdot X_{t,at}(s,k) \geq X_{t,t}(s,k) \). Thus, \( R_X(t,k) = \limsup_{s \to \infty} \frac{X_{t,t}(s,k)}{sk/t} \leq \limsup_{s \to \infty} \frac{a \cdot X_{t,at}(s,k)}{sk/t} = \limsup_{s \to \infty} \frac{X_{t,at}(s,k)}{sk/(at)} = R_X(a \cdot t,k) \).
Chapter 7

Locality Codes

In this section we study a new family of array codes which is a special case of functional PIR array codes in the sense that each recovering set is of size at most \( r \) and all the cells of each bucket can be read, i.e., \( \ell = t \). This new family of array codes will be called *locality functional array codes*. In order to find lower bounds and constructions for locality functional array codes we will use codes and designs in subspaces and covering codes.

### 7.1 Definitions and Basic Constructions

This section is studying the following family of codes.

**Definition 7.1.** An \((s, k, m, t, r)\) *locality functional array code* over \( \Sigma \) is defined by an encoding map \( \mathcal{E} : \Sigma^s \to (\Sigma^t)^m \) that encodes \( s \) information bits \( x_1, \ldots, x_s \) into a \( t \times m \) array and a decoding function \( D \) that satisfies the following property. For any request of a linear combination \( \nu \) of the information bits, there is a partition of the columns into \( k \) recovering sets \( S_1, \ldots, S_k \subseteq [m] \) where \( |S_j| \leq r \) for any \( j \in [k] \).

We denote by \( D(s, k, t, r) \) the smallest number of buckets \( m \) such that an \((s, k, m, t, r)\) locality functional array code exists. For the rest of the section, assume that the parameters \( s, k, t \) and \( r \) are positive integers such that \( t \leq s \). The following theorem summarizes several results on \( D(s, k, t, r) \) based upon basic bound and constructions.

**Theorem 7.1.**

1. \( D(s, k, t, r) \geq m^* \), where \( m^* \) is the smallest positive integer such that
   \[
   \sum_{i=1}^{\min \{r, m^*-k+1\}} \binom{m^*}{i} (2^t-1)^i \geq k(2^s-1).
   \]
2. For any integer \( a \) where \( 1 \leq a < t \), \( D(s, k, t, r) \leq D(s - a, k, t - a, r) \).

3. For every positive integers \( s_1, s_2, r_1, r_2 \) and \( p \), \( D(s_1 + s_2, k, t, r_1 + r_2) \leq D(s_1, k, t, r_1) + D(s_2, k, t, r_2) \). In particular, \( D(ps, k, t, pr) \leq p \cdot D(s, k, t, r) \).

Proof

1. Similar to the proof of Theorem 3.3 but with minor changes. Here, all cells from each bucket can be read. Hence, for any positive integer \( n \), there are \( (2^t - 1)^n \) nonzero linear combinations that can be obtained from \( n \) buckets while using all the \( n \) buckets. Also, each recovering set must be of size at most \( \min\{r, m^s - k + 1\} \). Thus, we get that \( \sum_{i=1}^{\min\{r, m^s - k + 1\}} (m^s_i)(2^t - 1)^i \geq k(2^s - 1) \).

2. Let \( C \) be an \((s - 1, k, m, t - 1, r)\) locality functional array code with \( m \) buckets such that each bucket has \( t - 1 \) cells. For the \( s \) information bits \( x_1, \ldots, x_s \), we encode the first \( s - 1 \) bits using the encoder of \( C \) to get \( m \) buckets where each bucket has \( t - 1 \) cells. For each bucket, a new cell that stores \( x_s \) is added. Assume that \( R \) is the request which is a linear combination of the \( s \) information bits. Let \( R_1 \) be the part of the request which is a linear combination of the first \( s - 1 \) information bits. From the properties of \( C \), for the request \( R_1 \), there exist \( k \) disjoint recovering sets \( \{S_1, S_2, \ldots, S_k\} \) such that \( |S_j| \leq r \) for any \( j \in [k] \). If \( R = R_1 \), then the same \( \{S_1, S_2, \ldots, S_k\} \) are recovering sets for \( R \). If \( R = x_s \), we can take the first \( k \) buckets as \( k \) recovering sets each of size 1. If \( R \) includes \( x_s \), then the same \( \{S_1, S_2, \ldots, S_k\} \) are recovering sets for \( R \), where we can read \( x_s \) from one of the buckets in each \( S_j \). Thus, \( D(s, k, t, r) \leq D(s - 1, k, t - 1, r) \) and we can get that \( D(s, k, t, r) \leq D(s - a, k, t - a, r) \) by induction on \( a \).

3. Let \( C_1 \) be an \((s_1, k, m_1, t, r_1)\) locality functional array code and \( C_2 \) be an \((s_2, k, m_2, t, r_2)\) locality functional array code. The codes \( C_1 \) and \( C_2 \) are used to construct an \((s_1 + s_2, k, m_1 + m_2, t, r_1 + r_2)\) locality functional array code by encoding the first \( s_1 \) bits using the encoder of \( C_1 \) and the last \( s_2 \) bits using the encoder of \( C_2 \). Assume that \( R \) is the request which is a linear combination of the \( s_1 + s_2 \) information bits. Let \( R_1, R_2 \) be the part of \( R \) which is a linear combination of the first \( s_1 \), last \( s_2 \) information bits, respectively. According to \( C_1, C_2 \), there exist \( k \) recovering sets \( \{S_1^1, S_2^1, \ldots, S_k^1\}, \{S_1^2, S_2^2, \ldots, S_k^2\} \) for \( R_1, R_2 \) such that each recovering set has size at most \( r_1, r_2 \), respectively. Then, the set \( S_j^1 \cup S_j^2 \) for any \( j \in [k] \) is a recovering set for \( R \) with size at most \( r_1 + r_2 \). Therefore, the sets \( \{S_1^1 \cup S_1^2, S_2^1 \cup S_2^2, \ldots, S_k^1 \cup S_k^2\} \) are \( k \) recovering sets for \( R \) such that the size of each recovering set is at most

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In this section we show connections between the problem of finding the minimal number of buckets for locality functional array codes and several problems in subspaces. Subspaces were used in [31] to construct array codes and to examine their locality and availability. The family of array codes that was defined in [31] is a linear subspace of $b \times n$ matrices over $\mathbb{F}_q$ such that each codeword is a $b \times n$ matrix where each entry is called a symbol. The weight of each codeword was defined to be the number of nonzero columns in the codeword and the distance of the code is the minimal weight of a nonzero codeword.

The problem that was presented in [31] was to examine locality and availability of array codes where two types of locality were defined. The first one is node locality. A codeword column $j \in [n]$ has node locality $r_{nd}$ if it can be recovered by a linear combination of the symbols of the columns in a recovering set of size $r_{nd}$. If all codeword columns have node locality $r_{nd}$, then $r_{nd}$ is also called the node locality of the array code. The second type is symbol locality $r_{sb}$, which is similar to node locality but instead of recovering the whole column, here only one symbol (entry of the codewords matrices) is needed to be recovered. Similarly, there are two types of availability. The node, symbol availability, denoted by $t_{nd}, t_{sb}$ is the number of pairwise disjoint recovering sets of size at most $r_{nd}, r_{sb}$ for any codeword column, symbol, respectively.

