Discrete Time Gathering of Agents with Bearing Only and Limited Visibility Sensors

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Abstract

We analyse a gathering process for a group of mobile robotic agents, identical and indistinguishable, with no memory and no common frame of reference (neither absolute location nor a common orientation). The agents are assumed to have bearing only sensing within a limited visibility range. We prove that such robots can gather to a small disk in the \( \mathbb{R}^2 \)-plane within a finite expected number of time-steps, implementing a randomized visibility preserving motion law. In addition, we analyse the dynamics of the cluster of agents after gathering, and show that the agent-cluster performs a “random-walk” in the plane.

1 Introduction

Gathering is a basic task in multi-agent systems and a lot of research is devoted to the development of algorithms for accomplishing it, under various assumptions on the agents’ motion and sensing capabilities [1, 2, 3, 4]. Here we address the problem of achieving gathering with oblivious, anonymous (identical and indistinguishable) and non-communicating robots, lacking a joint frame of reference in space that are capable of only sensing their neighbours' bearing within a limited visibility range.

This was first addressed in the discrete time framework in [5][6]. The main result, proved in [6], is that a randomized rule of motion according to which each agent jumps a limited distance (\( \sigma \)) to a random location inside a region determined by the neighbours’ directions, achieves both cohesion of the swarm and gathering to a constellation within a disk of diameter equal to the visibility range (\( V \)), in finite expected time. Experimentally it was observed that, due to bearing only sensing, the agents cluster was in fact of a smaller size, of the order of the step \( \sigma \), and was drifting in the plane in what seemed to be a random walk.
In this paper, we slightly modify the motion law discussed in [6], and prove that it gathers the agents of the system to a disk with a radius equal to the agents’ maximal step size $\sigma$ within a finite expected number of time steps. Furthermore, we prove that as time tends to infinity, the distribution of the agents’ centroid converges in probability to the distribution of a random-walk in the plane.

We start by presenting the dynamics of a multi-agent system of agents having unlimited visibility, applying a non-randomized motion-law. We proceed with an explanation of the necessity for randomization in the motion law, when applying it to a system with agents having limited-visibility. Then, we formally define the adjusted randomized motion law, and prove that it gathers the agents to a disk with a radius equal to the length of the maximal allowed step, $\sigma$. We end this paper proving that the average position of the agents performs a random motion converging in probability to the distribution of a random-walk.

Note that by "gathering", we mean that the system actually reaches the goal state, which is a closely clustered constellation of agent locations, within a finite time or number of time-steps, while the meaning of "converging" implies an asymptotic approach to the goal state as time progresses, without necessarily reaching it within a finite time.

## 2 Preliminaries

We consider a system of $n$ identical, anonymous, and oblivious agents in the $\mathbb{R}^2$ plane specified by their time varying locations $\{p_i(k)\}_{i=1,2,...,n}$. We assume that the agents are able to sense the direction to their neighbours (i.e. bearing only sensing), so that their “knowledge” about neighbours is partial, and their motions are determined by the set of unit vectors pointing from their current location to their neighbours. The neighbours are defined for each agent $i$ at time-step $k$ as a set of agents located within a given visibility range $V$ form its position, $p_i(k)$, and are denoted by $N_i(k)$.

The neighbourhood relation between agents is usually described by a time dependent visibility-graph. Notice that when dealing with unlimited visibility, the set $N_i(k)$ comprises all the agents, and the visibility-graph is complete, i.e. all agents sense each other.

The proofs in this paper require the use of some results from basic geometry and the theory of random-processes which can be found in Appendix 1.

## 3 Unlimited visibility

Since the agents can not estimate the distance to their neighbours, they can not determine the relative positions to their neighbours. However, each agent can readily figure out whether it is located at a corner of the convex-hull of the agents’ constellation. The agents located at such corners “know” in which direction they should move in order to enter, or to go through the agent constellation’s convex-hull. If they move in a
good direction, they have a chance to decrease the convex-hull’s area and perimeter. In the sequel, we use this ability of the agents to postulate a motion law which gathers the agents to a small region.

Since only the agents located at the convex-hull corners can estimate a "good" direction of movement, we dictate that only they should move. An agent $i$ is located at a convex-hull corner only if $\psi_i(k)$, the angle of the current minimal angular-sector anchored at agent $i$’s position and containing all its neighbours, is less than $\pi$. We set the motion law for the agents as follows: an agent located at a convex-hull corner moves in the direction of $\hat{\psi}_i(k)$, the unit vector in the direction of the bisector of angle $\psi_i(k)$ a step size determined by a parameter and the cosine of $\psi_i(k)/2$, while agents inside the convex-hull stay put. Note that, in some cases the agents may cross the current convex-hull and leave it, to find themselves outside of it. In the sequel we consider the minimal enclosing circle of the agents constellation for analysing the system’s dynamics, and do not attempt to rely on properties of the convex-hull.

We simplify the analysis of the swarm’s dynamics by defining a motion law ensuring that the agents’ average position is a system invariant.

The law of motion:
Each agent $i$ located at a convex-hull’s corner jumps in the direction of the unit vector $\hat{\psi}_i$, a distance proportional to $\cos(\psi_i(k)/2)$, i.e. half the sum of the unit vectors pointing from $p_i(k)$ to its extremal left and right neighbours. Considering that our agents are capable of jumping steps of length at most $\sigma > 0$, the new motion law for an agent $i$ is

$$p_i(k+1) = p_i(k) + \begin{cases} \sigma \hat{\psi}_i(k) \cos(\psi_i(k)/2) & \psi_i(k) < \pi \\ 0 & \text{o.w.} \end{cases} \quad (1)$$

The trajectories of the agents applying this motion law may be seen in Figure 1

Lemma 1. In a multi-agent system with dynamics given by (1), $\bar{p}(k)$, the average position of the agents, is invariant.

Proof. Let $U_i^-(k)$ and $U_i^+(k)$ be the unit vectors pointing from the position of agent $i$ to the current extremal left and right agents defining the current minimal angular-sector anchored at $p_i(k)$ and containing all its neighbours.

Let $CH(k)$ and $\partial CH(k)$ be the convex-hull of the agents’ constellation and the set of agents located at its corners. Notice that for any agent $q \in \partial CH(k)$, located at a corner of the convex-hull, the associated unit vectors $U_q^-(k)$ and $U_q^+(k)$ are pointing to the next left and right corners of the convex-hull, and consequently the associated angle $\psi_q(k)$ is less then $\pi$. Furthermore, for each agent $q \notin \partial CH(k)$, we have that $\psi_q(k) \geq \pi$, hence it stays put.

Let us number the agents of $\partial CH(k)$ in an ascending order of indices choosing an arbitrary agent to be 1, so that the agent occupying the next left corner (in the clockwise direction) of the convex-hull is marked by 2 and so on. Then, the unit vectors $U_i^-(k)$ and $U_i^+(k)$ are pointing from $p_i(k)$ to $p_{i-1}(k)$ and $p_{i+1}(k)$ respectively. Hence, $U_i^-(k)$, the unit vector
in the direction of the left border of an agent $i$’s angular-sector, is directed opposite to the direction of $U^-_i(k)$, i.e.

$$U^+_{i+1}(k) = -U^-_i(k)$$

Rewriting the motion law (1) for an agent $i \in \partial CH(k)$ using the unit vectors $U^-_i(k)$ and $U^+_i(k)$, we have that

$$p_i(k+1) = p_i(k) + \sigma \hat{\psi}_i(k) \cos(\psi(k)/2) = p_i(k) + \sigma \frac{U^-_i(k) + U^+_i(k)}{2}$$

therefore since the agents $j \notin \partial CH(k)$ stay put, the average position of the agents at time step $k+1$ is

$$\bar{p}(k+1) = \frac{1}{n} \sum_{i=1}^{n} p(k+1) = \bar{p}(k) + \frac{\sigma}{n} \sum_{i \in \partial CH(k)} \frac{U^-_i(k) + U^+_i(k)}{2} =$$

$$= \bar{p}(k) + \frac{\sigma}{2n} \left( \sum_{i \in \partial CH(k)} U^-_i(k) - \sum_{i \in \partial CH(k)} U^+_i(k) \right) = \bar{p}(k) + 0$$

hence $p_i(k+1) = p_i(k)$ for any time-step $k$, proving Lemma 1. \qed
We next prove that this system gathers to a disk of radius $\sigma$ within a finite number of time-steps for any initial constellation $P(0)$. Our proof is based on the decrease rate of $R(k)$, the radius of the smallest enclosing circle of the agents. We shall show that, if $R(k)$ is greater than $\sigma$, it decreases to $\sigma$ within a finite number of time-steps, and once it is equal to or smaller than $\sigma$, it remains that way. We do so by showing that an agent located at a distance of less than $\sigma/2$ from $C(k)$, the center of the smallest enclosing circle, will never jump to a location farther than $\sigma$ from $C(k)$, and an agent located at a distance greater than or equal to $\sigma/2$ from $C(k)$ will never jump to a farther location from $C(k)$ (Lemma 2). In addition we show that, if $R(k)$ is greater than $\sigma$, at least two agents that lie on the circumference of that circle or within a close proximity to it will jump to locations closer to $C(P(k))$ by a strictly positive length (Lemma 3). Therefore, if $R(k)$ is greater than $\sigma$, it decreases in no more than $[n/2]$ time-steps by a quantity bounded away from zero by a constant.

In order to simplify our proof, without loss of generality, we let $C(k)$ to be the location of the origin of the $\mathbb{R}^2$-plane.

