On Optimal Disc Covers and a New Characterization of the Steiner Center

Yael Yankelevsky and Alfred M. Bruckstein

Technion - Israel Institute of Technology
Haifa 32000, Israel

September 23, 2013

Abstract
Given $N$ points in the plane $P_1, P_2, ..., P_N$ and a location $\Omega$, the union of discs with diameters $[\Omega P_i]$, $i = 1, 2, ..., N$ covers the convex hull of the points.

The location $\Omega_s$ minimizing the area covered by the union of discs, is shown to be the Steiner center of the convex hull of the points. Similar results for $d$-dimensional Euclidean space are discussed.

1 Introduction

In this paper we discuss a sphere coverage problem and, in this context, we propose an optimal coverage criterion defining a center for a given set of points in space.

Suppose that a constellation of $N$ points $\{P_1, P_2, ..., P_N\}$ in $\mathbb{R}^d$ (the $d$-dimensional Euclidean space) is given. An arbitrary point $\Omega \in \mathbb{R}^d$ is selected and the spheres $S_{P_i}(\Omega)$, having $[\Omega P_i]$ as diameters, are defined. Hence the centers of $S_{P_i}(\Omega)$ are at $\frac{1}{2}(\Omega + P_i)$ and their radii are $\frac{1}{2}\|\Omega - P_i\|$, $\forall i = 1, 2, ..., N$.

Consider the union of these spheres $S_{P_i}(\Omega)$, their surface "anchored" at $\Omega$. First we prove that the resulting $d$-dimensional shape always covers the convex hull of the given points $P_1, P_2, ..., P_N$, hence its volume exceeds the volume of $CH \{P_1, P_2, ..., P_N\}$ for all $\Omega \in \mathbb{R}^d$. This leads to the following natural question: what is the location $\Omega^*$ which minimizes the excess (or overflow) volume and hence the total volume of the shape, $\Sigma(\Omega) = \bigcup_{i=1}^N S_{P_i}(\Omega)$?

Such a location, we claim, would be a natural candidate as a "center" for the constellation of points $\{P_1, P_2, ..., P_N\}$.

The problem of determining the point that gives the tightest cover with spheres, minimizing the excess volume beyond the convex hull, is solved here for the planar case (i.e. $d = 2$). The result is the following: the optimal location $\Omega^*$, is the so called Steiner center of the convex hull of the given
points \( \{P_1, P_2, ..., P_N\} \in \mathbb{R}^2 \). Hence, the Steiner center \( \Omega_s \) of a convex polygon \([V_1V_2...V_k]\) is also characterized as the point that yields the tightest disc cover with discs having \([\Omega_sV_j]\) as diameters \((j = 1, 2, ..., k)\).

For the \(d\)-dimensional case we conjecture that a similar result holds, however we have no proof (yet) for this.

Finding meaningful centers for a collection of data points is a fundamental geometric problem in various data analysis and operation research/facility location applications. The Steiner center, along with the center of gravity, the centroid of the convex hull and the Weber-Fermat median, were all subject to intense research (see e.g. [5], [2], [3], [1], [7], [12]). All these points are characterized by various optimization criteria, such as (weighted) sums of distances (or functions of distances) to the given points. Minimax criteria, such as locations minimizing the maximal distance to a set of given points, with various metrics yielding so called \(\delta\)-centers, among them the center of the smallest covering sphere, as optimal centers for point constellations were also subject to extensive investigations over the years.

However, we have never encountered a "center" location optimization criterion expressed as the area of a union of shapes defined in terms of the variable point \(\Omega\) and the points of the given data set. We note that the problem of covering the convex hull of a set of points with unions of spheres, \(\Sigma(\Omega) = \bigcup_{i=1}^{N} S_{Pi}(\Omega)\), arose in the analysis of monitoring threshold functions over distributed data streams, in the work of Sharfman, Schuster and Keren [10]. They provide a proof of the coverage result based on a variant of Carathéodory’s theorem, using induction on the dimensionality \(d\). Our proof is simpler and direct, and does not rely on any results beyond the definition of convexity.

The rest of the paper is organized as follows: Section 2 proves the theorem on coverage of the convex hull in \(\mathbb{R}^d\), then Section 3 analyzes the problem for the plane \((d = 2)\) and presents an even simpler argument proving convex hull coverage and shows that the optimal \(\Omega\) is the Steiner point of the convex hull of a planar constellation of points. In Section 4 we follow with a discussion on the Steiner point and its other nice properties and provide some numerical evidence that the Steiner center is also optimally tight in covering the convex hull of the data points in 3-dimensions. Finally, Section 5 offers some concluding remarks.

