Abstract—We present a novel probabilistic gathering algorithms for agents that can only detect the presence of other agents in front or behind them. The agents act in the plane and are identical and indistinguishable, oblivious and lack any means of direct communication. They do not have a common frame of reference in the plane and choose their orientation (direction of possible motion) at random. The analysis of the gathering process assumes that the agents act synchronously in selecting random orientations that remain fixed during each unit time-interval. Two algorithms are discussed. The first one assumes discrete jumps based on the sensing results given the randomly selected motion direction and in this case extensive experimental results exhibit probabilistic clustering into a circular region with radius equal to the step-size in time proportional to the number of agents. The second algorithm assumes agents with continuous sensing and motion, and in this case we can prove gathering into a very small circular region in finite expected time.

I. INTRODUCTION

This paper deals with gathering of multi-agent systems, based on a decentralized control law. Agents move according to local information provided by their sensors. The agents are assumed to be identical and indistinguishable, memoryless (oblivious), with no explicit communication between them. They do not have a common frame of reference (i.e. agents are not equipped with GPS sensors or compasses).

A wealth of gathering algorithms were described and analyzed in the multi-agent robotics literature. They differ in the assumptions made on the sensing that is performed by the agents, in the assumptions on the possible moves that can be made and in the computational requirements for the decision process that leads to the motion response [1] - [12].

A recent report on gathering [13] surveyed in detail gathering or clustering algorithms under the assumptions of limited or unlimited visibility sensing, complete relative position information sensing vs. bearing only information provided by the sensors, and discrete vs. continuous motion schedules for the agents.

We here present a novel gathering, or geometric consensus algorithm based on a simple randomized rule of motion for agents that can only sense the presence or absence of other agents in front or behind them. We assume that the agents have orientation and they may move only forward, but each agent can, at various instances, select a new heading at random, uniformly distributed over $[0, 2\pi]$ with respect to an absolute frame of reference. This is accomplished by doing an independent and uniformly distributed turn from $[0, 2\pi]$ to their current orientation.

Under these assumptions, we consider two different gathering algorithms. The first one assumes that agents make a forward jump of size 1 whenever there are no agents behind them, i.e. in their back half-plane. The agents act synchronously and new headings are selected at random and independently at each unit-time, then the "back" sensor's reading tells the agent whether to jump forward or stay put.

Extensive experimental results with this process shows that probabilistic gathering to a small region occurs in time proportional to the number of agents. We can not prove this result yet.

The second gathering algorithm we discuss is a continuous version of this process. Here we assume that the sensing is continuously done, and the forward motion of agents during each unit time-interval is continuously controlled by the absence of agents in the sensing area behind. As we shall see, in this case we also need to assume a blind-zone for the backwards sensing, in order to avoid dead-lock situations preventing gathering. Under these assumptions we can prove gathering in finite expected time.

II. THE DISCRETE ALGORITHM

Consider a system of $n$ identical, anonymous, and memoryless agents specified by their time varying locations in the plane \( \{p_i(k)\}_{i=1,2,\ldots,n} \in \mathbb{R}^2 \) and heading vectors \( \{\theta_i(k)\}_{i=1,2,\ldots,n} \) which are unit vectors randomly selected on the unit circle. These quantities are unknown to the agents themselves as they lack global position and orientation sensors. We define "heading" as the direction where the agent’s nose is pointing, i.e. its current direction of (possible) motion. The agents implicitly interact with each other in such a way that an agent next position after one time unit \( p_i(k+1) \) is determined by the constellation of all the agents in the system.

A. Sensing

Each agent is equipped with an onboard sensing device, aimed in the opposite direction to the agent’s heading \( \theta_i(t) \), covering the back half-plane (with 180° field of view). We may call this sensing device a "Backward Looking Binary Sensor". If there is no other agent in the field of view of agent \( i \), the output signal is \( s_i(k) = 1 \), else \( s_i(k) = 0 \).