**Lemma 2.** In a multi-agent system with dynamics (1), if $\|p_i(k)\|$, the current distance between the position of agent $i$ and $C(k)$ is greater than or equal to $\sigma/2$, then at the next time step $\|p_i(k+1)\|$ will be less than or equal to $\|p_i(k)\|$. Otherwise, $\|p_i(k+1)\|$ will be less than or equal to $\sigma$.

**Proof.** Let $\theta_i(k)$ be the angle between the movement direction of an agent $i \in \partial CH(k)$ and the vector pointing from $p_i(k)$ to $C(k)$. Let us divide the current minimal enclosing circle into two half circles by a line defined by the points $p_i(k)$ and $C(k)$ (the dashed line in Figure 2).

![Figure 2: Minimal enclosing circle partition. The dashed line divides the current minimal enclosing circle into two half circles.](image)

By Proposition 4 (See Appendix), there is at least one agent lying on each one of those half circles, therefore the angles $\theta_i(k)$ and $\psi_i(k)$, which
associate with the movement of agent $i \in \partial CH(k)$, are bounded as follows:

$$0 \leq \theta_i(k) \leq \frac{\pi}{2} \quad \text{and} \quad 2\theta_i(k) \leq \psi_i(k) \leq \pi$$

Then, by the motion law (1) we have that,

$$|p_i(k + 1)| = \sqrt{|p_i(k)|^2 + \left(\sigma \cos \left(\frac{\psi_i(k)}{2}\right)\right)^2 - 2|p_i(k)|\sigma \cos \left(\frac{\psi_i(k)}{2}\right) \cos(\theta_i(k)) \leq \sqrt{|p_i(k)|^2 + \sigma^2 \cos \left(\frac{\psi_i(k)}{2}\right) \cos(\theta_i(k)) - 2|p_i(k)| \sigma \cos \left(\frac{\psi_i(k)}{2}\right) \cos(\theta_i(k)) = \sqrt{|p_i(k)|^2 + \sigma \cos \left(\frac{\psi_i(k)}{2}\right) \cos(\theta_i(k)) (\sigma - 2|p_i(k)|)}$$

(2)

Since, $\sigma \cos (\psi_i(k)/2) \cos(\theta_i(k)) \geq 0$, we have that if $|p_i(k)| \geq \sigma/2$, then

$$|p_i(k + 1)| \leq |p_i(k)|$$

Otherwise (if $|p_i(k)| < \sigma/2$),

$$|p_i(k + 1)| \leq \sqrt{|p_i(k)|^2 + \sigma \cos \left(\frac{\psi_i(k)}{2}\right) \cos(\theta_i(k)) (\sigma - 2|p_i(k)|) \leq \sqrt{|p_i(k)|^2 + \sigma (\sigma - 2|p_i(k)|) = |p_i(k)| - \sigma \leq \sigma}$$

Hence, $|p_i(k + 1)|$ is bounded as claimed in Lemma 2.

**Lemma 3.** For a strictly positive $\delta < \sigma/2$, in a multi-agent system with dynamics (1), if $R(P(k))$ is greater than or equal to $\sigma$, then there are at least two agents located within the range $\delta$ from the circumference of the agent constellation’s minimal enclosing circle that will jump to a distance closer to $C(P(k))$ by lengths bounded away from zero by a constant.

**Proof.** By Proposition 5 in Appendix 1, we have that for a $\delta < R(P(k))$ there are at least two agents $s_{1,2}$ located within the range $\delta$ from the circumference of the minimal enclosing circle, and at different convex-hull corners with inner angles $\psi_{s_{1,2}}(k)$ bounded away bellow $\pi$ by a constant as follows:

$$\psi_{s_{1,2}}(k) \leq \varphi(R(0), \delta) = \pi - \frac{2\arctan \left(\frac{\delta}{\sqrt{R(0)^2 - \delta^2}}\right)}{m}$$

If $|p_{s_{1,2}}(k)| \geq \sigma$, then by (2) we have that $|p_{s_{1,2}}(k + 1)|$ is bounded bellow $|p_{s_{1,2}}(k)|$ by a constant as follows:

$$|p_{s_{1,2}}(k + 1)| = \sqrt{|p_{s_{1,2}}(k)|^2 + \sigma \cos \left(\frac{\psi_{s_{1,2}}(k)}{2}\right) \cos(\theta_{s_{1,2}}(k)) (\sigma - 2|p_{s_{1,2}}(k)|) \leq \sqrt{|p_{s_{1,2}}(k)|^2 - \sigma^2 \cos^2 \left(\frac{\psi(R(0), \delta)}{2}\right)}$$
\[ \|p_{s1.2}(k)\| - \|p_{s1.2}(k)\| + \|p_{s1.2}(k)\| \sin \left( \frac{\varphi(R(0), \delta)}{2} \right) \leq \|p_{s1.2}(k)\| - \|p_{s1.2}(k)\| \left(1 - \sin \left( \frac{\varphi(R(0), \delta)}{2} \right) \right) \leq \|p_{s1.2}(k)\| - \sigma \left(1 - \sin \left( \frac{\varphi(R(0), \delta)}{2} \right) \right) \]

i.e.

\[ \|p_{s1.2}(k + 1)\| \leq \|p_{s1.2}(k)\| - \sigma \left(1 - \sin \left( \frac{\varphi(R(0), \delta)}{2} \right) \right) \]

proving Lemma 3.

\[ \square \]

**Theorem 1.** A multi-agent system with dynamics (1) gathers to a disk of radius \( \sigma \) within finite number of time steps.

**Proof.** By Lemma 2, no agent located at a distance greater than \( \sigma \) from \( C(k) \) can jump to a farther distance, and we have that all agents located within a range of \( \sigma \) from \( C(P(k)) \) remains within this range at the next time-step. Furthermore, by Lemma 3, we have that if \( R(P(k)) \) is greater than \( \sigma \), then there are at least two agents, located on the circumference of the smallest enclosing circle or at a distance less than \( \delta \) from it jumping to positions closer to \( C(P(k)) \) by at least a constant quantity as discussed next. If \( R(k) \) is greater than \( \sigma \), after at most \( \lceil n/2 \rceil \) consecutive time-steps all the agents of the system will fit into a smaller disk centered at \( C(P(k)) \) of a radius less than or equal to \( \max\{\sigma, R(P(k)) - \sigma \left(1 - \sin \left( \frac{\varphi(R(0), \delta)}{2} \right) \right) \} \), hence the radius of the minimal enclosing circle will decrease within every sequence of \( \lceil n/2 \rceil \) time-steps by at least \( \sigma \left(1 - \sin \left( \frac{\varphi(R(0), \delta)}{2} \right) \right) \), until it reaches \( \sigma \). As a consequence, all agents will gather to a disk of radius \( \sigma \) within a finite number of time-steps.

Recall that, by Lemma 1 the average position of the agents is invariant, therefore we may claim that the system gathers to a static disk of the radius \( 2\sigma \) centred at \( \bar{p} \).

## 4 Limited visibility

We next assume that the agents have limited visibility: an agent can “see” only agents located within its visibility range \( V \). Contrary to agents in the former section, which could “figure out” whether they are located at corners of the convex-hull or not, here the agents can not decide on this. However, despite the agents’ lack of information, they will still be able to perform the basic task of preserving the visibility to their neighbours while moving, and hence they will be able to ensure connectivity, and under a randomized motion rule even ensure a monotone evolution of the visibility graph of the system toward a complete visibility graph.
We next present a motion rule, improving the one considered by Gordon et al. in [6], which is based on regions the agents may move to, and prevent them from losing visibility to their neighbours. According to [6], an agent $i$ may move only into an allowable region $AR_i(k)$, defined below:

Let $D_r(c)$ be a disc of radius $r$ centered at point $c$, and let $c_{ij}(k)$ be a point at a distance $V/2$ from $p_i(k)$ in the relative direction of $p_j(k)$, i.e.

$$c_{ij}(k) = p_i(k) + \frac{V}{2} \frac{p_j(k) - p_i(k)}{\|p_j(k) - p_i(k)\|}$$

Then, the allowable region of an agent $i$ with $N_i(k)$ as its current set of neighbours is

$$AR_i(k) = \left( \bigcap_{j \in N_i(k)} D_{\psi}(c_{ij}(k)) \right) \cap D_{\psi}(p_i(k))$$  \hspace{1cm} (3)

see Figure 3.

**Lemma 4.** If all agents move inside their allowable regions, none of them will lose visibility to its neighbours.

**Proof.** Considering an agent $i$, we realize that if it “sees” an agent $j$ in a given direction, agent $j$ will be somewhere at a distance less than $V$ from it. If the agent is at a distance $V$, then clearly both $i$ and $j$ can move into a disc of radius $V/2$ centered at their average location $(p_i(t) + p_j(t))/2$ without losing mutual visibility. If $j$ will be at a distance less than $V$ from $i$ then they can again move into a disk of a radius $V/2$ centered at the average of their locations. Hence we have that the intersection of all possible moves for agent $i$ due to possible locations of agent $j$ within a distance $r < V$ from agent $i$, in the direction to $j$ (known to $i$) is given by

$$AR_{ij}(k) = \bigcap_{r=0}^{V} D_{\psi} \left( p_i(k) + \frac{1}{2} \frac{p_j(k) - p_i(k)}{\|p_j(k) - p_i(k)\|} \right) =$$

$$= D_{\psi}(p_i(k)) \bigcap D_{\psi} \left( p_i(k) + \frac{1}{2} \frac{p_j(k) - p_i(k)}{\|p_j(k) - p_i(k)\|} \right)$$

The allowable region for $i$ to jump to will be

$$AR_i(k) = \bigcap_{j \in N_i(k)} AR_{ij}(k)$$

hence we obtain formula (3).