2 \(d\)-dimensional sphere covers

Given a set of points in \(\mathbb{R}^d\), denoted by \(\{P_1, P_2, ..., P_N\}\), for any \(\Omega \in \mathbb{R}^d\) define the spheres \(S_{Pi}(\Omega)\) with center at the midpoint of the segment \([\OmegaP_i]\) and radius \(\frac{1}{2}\|\OmegaP_i\|\). We prove the following:

**Theorem 1.**

\[
\text{CH}\{P_1, P_2, ..., P_N\} \subset \bigcup_{i=1}^{N} S_{Pi}(\Omega) \quad (1)
\]
where \( CH \{P_1, P_2, ..., P_N\} \) denotes the convex hull of the points \( P_1, P_2, ..., P_N \).

Without loss of generality, we choose the coordinate system such that \( \Omega \) is the origin, i.e. \( \Omega = (0,0,...0) \in \mathbb{R}^d \). Denote a general point in the convex hull of \( \{P_1, P_2, ..., P_N\} \) by \( Q = \sum_{i=1}^{N} \lambda_i P_i \) (with \( \lambda_i \geq 0, \sum_{i=1}^{N} \lambda_i = 1 \)).

To prove the inclusion of the convex hull in the union of the spheres \( S_{P_i}(\Omega) \) we must show that:

\[
\exists i \in \{1, 2, ..., N\} \text{ s.t. } d(Q, \frac{1}{2} P_i) \leq d(\Omega, \frac{1}{2} P_i) \tag{2}
\]

hence \( Q \) is inside at least one of the spheres, being closer to the sphere center than its radius. This clearly implies that:

\[
Q \in CH \{P_1, P_2, ..., P_N\} \Rightarrow Q \in \bigcup_{i=1}^{N} S_{P_i}(\Omega) \tag{3}
\]

Proof of Theorem 1. Assume that

\[
d(Q, \frac{1}{2} P_i) > d(\Omega, \frac{1}{2} P_i) \quad \forall i \in \{1, 2, ..., N\} \tag{4}
\]

Hence we have

\[
d^2(Q, \frac{1}{2} P_i) > d^2(\Omega, \frac{1}{2} P_i)
\]

\[
(Q - \frac{1}{2} P_i)^T (Q - \frac{1}{2} P_i) > \frac{1}{2} P_i^T \frac{1}{2} P_i
\]

\[
Q^T Q - Q^T P_i > 0 \quad \forall i \in \{1, 2, ..., N\} \tag{5}
\]

This means that the projections of all the vectors from \( Q \) to \( P_i \) (= \( \vec{P}_i - \vec{Q} \)), on the vector from \( \Omega \) to \( Q \) (= \( \vec{Q} \)) are strictly negative (see Figure 1). But this is impossible since \( Q \in CH\{P_1, P_2, ..., P_N\} \) and this implies that \( CH\{P_1, P_2, ..., P_N\} \) cannot project on the line \( \Omega Q \) on ”one side” of \( Q \).

![Figure 1: Strictly negative projections](technion.png)
The contradiction to the assumption in (4) proves that we must have for some $i$:

$$d(Q, \frac{1}{2}P_i) \leq d(\Omega, \frac{1}{2}P_i)$$

(6)

Hence

$$Q \in CH \{P_1, P_2, ..., P_N\} \Rightarrow Q \in \bigcup_{i=1}^{N} S_{P_i}(\Omega)$$

(7)

As stated in the introduction, Theorem 1 is due to D. Keren and his coworkers (see [10],[6]). Their proof is rather complex, being based on Carathéodory’s theorem on convex sets and induction on the dimensionality $d$.

Since $CH \{P_1, P_2, ..., P_N\} \subset \bigcup_{i=1}^{N} S_{P_i}(\Omega)$, we have that

$$Volume \left( \bigcup_{i=1}^{N} S_{P_i}(\Omega) \right) = Volume \left( CH \{P_1, ..., P_N\} \right)$$

$$+ Volume \left( \bigcup_{i=1}^{N} S_{P_i}(\Omega) \setminus CH \{P_1, P_2, ..., P_N\} \right)$$

It therefore makes sense to ask what is the location $\Omega^*$ that minimizes the volume of the union of spheres $S_{P_i}(\Omega)$, hence also the excess volume beyond the convex hull of the data points. In the next section we solve this problem for the important planar case. Surprisingly, the optimal location turns out to be a well-known center for planar convex shapes.

3 A discovery on disc covers

In this section, we analyze the planar disc covering problem, first providing an even simpler proof of the convex hull coverage result and then determining the location of $\Omega$ that results in the tightest cover.

Given $N$ points in the plane, $\mathbb{R}^2$, we shall prove that:

1. The discs $S_{P_i}(\Omega)$ cover their convex hull,

2. The optimal location for $\Omega^*$, yielding the tightest cover, is the Steiner center of the convex hull polygon.

The second result provides a novel characterization for the Steiner point of a convex polygon in the plane, namely:

Given $\{V_1, V_2, ..., V_k\}$ the vertices of a convex polygon in $\mathbb{R}^2$, the Steiner point $\Omega_s$ is the solution of

$$\Omega_s = \arg\min_{\Omega} \left\{ Area \left( \bigcup_{i=1}^{k} S_{V_i}(\Omega) \right) \right\}$$

(8)
A traditional characterization of the Steiner point is

$$\Omega_s = \arg \min_\Omega \sum_{i=1}^k \theta_i d^2(V_i, \Omega)$$

(9)

yielding explicitly

$$\Omega_s = \frac{1}{2\pi} \sum_{i=1}^k \theta_i V_i$$

(10)

where \(\theta_i\) are the external turn angles at the vertices \(V_i\) of the convex polygon, summing to \(2\pi\) (see Figure 2).