B. Timing and the Motion Law

At time \( k = 0 \) the agents are in an arbitrary initial constellation with randomly selected headings, and perform forward jumps if their sensor reading is 1. Then at each time-step \( k \) every agent changes its heading direction by choosing a uniformly distributed random direction \( 0 \leq \chi_i(k) < 2\pi \), and then, if its back closed half-plane is empty, it jumps forward.
a fixed step-size $d = 1$. Otherwise it stays put until the next time-step.

The dynamic motion law is formally described as follows. Let \{$p_i(k), p_2(k), \ldots, p_n(k)$\} be the locations of the \(n\) agents at time-step \(k\). Then:

\[
p_i(k + 1) = p_i(k) + \left[ \cos(\theta_i(k)) \right] s_i(k)
\]

\[
\theta_i(k) = \sum_{k=1}^{\infty} \chi_k \Delta_k(t)
\]

where \(\chi_k\) are iid uniformly distributed over \([0, 2\pi]\)

\[
1\Delta_k(t) = \begin{cases} 
1, & \text{for } t \in [k, k + 1) \\
0, & \text{otherwise}
\end{cases}
\]

\[
s_i(k) = \begin{cases} 
0, & \exists \ j: \hat{\theta}_i(k)[p_j(k) - p_i(k)] \leq 0 \\
1, & \text{otherwise}
\end{cases}
\]

where \(s_i(k)\) is the binary output from the sensor as seen in Figure 1, and \(\hat{\theta}_i(k) = [\cos(\chi_i(k)), \sin(\chi_i(k))]^T\) is the agent's random heading.

Fig. 1. Sensing geometry: The heading of an agent is presented by an arrow, and the product \(\hat{\theta}_i[p_j - p_i]\) is marked in bold line. If (a) agent \(j\) is located in front of agent \(i\) the product is positive, and if \(j\) is in its back (b) the product is negative. If all products are positive \(s_i(k) = 1\) otherwise \(s_i(k) = 0\).

### C. Simulation Results and System Behavior

Typical simulation results of gathering are shown in Figure 2 and 3. In all the simulations we ran, a number of agents is randomly placed in the plane, and in all cases the system converges to an area within circular region of radius \(1\), whose "center" wanders at random in the plane.

Figure 4 summarizes 75 simulation runs with different number of agents, spread uniformly over the same initial area. Notice that the effect of the number of agents on the convergence time of the system is linear.

As shown in Figure 5, only the agents occupying the corners of the convex-hull of the constellation may select orientation with empty back half-planes, thus only they may jump, while the inner agents necessarily stay put until they become external. Since agents' headings change randomly, agents on the convex-hull of the system have a high probability to jump towards all other agents, hence the system tends to gather, as shown in the simulations. But this simplified account of the system's dynamics is not accurate, as discussed in the sequel.

An adversarial argument proves, however, that a system like we defined may even diverge, however this happens with very low probability.

Consider a system of two agents, \(n = 2\), and assume that
the agents’ headings are (almost) perpendicular to the line they define and oriented in opposite directions (so that we have $0 < \hat{\theta}_i(k) \cdot [p_j(k) - p_i(k)] < 1$ and $\hat{\theta}_j(k) = -\hat{\theta}_i(k)$). Since both their back half planes are empty they both jump forward, and obviously we have that the distance between them increases.

We next suggest a modified gathering algorithm with piecewise continuous-time dynamics, for which we can actually prove gathering to a very small region in finite expected time.

III. PIECEWISE CONTINUOUS-TIME DYNAMICS

In this system, once in a unit time-interval $\Delta t = 1$, simultaneously, each agent changes its heading direction $\hat{\theta}_i(t)$ to a uniformly distributed random angle between 0 and $2\pi$. Here too, the sensor is aimed backwards (at $-\hat{\theta}_i(t)$) but it includes a “blind-zone” half disc area of radius $\delta$ << 1. During the time-interval $\Delta t$ an agent keeps its heading direction, and if its sensing area is empty it moves forward with a fixed velocity $v = 1$, otherwise it stops. Note that we assume that the agent may move and stop during $\Delta t$ according to the changing constellation of the system.