Therefore, for any pair of neighbours $i$ and $j$, if both $i$ and $j$ move into their allowable region, we have that $AR_i(k)$ and $AR_j(k)$ are contained in $D_{\psi/2}(p_i(k) + p_j(k))/2$, hence the distance between them remains within $V$.

Note that if the agents of the set $N_i(k)$ surround the position of agent $i$ (i.e. $\psi_i(k) > \pi$), its allowable region shrinks to a point located at its own position, hence it may not move at all without risking losing visibility with some of its neighbours. Otherwise, due to simple geometrical considerations, its allowable region is actually defined only by its extreme
Figure 3: Allowable regions for agent $i$. (a) Single neighbour. (b) Intersection between the extreme left agent’s disc, the extreme right agent’s disc, and the disc $D_{\psi}(p_i(k))$. (c) No allowable region since the intersection yields an empty region.

right and extreme left neighbours, denoted by $i^-$ and $i^+$. Then we have that

$$AR_i(k) = AR_{i^-}(k) \cap AR_{i^+}(k)$$

(4)

From the above result it is clear that under the motion law (1), when $\sigma < V/2$, the connectivity of the system’s visibility graph is maintained.

**Corollary 1.** Given that $\sigma \leq V/2$, in a system where all agents move according to the dynamic law (1), none of the agents lose visibility with their neighbours, hence the connectivity of the visibility graph is preserved.

**Proof.** For each agent $i$ which currently sees all its neighbours in an angular section of angle $\psi_i(k) < \pi$, let $\theta_{ij}(k)$ be the angle between the vectors $\psi_i(k)$ and $p_j(k) - p_i(k)$ where $j \in N_i(k)$ (see Figure 4), and let Limit$_i(k)$
and $Limit_{ij}(k)$ be the lengths of the section segments crossing $AR_i(k)$ and $AR_j(k)$ respectively, starting at $p_i(k)$ in the direction of $\psi_i(k)$. We have

$$Limit_{ij}(k) = \min \{ V/2, V \cos(\theta_{ij}(k)) \}$$

Since, $\dot{\psi}_i(k)$ is in the direction of the bisector of angle $\psi_i(k)$, we have that for each $j \in N_i(k)$ the angle $\theta_{ij}(k) < \psi_i(k)/2 < \pi/2$. Therefore, for a $\sigma \leq V/2$ we have that the step size of agent $i$ is bounded as follows:

$$|p_i(k + 1) - p_i(k)| = \sigma \cos(\psi_i(k)/2) \leq \frac{V}{2} \cos(\psi_i(k)/2) \leq Limit_{ij}(k)$$

Hence, an agent $i$ takes a step inside $AR_{ij}(k)$ for all $j \in N_i(k)$, and therefore, clearly, takes a step inside $AR_i(k)$.

$$p_i(k + 1) = p_i(k) + \sigma \dot{\psi}_i(k) \cos(\psi_i(k)/2) \in AR_i(k)$$

Therefore, by Lemma 4 all agents of the system maintain visibility with their neighbours, as claimed in Corollary 1.

We have shown that the motion law (1) maintains the connectivity of the visibility graph. However, due to the agents’ limited-visibility, the constellation of the agents may get stuck in cyclic sequences of time-steps, without gathering. We next give an example of such a situation.

Consider the constellation of agents presented in Figure 5. In this figure the agents 1, 2, 3 and 4, 5, 6 are located on parallel lines so that $\bar{p}_1 \parallel \bar{p}_3 \parallel \bar{p}_4 \parallel \bar{p}_6$. These parallel lines are at a distance $V$ from each other, and...
only \( p_2p_5 \) is perpendicular to \( p_1p_3 \) (and to \( p_4p_6 \)), so that \( \| p_2 - p_5 \| = V \). Assume \( \| p_1 - p_3 \| = \| p_4 - p_6 \| = \sigma < V \). Both \( p_1 \) and \( p_3 \) are not visible to \( p_4, p_5 \) and \( p_6 \) since they are distanced more than \( V \) from them, and both 4 and 6 are not visible to 1, 2 and 3. Considering the dynamic rule (1), where all agents are active at each time step, we have that at time-step \( k \) the wedge angles of 2 and 5 are \( \psi_2(k) = \psi_5(k) = \pi \), therefore both 2 and 5 are locked. At time-step \( k + 1 \) both agents 1 and 3 must move a step of size \( \sigma \) towards each other, so that they switch positions, and so do 4 and 6. The same switching phenomenon occurs over and over again simultaneously, leaving 2 and 5 locked forever, preventing the system from gathering.

This example shows that the deterministic schedule of motion (1) may lead to non-gathering constellations, hence some randomization is absolutely necessary. Indeed adding randomization to the motion schedule will break such “locked” situations and “free” the agents to move. For example, in the constellation above, if, once in a while, an agent "sleeps" and doesn’t move (resulting, due to the jumps of 1 to 3 while 3 sleeps or due to the jump of 4 while 6 sleeps, in \( \psi_2(k) = \pi/2 \) or \( \psi_5(k) = \pi/2 \)), agents 2 and 5 will approach each other, and eventually more agents will become visible to each other.

Gordon and Bruckstein in [6], suggested a randomized rule of motion according to which each agent jumps a limited distance (\( \sigma \)) to a random location inside a region determined by the neighbours’ directions. This rule achieves gathering to a constellation within a disk of diameter equal to the visibility range (\( V \)), in a finite expected time. However, experimentally it was observed that the agents’ cluster was always of a smaller size, of the order of the step size \( \sigma \), and was drifting in the plane in what seemed
to be a random-walk.

We next modify the motion law discussed in [6], and prove that it gathers agents of the system to a disk with a radius equal to the agents' maximal step size $\sigma$ within a finite expected number of time steps. Furthermore, as time tends to infinity the distribution of the agents' average position converges in probability to the distribution of a 2D random-walk.

Let us define $ar_i(k)$, a new allowable region for agent $i$, which is contained in $AR_i(k)$, the allowable region given in [5, 6]

$$ar_i(k) = D_{\frac{\sigma}{2}} \left( p_i(k) + \frac{\sigma}{2} U_i^* \right) \cap D_{\frac{\sigma}{2}} \left( p_i(k) + \frac{\sigma}{2} U_i^* \right)$$

where $\sigma < V/2$ (see Figure 6). Recall that, if all agents take steps into their allowable regions ($AR_i(k)$), they all maintain visibility with their neighbours. Hence, the same results concerning connectivity preservation apply to the new allowable regions ($ar_i(k)$).

![Figure 6: Allowable region according to (5). The dashed area created by the intersection of the circles of radius $V/2$ with the circle of radius $\sigma$ is the allowable region $AR_i(k)$ under maximal allowed step size ($\sigma$) restriction. The bold-line bordered shape created by the intersection of the circles of diameter $\sigma < V/2$ is the new allowable region $ar_i(k)$.](image)

We next show that, if the agents of the system jump to uniformly distributed random points in their (new!) allowable regions, they gather
to a disk of radius $\sigma$. We have already proved gathering to a disk of radius $\sigma$ for a constellation with a complete visibility graph when unlimited visibility was assumed. Here, we first need to prove that, from an initial constellation corresponding to an arbitrary, but connected, visibility graph, the system reaches constellation with a complete visibility graph within a finite expected time, and subsequently remains that way. To show this, we follow [6][7], for a timing model of the system that is semi-synchronised. Hence, the agents’ assumed motion law is that, at any time-step $k$, each agent $i$ has a strictly positive probability $\delta < 1$ to be active, and each active agent jumps to a uniformly distributed random point inside, its current allowable region, $ar_i(k)$.

$$p_i(k + 1) = \begin{cases} p_i(k) & \text{if } \psi_i(k) \geq \pi \text{ or } \chi_i(k) = 0 \\ \text{a random point in } ar_i(k) & \text{if } \psi_i(k) < \pi \text{ and } \chi_i(k) = 1 \end{cases}$$

$$\chi_i(k) = \begin{cases} 1 & \text{w.p. } \delta \\ 0 & \text{w.p. } 1 - \delta \end{cases}$$

(6)

The trajectories of the agents applying this motion law may be seen in Figure 7.

Figure 7: Gathering according to the motion rule given by (6). Initial locations are the empty squares, and the trajectories to the current locations (full squares) are drawn. The agents gather to a disk of a radius equal to $\sigma$ (as drawn), within a finite number of time steps, and then remain confined to a disk of this size.
As mentioned above, we will prove that our system gathers to a disk of radius $\sigma$, within a finite expected number of time-steps. First, we shall show that any constellation having a connected visibility graph reaches a constellation with a complete visibility graph within a finite expected number of time-steps. Then, we show that, in a constellation having a complete visibility graph, if the radius of the minimal enclosing circle of the system is greater than $\sigma$, the radius cannot increase and significantly decreases within every batch of finite expected number of time steps until it reaches $\sigma$.