To simplify the problem, they flattened each $b \times n$ codeword into a vector of length $bn$ by reading the symbols of the codeword column by column from first to last entry. The $M \times bn$ generator matrix $G$, where each row is a flattened codeword, can represent the array code $C$, where the columns $(j-1)b+1, \ldots, jb$ of $G$ correspond to the symbols of the $j$-th codeword column of $C$ and these columns are called the $j$-th thick column of $G$. By this way, the $j$-th thick column of $G$ which corresponds to the $j$-th codeword column of $C$, can be represented by $V_j$ which is a $b$-subspace of $\mathbb{F}_q^M$. Thus, an equivalent constraints of node and symbol locality can be formed using subspaces as stated in [31] Lemma 3], where a subset $S = \{j_1, \ldots, j_p\} \subseteq [n] \setminus \{j\}$ is a recovering set for the codeword column $j \in [n]$, if and only if $V_j \subseteq V_{j_1} + \cdots + V_{j_p}$. Similarly, $S$ is a recovering set for the symbol $(i, j), i \in [b], j \in [n]$ if and only if $g_{(j-1)b+i} \in V_{j_1} + \cdots + V_{j_p}$, where $g_{(j-1)b+i}$ is the $i$-th column in the $j$-th thick column of $G$ that corresponds to the $i$-th entry in the $j$-th codeword column of $C$.

In our work we are interested in the problem of recovering the requests which
are all possible linear combinations of the information bits, which is different from the problem in [31] where the nodes or symbols that are part of the code are needed to be recovered. We can apply some of the results and constructions from [31] in our case. Recall that we defined $\Sigma = \mathbb{F}_2$. Let $\Sigma^s$ be a vector space of dimension $s$ over $\Sigma$. We can consider each bucket which has $t$ cells, as a subspace of $\Sigma^s$ with dimension $t$ and denote a subspace of dimension $t$ as a $t$-subspace. The following claim is motivated by [31] Lemma 3.

Claim 7.2. The value of $D(s,k,t,r)$ is the smallest number $m$ of $t$-subspaces of $\Sigma^s$ such that there exists a partition of the subspaces into $k$ subsets, $S_1, \ldots, S_k$, that satisfies the following property. The size of each subset $S_i$ is at most $r$ and for every request $R$, which can be represented by a $1$-subspace $W$, it holds that for each $S_i$, $W \subseteq \bigcup_{j=0}^{t'} S_{ij}$ where $S_{ij}$ is the $j$-th subspace in $S_i$ and $|S_i| = r' \leq r$.

Let $x = (x_1, x_2, \ldots, x_s)$ be the vector of dimension $1 \times s$ with the $s$ information bits and let $V$ be a $t$-subspace of $\Sigma^s$. It is said that a bucket with $t$ cells stores a $t$-subspace $V$ if for a given basis $B = \{v_1, v_2, \ldots, v_t\}$, the $i$-th cell $i \in [t]$ of the bucket stores the linear combination $\langle v_i, x \rangle$. Note that the choice of the basis $B$ does not matter and we can choose any basis of $V$. Each request $R$ which is a linear combination of the $s$ information bits can be represented by a $1$-subspace $W$ of $\Sigma^s$. It is said that a request is contained in a bucket $b$ if the set $\{b\}$ is a recovering set for the request. Note that if $W$ is contained in a $t$-subspace $V$ then the request $R$ is contained in the bucket that stores $V$.

Let $G_q(s,t)$ denote the set of all $t$-dimensional subspaces of the vector space $\mathbb{F}_q^s$. The set $G_q(s,t)$ is often called the Grassmannian [16]. It is well known that

$$|G_q(s,t)| = \binom{s}{t}_q := \frac{(q^s - 1)(q^{s-1} - 1) \cdots (q^{s-t+1} - 1)}{(q^t - 1)(q^{t-1} - 1) \cdots (q - 1)},$$

where $\binom{s}{t}_q$ is the $q$-ary Gaussian coefficient [33]. The following is a definition of spreads from [19] which are partitions of vector spaces.

Definition 7.2. Let $s = at$. Then a set $S \subseteq G_q(s,t)$ is called a $t$-spread if all elements of $S$ intersect only trivially and they cover the whole space $\mathbb{F}_q^s$.

It is known that the size of a $t$-spread of $\mathbb{F}_q^s$ is $\frac{q^s - 1}{q^t - 1}$ when $s$ is a multiple of $t$ [19]. It is also follows that spreads do not exist when $t$ does not divide $s$. In case $s$ is not a multiple of $t$ there is a notion of partial spreads, where a partial $t$-spread of $\mathbb{F}_q^s$ is a collection of mutually disjoint $t$-subspaces. For the problem we are
studying in this section, partial spreads cannot be used due to the fact that they do not necessarily cover the whole space. Thus, in order to deal with the cases when \( t \) does not divide \( s \) we use covering designs which are defined as follows \[15\].

**Definition 7.3.** A covering design \( C_q(s, t, a) \) is a subset \( S \subseteq G_q(s, t) \) such that each element of \( G_q(s, a) \) is contained in at least one subspace from \( S \).

The covering number \( C_q(s, t, a) \) is the minimum size of a covering design \( C_q(s, t, a) \). From \[15, Th. 4.6\] we get that for any \( 1 \leq t \leq s \),

\[
C_q(s, t, 1) = \left\lceil \frac{q^s - 1}{q^t - 1} \right\rceil \quad (7.1)
\]

Note that when \( t \mid s \), an optimal covering design \( C_q(s, t, 1) \) is exactly a \( t \)-spread of \( F_{qs} \). Now, we will define another family of partitions and another family of codes that can be used to construct locality functional array codes. The following is a definition of \( \lambda \)-fold partitions from \[14\].

**Definition 7.4.** Let \( \lambda \) be a positive integer. A \( \lambda \)-fold partition of the vector space \( V = F_{qs} \) is a multiset \( S \) of subspaces of \( V \) such that every nonzero vector in \( V \) is contained in exactly \( \lambda \) subspaces in \( S \).

Note that a 1-fold partition of \( F_{qs} \) that does not contain a subspace with dimension larger than \( t \) is also a covering design \( C_q(s, t, 1) \). Denote by \( A_q(s, t, \lambda) \) the minimum size of a \( \lambda \)-fold partition of \( F_{qs} \) that does not contain a subspace with dimension larger than \( t \). In \[14\], it is also possible to find results on \( \lambda \)-fold partitions. For example, there exists a construction of a \( \left\lceil \frac{2^t - 1}{2^p - 1} \right\rceil \)-fold partition of \( \Sigma^s \) with \( \frac{2^t - 1}{2^p - 1} \) subspaces where \( p = \gcd(s, t) \). Therefore, \( A_2(s, t, \frac{2^t - 1}{2^p - 1}) \leq \frac{2^t - 1}{2^p - 1} \).

Lastly, the following is a definition of covering Grassmannian codes from \[16\].

**Definition 7.5.** For every positive integers \( \alpha \) and \( \delta \) where \( \delta + t \leq s \), an \( \alpha-(s, t, \delta)_q^c \) covering Grassmannian code \( C \) is a subset of \( G_q(s, t) \) such that each subset of \( \alpha \) codewords of \( C \) spans a subspace whose dimension is at least \( \delta + t \) in \( F_{qs} \).

The value \( B_q(s, t, \delta; \alpha) \) will denote the maximum size of an \( \alpha-(s, t, \delta)_q^c \) covering Grassmannian code. The following theorem summarizes some bounds on \( D(s, k, t, r) \) using spreads, covering designs, \( \lambda \)-fold partitions, and covering Grassmannian codes.