Denote by $d_{ij}(t) = \|p_i(t) - p_j(t)\|$ the distance between agents $i$ and $j$ at time $t$, so that if $d_{ij} < \delta$, the agents are too close to potentially see each other, otherwise we call them “separated”. The dynamic law in piece-wise continuous-time is:

$$\dot{p}_i(t) = \begin{bmatrix} \cos(\theta_i(t)) \\ \sin(\theta_i(t)) \end{bmatrix} s_i(t)$$
$$\dot{\theta}_i(t) = \sum_{k=1}^{\infty} \chi^{(i)}_k 1_{\Delta_k}(t)$$

where

$$\chi^{(i)}_k = \text{ iid uniformly distributed over } [0, 2\pi]$$

$$1_{\Delta_k}(t) = \begin{cases} 1, & \text{for } t \in [k, k+1) \\ 0, & \text{otherwise} \end{cases}$$

$$s_i(t) = \begin{cases} 0, & \exists j : d_{ij} > \delta \text{ and } \hat{\theta}_i(t)[p_j(t) - p_i(t)] \leq 0 \\ 1, & \text{otherwise} \end{cases}$$

(2)

In the following proofs we often omit the time index $t$.

Lemma 1: The distance between two “separated” agents never increases.

Proof: Suppose agents $i$ and $j$ are “separated” at time $t$ so that $d_{ij} > \delta$. Denote by $\theta_{ij}$ the (current) small angle between vector $p_j - p_i$ and the heading direction of agent $i$ as shown in Figure 6.

The derivative of the distance between $p_i$ and $p_j$ (that is the inverse of their approach speed) is given by

$$\frac{d}{dt} d_{ij} = \frac{d}{dt} \|p_j - p_i\| = -\left( \dot{p}_i \cdot \frac{p_j - p_i}{\|p_j - p_i\|} + \dot{p}_j \cdot \frac{p_i - p_j}{\|p_i - p_j\|} \right)$$

$$= -\left( \|p_i\| \cos \theta_{ij} + \|p_j\| \cos \theta_{ji} \right) = -(s_i \cos \theta_{ij} + s_j \cos \theta_{ji})$$

(3)

By the dynamic law if the sensor coverage area of agent $i$ is not empty (i.e. $s_i = 0$) it does not move. In this case its speed is $\|\dot{p}_i\| = s_i = 0$. Otherwise, if all other agents are in front of it, necessarily $-\pi \leq \theta_{ij} \leq \pi$, i.e. $0 < \cos \theta_{ij} \leq 1$, and then it moves forward at $\|\dot{p}_i\| = s_i = 1$. Similar arguments hold for all agents, e.g. (see Figure 6) agent $p_j$ with $0 < \cos \theta_{ij} \leq 1$ and $\|\dot{p}_i\| = v = 1$, and agent $p_k$ with $\|\dot{p}_k\| = 0$. Hence we have that

$$d_{ij} > \delta \implies \frac{d}{dt} d_{ij} \leq 0$$

(4)

Corollary 1: Agents within range $\delta$ at time $t'$ remain within range $\delta$ at all $t \geq t'$.

Proof: This is a direct consequence of Lemma 1 and (4). Note that if an agent $j$ is closer than $\delta$ to agent $i$ (so that $d_{ij} < \delta$), in order for $d_{ij}$ to increase above $\delta$, it first have to reach the value $d_{ij} = \delta$ since when an agent moves, it moves in a continuous motion, but by Lemma 1 the value $d_{ij} = \delta$ can not increase.

Next we show that if not all agents are confined inside a circle of radius $\delta$ there is a strictly positive probability for the distance between pairs of agents to decrease at a positive and bounded away from zero rate, until all agents are confined in a circle of radius $\delta$.

A. Proof of convergence

Theorem 1: Piece-wise continuous dynamics with agents acting according to the motion law given in (2), converges to a region of radius $\delta$ in finite expected time.