The idea of the first part of the proof is that at each time-step there is a strictly positive probability for a specific agent, located at the sharpest corner of the convex-hull, to be the only active agent, and to reduce its distance from $\bar{p}(k)$ (the current average position of the agents) by a strictly positive quantity $s^*$. As a consequence the sum of all agents’ squared distances of from $\bar{p}(k)$ is reduced by at least $s^*/n$ with a strictly positive probability. Hence, as long as the agents’ interconnection graph is not complete, there is a bounded away from zero probability that it becomes complete within a finite number of time-steps, and therefore it becomes complete after a finite expected number of time steps. As a consequence of Lemma 4, once the agents visibility graph is complete it remains complete, i.e. all the agents are henceforth confined to a disc of diameter $V$.

Let $CH(P(k))$ and $\partial CH(P(k))$ be the convex-hull of the agents’ locations and the set of agents defining it (located at its corners). Let $\varphi_i(k)$ be the internal angle of the convex-hull’s corner associated with an agent $i \in \partial CH(P(k))$, and let $D(P(k))$ be the diameter of the convex-hull, i.e.

$$D(P(k)) \triangleq \max_{i,j} ||p_i(k) - p_j(k)||$$

**Lemma 5.** For any agent $i \in \partial CH(P(k))$, the distance between $p_i(k)$ and $\bar{p}(k)$ is bounded as follows:

$$||p_i(k) - \bar{p}(k)|| \geq \frac{D(P(k))}{2n} \cos(\varphi_i(k)/2)$$

**Proof.** Any agent $i \in \partial CH(P(k))$, either defines the convex-hull diameter together with another agent $j$ so that

$$D(P(k)) = ||p_j(k) - p_i(k)||$$

or there are two other agents $j_1$ and $j_2$ defining its diameter, so that

$$D(P(k)) = ||p_{j_1}(k) - p_{j_2}(k)||$$

By the general triangle inequality, we have that

$$\max\{||p_i(k) - p_{j_1}(k)||, ||p_i(k) - p_{j_2}(k)||\} \geq \frac{D(P(k))}{2}$$

(7)

and we also have that:

$$||\bar{p}(k) - p_i(k)|| = \left\| \frac{1}{n} \sum_j p_j(k) - p_i(k) \right\| = \frac{1}{n} \sum_j (p_j(k) - p_i(k))$$
Figure 8: Since each angle of the convex-hull is smaller than $\pi$, any angle defined by an internal agent, a convex-hull corner and its associated bisector is smaller than $\pi/2$.

Let $\theta_{ij}(k)$ be the angle between the vectors $p_j(k) - p_i(k)$ and $U_i$, a unit vector in the direction of the bisector of $\varphi_i(k)$.

Since $\cos(\theta_{ij}(k)) > 0$ (see Figure 8), we have that

$$\|\bar{p}(k) - p_i(k)\| \geq \frac{1}{n} \sum_j |p_j(k) - p_i(k)| \cos \theta_{ij}(k)$$

Using (7), and that $\theta_{ij}(k) \leq \varphi_i(k)/2$ we have:

$$\|\bar{p}(k) - p_i(k)\| \geq \frac{1}{n} \sum_j |p_j(k) - p_i(k)| \cos \theta_{ij}(k) \geq \frac{D(P(k))}{2n} \cos (\varphi_i(k)/2)$$

as claimed.

Using Lemma 5, let us show that if the diameter of the convex-hull is bounded away from zero, it has at least one corner farther from $\bar{p}(k)$ by a bounded away from zero value.

**Corollary 2.** If $D(P(k))$ is bounded away from zero, the distance between $\bar{p}(k)$ and the position of $s$, the agent located at the sharpest corner of the system’s convex-hull, is bounded away from zero as well.

**Proof.** By Proposition 6, we have $\varphi_s(k) \leq \varphi_* = \pi(1 - 2/n)$, and by Lemma 5, we have that

$$|p_s(k) - \bar{p}(k)| \geq \frac{D(P(k))}{2n} \cos(\varphi_s(k)/2)$$

Hence,

$$\|p_s(k) - \bar{p}(k)\| \geq \frac{D(P(k))}{2n} \cos(\varphi_*/2)$$
i.e. the distance between $\bar{p}(k)$ and $p_{s}(k)$ is bounded away from zero as claimed.

\[\square\]

**Lemma 6.** There exist strictly positive constants $\rho^*$ and $s^*$, so that for any constellation $P(k)$, while $D(P(k))$ is bounded away from zero, if agent $s$ is active, the probability that at the next time-step it will be closer to $\bar{p}(k)$ by a distance greater than or equal to $s^*$ is at least $\rho^*$.

**Proof.** Let $\psi_s(k)$ be the angle of the minimal sector anchored at agent $s$'s position and containing all its neighbours, so that $\psi_s(k) \leq \varphi_s(k) \leq \varphi^*$. Then, we have that the area of the allowable region of agent $s$ is

$$|ar_s(k)| = \left(\frac{\sigma}{2}\right)^2 (\pi - \psi_s(k) - \sin\left(\frac{\psi_s(k)}{2}\right))$$

Angle $\psi_s(k)$ is upper bounded by $\varphi^*$, hence $|ar_s(k)|$ is bounded away from zero by a constant.

Let $D_{[\bar{p}(k) - p_{s}(k)] - s^*}(\bar{p}(k))$ be a disk centered at $\bar{p}(k)$ with the radius of $|\bar{p}(k) - p_{s}(k)| - s^*$, where $s^*$ is a small but bounded away from zero value, so that if agent $s$ jumps inside that disk, it is guaranteed to be closer to $\bar{p}(k)$ (comparing to where it was before the jump) by at least $s^*$.

The current agents’ average position $\bar{p}(k)$ is located inside $CH(P(k))$, hence for any agent $i \neq s$ the angle $\angle p_{i}(k)p_{s}(k)\bar{p}(k)$ is smaller than or equal to $\varphi_s(k) \leq \varphi^*$. Furthermore, by Corollary 2, if the diameter of the system is bounded away from zero, the distance between $p_{s}(k)$ and $\bar{p}(k)$ is bounded away from zero, therefore for a small enough but significant $s^*$, the area of the intersection of $ar_s(k)$ and $D_{[\bar{p}(k) - p_{s}(k)] - s^*}(\bar{p}(k))$ is bounded away from zero as well (see Figure 9). Denote this intersection region by $F(k)$

$$F(k) \triangleq ar_s(k) \cap D_{[\bar{p}(k) - p_{s}(k)] - s^*}(\bar{p}(k))$$

Then, we have that the probability that agent $s$ (if active) moves inside this region is strictly positive. We denote this probability by $\rho^*$, as follows:

$$\rho^* = \frac{\|F(k)\|}{|ar_s(k)|}$$

where $\|F(k)\|$ and $|ar_s(k)|$ are the areas of regions $F(k)$ and $ar_s(k)$, respectively, and argue that this value is strictly positive and bounded away from zero by $\rho^*$, a strictly positive constant, for all $k$, while $D(k) > V$ (see Appendix 2). Hence, we have that whenever agent $s$ is active and moves into area $F(k)$, it moves closer to $\bar{p}(k)$ by at least $s^*$. Therefore, the probability that agent $s$ will be closer to $\bar{p}(k)$, is bounded away from zero by $\rho_s$ as claimed.

\[\square\]

Let $L(P(k))$ be the sum of squared distances of all agents from their current average position, i.e.

$$L(P(k)) = \sum_{i=1}^{\gamma} |p_{i}(k) - \bar{p}(k)|^2$$

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Figure 9: Given that agent $s$ is active, the probability that the current distance of agent $s$ from $\bar{p}$ will decrease in the next time-step is the proportion between the grey area $F$ and the current allowable region $ar_s$ of agent $s$.

Lemma 7. There is a probability of at least $\delta(1 - \delta)^{n-1}$ for a bounded away from zero $\mathcal{L}(P(k))$ to decrease by at least $\|s^*\|^2/n$ at each time-step.

Proof. The probability that at some time-step $k$ only the agent $s$ becomes active is $\delta(1 - \delta)^{n-1}$. The probability that agent $s$ takes a step $\Delta_p_s(k)$ of a size $\tilde{s} \geq s^*$ inside the region $F(k)$ is at least $\rho^*$, as shown in Lemma 6. In this case the value of $\mathcal{L}(P(k))$ decreases as follows:

$$L(P(k+1)) - L(P(k)) = \tilde{s}(\tilde{s} - 2\|p_s(k) - \bar{p}(k)\| \cos(\theta_s(k)) - \frac{s^2}{n}) \tag{8}$$

where $\theta_s(k) = \angle \bar{p}(k)p_s(k)p_s(k+1)$.

To prove this we proceed as follows:

$$\mathcal{L}(P(k+1)) = \sum_{i=1}^{n} \|p_i(k+1) - \bar{p}(k+1)\|^2 =$$

$$\sum_{i=1}^{n} \|p_i(k) - (\bar{p}(k) + \frac{\Delta p_s(k)}{n})\|^2 + \|p_s(k) + \Delta p_s(k) - (\bar{p}(k) + \frac{\Delta p_s(k)}{n})\|^2 =$$

$$\mathcal{L}(P(k)) + 2\frac{1}{n} \Delta p_s^2(k) \left( (n-1)(p_s(k) - \bar{p}(k)) - \sum_{i=1}^{n}(p_i(k) - \bar{p}(k)) \right) +$$
we have a disk of radius $\sigma$.

We analyse the dynamics of the system considering the $M$ steps. At the end of each series of $M$ steps, the probability that $\mathcal{L}(P(k)) \leq (n-1)V$ gets this maximal value when the agents are evenly distributed along a straight line, with a distance $V$ between neighbours. Therefore $\mathcal{L}(P(k)) < n((n-1)V)^2$.