**Theorem 7.3.** For each \( s, t, k \) and \( r \) positive integers

1. \( D(s, 1, t, 1) = C_2(s, t, 1) = \left\lceil \frac{2^t - 1}{2^{t-1}} \right\rceil \).
2. \( D(s, 1, t, r) \leq r \cdot \left\lfloor \frac{2^s - 1}{2^t - 1} \right\rfloor \), where \( r \mid s \).

3. \( D(s, k, t, 1) \leq A_2(s, t, k) \).

4. \( D(s, \lfloor B_2(s, t, s - t; r) / r \rfloor, t, r) \leq B_2(s, t, s - t; r) \).

5. \( D(s, \left\lfloor \frac{s - 1}{t - 1} \right\rfloor, t, 1) \leq \left\lfloor \frac{s}{t} \right\rfloor \), where \( t > 1 \).

6. \( D(s, \left\lfloor \frac{2^s - 2^t}{r^2 - r} \right\rfloor + 1, t, r) \leq \frac{2^s - 1}{2^t - 1} \), where \( s = rt \).

**Proof**

1. To prove this part we use a construction motivated by [31, Construction 2]. Let \( \mathbb{C} \) be a \( C_2(s, t, 1) \) covering design with \( C_2(s, t, 1) \) \( t \)-subspaces. To construct an \( (s, C_2(s, t, 1), t, 1) \) locality functional array code, we take \( C_2(s, t, 1) \) buckets where each bucket stores one of the \( t \)-subspace from \( \mathbb{C} \). From Definition 7.3, every 1-subspace of \( \Sigma^s \) is contained in at least one \( t \)-subspace from \( \mathbb{C} \). Thus, each request \( R \) which can be represented by a 1-subspace of \( \Sigma^s \), is contained in at least one bucket. Therefore, by using Equation (7.1) we get that \( D(s, 1, t, 1) \leq C_2(s, t, 1) = \left\lfloor \frac{2^s - 1}{2^t - 1} \right\rfloor \).

   For the other direction, assume that \( \mathbb{C} \) is an \( (s, m, t, 1) \) locality functional array code with \( m \) buckets. We construct a \( C_2(s, t, 1) \) covering design with \( m \) \( t \)-subspaces of \( \Sigma^s \) that are stored in the \( m \) buckets of \( \mathbb{C} \). Let \( W \) be a 1-subspace of \( \Sigma^s \) that represents a request \( R \) for the code \( \mathbb{C} \). From the property of the code \( \mathbb{C} \), there exists one bucket that contains \( R \). Therefore, there exists one \( t \)-subspace in \( \mathbb{C} \) that contains \( W \). Thus, \( C_2(s, t, 1) \leq D(s, 1, t, 1) \).

2. This result holds from part [1] in this theorem and Theorem [7.1,3].

3. Let \( \mathcal{S} \) be a \( k \)-fold partition of \( \Sigma^s \) that does not contain a subspace with dimension larger than \( t \). Assume that \( |\mathcal{S}| = m \). To construct a locality functional array code, we take \( m \) buckets where each bucket stores one of the subspaces from \( \mathcal{S} \). Assume that \( R \) is the request which can be represented by a vector \( u \) of \( \Sigma^s \). Then, from the property of the multiset \( \mathcal{S} \), the vector \( u \) is contained in exactly \( k \) subspaces in \( \mathcal{S} \). Therefore, \( R \) is contained in exactly \( k \) buckets. Thus, the \( m \) buckets form an \( (s, k, m, t, 1) \) locality functional array code, and hence, \( D(s, k, t, 1) \leq A_2(s, t, k) \).

4. Let \( \mathbb{C} \) be an \( r \cdot (s, t, s - t) \) covering Grassmannian code with \( m \) \( t \)-subspaces of \( \Sigma^s \). We take \( m \) buckets where each bucket stores one of the \( t \)-subspaces from \( \mathbb{C} \). Let \( R \) be the request. From the property of the code \( \mathbb{C} \), every
subset of \( r \) \( t \)-subspaces of \( \mathbb{C} \) spans the whole space \( \Sigma^s \). Hence, every subset of \( r \) buckets contains \( R \). Therefore, we can partition the \( m \) buckets into \( \lfloor m/r \rfloor \) parts, where each part contains \( R \), and hence, there exist \( \lfloor m/r \rfloor \) recovering sets for \( R \). Thus, the construction with the \( m \) buckets forms an \((s, \lfloor m/r \rfloor, m, t, r)\) locality functional array code.

5. To prove this part we use a construction motivated by [31, Construction 1]. We construct an \((s, \lfloor s - 1 \rfloor, \lfloor t \rfloor, t, 1)\) locality functional array code by taking \( \lfloor s \rfloor \) buckets where each bucket has \( t \) cells and stores one of the \( t \)-subspaces of \( \Sigma^s \). Every \( 1 \)-subspace of \( \Sigma^s \) is contained in exactly \( \lfloor s - 1 \rfloor \) \( t \)-subspaces. Therefore, every request \( R \) which can be represented by a \( 1 \)-subspace is contained in exactly \( \lfloor s - 1 \rfloor \) buckets. Thus, we get that

\[
D(s, \lfloor s - 1 \rfloor, t, 1) \leq \lfloor s \rfloor.
\]

6. Let \( s = rt \) and \( S \) be a \( t \)-spread of \( \Sigma^s \) such that \( |S| = \frac{2^s - 1}{2^t - 1} \). To construct a locality functional array code we store each \( t \)-subspace in \( S \) in a bucket with \( t \) cells. Assume that \( R \) is the request which can be represented by a \( 1 \)-subspace \( W \) of \( \Sigma^s \). From the property of spreads, there exists a subspace in \( S \) that includes \( W \). Therefore, there exists a bucket that contains \( R \) which can form a recovering set of size \( 1 \). Then, partition the remaining \( \frac{2^s - 1}{2^t - 1} - 1 = \frac{2^s - 2^t}{2^t - 1} \) buckets into \( \left\lfloor \frac{2^s - 2^t}{2^t - r} \right\rfloor \) parts where each part has size \( r \). Each part \( \mathcal{P}_i \) has \( r \) mutually disjoint \( t \)-subspaces \( U_{i1}, U_{i2}, \ldots, U_{ir} \). Hence, \( \Sigma'_{j=1} U_{ij} = \Sigma^s \). Thus, each part \( \mathcal{P}_i \) is a recovering set of \( R \) of size \( r \). Then, there exist \( 1 + \left\lfloor \frac{2^s - 2^t}{2^t - r} \right\rfloor \) recovering sets each of size at most \( r \) and the code is an \((s, \left\lfloor \frac{2^s - 2^t}{2^t - r} \right\rfloor + 1, \frac{2^s - 1}{2^t - 1}, t, r)\) locality functional array code.

The following is an example of Theorem 7.3(3).

**Example 3.** In this example we will use an example of a 2-fold partition from [14] in order to construct a locality functional array code. Let \( s = 3 \). The following multiset \( S \) of subspaces of \( \Sigma^3 \) is a 2-fold partition that does not contain a subspace with dimension larger than \( t = 2 \).

\[
S = \{ \{100, 011, 111\}, \{010, 001, 011\}, \{001, 110, 111\}, \{110, 010, 100\}, \{101\}, \{101\} \}.
\]
We represent each element in $\Sigma^3$ as a binary vector of length 3 and every subspace in $S$ by its elements except the zero vector. It holds that any binary vector of length 3 is contained in exactly two subspaces in $S$, and hence, $A_2(3, 2, 2) \leq 6$. We construct a $(3, 2, 6, 2, 1)$ locality functional array code with the following buckets that are obtained from $S$.