Proof: We know by Corollary 1 that pairs of agents within range $\delta$ from each other at some time $t$ will remain
so forever. We shall next show that while in the agents’
configuration there exists pairs of agents at distance bigger
than δ from each other, there is a finite probability that there
will be a significant (bounded away from zero by a constant)
decline in the distance between them.

Lemma 2: As long as not all agents are confined in a
circle of radius δ, there is an agent with a strictly positive
probability to move a distance bounded away from zero by a
constant during ∆t.

Proof: The sum of corner angles of any convex polygon
of m corners is given by π(m − 2), therefore the sharpest
corner of a convex polygon of m corners is bounded from
above by \( \frac{\pi(m-2)}{m} = \pi(1 - \frac{2}{m}) \). Since the maximal number
of corners of the convex-hull of a system of n agents is n,
we have that \( \alpha_s \), the sharpest corner of the convex-hull of a
system on n agents, is bounded by

\[
\alpha_s \leq \pi(1 - \frac{2}{n}) \tag{5}
\]

Let \( \alpha_i = \angle p_{i-1}p_ip_{i+1} \) be the convex-hull angle at a corner
\( p_i \), and let \( p_{i-1} \) and \( p_{i+1} \) be the locations of the corners of
the convex-hull adjacent to \( p_i \) (see Figure 7).

Denote by \( \beta_i = \pi - \alpha_i \), the smaller angle between the two
lines perpendicular to the sides of \( \alpha_i \) so that if the random
heading of agent \( i \) falls inside \( \beta_i \) it can move. Note that if
the heading of \( p_i \) is precisely along one of the sides of \( \beta_i \)
(and not explicitly inside \( \beta_i \)), agent \( i \) may not move (as then
\( p_{i-1} \) or \( p_{i+1} \) may be located in its sensing area). Therefore
consider the symmetric central half of \( \beta_i \) about the bisector of
\( \beta_i \) (and \( \alpha_i \) too) denoted by \( \beta_i^* = \frac{1}{2} \beta_i = \frac{1}{2}(\pi - \alpha_i) \). The
probability for the heading of agent \( i \) to fall inside \( \beta_i^* \) is
\( \frac{1}{2\pi} \beta^*_i = \frac{1}{2\pi} \beta_i = \frac{1}{2\pi}(\pi - \alpha_i) \). Hence the probability for agent
(defined the convex-hull) to move is lower bounded as

\[
Pr(\text{agent } i \text{ moves}) > \frac{1}{4\pi}(\pi - \alpha_i) \tag{6}
\]

To further bound this probability independently of the
constellation, let us consider agent at the sharpest corner of
the convex-hull, so that by (5) we know that \( \alpha_s \leq \pi(1 - \frac{2}{n}) \).

Hence the probability of \( s \), the agent at the sharpest corner
of the convex-hull \( p_s \) to move is lower bounded by

\[
Pr(\text{agent } s \text{ moves}) > \frac{1}{4\pi}(\pi - \pi(1 - \frac{2}{n})) = \frac{1}{2n} \tag{6}
\]

We shall next prove that if agent \( s \) moves, it moves a distance
bounded away from zero by a constant. Let us define the case
where the heading of agent \( s \) is inside its associated \( \beta_s^* \) (at the
beginning of a time-interval) as a ”successful time-interval”,
and its associated bound (6) as “the minimal probability for a
successful time-interval” at the beginning of \([k, k+1]\).

To find a bound on the minimal distance that agent \( s \)
will surely travel if a successful heading was selected (with
probability \( \frac{1}{2n} \)), we must compute the distance that agent \( s \)
can move ahead unimpeded by any other agent entering its
sensing area.

Assume the heading of an agent \( s \) at the sharpest corner
of the convex-hull is along one side of its associated \( \beta_s^* \) as
shown in Figure 7. Agent \( s+1 \) (adjacent to \( s \) on the convex-
hull) and agent \( s \) define a side of the convex-hull, therefore
\( p_{s+1} \) is located somewhere along this side of \( \alpha_s \).

The geometry of the first possible encounter is as seen in
Figure 8.

From simple considerations we see that the earliest possible
encounter between the forward moving sensing region of
agent \( s \) and the expansion of the convex-hull of the agents
dilation with the same speed.