In addition, if $\mathcal{L}(P(k)) \leq (V/2)^2$ the agents’ interconnection graph is necessarily complete, since the maximal distance of an agent from $\bar{p}(k)$ is $V/2$, and hence all inter-agent distances are necessarily less than $V$.

Therefore, the transition from any constellation comprising a connected visibility graph to a complete visibility graph constellation may be achieved by a finite number of possible steps $M$, where

$$M < \frac{n((n-1)V)^2 - V^2/4}{\|s^*\|^2/n} + 1$$

Let us examine the evolution of the agents’ constellation every $M$ steps. At the end of each series of $M$ steps, the probability that $\mathcal{L}(P(k + M))$ will be less than $V^2/4$ is at least $(\delta(1 - \delta)^{n-1}\rho^*)^M$. Therefore, by Proposition 7 the expected number of time-steps for gathering to a complete visibility graph is at most:

$$M \leq \frac{1}{(\delta(1 - \delta)^{n-1}\rho^*)^M}$$

By Lemma 4, once a complete visibility graph constellation is reached the system remains in such a constellation. Therefore, gathering to a disk of diameter $V$ is achieved within a finite expected number of time-steps.

From this point on, we shall prove that the agents further gather to a disk of radius $\sigma$, after having reached a constellation with complete visibility graph. We analyse the dynamics of the system considering the
minimal enclosing circle of the agents locations, its radius and center being denoted by \( R(k) \) and \( C(k) \). We show that any agent located at a distance greater than \( \sigma/2 \) from \( C(k) \) can not jump to a greater distance from it, and if an agent is located at a distance smaller than or equal to \( \sigma/2 \) from \( C(k) \), it can not jump to a distance greater than \( \sigma \) from \( C(k) \). Therefore, if \( R(k) > \sigma \), it cannot increase. Furthermore, we show that there are at least two agents located on the circumference of the enclosing circle or within infinitesimal distances from it, that will most likely jump to positions closer to \( C(k) \). Hence, if \( R(k) > \sigma \), the radius of the smallest enclosing circle drops significantly within a batch of \( [n/2] \) time-steps with a strictly positive probability. As a consequence, if \( R(k) > \sigma \), it decrease significantly within a finite expected number of time steps. Due to the fact that it cannot increase, after finite number of occurrences of the event where \( R(k) \) being significantly decrease, \( R(k) \) will reach \( \sigma \), and cannot exceed it.

Without loss of generality, let \( C(k) \) be at the origin of the \( \mathbb{R}^2 \)-plane, and let \( D_{|p_i(k)|}(0) \) and \( D_\sigma(0) \) be disks centered at \( C(k) = 0 \) of radii \( ||p_i(k)|| \) and \( \sigma \).

**Lemma 8.** If all the agents of the system are within visibility range of each other, the allowable region of an agent \( i \) located at a distance greater than \( \sigma/2 \) from \( C(k) \) is contained in \( D_{|p_i(k)|}(0) \), and the allowable region of an agent located at a distance smaller than or equal to \( \sigma/2 \) from \( C(k) \) is contained in \( D_\sigma(0) \).

**Proof.** Let us focus on the allowable regions of the agents located at the convex-hull’s corners (since only they have non-zero allowable regions), and let us divide the current minimal enclosing disk into two half disks by a line defined by the points \( p_i(k) \) and \( C(P(k)) = 0 \). Since all agents ”see” each other, by Proposition 4, each one of the unit vectors \( U_i^-(k) \) and \( U_i^+(k) \) are necessarily pointing from \( p_i(k) \) toward agents located at different half-disks (in some rare situations, where all agents lie on a line-segment with \( i \) located at its edge, both unit vectors are pointing towards the far end of the segment). Therefore, the intersection of the two disks of diameter \( \sigma \) defining \( i \)'s allowable region, is necessarily contained in a disk of radius \( \sigma/2 \) centered at a distance \( \sigma/2 \) from \( p_i(k) \) in the direction to the center of the enclosing disk (See figure 10), i.e.

\[
\text{ar}_i(k) \subset D_{\frac{\sigma}{2}}(p_i(k) + \frac{C(k) - p_i(k)}{||C(k) - p_i(k)||})
\]

Hence clearly, if \( ||p_i(k)|| > \sigma/2 \), agent \( i \)'s allowable region is contained in the disk \( D_{|p_i(k)|}(0) \), and if \( ||p_i(k)|| \leq \sigma/2 \), it’s allowable region is contained in \( D_\sigma(0) \), proving Lemma 8.

**Lemma 9.** If all agents are within visibility range of each other, and \( R(k) \) is greater than \( \sigma \), then there exist strictly positive constants \( \rho^{**}, s^{**} \) and \( \alpha < \sigma/2 \), for any constellation \( P(k) \), so that at least two agents \( s_{**} \) located within range of \( \delta \) from the circumference of the smallest enclosing circle, have probabilities of at least \( \rho^{**} \) to jump to positions closer to \( C(k) \) by distances of at least \( s^{**} \).

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Figure 10: The allowable region of agent $i$ (bordered by the thick line), is contained in a disk of radius $\sigma/2$ centered at the distance $\sigma/2$ from $p_i(k)$ in the relative direction to $C(k)$ (the doted area), due to the fact that the two intersecting disks (bordered by thin line circles), which determine $ar_i$, are centered at opposite sides of the line defined by points $p_i(k)$ and $C(k)$ (dashed line).

**Proof.** By Proposition 5, there exist two agents located on/or within a range of $\alpha$ from the circumference of the minimal enclosing circle and at corners of the convex-hull with inner-angles bounded away below $\pi$ by a constant ($\varphi(\alpha, R(0))$). We refer to any one of these two agents as agent $s$. Due to the fact that the angle $\psi_s(k)$ is upper bounded by $\varphi(\alpha, R(0))$, the allowable region $ar_s(k)$ has an area bounded away from zero by a constant. We deal with $R(k) \geq \sigma$, and since $\alpha < \sigma/2$, we have that $|p_s(k)| > \sigma/2$. Then, by Lemma 8, the allowable region $ar_s(k)$ is contained in the disk $D_{|p_s(k)|}(C(k))$. Therefore, for a small enough but strictly positive constant $s^{**}$, we have that each one of the agents denoted by $s$ has strictly positive probability to jump inside the disk $D_{R(k)-s^{**}}(0)$ (see Figure 11).

Furthermore, each one of these agents has a strictly positive probability to be active, $\delta$, hence a strictly positive probability to jump inside the disk $D_{R(k)-s^{**}}(0)$, and as a consequence to significantly reduce its distance.
from $C(k)$. We denote this probability by $\rho_s(k)$, i.e.

$$\rho_s(k) = \frac{|a_{s^*} \cap D_{R(k)-s^*}(C(k))|}{|a_{s}|}$$

where operator $|\cdot|$ returns the area of the region $\cdot$, and argue that $\rho_s(k)$ is strictly positive and bounded away from zero by a constant $\rho^{**}$, in each time step $k$ while $R(k) > \sigma$, where

$$\rho^{**} = \frac{\frac{\pi}{2}(\frac{s^{**}}{2})^2}{\frac{\pi}{2}(\frac{s^{**}}{2})^2(\pi - \varphi(\alpha, R_0) + \sin(\varphi(\alpha, R_0)))}$$

and

$$s^{**} = \frac{\sigma}{2} \left( 1 - \sin \left( \frac{\varphi(\alpha, R_0)}{2} \right) \right)$$

For the geometric derivation of these values, see Appendix 2. □

![Figure 11](image_url)

Figure 11: The allowable region of agent $s$ has a significant area inside disk $D_{R(k)-s^*}(C(k))$, marked by the dashed area. Disks $D_{R(k)}(C(k))$ and $D_{R(k)-s^*}(C(k))$ are defined by the full-line circle and dashed circle respectively, and $a_{r_i}(k)$ is the region bounded by the thick line.

**Theorem 3.** If all the agents are within visibility range of each other, they will gather to a disk of radius $\sigma$ within a finite expected number of time-steps.

**Proof.** By Lemma 6 the interior agents of disk $D_\sigma(0)$ can not jump out of it, and the exterior agents can not jump to distances farther from $C(k)$. 21
Furthermore, by Lemma 11, there is a probability of at least \((\delta \rho^{**})^2\) that at least two agents, located on the circumference of the minimal enclosing circle or within \(\alpha\) distance from it, will jump to positions closer to \(C(k)\) by at least \(s^{**}\). By Proposition 7 of the Appendix, the expected number of time-steps for that event to occur is \((\delta \rho^{**})^{-2}\). Therefore, if the radius of the minimal enclosing circle is greater than \(\sigma\), the expected number of time-steps for the radius of the minimal enclosing circle to decrease by at least \(s^{**}\) is at most \([n/2][(\delta \rho^{**})^{-2}]\), and consequently the expected number of time-steps for the radius of the minimal enclosing circle to decrease to \(\sigma\) is at most

\[
\frac{R(k) - \sigma}{s^{**}} [n/2][(\delta \rho^{**})^{-2}] \leq \frac{V/2 - \sigma}{s^{**}} [n/2][(\delta \rho^{**})^{-2}]
\]

where \(s^{**}\) and \(\rho^{**}\) are given in (9). Once all agents are gathered to a disk of radius \(\sigma\), by Lemma 8, they will remain confined to such a disk, hence the system gathers as claimed in Theorem 3.

\[\square\]

5 Random dynamics analysis

We see that the agents of the system gather to a disk of radius \(\sigma\) within a finite expected number of time steps. Extensive simulations reveal that the "minimal confining disk" defined by the agents’ constellation moves randomly in the plane.