For example, if the request is $x_1 + x_2$, then the recovering sets are $\{\{1\}, \{2\}\}$.

The following is an example of Theorem 7.3.

**Example 4.** For $s = 4$, $t = 2$ and $r = 2$, the following set $S$ is a 2-spread of $\Sigma^4$ of size $\frac{2^4-1}{2-1} = 5$.

$S = \{\{0001, 0010\}, \{0100, 1000\}, \{0101, 1010\}, \{1001, 0111\}, \{0110, 1011\}\}$.

We represent each element in $\Sigma^4$ as a binary vector of length 4 and every 2-subspace as a basis with 2 vectors. We construct a $(4, 3, 5, 2, 2)$ locality functional array code with the following buckets that are obtained from $S$.

For example, if the request is $x_1 + x_2$, then the recovering sets are $\{\{1\}, \{2, 3\}, \{4, 5\}\}$.

### 7.3 Bounds and Constructions based upon Covering Codes

In this section we show how covering codes are used to construct locality functional array codes and to get lower bounds for $D(s, k, t, r)$. For the rest of the section we assume that $x = (x_1, x_2, \ldots, x_s)$ is the vector of dimension $1 \times s$ with the $s$ information bits. For the case of $t = 1$ the following result can be obtained. Remember that $h[s, r]_q$ is the smallest length of a linear covering code over $\mathbb{F}_q$ with covering radius $r$ and redundancy $s$.

**Theorem 7.4.** $D(s, 1, 1, r) = h[s, r]$. 


Proof There exists an \([h[s,r], h[s,r] - s,r]\) linear covering code with some parity check matrix \(H\). To construct a locality functional array code we store in each bucket the linear combination \(\langle h_i, x \rangle\) where \(h_i\) is the \(i\)-th column of \(H\). Assume that \(R\) is the request which can be represented by a binary vector \(u \in \Sigma^s\). From Property [4.3] we know that the vector \(u\) can be represented as the sum of at most \(r\) columns of \(H\). Therefore, there exists a recovering set of size at most \(r\) for the request \(R\). The number of buckets is the number of columns of \(H\) which is \(h[s,r]\). Thus, \(D(s,1,1,r) \leq h[s,r]\). The lower bound can be obtained from Corollary [7.13] which will appear later.

We can generalize the connection of covering codes and locality functional array codes with general \(t\). We start by defining a partition of matrices.

Definition 7.6. A \(t\)-partition of a matrix \(H\) is a collection \(\mathcal{P}\) of subspaces of dimension \(t\) with the property that every column vector of \(H\) is contained in at least one member of \(\mathcal{P}\). A \(t\)-partition is called strict if every column vector of \(H\) is contained in exactly one member of \(\mathcal{P}\).

The next theorem shows the connection between covering codes and locality functional array codes with \(k = 1\).

Theorem 7.5. Let \(H\) be a parity check matrix for an \([n, n-s, r]\) covering code, and let \(p\) be the smallest size of a \(t\)-partition of \(H\). Then, \(D(s,1,t,r) \leq p\).

Proof Let \(H\) be a parity check matrix of a given \([n, n-s, r]\) covering code. Let \(\mathcal{P}\) be a \(t\)-partition of \(H\), that contains \(p\) subspaces of dimension \(t\). We construct an \((s,1,p,t,r)\) locality functional array code \(C\) by storing each \(t\)-subspace from \(\mathcal{P}\) in one bucket with \(t\) cells. Let \(u \in \Sigma^s\) be a request which represents the linear combination \(\langle u, x \rangle\) of the \(s\) information bits. From Property [4.3] we know that there exists a vector \(y \in \Sigma^n\) such that \(H \cdot y = u\), where \(w = w_H(y) \leq r\). If \(w_H(y) = r' \leq r\), then the request \(u\) is equal to the sum of \(r'\) columns of \(H\) and denote them by \(h_{i_1}, h_{i_2}, \ldots, h_{i_{r'}}\). We know that each 1-subspace with a basis \(\{h_{i_j}\}, j \in [r']\) is contained in one subspace from the partition \(\mathcal{P}\), and hence, the vector \(h_i\) is contained in one bucket of \(C\). Thus, we can get all the \(r'\) columns from at most \(r' \leq r\) buckets.

Now, the method to get locality functional array codes from covering codes over \(\mathbb{F}_q\) is established. We follow an example from [6] and for that we use the following definition in the rest of this section.
Definition 7.7. Let $B = \{1, e, e^2, \ldots, e^{w-1}\}$ be a basis for $\mathbb{F}_{2^w}$ over $\Sigma$ where $e$ is a primitive element of $\mathbb{F}_{2^w}$. For each $i \in [0, 2^w - 2]$ let $(e^i)_w$ be the binary column vector of length $w$ that represents the element $e^i$ of $\mathbb{F}_{2^w}$ with respect to the basis $B$. Let $U_0$ be the binary matrix of size $(w \times (2^w - 1))$ that has in column number $i, i \in [0, 2^w - 2]$ the vector $(e^i)_w$. For each $i \in [0, 2^w - 2]$, let $U_i$ be the matrix which is obtained from $U_0$ by cyclically rotating its columns $i$ places to the left. Note that for each $i \in [0, 2^w - 2]$ the first column in matrix $U_i$ is the vector $(e^i)_w$.

For an element $e^i$ over $\mathbb{F}_{2^w}$ let $\mathcal{T}(e^i) = U_i$ be a matrix over $\Sigma$ of size $(w \times (2^w - 1))$ and let $\mathcal{T}(0)$ be the $(w \times (2^w - 1))$ zeros matrix. We define the same transformation for vectors and matrices, where for a matrix $M_1$ of size $(a \times b)$ over $\mathbb{F}_{2^w}$ let $\mathcal{T}(M_1) = M_2$ be the matrix over $\Sigma$ of size $(aw \times b(2^w - 1))$ that is obtained from $M_1$ by replacing each element $\alpha$ of $\mathbb{F}_{2^w}$ in the matrix $M_1$ by its appropriate $(w \times (2^w - 1))$ matrix $\mathcal{T}(\alpha)$.

The following is an example to demonstrate Definition 7.7

Example 5. Let $B = \{1, e^1, e^2\}$ be a basis for $\mathbb{F}_{2^3}$ over $\Sigma$, where $e$ is a primitive element of $\mathbb{F}_{2^3}$ chosen to satisfy the primitive polynomial $x^3 + x + 1$, and hence, $e^3 = e + 1$. Then, the coordinates of the successive powers of $e$ with respect to $B$ are the columns of the matrix $U_0$

$$U_0 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$ 

For example, the following matrix is $\mathcal{T}(e^1)$

$$\mathcal{T}(e^1) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$ 

We show that the transformation defined in Definition 7.7 is a linear transformation.

Lemma 7.6. The transformation $\mathcal{T} : \mathbb{F}_{2^w} \rightarrow \mathbb{F}_{2^w}^{w \times (2^w - 1)}$ is a linear transformation.