By our assumption the agents of the system reside in a
convex region contained in the front half plane of the moving
agent \( s \). The geometry of the convex region may change in
time as some other agents can possibly move. However we
have that, due to the finite limit on the speed of all agents
the region where all agents reside at some time \( \Delta t \) after the
motion of agent \( s \) started, will not exceed a dilation of the
original convex-hull by a disc of radius \( \Delta t \).

Due to this fact we can compute the earliest time when
any agent (different from \( s \)) can enter the sensing region of
the moving agent \( s \), thereby stopping its progress.

In the Figure 8 we see that this may happen when the
forward moving front line of the sensing range will intersect
the dilating convex-hull of the agents as it looked at the
beginning of agent \( s \)’s motion. This may happen at the point
\( M \) which is located a distance of \( |BM| = \delta \tan \frac{\pi}{4n} \), where 
\( \gamma_s = \frac{\pi}{2} - \theta_s \).

\[
\theta_s = \frac{1}{2}(\alpha_s + \beta_s^*) = \frac{1}{2}(\alpha_s + \frac{1}{2}(\pi - \alpha_s)) = \frac{1}{4}(\alpha_s + \pi)
\]

We have by (5) that \( \alpha_s \leq \pi(1 - \frac{2}{n}) \), therefore

\[
\theta_s \leq \frac{1}{4}(\pi(1 - \frac{2}{n}) + \pi) = \frac{1}{4}(2\pi - \frac{2\pi}{n}) = \frac{1}{2}(\pi - \frac{\pi}{n}) = \frac{\pi}{2}(1 - \frac{1}{n}).
\] (7)

And since \( \gamma_s = \frac{\pi}{2} - \theta_s \), we have that \( \gamma_s \geq \frac{\pi}{2} - \frac{\pi}{2}(1 - \frac{1}{n}) = \frac{\pi}{n} \), and therefore (as \( 0 < \frac{2\pi}{n} < \frac{\pi}{2} \)) we have that

\[
|BM| \geq \delta \tan \frac{\pi}{4n}
\]

This shows that the sharpest corner agent, if its heading falls inside \( \beta_s^* \), will traverse a distance of at least

\[
\text{Step}_s > \delta \tan \frac{\pi}{4n}
\]

before possibly being stopped by the motion law we defined.

Note that the since \( \theta_s \) is bigger than half \( \alpha_s \), the angle of convex-hull corner at \( s \), the bound holds for the case where there are agents on the other side of \( p_s \) that might impede its forward motion.

The minimal travel that the agent defining the sharpest corner of the convex-hull can move during one time-interval is bounded by the smallest value between \( \text{Step}_s \) above and the physical limit due to the travel speed \( v = 1 \), i.e. \( v \Delta t = 1 \), hence we have that the bound on the travel of the agent at the sharpest corner of the convex-hull is

\[
\text{Step}_s = \min\{ \delta \tan \frac{\pi}{4n}; 1 \}
\] (8)

B. The Lyapunov function

A function is called Lyapunov if it maps the state of the system to a non-negative value in such a way that the system dynamics causes a monotonic decrease of this value. If the Lyapunov function reaches zero only at desirable states of the system and we prove that the dynamics leads the Lyapunov function to zero, we can argue that the system converges to a desirable state.

For the proof of system convergence, let us define variables \( l_{ij}(t) \) as follows:

\[
l_{ij}(t) = \begin{cases} 0, & 0 \leq d_{ij}(t) \leq \delta \\ d_{ij}(t), & d_{ij}(t) \geq \delta \end{cases}
\] (9)

and a global variable \( c(t) \):

\[
c(t) = \begin{cases} 0, & \exists p_i \in \mathbb{R}^2 \forall i : \| p_i(t) - p_s(t) \| < \delta \\ 1, & \text{otherwise} \end{cases}
\] (10)

so that if \( c(t) = 0 \) we have that the system is confined in a disc of radius \( \delta \) in the plane.