We next prove that the centroid of the agents’ constellation performs a random motion, and its location converges in probability to the distribution of a random-walk as \(k\) tends to infinity. To do so, we rely on a theorem from Reference [8] on the convergence in probability of random variables that are sums of uniformly bounded random increments. Given a random variable \(X_k\) so that

\[X_k = \sum_{i=1}^{k} Y_i,\text{ i.e. } Y_k = X_k - X_{k-1}\]

If \(Y_k\) satisfies

\[\mathbb{E}[Y_k | \mathcal{F}_{k-1}] = 0\]

where \(\mathcal{F}_{k-1}\) is the sigma-field induced by the prior realization of the process and

\[\sum_{i=1}^{k} \text{Var}(Y_i) \xrightarrow{a.s.} \infty\]

(i.e. the variances sum tends to infinity with probability 1) then

\[
\frac{X_{\tilde{k}}}{\sqrt{\nu}} \xrightarrow{p} N(0, 1)
\]

where \(\tilde{k}\) is the stopping time as \(\nu\) goes to infinity.

\[
\tilde{k} = \min\{k : \sum_{\nu=1}^{k} \mathbb{E}[\|Y_{\nu}\|^2] > \nu\}
\]
Let us analyse the long term behaviour of the random variable vectors $\vec{p}(k) = \frac{1}{n} \sum_{i=1}^{n} p_i(k)$. Denote the displacement of agent $i$ at time step $k$ by $\Delta p_i(k) = p_i(k+1) - p_i(k)$. Then, $\vec{p}(k)$ obeys

$$\vec{p}(k + 1) = \frac{1}{n} \sum_{i=1}^{n} p_i(k + 1) = \frac{1}{n} \sum_{i=1}^{n} (p_i(k) + \Delta p_i(k)) = \vec{p}(k) + \frac{1}{n} \sum_{i=1}^{n} \Delta p_i(k)$$

Therefore, we have to consider the sum of the displacements of the agents at each time-step.

Recall that we have intentionally designed the motion law (6) so as to have, in case the agents’ constellation has a complete visibility graph, that $\mathbb{E}\{\vec{p}(k + 1)|\vec{p}(k)\}$ is equal to $\vec{p}(k)$.

Let $\vec{a}r_i(k)$ be the mean position of the current allowable region of agent $i$. If agent $i$ is not located at a corner of the system’s convex-hull, then $\psi_i(k) \geq \pi$, and therefore cannot jump, i.e. $p_i(k+1) = p_i(k)$. Otherwise, $\psi_i(k)$ is equal to the inner-angle of the convex-hull corner occupied by agent $i$. The inner-angle of the convex-hull corner is zero. Let $\bar{p}_i(k)$ be the projection of the current allowable region of agent $i$ on a unit vector with an arbitrary direction $\hat{\psi}_i(k)$.

Then by (5), $\bar{a}r_i(k)$ is located at the center of $\bar{a}r_i(k)$, given by:

$$\bar{a}r_i(k) = \int_{v \in \partial CH(k)} v dv = p_i(k) + \frac{\sigma}{2} \cos\left(\frac{\psi_i(k)}{2}\right) \hat{\psi}_i(k)$$

Since, an agent $i$ located at a corner of the convex-hull stays put with probability $1 - \delta$ and jumps with probability $\delta$ to a uniformly distributed random point in $ar_i(k)$, its expected position at the next time-step is

$$\mathbb{E}(p_i(k+1)) = p_i(k)(1 - \delta) + \left(p_i(k) + \frac{\sigma}{2} \cos\left(\frac{\psi_i(k)}{2}\right) \hat{\psi}_i(k)\right) \delta$$

Hence, the agents’ expected average position at the next time step is

$$\mathbb{E}(\vec{p}(k+1)) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(p_i(k+1)) = \frac{1}{n} \sum_{i \in \partial CH(k)} p_i(k) + \frac{1}{n} \sum_{i \in \partial CH(k)} \left(p_i(k) + \delta \frac{\sigma}{2} \cos\left(\frac{\psi_i(k)}{2}\right) \hat{\psi}_i(k)\right) = \frac{1}{n} \sum_{i \in \partial CH(k)} p_i(k) + \delta \frac{\sigma}{2n} \sum_{i \in \partial CH(k)} \cos\left(\frac{\psi_i(k)}{2}\right) \hat{\psi}_i(k)$$

We have seen already in the proof of Lemma 1 that the sum of cosines of half the inner-angles of a convex-hull is zero.

$$\sum_{i \in \partial CH(k)} \cos\left(\frac{\psi_i(k)}{2}\right) \hat{\psi}_i(k) = 0$$

hence we have that

$$\mathbb{E}\{\vec{p}(k+1)|\vec{p}(k)\} = \vec{p}(k)$$

Let $\Delta \vec{p}(k)$, be the constellation’s centroid displacement at time step $k$, i.e $\Delta \vec{p}(k) = \vec{p}(k+1) - \vec{p}(k)$. Let $S_k$ and $X_k$ be the projections of the distribution of $\vec{p}(k)$ and $\Delta \vec{p}(k)$ on a unit vector with an arbitrary direction $U$. Then, by (12), we have that $\mathbb{E}\{X(k+1)|X(k)\} = 0$. 

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Clearly, the increments $X(k)$ are uniformly bounded by $n\sigma$, and we next prove that the sum of their variances tends to infinity with probability 1. Let $Var(A)$ be the variance of a random variable $A$. Then, due to the fact that the random variables $\Delta p_i(k)$ and $\Delta p_i(k)$ are conditionally independent for $i \neq j$, we have that

$$Var(X_k|P(k)) = Var(U^T\Delta p(k)|P(k)) =$$

$$Var(\frac{1}{n} \sum_i \Delta p_i(k) \geq \frac{1}{n^2} Var(U^T\Delta p_i(k)|P(k))$$

where $P(k) = \{p_1(k), p_2(k), \ldots, p_1(k)\}$, and $s$ is the agent located at the sharpest corner of the constellation’s convex-hull.

The minimal value $Var(U^T\Delta p_i(k))$ can assume is for a unit vector $U$ orthogonal to $\psi_s(k)$, i.e $U^T\psi_s(k) = 0$. Then, we have that:

$$Var(U^T\Delta p_s(k)) = 2\delta^2 \frac{\int_0^{4\pi} (x - \frac{\sigma}{2} \sin(\frac{\psi_s(k)}{2}))^2 \sqrt{\left(\frac{\sigma}{2}\right)^2 - x^2} dx}{\left(\frac{\sigma}{2}\right)^2 \left(\frac{\pi - \psi_s(k)}{2} + \sin(\frac{\psi_s(k)}{2})\right)} =$$

$$-2\delta^2 \left(\frac{\sigma}{2}\right)^2 \cos^4\left(\frac{\psi_s(k)}{2}\right) =$$

$$\delta^2 \left(\frac{\sigma}{2}\right)^2 \frac{1 - \cos^4\left(\frac{\pi - \psi_s(k)}{2}\right)}{\frac{\pi - \psi_s(k)}{2} - \frac{1}{2} \sin(\pi - \psi_s(k))}$$

By Proposition 6, $\psi_s(k)$ is upper bounded by $\varphi_s = \pi(1 - 2/n)$, hence we have that

$$Var(U^T\Delta p_s(k)) \geq \delta^2 \left(\frac{\sigma}{2}\right)^2 \frac{1 - \cos^4\left(\frac{\pi - \psi_s(k)}{2}\right)}{\frac{\pi - \psi_s(k)}{2} - \frac{1}{2} \sin(\pi - \psi_s(k))} = Var^*$$

Hence, $Var(X_k)$ is bounded away from zero by the strictly positive constant $Var^*$, and as a consequence

$$\sum_{k=1}^{\infty} Var\{X_k\} \to \infty$$

Hence, from Theorem 35.11 in [8], we have that

$$\frac{S\tilde{k}}{\sqrt{\nu}} \overset{p}{\to} N(0,1)$$

where $\tilde{k}$ is defined as the stopping time, as $\nu$ goes to infinity

$$\tilde{k} = \min\{t: \sum_{k=1}^{t} Var(X_k) > \nu\}$$

Recall that all the variables $X_k$ are uniformly bounded by $n\sigma$. Hence, their variances are bounded by $(n\sigma)^2$, and we have that the mean value
of the (increments) $X_k$'s variances converges to a finite constant value $\eta^2$, i.e
\[
\frac{1}{k} \sum_k \text{Var}(X_k) \rightarrow \eta^2
\]
Then, we have that
\[
\frac{S_k}{\sqrt{k\eta}} \overset{p}{\rightarrow} N(0,1)
\]
i.e the distribution of $S_k$ converges in probability to the distribution of a random-walk with steps of the size $\eta$, so that the projection of the random vector $\tilde{\rho}(k)$ on an arbitrary (constant) direction $U$ converges to a normal distribution with variance $k\eta^2$.

Extensive simulations were performed, and the results were exploited to estimate $\eta$. We present part of the simulation results in Figure 12, and the experimental evaluation of $\eta$ in Figure 13. Interestingly, the average “random walk” step size, $\eta$, is inversely dependent on the numbers of agents
\[
\eta \propto \frac{1}{n}
\]

Figure 12: Gaussian fit for the average position of the systems' constellations at time-step 1000. The results of 10000 simulations with random initial constellations of $\{3, 10, 30\}$ mutually visible agents. Note that we set the centroids of the initial constellations at the origin.