Proof. We want to show that for any $e^{i_1}, e^{i_2} \in \mathbb{F}_{2^w}$, $\mathcal{T}(e^{i_1}) + \mathcal{T}(e^{i_2}) = \mathcal{T}(e^{i_1} + e^{i_2})$. Assume that $e^{i_1} + e^{i_2} = e^{i_3}$. From Definition 7.7, we know that $\mathcal{T}(e^{i_1}) + \mathcal{T}(e^{i_2}) = U_{i_1} + U_{i_2}$. From Definition 7.7, for every $j \in [2^w - 1]$, the $j$-th column of $U_{i_1}, U_{i_2}, U_{i_3}, U_{i_1} + U_{i_2}$ is $(e^{i_1+j})_w, (e^{i_2+j})_w, (e^{i_3+j})_w, (e^{i_1+j} + e^{i_2+j})_w$, respectively. Also, $e^{i_1+j} + e^{i_2+j} = e^j(e^{i_1} + e^{i_2}) = e^{i_3+j}$. Thus, the $j$-th col-

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column of \( \mathcal{U}_1 + \mathcal{U}_2 \) for all \( j \in [2^w - 1] \). Thus, 
\[ T(e^{i_1}) + T(e^{i_2}) = \mathcal{U}_1 + \mathcal{U}_2 = \mathcal{U}_3 = T(e^{i_1} + e^{i_2}). \]

The same transformation \( T \) that was defined for vectors and matrices in Definition 7.7 is also a linear transformation following similar proof as for Lemma 7.6. The following result can be found in [6, Lemma 3.1], but we want to prove it in a different way, by constructing a specific parity check matrix in order to use it in other claims.

**Lemma 7.7.** Let \( H \) be a parity check matrix of an \([n, n - s, r]_{2^w}\) covering code. Then, the matrix \( T(H) \) is a parity check matrix of a binary \([ (2^w - 1)n, (2^w - 1)n - ws, r ]\) covering code. In particular, \( h[ws, r] \leq (2^w - 1) \cdot h[s, r]_{2^w} \).

**Proof** Let \( C \) be an \([n, n - s, r]_{2^w}\) covering code and let \( H \) be a parity check matrix of the code \( C \) of size \((s \times n)\). We want to show that the matrix \( H' = T(H) \) is a parity check matrix of a binary \([ (2^w - 1)n, (2^w - 1)n - ws, r ]\) covering code. The size of \( H' \) is \((ws \times (2^w - 1)n)\). Given a binary column vector \( u \) of length \( ws \), we show that there are at most \( r \) columns of \( H' \) that their sum is \( u \).

The vector \( u \) can be partitioned into \( s \) vectors each vector of length \( w \) where \( u = (u_1, u_2, \ldots, u_s) \). Each vector \( u_i \) of length \( w \) can represent an element of \( \mathbb{F}_{2^w} \) according to the basis \( B \) from Definition 7.7. Therefore, we can present \( u \) as \( u = ((e_1)^{i_1}_w, (e_2)^{i_2}_w, \ldots, (e_s)^{i_s}_w) \) and from the \( s \) elements we can get a column vector \( v = (e_1, e_2, \ldots, e_s) \) of dimension \( s \times 1 \) over \( \mathbb{F}_{2^w} \). The first column in each \( \mathcal{U}_i, i \in [0, 2^w - 2] \) is the vector \( (e_i)_w \). Then, from the construction of \( T(v) \), the first column of the matrix \( T(v) \) is the vector \( u \).

From the property of the code \( C \), it is known that there exists a vector \( y \in \mathbb{F}_{2^w}^n \) such that \( H \cdot y = v \), where \( w_H(y) \leq r \). Let \( A = \{ i : i \in [n], y_i \neq 0 \} \) and note that \( |A| \leq r \). Let \( h_i \) be the \( i \)-th column of \( H \). Then, \( \sum_{i \in A} y_i h_i = v \). For each \( i \in A \) we define \( h'_i = y_i h_i \) and from the linearity of the transformation \( T \) we have \( T(v) = T(\sum_{i \in A} h'_i) = \sum_{i \in A} T(h'_i) \). Thus, the vector \( (\sum_{i \in A} T(h'_i))_1 = \sum_{i \in A} T(h'_i)_1 = u \), where \( T(h'_i)_1 \) is the first column of the matrix \( T(h'_i) \).

For each \( i \in A \), assume that \( y_i = e^{b_i} \). Then, the first column of the matrix \( T(h'_i) \) is the \( j_i \)-th column of the matrix \( T(h_i) \). Thus, \( \sum_{i \in A} T(h_i)_j = u \), where \( T(h_i)_j \) is the \( j \)-th column of the matrix \( T(h_i) \). For each \( i \in A \), the matrix \( T(h_i) \) has size \((ws \times (2^w - 1))\) and it is a sub matrix of \( H' \) that starts in the column number \((2^w - 1)(i - 1) + 1\) of \( H' \). Hence, the \( j_i \)-th column of the matrix \( T(h_i) \) is the column number \((2^w - 1)(i - 1) + j_i \) of the matrix \( H' \). Therefore, \( \sum_{i \in A} T(h'_i)(2^w - 1)(i - 1) + j_i = u \), where \( h'_i \) is the \( i \)-th column of \( H' \). Thus, the vector \( u \) is a sum of \( |A| \leq r \) columns of \( H' \) and the matrix \( H' \) is a parity check matrix of a binary \([ (2^w - 1)n, (2^w - 1)n - ws, r ]\) covering code.
An upper bound on the value of $D(s, 1, t, r)$ can be obtained in the next theorem using non-binary covering codes.

**Theorem 7.8.** For any positive integer $w$ such that $t|w$,

$$D(ws, 1, t, r) \leq \frac{(2^w - 1)h[s, r]2^w}{2^t - 1}.$$  

**Proof** Let $C$ be an $[n, n-s, \frac{n-s}{2}]$ covering code over $\mathbb{F}_2^w$, where $n = h[s, r]2^w$. Let the matrix $H$ be a parity check matrix of $C$ of size $(s \times n)$. From Lemma 7.7 we get that there exists a binary $\left(\frac{2^w - 1}{2^t - 1}\right)n - ws, r]$ covering code with parity check matrix $H^t = T(H)$. We want to find the smallest size of a $t$-partition of $H^t$.

Let $j_1, j_2, j_3 \in [0, 2^w - 2]$ be such that $e^1 + e^{j_2} = e^h$. Then, in the matrix $U_0$ from Definition 7.7 it holds that the sum of the $j_1$-th and $j_2$-th columns is the $j_3$-th column. In the matrix $U_i, i \in [0, 2^w - 2]$ the $j_1$-th, $j_2$-th, $j_3$-th column is $(e^i \cdot e^h)_{\mathbb{F}_2}, (e^i \cdot e^{j_2})_{\mathbb{F}_2}, (e^i \cdot e^h)_{\mathbb{F}_2}$, respectively. It holds that $e^i \cdot e^h + e^i \cdot e^{j_2} = e^i \cdot (e^h + e^{j_2}) = e^i \cdot e^h$. Thus, we can conclude that in the matrix $U_i, i \in [0, 2^w - 2]$ it also holds that the sum of the $j_1$-th and $j_2$-th columns is the $j_3$-th column. Let $(U_i)$ be the $j$-th column of $U_i$. Assume that a basis that includes the columns $\{(U_0)_{j_1}, (U_0)_{j_2}, \ldots, (U_0)_{j_{t-1}}\}$ spans the columns $\{(U_0)_{j_1}, (U_0)_{j_2}, \ldots, (U_0)_{j_{t-1}}\}$ of the matrix $U_0$. Then, the basis that includes the columns $\{(U_i)_{j_1}, (U_i)_{j_2}, \ldots, (U_i)_{j_{t-1}}\}$ spans the columns $\{(U_i)_{j_1}, (U_i)_{j_2}, \ldots, (U_i)_{j_{t-1}}\}$ of the matrix $U_i, i \in [0, 2^w - 2]$.