Let us define the following Lyapunov function:

\[
\mathcal{L}(P(t)) = c(t) \sum_{i=1}^{n} \sum_{j=1}^{n} l_{ij}(t)
\] (11)

By Corollary 1 we have that \( l_{ij} \)’s can never increase, hence \( \mathcal{L}(P(t)) \) never increases. We shall next prove that with a probability which is finite and bounded away from zero by a constant, \( \mathcal{L}(P(t)) \) decreases by a positive and bounded away from zero quantity, until it reaches the value zero (within a finite expected time). We show the sum \( \sum_{j=1}^{n} l_{s_j}(t) \) become zero in finite expected time, where \( s \) is the agent currently at the sharpest angle of the convex-hull, and therefore \( \mathcal{L}(P(t)) \) also become zero in finite expected time.

Lemma 3: If at time \( t' \), the beginning of a time-interval, there is an agent \( j \) distant more than \( \delta \) from the agent \( s \), currently located at the sharpest corner of the convex-hull, the probability that \( l_{s_j}(t') \rightarrow l_{s_j}(t' + 1) \) is at least a constant bounded away from zero, is higher than \( \frac{1}{8n} \).

Note that by Lemma 3, we have that in finite expected time all agents will necessarily be confined inside a disc of radius \( \delta \).

Proof: In order to evaluate the influence of agent \( s \)’s motion on the Lyapunov function defined by (11), we need to see how a step bigger than or equal to \( \text{Step}_s = \delta \tan \frac{\pi}{4n} \) can influence the \( l_{ij}(t) \)’s in the sum defining \( \mathcal{L}(P(t)) \).

If all agents are confined inside a disc of radius \( \delta \) then gathering has already been achieved. Therefore we need to consider the case where there exists at least one other agent \( j \) farther than \( \delta \) from \( s \) at the beginning of motion of agent \( s \). Suppose agent \( j \) is located somewhere in the region shown in Figure 9 as 1 and 2. We can easily lower bound the probability that it will remain stationary during the entire motion of agent \( s \) as follows.

![Fig. 9.](image-url)
then \( l_{s_j} \) and \( l_{j_s} \) will decrease by at least \( \delta \) each. 

2) remain stationary at a distance bigger than \( \delta \) from \( s \) for the entire motion of \( s \), and in this case we can bound the decrease of \( l_{s_j} \) and \( l_{j_s} \) as follows:

Let us consider

\[
Shrink_{s_i} = d_{s_j} - \sqrt{d_{s_j}^2 + Step_s^2 - 2d_{s_j}Step_s cos \theta_{s_j}}
\]  

(12)

where (see Figure 9) \( d_{s_j} \) is the mutual distance of agents \( s \) and \( j \) at the beginning of a time-interval, and \( \sqrt{d_{s_j}^2 + Step_s^2 - 2d_{s_j}Step_s cos \theta_{s_j}} \) is their mutual distance at the end of that time-interval.

Let us consider the function \( Shrink_{s_i} \) of three variables \( d, St, \theta \):

\[
Sh(d, St, \theta) \triangleq d - \sqrt{d^2 + St^2 - 2dSt cos \theta}
\]

We have that

\[
\frac{\partial Sh}{\partial d} = 1 - \frac{d - St cos \theta}{\sqrt{d^2 + St^2 - 2dSt cos \theta}} > 0 \\
\forall d > 0; St > 0; \theta \in [0, \frac{\pi}{2})
\]

\[
\frac{\partial Sh}{\partial St} = -\frac{dSt sin \theta}{\sqrt{d^2 + St^2 - 2dSt cos \theta}} < 0 \\
\forall \theta \in [0, \frac{\pi}{2}); d > 0; St > 0
\]

\[
\frac{\partial Sh}{\partial St} = -\frac{St - d cos \theta}{\sqrt{d^2 + St^2 - 2dSt cos \theta}} > 0 \\
\forall St > d cos \theta; d > 0; \theta \in [0, \frac{\pi}{2})
\]

and

\[
d \in [\delta, \infty) ; \theta \in [0, \theta_\alpha) ; St \in [St_{min}, d cos \theta_\alpha)
\]