6 Discussion

We proved that a system of identical, anonymous and oblivious agents having limited visibility, bearing only sensing, and starting with an initial constellation with a connected visibility graph gathers to a disk with
Figure 13: Estimation of $\eta$ from simulation results. Analysis of the results of 10000 simulations with random initial constellations (centroids set at the origin) of $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 25, 30\}$ agents having complete visibility graphs. We preformed Gaussian fit for the average position at time-step 1000, for all results having the same number of agents.

radius equal to the agents’ maximal allowed step size within a finite expected number of time steps. The agents motion law is a modification of one considered in [6], ensuring the stationarity of the expected centroid of the constellation, once the visibility graph is complete. Furthermore, we proved that the distribution of the random centroid location converges to that of a “random-walk”.

We may also generalize the motion law to one preforming convergence in distribution to a random-walk dynamics, as follows: Each agent $i$ jumps to a random point with a general, non-uniform 2D distribution function bounded inside $AR_i(k)$, the original allowable region presented in [5][6], with average location centroid at the point $p_i(k) + \alpha \psi_i(k) \cos(\psi_i(k)/2)$, where $\alpha$ is a strictly positive constant. Recall that

$$AR_i(k) = \left(\bigcap_{j \in N_i(k)} D_{\psi_i} (c_{ij}(k))\right) \cap D_{\psi_i} (p_i(k))$$

and $c_{ij}(k) = p_i(k) + V/2(p_j(k) - p_i(k))/|p_j(k) - p_i(k)|$. In this case we can prove results similar to the ones described above.

An interesting result of the above analysis is the influence on the swarm mobility, or the variance of the "random-walk" distribution of the number agents in the swarm. The swarm mobility decreases as $1/n$ where $n$ is the size of the swarm, showing that larger swarms have greater "inertia" in their collective motion.

The proposed gathering algorithm shows that a large group of identical agents with limited visibility and bearing-only sensing of neighbours can perform gathering to a small region defined by the size of the steps the agents can make. Subsequently in absence of exogenous controls the
agents wander in the plane their centroid performing a random-walk like motion with localization becoming more precise as the size of the swarm increases.

The results presented herein exhibit the emergence of an interesting global cohesive behaviour of a swarm, based on very simple interactions between agents with limited capabilities and sets the stage for the design of externally controlled cohesive swarms composed of such agents that will be able to jointly perform a variety of patrolling, area surveillance, mapping, intruder detection tasks.
Appendix 1 - Geometry and Probability

Results

Proposition 4. Let $O(P)$ be the minimal enclosing circle of $P$ a set of points in $\mathbb{R}^2$. Then, any partition of $O(P)$ into two by a line passing through its center results in half circles with at least one point of $P$ on each.

Proof. The minimal enclosing circle is defined by only one of the two following options:

(1) By two points located on the circumference of $O(P)$ which lie on two sides of its diameter. The arcs between these two points are both of the size $\pi$.

(2) By three points located on the circumference which define an acute triangle. The arcs between these points are all less than $\pi$. (the proof of this property is given in [9]).

Therefore, any partition of $O(P)$ into two arcs with angular-sizes equal to $\pi$, result in at least one of these points on each one of the arcs. \(\square\)

Proposition 5. Given a constellation with a finite number of points $n > 1$ in the $\mathbb{R}^2$-plane, and a strictly positive diameter $D$. There are at least two points within a distance $\alpha > 0$ from the circumference of the constellation’s smallest enclosing circle, at corners of the constellation’s convex-hull with inner angles bounded away below $\pi$ by a strictly positive value (dependent on $\alpha$).

Proof. There are at least two corners of the constellation’s convex-hull located on the circumference of the minimal enclosing circle. Let us denote the points located at these corners by $p$. Let $c$ be the center of the enclosing circle, let $l$ be a line passing trough the point $p + \alpha(c - p)/|c - p|$ and perpendicular to the vector $c - p$. Let $F$ be the region created by the intersection of the smallest enclosing circle and the half plane bounded by $l$ which includes point $p$, and denote the set of the constellation’s points located in $F$ by $P_\alpha$ (see Figure 14).

Assuming the number of points in $P_\alpha$ is $m \leq n - 1$, if we cut the constellation’s convex-hull with $l$ into two convex polygons, the sum of the inner-angles of the polygon which includes point $p$ is upper bounded by $\pi(m + 2 - 2)$, since this polygon is comprised by the points of the set $P_\alpha$ and the two points created by the intersection with $l$, denoted by $\pm$. The inner-angles of the above mentioned polygon corners at the points $\pm$ are lower bounded by $\beta = \text{atan} \left( \frac{\alpha}{\sqrt{2\alpha R - \alpha^2}} \right)$ (see Figure 14). Hence, the average value of the inner-angles associated with the points of the set $P_\alpha$ is upper bounded by

$$\varphi(R, \alpha) = \frac{\pi(m + 2 - 2) - 2\beta}{m} = \pi - \frac{2\beta}{m}$$

and therefore the inner-angle of the sharpest corner of the convex-hull of the set $P_\alpha$ is upper bounded as well by

$$\varphi(R, \alpha) = \pi - \frac{2\text{atan} \left( \frac{\alpha}{\sqrt{2\alpha R - \alpha^2}} \right)}{m}$$
Figure 14: Upper bound of the two angles of corners of the constellation’s convex-hull which are located within a strictly positive distance \( \alpha < R \) from the circumference of the constellation’s minimal enclosing circle. Point \( p \) is located on the circumference of the minimal enclosing circle centred at point \( c \), and the angles \( \varphi \) are lower bounded by \( \beta \), as may be seen in the figure.

Since at least two corners of the constellation’s convex-hull are located on the circumference of the minimal enclosing circle, we have that, there are at least two corners of the convex-hull located within a strictly positive distance of \( \alpha < R \) from the circumference of the minimal enclosing circle with angles bounded away from \( \pi \) by a value equal to \( \varphi(R, \alpha) \).

Proposition 6. The sharpest corner of the convex-hull of any \( n \) points in \( \mathbb{R}^2 \) is upper bounded by \( \varphi_s = \pi(1 - 2/n) \).

Proof. For any convex polygon with \( m \leq n \) corners, the sum of the corners’ inner angles is \( \pi(m-2) \), and the average inner-angle is \( \pi(1 - \frac{2}{m}) \). Therefore, \( \varphi_s \), the interior angle of the polygon’s sharpest corner is necessarily smaller than or equal to \( \pi(1 - \frac{2}{m}) \). Since, we deal with the convex-hull of \( n \) points, we have that

\[
\varphi_s = \pi(1 - 2/n) \geq \pi(1 - 2/m) = \varphi_s
\]

Proposition 7. If at each time-step an event occurs with probability \( p < 1 \), the expected number of time-steps for the first event to occur is \( p^{-1} \).

Proof. The probability that the first event occurs at exactly time-step \( k \) is \( (1-p)^{k-1}p \). Therefore, the expected number of time-steps for the first event to occur is

\[
\sum_{k=1}^{\infty} k(1-p)^{k-1} p = -p \frac{d}{dp} \sum_{k=1}^{\infty} (1-p)^k = -p \frac{d}{dp} \left( \sum_{k=0}^{\infty} (1-p)^k - 1 \right) = -p \frac{d}{dp} \frac{1}{p} = \frac{1}{p}
\]
Appendix 2 - Bounds on the various constants

In the following section we derive the lower bounds of the probabilities \( \rho^*, \rho^{**} \) that the agents marked by \( s \) in Lemmas 6, 9 will jump to locations closer to \( \bar{p}(k), C(k) \) by values \( s^*, s^{**} \) bounded away from zero.

In order to simplify our calculations, we define the following quantities (see Figures 15 and 16):

- \( |ar_s(k)| = (\sigma/2)^2(\pi - \psi_s(k) - \sin(\psi_s(k))/2) < \pi(\sigma/2)^2 \) the area of agent \( s \)'s allowable region
- \( p^s(k) = p_s(k) + \hat{\psi}_s(k) \sigma \cos(\psi_s(k)/2). \) The allowable region \( ar_s(k) \) is bounded by two arcs meeting at two points. The first point is \( p_s(k) \), and the second is \( p^s(k) \)
- \( v(\psi_s(k)) = \frac{1}{2}\sigma(1 - \sin(\psi_s(k)/2)) \) half the distance between the middle points of the two arcs mentioned above
- \( h(\psi_s(k)) = \frac{1}{2}\sigma \cos(\psi_s(k)/2) \) half the length of the line segment \([p_s(k), p^s(k)]\)
- \( R = \|p_s(k) - \bar{p}(k)\| \) the distance between agent \( s \) and the current constellation's centroid
- \( R' = \|p_s(k) - p^s(k) - \bar{p}(k)\| \) the distance between the centroid of \( ar_s(k) \) and the constellation's centroid
- \( \theta \leq \psi_s(k)/2 - \) the angle between \( \hat{\psi}_s(k) \) and the vector pointing from \( p_s(k) \) to \( \bar{p}(k) \). Due to \( \bar{p}(k) \) being located inside the convex-hull, \( \theta \) is bounded by \( \psi_s(k)/2 \leq \varphi_\ast/2 \) (\( \varphi_\ast = \pi(1 - 2/n) \) by Proposition 6)
- \( R^i = R \sin(\theta) \) the length of the line segment starting at point \( \bar{p}(k) \) in the direction perpendicular to \( \hat{\psi}_s(k) \) until it meets the line going through the segment \([p_s(k), p^s(k)]\) (see Figure 16).
- \( \bar{R} = \|p_s(k) - C(k)\| \) the distance between agent \( s \) and \( C(k) \), the center of constellation's current minimal enclosing circle
- \( \bar{R}' = \|p_s(k) - p^s(k)/2 - C(k)\| \) the distance between the centroid of \( ar_s(k) \) and \( C(k) \)
- \( \hat{\theta} \leq \psi_s(k)/2 - \) the angle between \( \hat{\psi}_s(k) \) and the vector pointing from \( p_s(k) \) to \( C(k) \). By Propositions 4 and 5, \( \psi_s(k)/2 \) in upper bounded by \( \varphi(V/2, \alpha)/2 \), where \( V/2 \) is the maximal radius that the minimal enclosing circle my get in view of Lemma 9, and \( \varphi(V/2, \alpha) = \pi - 2\arctan\left(\frac{\alpha}{\sqrt{V^2 - \alpha^2}}\right) \)

The bounds \( s^* \) and \( \rho^* \) in Lemma 6

Our calculations address two cases. In the first case point \( p^s(k) \) is inside the disk \( D_R(\bar{p}(k)) \), and in the second case this point is not in the disk \( D_R(\bar{p}(k)) \).