The matrix $U_0$ includes all the nonzero column vectors of length $w$, which means that it includes the space $\mathbb{F}_2^w \setminus \{0\}$. It is given that $t|w$. Hence, there exists a $t$-spread of $\mathbb{F}_2^w$. Thus, there exists a strict $t$-partition $\mathcal{P}$ of $U_0$ with $p = \frac{2^w - 1}{2^t - 1}$ $t$-subspaces. Each subspace of $\mathcal{P}$ is represented by a basis of $t$ column vectors of $U_0$ and denote them by $\{(U_0)_{j_1}, (U_0)_{j_2}, \ldots, (U_0)_{j_{t-1}}\}$. The $p$ $t$-subspaces $\{(U_1)_{j_1}, (U_1)_{j_2}, \ldots, (U_1)_{j_{t-1}}\} \ldots, \{(U_p)_{j_1}, (U_p)_{j_2}, \ldots, (U_p)_{j_{t-1}}\}$ form a strict $t$-partition of $U_0$. For each $i \in [n]$ let $h_i$ be the $i$-th column of the matrix $H$. The matrix $T(h_i)$ includes $s$ matrices of size $(w \times (2^w - 1))$ that all have the same partition regarding the column numbers. Therefore, the partition $\{(U_i)_{j_1}, (U_i)_{j_2}, \ldots, (U_i)_{j_{t-1}}\} \ldots, \{(U_i)_{j_1}, (U_i)_{j_2}, \ldots, (U_i)_{j_{t-1}}\}$ is a strict $t$-partition of $T(h_i)$ with $p = \frac{2^w - 1}{2^t - 1}$ $t$-subspaces. Therefore, there exists a strict $t$-partition of the matrix $H^t$ with $\frac{(2^w - 1)n}{2^t - 1}$ $t$-subspaces. Thus, By using Theorem 7.5 we get that

$$D(ws, 1, t, r) \leq \frac{(2^w - 1)h[s, r]2^w}{2^t - 1}.$$
We can use Theorem 7.8 to find upper bounds on the value of $D(s, 1, t, r)$ by using previous bounds on the size of non-binary covering codes.

**Example 6.**

1. In [12] a $[1097, 1097 - 8, 2]_{2^3}$ covering code is provided. Thus, $h[8, 2]_{2^3} \leq 1097$. Therefore, from Theorem 7.8 $D(3 \cdot 8, 1, 3, 2) = D(24, 1, 3, 2) \leq \frac{2^{3^3} - 1}{2^3} h[8, 2]_{2^3} = 1097$. For a lower bound, we can use Theorem 7.11 to get $D(24, 1, 3, 2) \geq 828$.

2. For $r = 3$, the following result can be obtained from [11, Theorem 4.3]. For $q = 4$ and $p = 3$, $h[s = 3p + 2, 3]_q \leq (9 \cdot q^2 + 2^{q^2 - 1}) = 154$. Hence, $h[11, 3]_{2^2} \leq 154$. From Theorem 7.8, $D(22, 1, 2, 3) \leq 154$. For a lower bound, we can use Theorem 7.11 to get $D(22, 1, 2, 3) \geq 99$.

The following is another use of Theorem 7.8 to find bounds on the value of $D(s, 1, t, r)$ using another general family of non-binary covering codes.

**Corollary 7.9.** For any positive integers $w$ and $t$, where $t | w$, $D(4w, 1, t, 2) \leq \frac{(2^w - 1)(2^{w+1} + 1)}{2^t - 1}$.

**Proof.** In [6, Theorem 3.2] there exists a construction of a $(4 \times (2^w + 1))$ parity check matrix $H$ of a $[2^w + 1, 2^w + 1 + 1 - 4, 2]_{2^w}$ covering code over $\mathbb{F}_{2^w}$. Therefore, $h[4, 2]_{2^w} \leq 2^{w+1} + 1$. From Theorem 7.8, we get $D(4w, 1, t, 2) \leq \frac{(2^w - 1)(2^{w+1} + 1)}{2^t - 1}$.

For any positive integers $w$ and $t$, where $t | w$ we have $D(4w, 1, t, 2) \leq 2 \cdot \frac{2^{w-1}}{2^t - 1}$ from Theorem 7.2, and from Corollary 7.9 we get $D(4w, 1, t, 2) \leq \frac{(2^w - 1)(2^{w+1} + 1)}{2^t - 1}$. Thus, we can save $2 \cdot \frac{2^{w-1}}{2^t - 1} - \frac{(2^w - 1)(2^{w+1} + 1)}{2^t - 1} = \frac{2^{w-1}}{2^t - 1}$ buckets.

The following is an example of a locality functional array code that is obtained from Corollary 7.9.

**Example 7.** For the case of $w = 4$ and $t = 2$, we have $s = 4w = 16$. Let $V$ be $\mathbb{F}_4^4 = \mathbb{F}_4^4$. To get a basis for $V$ as a vector space over $\mathbb{F}_4$, we first choose a basis $B = \{1, \varepsilon, \varepsilon^2, \varepsilon^3\}$ for $\mathbb{F}_{16}$ over $\mathbb{F}_4$ where $\varepsilon$ is a primitive element of $\mathbb{F}_{16}$ chosen to satisfy the primitive polynomial $x^4 + x + 1$. It holds that $\varepsilon^4 = \varepsilon + 1$. Then, the coordinates of the successive powers of $\varepsilon$ with respect to the basis $B$ are the
columns of the matrix

\[ U_0 = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}. \]

In [6, Theorem 3.2], there exists a construction of a parity check matrix of a \([33, 33 - 4, 2]_{2^w}\) covering code.

\[ H = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & e^1 & e^2 & \cdots & e^{14} & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & e^2 & e^4 & \cdots & e^{13} & 0 & 0 & 0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & e^1 & \cdots & e^{14}
\end{bmatrix}. \]

Let \( (U_i)_j \) be the \( j \)-th column of \( U_i \). The following is a strict \( t \)-partition of \( U_i \), \( P_i = \{ \{(U_i)_1, (U_i)_6, (U_i)_{11}\}, \{(U_i)_2, (U_i)_7, (U_i)_{12}\}, \{(U_i)_3, (U_i)_8, (U_i)_{13}\}, \{(U_i)_4, (U_i)_9, (U_i)_{14}\}, \{(U_i)_5, (U_i)_{10}, (U_i)_{15}\}\} \), where we represent every subspace in \( P \) by its elements except the zero vector. In addition, each subspace can be represented by a basis of two vectors.

From Lemma [7,7] we get that \( H' = T(H) \) is a parity check matrix of a binary \([495, 495 - 16, 2]\) covering code. Recall that in the transformation \( T \), each element of \( \mathbb{F}_{16} \) is replaced with an appropriate matrix \( U_i \) of size \((4 \times 15)\). Each column in \( H \) has 4 elements of \( \mathbb{F}_{16} \) and is replaced with 4 matrices such that each matrix \( U_i \) has a strict \( t \)-partition \( P_i \) with 5 subspaces. Each column in \( H \) is a \((16 \times 15)\) matrix in \( H' \), which can be stored in 5 buckets such that each bucket stores one subspace from the partition, and hence, the 33 columns of \( H \) can be stored in \( 33 \times 5 = 165 \) buckets. Thus, we get that \( D(16, 1, 2, 2) \leq 165 \).

Next, another possible way to obtain locality functional array codes from covering codes is presented. First, we define a possible modification for matrices that we will use in order to construct new parity check matrices for covering codes from given parity check matrices.