If there is an agent \( j \), distant more than \( \delta \) from agent \( s \), it is located either in region 1 or 2 shown in Figure 9. (As we have seen before, if it were in region 1 we have by (9) that \( l_{s_j}(t) - l_{s_j}(t+1) \geq \delta \), and \( l_{j_s}(t) - l_{j_s}(t+1) \geq \delta \), i.e. there is a significant drop in the Lyapunov function of at least \( 2 \delta \). If agent \( j \) is in region 2, since \( \frac{\partial Sh}{\partial St} > 0 \), we have

\[
Sh \geq Sh(d_{min}, \theta_{max}, St_{min})
\]

But we can compute

\[
Sh(d_{min}, \theta_{max}, St_{min}) = \delta - \sqrt{\delta^2 - \delta^2 tan^2 \frac{\pi}{4n} - 2\delta^2 tan \frac{\pi}{4n} sin \frac{\pi}{4n}}
\]

hence

\[
Shrink_{min} > \delta (1 - \sqrt{1 - tan^2 \frac{\pi}{4n}})
\]

(16)

To lower bound the probability for such "successful" time-interval, recall that if \( Q \) is an event that occurs in a trial with probability \( q \), we have that the mathematical expectation \( E \) of number of trials \( k \) to first occurrence of \( Q \) in a sequence of trials is

\[
E[k]_Q = q + (1-q)q + (1-q)^2 q + ... = \sum_{k=1}^{\infty} k(1-q)^{k-1} q = \frac{1}{q}
\]

In our case the probability for "successful time-interval", given in (6), is lower bounded by \( \frac{1}{2n} \), and the probability that agent "\( j \)" stays stationary while "\( s \)" moves is lower bounded by \( \frac{1}{4} \), therefore since these events are independent, their joint probability equals the product of their probabilities, i.e. lower bounded by \( \frac{1}{8n} \). Therefore the expected number of time-intervals for "shrink_{min}" event to occur is upper bounded by \( 8n \).

To complete the proof of Theorem 1, the Lyapunov function \( L(P(t)) \) will become zero when there is a point in \( \mathbb{R}^2 \) whose distance to all agents is smaller then \( \delta \). Hence, the expected number of time-intervals to convergence inside a disc of radius \( \delta \) is therefore upper bounded by

\[
\frac{L(P(0))}{Shrink_{min}} \leq 8n
\]

Clearly if the Lyapunov function equals zero, all agents are confined in a region of radius \( \delta \). Since the initial value of the Lyapunov function is less than \( n(n-1)d_{max}(0) \) (since in the chosen Lyapunov each "edge" is counted twice), therefore by (16), the expected number of time-intervals to convergence of the system is

\[
E(t)_{convergence} \leq \frac{8n^3}{(1 - \sqrt{1 - tan^2 \frac{\pi}{4n}})} \frac{d_{max}(0)}{\delta}
\]

(17)

which is finite, and dependent on the initial constellation, the number of agents \( n \), and the radius of the blind-zone \( \delta \).

IV. SUMMARY

We proposed and analyzed two randomized gathering processes for identical, anonymous, oblivious mobile agents, only capable to sense the presence of other agents behind their motion orientations. The agents act synchronously, and at unite time-intervals they randomly select new forward motion orientations. We proved that the "continuous version" of the process ensures gathering to within a region of diameter \( 2 \delta \) where \( \delta \) is a parameter setting a "blind spot" in sensing nearby agents. Gathering happens in finite expected time, proportional to \( \delta^{-1} \). This result also shows that the blind spot is absolutely necessary for finite expected time convergence.

The fully discrete model, in which agents perform unit jumps forward if no agents are detected behind them, was also found experimentally to gather the agents to a radius 1 minimal enclosing circle, in time proportional to the number of agents. This happens in all cases we tested, however the proof of this result will certainly involve probabilistic convergence arguments. We are currently considering ways to prove that the agents will gather to a small region and remain in a cluster that wanders at random in the plane.
REFERENCES