If \( p^s(k) \) is inside the disk \( D_R(\bar{p}(k)) \), we have (see Figure 15 (a))

\[
R' \leq \sqrt{R^2 - h^2(\psi_s(k))} \leq \sqrt{R^2 - h^2(\varphi_\ast)}
\]
Then, we have that at least half of a disk of diameter $v(\psi_s(k))$ is inside $ar_s(k)$ and also inside the disk of radius $R'$ centered at $\bar{p}(k)$ (see Figure 15 (b)). The area of the half disk is

$$\frac{1}{2} \pi \left( \frac{v(\psi_s(k))}{2} \right)^2 \geq \frac{1}{2} \pi \left( \frac{v^*(\bar{p}(k))}{2} \right)^2$$

The area $\|ar_s(k)\|$ is upper bounded by $\pi (\sigma/2)^2$. Therefore, the probability of agent $s$ to jump inside $ar_s(k)$ and inside the disk of radius $R'$ centered at $\bar{p}(k)$ is higher than (i.e. bounded below!)

$$\rho^* = \frac{1}{2} \pi \left( \frac{v^*(\bar{p}(k))}{2} \right)^2 = \frac{1}{2} \left( 1 - \sin \left( \frac{\varphi^*}{2} \right) \right)^2$$

Jumping into the half disk mentioned, results in agents $s$ getting closer to $\bar{p}(k)$ by at least

$$R - R' \geq \sqrt{R^2 - h^2(\varphi^*)} = R \left( 1 - \sqrt{1 - \frac{\sigma \cos(\varphi^*/2)}{2R}} \right) \geq$$

$$\frac{\sigma}{2} \left( 1 - \sqrt{\frac{\sigma \cos(\varphi^*/2)}{V(n - 1)}} \right) \triangleq s^*$$

Note that in this case $R \geq \sigma/2$ due to $p^h_s(k)$ located inside the disk $D_R(\bar{p}(k))$, and $R \leq (n - 1)V/2$ since $(n - 1)V$ is the maximal allowed diameter of a connected constellation.

If point $p^h_s(k)$ is outside the disk $D_R(\bar{p}(k))$, let us define the $h^i$ as the meeting point of the line through the point $\bar{p}(k)$ and orthogonal to $\psi_s(k)$, with the segment $[p_s(k), p^h_s(k)]$. Then we have that $|h^i - p_s(k)| = R \cos(\theta) \geq R \cos(\psi_s(k)/2)$, and that $R^i = |h^i - \bar{p}(k)| = R \sin(\theta) \leq R \sin(\psi_s(k)/2)$ (see Figure 16).

The part of the line segment $[h^i, \bar{p}(k)]$ located inside $ar_s(k)$ is at least the length of

$$\begin{align*}
\frac{R \cos(\theta)}{h(\psi_s(k))} v(\psi_s(k)) &\geq \frac{R \cos(\psi_s(k)/2)}{h(\psi_s(k))} v(\psi_s(k)) = R \left( 1 - \sin \left( \frac{\psi_s(k)}{2} \right) \right) \\
R \left( 1 - \sin \left( \frac{\varphi^*}{2} \right) \right) &\geq \frac{D(P(k))}{2n} \cos \left( \frac{\varphi^*}{2} \right) \left( 1 - \sin \left( \frac{\varphi^*}{2} \right) \right) \\
\frac{V}{2n} \cos \left( \frac{\varphi^*}{2} \right) \left( 1 - \sin \left( \frac{\varphi^*}{2} \right) \right) &\geq 2r
\end{align*}$$

Note that $R \geq \frac{D(P(k))}{2n} \cos(\varphi^*/2)$ by corollary 2, where $D(P(k)) = \max_{i,j} ||p_i(k) - p_j(k)||$, and $D(P(k)) \geq V$ in view of Lemma 6.

Hence, we have that there is at least half disk of radius $r$ inside $ar_s(k)$ and inside the disk of radius $R^i$ centered at $\bar{p}(k)$ (see Figure 16). Therefore, we have that the probability of agent $s$ to jump inside $ar_s(k)$ and inside the disk of radius $R'$ centered at $\bar{p}(k)$ is bounded by

$$\rho^* = \frac{1}{2} \pi r^2$$
Figure 15: The bounds $s^*$ and $ρ^*$ in case point $p_s^h(k)$ is inside the disk $D_R(\bar{p}(k))$. (a) $R' \leq \sqrt{R^2 - h^2(ψ_s(k))} \leq \sqrt{R^2 - h^2(ϕ^{*})}$ (b) the half disk (dashed area) is contained inside $ar_s(k)$ and the disk of radius $R'$ centered at $\bar{p}(k)^*$

and then $s$ will get closer to $\bar{p}(k)$ by at least

$$R - R' = R(1 - \sin(\theta)) \geq R \left( 1 - \sin \left( \frac{ϕ^*}{2} \right) \right) \geq \frac{V}{2n} \cos \left( \frac{ϕ^*}{2} \right) \left( 1 - \sin \left( \frac{ϕ^*}{2} \right) \right) \geq s^*$$

The bounds $s^{**}$ and $ρ^{**}$ in Lemma 9
We use the result of Lemma 8, which states that if an agent $s$ is located with a distance greater then or equal to $σ/2$ from $C(k)$, the center of the minimal enclosing circle of the current constellation, its allowable region, $ar_s(k)$, is contained inside the disk $D_{|p_s(k) - C(k)|}(C(k))$ (Note that $\|p_s(k) - C(k)\| = \tilde{R}$).

Since $ar_s(k)$, is contained inside the disk $D_{\tilde{R}}(C(k))$, we have that $\tilde{R}'$, the distance between points $C(k)$ and the centroid of $ar_s(k)$, is bounded as follows:

$$\tilde{R}' \leq \sqrt{R^2 - h^2(ψ_s(k))} \leq \sqrt{R^2 - h^2(ϕ(\alpha_V, \alpha))}$$

(The same as in Figure 15 (a))

Then, we have that at least half of a disk of diameter $v(ψ_s(k))$ is inside $ar_s(k)$ and also inside the disk of radius $\tilde{R}'$ centered at $C(k)$ (see Figure 17). The half disk area is

$$\frac{1}{2} \pi \left( \frac{v(ψ_s(k))}{2} \right)^2 \geq \frac{1}{2} \pi \left( \frac{v(ϕ(V/2, α))}{2} \right)^2$$
The bounds $s^*$ and $\rho^*$: if $p^h_s(k)$ is located outside the disk $D_R(\bar{p}(k))$, then we have that at least the half disk, marked by the dashed area, is inside the allowable region $ar_s(k)$ and inside the disk $D_{\tilde{R}}(\bar{p}(k))$.

The area $\|ar_s(k)\|$ is upper bounded by $\pi(\sigma/2)^2$. Therefore, the probability of agent $s$ to jump inside $ar_s(k)$ and inside the disk of radius $\tilde{R}'$ centered at $C(k)$ is higher than

$$\rho^{**} = \frac{\frac{1}{2} \pi \left( \frac{\varphi(V/2, \alpha)}{2} \right)^2}{\pi \left( \frac{\sigma}{2} \right)^2} = \frac{1}{2} \left( 1 - \sin \left( \frac{\varphi(V/2, \alpha)}{2} \right) \right)^2$$

Jumping into the half disk mentioned, results in agent $s$ getting closer to $C(k)$ by at least

$$\tilde{R} - \tilde{R}' \geq \tilde{R} - \sqrt{\tilde{R}^2 - h^2(\varphi(V/2, \alpha))} = \tilde{R} \left( 1 - \sqrt{1 - \frac{\sigma \cos(\varphi(V/2, \alpha)/2)}{2\tilde{R}}} \right) \geq \sigma \left( 1 - \sqrt{1 - \frac{\sigma \cos(\varphi(V/2, \alpha)/2)}{V}} \right) \geq s^{**}$$

Recall that $\tilde{R} \leq V/2$ in view of Lemma 9.

**References**


Figure 17: The bounds $s^{**}$ and $\rho^{**}$: Half disk contained in $ar_s(k)$ and in $D_{\tilde{R}}(C(k))$.