**Definition 7.8.** Given a matrix \( H \) of size \((n \times s)\), its \( i \)-th **modified matrix** denoted by \( H^{(i)} \) of size \((n + 1 \times s)\) is the matrix that has the same rows of \( H \) except of row \( i \), where it has the complement of row \( i \) of \( H \), with an additional column with only 1 in row \( i \).

The next theorem shows that for a given parity check matrix of a covering code, the modified matrix is also a parity check matrix of another covering code. Even
though the following seems to be a basic property, we could not find its proof, and hence, we add the following proof for completeness.

**Theorem 7.10.** For a parity check matrix $H$ for a binary $[n, n - s, 2]$ covering code and an integer $i$, the $i$-th modified matrix $H^{(i)}$ is also a parity check matrix of a binary $[n + 1, n + 1 - s, 2]$ covering code.

**Proof** Let $H$ be a parity check matrix of an $[n, n - s, 2]$ covering code. For a given $i \in [s]$, let $H^{(i)}$ be the $i$-th modified matrix of $H$. The size of $H^{(i)}$ is $(s \times (n + 1))$. From Property 4.3, for each vector $v \in \Sigma^s$ there exists a vector $y \in \Sigma^n$ such that $H \cdot y = v$ where $w_H(y) \leq 2$. Let $h_i, h'_i$ be the $i$-th column of $H, H^{(i)}$, respectively. If $w_H(y) = 2$, assume that $v = h_{j_1} + h_{j_2}$. The column vector $h'_i$ is different from the column vector $h_j$ only in row $i$, where $h'_i$ has the complement of the element in row $i$ in $h_j$. Thus, it holds that $v = h'_{j_1} + h'_{j_2}$.

If $w_H(y) = 1$, assume that $v = h_j$. From the construction of $H^{(i)}$, it holds that $h_j = h'_j + h'_{n+1}$. Therefore, we can get $v$ as a sum of two columns of $H^{(i)}$. Thus, $H^{(i)}$ is a parity check matrix of a binary $[n + 1, n + 1 - s, 2]$ covering code.

One possible way to use Theorem 7.10 to get locality functional array codes is shown next.

**Theorem 7.11.** $D(7, 1, 2, 2) = 7$.

**Proof** From [18] Theorem 1 and the example after it, we can get a construction of a parity check matrix for a binary $[19, 19 - 7, 2]$ covering code. The following is a parity check matrix $H$ of the code.

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
$$

The following is the matrix $H^{(1)}$, the first modified matrix of $H$ where the first row is the complement of the first row of $H$ and a new column with only 1 in the first entry is added.
From Theorem 7.10, the matrix $H^{(1)}$ is a parity check matrix of a binary $[20, 20 - 7, 2]$ covering code. Note that the fourth column is all zero column which we can remove to get the following matrix $H^{(1)'}$ which is a parity check matrix of a binary $[19, 19 - 7, 2]$ covering code.

Let $h'_j$ be the $j$-th column of the matrix $H^{(i)'}$. We can find a 2-partition of the matrix $H^{(1)'}$. We will present the partition as a set of 7 2-subspaces such that each subspace is presented by a basis with two columns of $H^{(i)'}$. The following is a possible 2-partition of $H^{(1)'}$: $P = \{\{h'_7, h'_1\}, \{h'_8, h'_12\}, \{h'_9, h'_13\}, \{h'_10, h'_14\}, \{h'_4, h'_5\}, \{h'_1, h'_2\}, \{h'_3, h'_19\}\}$. We can see that 14 out of 19 columns form the bases. It can be verified that $h'_4 + h'_5 = h'_6$, $h'_7 + h'_11 = h'_15$, $h'_8 + h'_12 = h'_16$, $h'_10 + h'_14 = h'_17$ and $h'_9 + h'_13 = h'_18$. Therefore, $P$ is a 2-partition of $H^{(1)'}$ with size 7. Thus, from Theorem 7.5 we get that $D(7, 1, 2, 2) \leq 7$.

For the lower bound, assume by contradiction that there exists a $(7, 1, 6, 2, 2)$ locality functional array code. Then, from Theorem 7.12 we get that $h[7, 2] \leq 18$. But from [10] we have that $h[7, 2] = 19$, which is a contradiction. Thus, $D(7, 1, 2, 2) \geq 7$.

Next, we show how to construct covering codes using locality functional array codes.

**Theorem 7.12.** Let $C$ be an $(s, 1, m, t, r)$ locality functional array code. Then, $h[s, r] \leq m \cdot (2^t - 1)$.

**Proof** Assume that $C$ is an $(s, 1, m, t, r)$ locality functional array code which has $m$ buckets such that in each bucket stored at most $t$ linear combinations of the $s$ information bits. From the $t$ cells in each bucket we can get at most $(2^t - 1)$
different linear combinations. We can represent each linear combination as a binary vector of length $s$. Then, we construct an $(s \times m \cdot (2^t - 1))$ parity check matrix $H$ where we have all the vectors that we get from the linear combinations of all the $m$ buckets as columns of the matrix. Let $u \in \Sigma^s$ be a column vector of length $s$ which can represent a request for the code $C$. From the property of $C$, there exists a recovering set $S \subseteq [m]$ where $|S| \leq r$ that satisfies the request. Assume that $S = \{b_1, b_2, \ldots, b_{r'}\}$ where $r' \leq r$. From each bucket $b_i \in S$ we read a linear combination $v_i$ of the $t$ cells which is a linear combination of the $s$ information bits. From the construction of $H$, the column vector $v_i$ is a column in $H$. Then, $u = \sum_{i=1}^{r'} v_i$, and hence, the vector $u$ is a sum of at most $r$ columns of $H$. Thus, the matrix $H$ is a parity check matrix of a binary $[m \cdot (2^t - 1), m \cdot (2^t - 1) - s, r]$ covering code, and hence, $h[s, r] \leq m \cdot (2^t - 1)$.

Now we will use Theorem 7.12 to get a lower bound on the value of $D(s, 1, t, r)$.

**Corollary 7.13.** $D(s, 1, t, r) \geq \left\lceil \frac{h[s, r]}{2^t - 1} \right\rceil$.

**Proof** Assume by contradiction that $D(s, 1, t, r) = m < \left\lceil \frac{h[s, r]}{2^t - 1} \right\rceil$. The number of buckets $m$ is an integer. Then, $m < \frac{h[s, r]}{2^t - 1}$. From Theorem 7.12 we have $h[s, r] \leq m \cdot (2^t - 1) < \frac{h[s, r]}{2^t - 1} \cdot (2^t - 1) = h[s, r]$ which is a contradiction.

We can get upper bounds on the value $h[s, r]$ from [10]. For example, $h[2s - 1, 2] \geq 2^s - 1$ for any $s \geq 3$ and we can conclude that $D(2s - 1, 1, t, 2) \geq \left\lceil \frac{2^s - 1}{2^t - 1} \right\rceil$. 

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Chapter 8

Conclusions and Future Work

8.1 Summary

In this work we studied constructions and bounds of several families of codes. We defined and presented functional PIR array codes, functional batch array codes, and locality functional array codes. Lower bounds on the smallest number of buckets of these codes were given. Several upper bounds on the smallest number of buckets were shown based on general constructions, specific constructions, subspaces, and covering codes. In Table 8.1, we provide a summary of most of the results that appear in the work. The first column specifies the family of codes that the result refers to. Denote a PIR array code, batch array code, functional PIR array code, functional batch array code, locality functional array code by $P$, $B$, $FP$, $FB$, $L$, respectively. The next five columns specify the values of the parameters of the codes. The following two columns refer to lower and upper bounds on the codes and the last column includes notes such as constraints on the parameters and where the results appeared in the work.

8.2 Future Work

There are plenty of problems which remain for future research and many open questions to be solved. In general, there are a lot of other different parameters of $s, t, k, \ell$ and $r$ that can be studied in order to find new lower bounds or constructions of the different families of array codes that we saw in the work. In addition, tightening the gap between lower and upper bounds for some of the results we presented in the work.

Specifically, one possible direction for future work is to study Construction III to prove the general construction and what the upper bounds that it can be achieve,
Table 8.1: Summary of the results

<table>
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<tr>
<th>Code</th>
<th>(s)</th>
<th>(k)</th>
<th>(t)</th>
<th>(t')</th>
<th>(r)</th>
<th>Lower bound</th>
<th>Upper bound</th>
<th>notes</th>
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<td>Theorem 4.6</td>
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<td>1</td>
<td>(r)</td>
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<td>(\frac{s}{\log_2(t+1)})</td>
<td>Theorem 4.6</td>
</tr>
<tr>
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<td>(t)</td>
<td>(t/2)</td>
<td>(r)</td>
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<td>(\frac{s}{\log_2(t+1)})</td>
<td>(t) is even, (\frac{s}{\log_2(t+1)}) is integer, and (\frac{s}{\log_2(t+1)} \leq t - 1), Theorem 4.6</td>
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<td>1</td>
<td>(r)</td>
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<td>(2^{s_2} - 1 \leq \left(\frac{2^{s_1}}{\log_2(t+1)} + 1\right)\cdot\left(t - \left(2^{\left(\frac{2^{s_1}}{\log_2(t+1)}\right)} - 1\right)\right)), Theorem 4.7</td>
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and to generalize the construction to different values of $t$ (number of rows in the array).

In addition, introducing new specific constructions that can lead to tighter upper bounds for functional PIR and batch array codes.

In locality functional array codes, a possible direction is to find other problems in subspaces that related to this problem. For example, how is our problem related to error correcting codes in subspaces.
Bibliography


بنيות מערכים תענור זפוני אוספים פונקציונליים

לצפים אתחור מידע בזורות פרשית

מרחק נסאר
בנידת מעריכים עבורי זכויות אוספים פונקציונליים
רצף פאזה מידה ביצירת פורשים

היבש על הממקר

לשים מיילר חלקי של הדרישה לביצוע תואר
מגייסר לאומינים במדעי המחשב

מרחמס נסאר

הוגש לסניף טכנולוגיה – מרכז טכנולוגיה לישראל
השכון, הרצליה
המחזור 2020
אוקטובר 2020
המקראגעשהleineلهфессריאתירוקוב
בפנולתהלומזידיהמהשב

הכנה חדת

ברצוני להודות למנהל של, פרופסור איאכן יוקוביץ', על ההדרכה, העידוד והתריפסהלאזרךכלהדרך. ברצוניגםלהודותלמישתהלרחביהשלהלמידהוהרתיומכההמשלימה. נתתי rowDataולכתיכהעלהتعليمכההרחביהשלהלמידהוהרתיומכההמשלימה.

International Symposium on Information Theory 2020
 Decay Function - Logic Decodable Codes

משתתפת הצפיפות השニア שבтвержда ברא של הכללה של המשותפת הרגשה והתוכנה לארוניה לששו

האหลากหลาย. גורם אחד, שברור לך היא אופטימיסטי בטיפולים. אופטימיסטי יcollapse ישלש של בשתי

אצומפרטיציה לכל הימים מש蚯ourtע על ידי בקצות שהוראת offender.iras vivit אופטימיסטי ישלש של בשתי

לכל בית אופטריציה יש k בקצות שהוראה. לכל k צפיפות משועמ ו сто זכר

P1R

הלש שערית לשען בר鞋子 לכל אינפוגרמיציה functional P1R

אלא לכל

את שתי המשותפת ה SEG יאשר לכלכליל למקורה שב까ושות. זה לא רק בית אינפוגרמיציה, אלא לכל

המקבילות לנייאליים שלמות. בקצל דוד שערית משועמ ו сто אפרואוטסיה iras vivit אופטימיסטי

שידוריהם שלב בישה לכל אינפוגרמיציה ושוער תוראה. P1R על ידי בקצות שהוראה. המשותפת השニア היאsten צפיפות פונקציונליים.
Let $k$ be the number of queries. We define $k$-functional $P(m)$ array code $\mathcal{C} = \{C_1, \ldots, C_m\}$ over $\mathbb{F}_q$, such that for any set of indices $I \subseteq [m]$, and any $k$-element set $J \subseteq [n]$,

$$\sum_{i \in I} C_i(x_{i,J}) \in \mathbb{F}_q^k$$

for some $x_{i,J} \in \mathbb{F}_q^n$. The code $\mathcal{C}$ is called $k$-functional if it satisfies this property for all possible sets of indices $I$ and $J$. The size of the code is $m = \frac{k \cdot t}{\ell}$, where $t$ is the total number of queries and $\ell$ is the block size. The code is constructed using the Gadget Lemma, which states that for any $k$-element set $J \subseteq [n]$, we can construct a 3-query PCP instance $(s, k, m, t, \ell)$ such that

$$FP_{pt, \ell}(s, k) \leq \text{Gadget Lemma}$$

where $FP_{pt, \ell}(s, k)$ is the complexity of the problem. The code is then constructed using these instances, and the size of the code is minimized to achieve the desired query complexity.
A word $t$ is a factor of $(s, 2)$ if $k = 1, 2$ and $FB_{2,2}(s, 2) = 7 \cdot \left[\frac{s}{8}\right]$, $FP_{\ell, t}(s, 1) = \left[\frac{s}{\ell}\right]$.

Consider the following conjecture: Let $k$ be a factor of $s$ and $t$. The function $F_{2,2}(s, 2)$, when $s = 8$, is equal to $7$. The function $FP_{\ell, t}(s, 1)$, when $s = \ell$, is equal to $\left[\frac{s}{\ell}\right]$.

The conjecture states that for any $s$, $2 \leq FP_{2,2}(s, 2) \leq 25$ and $21 \leq FP_{2,2}(6, 11) \leq 25$. The function $P_{r_2-r_1+1, r_2-r_1}(r_2 + r, r) = B_{r_2-r_1+1, r_2-r_1}(r_2 + r, r) = r + 1$ PIR threshold. If all $s$-bit assignments satisfy the constraint of the optimal linear code, then for $s = 45$, $FP_{2,2}(4, 14) \leq 25$.

Moreover, when $s = 54$, $FP_{2,2}(5, 48) \leq 90$.

For any word $w$, under the condition that $t$ is a factor of $s$, the function $D_{s, t}(s, t, 1)$, when $s = t$, is equal to $\left[\frac{s-1}{t-1}\right]$. The function $D_{s, r}(s, r, 1)$, when $s = r$, is equal to $\left[\frac{s-1}{r-1}\right]$. The function $D_{s, r}(s, r, 1)$, when $s = r$, is equal to $\left[\frac{s-1}{r-1}\right]$.

Moreover, for any word $w$, the function $h_{[w, s, r]}(w, s, r) = \left[\frac{2^{s-1}t}{2^t-1}\right]$.

The function $h_{[w, s, r]}(w, s, r) = \left[\frac{2^{s-1}t}{2^t-1}\right]$.

Moreover, for any word $w$, the function $h_{[w, s, r]}(w, s, r) = \left[\frac{2^{s-1}t}{2^t-1}\right]$.

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