Figure 2: External turn angles

3.1 \(CH\{P_1, P_2, \ldots, P_N\}\) is covered by the union of discs \(\bigcup_i S_{P_i(\Omega)}\)

For the special case of \(\mathbb{R}^2\), we offer an additional proof of Theorem 1.

In 2D, each pair of discs \(i, j \in \{1, \ldots, N\}\) may have one of the following mutual positions:

1. The boundary circles are tangent to each other at the point \(\Omega\).

   It is readily seen from Figure 3 that in this case, the segment \([P_iP_j]\) is either entirely included in a single disc, or the common tangent line through \(\Omega\) is perpendicular to both diameters and so \([\Omega P_i], [\Omega P_j]\) are collinear, such that \([P_iP_j]\) consists of the 2 diameters and hence belongs to the union of the 2 discs.

2. The circles intersect at two points: \(\Omega\) and \(Q\) (\(Q \neq \Omega\))

   Since every inscribed angle that subtends a diameter is a right angle, we have \(\angle \Omega Q P_i = \angle \Omega Q P_j = \frac{\pi}{2}\). Hence either \(Q \in [P_iP_j]\) or \(Q\) is outside the segment \([P_iP_j]\) but on the same line. We clearly see that in both cases the segment \([P_iP_j]\), and in fact the triangle \(\triangle \Omega P_iP_j\), is covered by the union of the 2 discs (see Figure 4).
So far it was shown that for every pair of discs \( i, j \), the line segment \([P_i P_j]\), and in fact the triangle \( \triangle \Omega P_i P_j \), is covered by the union of the 2 discs.

The convex hull of a finite set of points in \( \mathbb{R}^2 \) is a convex polygon whose vertices are a subset of the point set \( \{P_1, P_2, ..., P_N\} \). Therefore the CH polygon edges are a subset of all possible segments \( \{[P_i P_j] \quad \forall i, j \} \). As each such segment, and hence each polygon edge, belongs to the union of 2 discs, it obviously belongs to the union of all discs.

Since all the discs intersect at \( \Omega \), the union of discs is a star-shaped region, i.e.

\[
\forall Q_0 \subset \bigcup_{i=1}^{N} S_{P_i}(\Omega), \quad [\Omega Q_0] \subset \bigcup_{i=1}^{N} S_{P_i}(\Omega)
\]

Due to this fact, together with the convexity of the CH polygon, the CH is completely covered by the union of triangles \( \bigcup_{i,j=1}^{N} \Delta P_i P_j \). Finally, since each such triangle is covered by the union of discs, it follows that

\[
\forall \Omega : \quad CH\{P_1, ..., P_N\} \subset \bigcup_{i=1}^{N} S_{P_i}(\Omega)
\]
3.2 The optimal location for $\Omega$

Next, let us determine the optimal location of $\Omega$ in the sense of minimizing the area difference between the union of discs $S_{P_i}(\Omega)$, $i = 1, 2, ..., N$ and the convex hull $CH \{P_1, P_2, ..., P_N\}$. Clearly this requires us to simply minimize the area of $\bigcup_{i=1}^{N} S_{P_i}(\Omega)$.

Denote by $\Delta S(\Omega)$ the ”overflow” region covered beyond $CH \{P_1, P_2, ..., P_N\}$, i.e.

$$\Delta S(\Omega) = \bigcup_{i=1}^{N} S_{P_i}(\Omega) \setminus CH \{P_1, P_2, ..., P_N\} \quad (13)$$

We consider possible locations for $\Omega^*$ only in the convex hull of the data points. Clearly, if $\Omega^*$ would be outside $CH \{P_1, P_2, ..., P_N\}$, we could add it to the data points and get a set $\{P_1, P_2, ..., P_N, \Omega^*\}$ for which the optimal location is a-priori known to be $\Omega^*$, a point located on the boundary of the convex hull of the points $\{P_1, P_2, ..., P_N, \Omega^*\}$. However, we shall next show that for a convex polygonal region, the optimal point is a (positively) weighted sum of the external points, hence it cannot be any one of them. From this it follows that $\Omega^*$ will have to be inside the convex hull of $\{P_1, P_2, ..., P_N\}$. Therefore we shall prove the following:

**Theorem 2.** The area of $\Delta S(\Omega)$ is minimized when $\Omega$ is located at the Steiner center of the convex hull of the data points in $\mathbb{R}^2$.

**Proof of Theorem 2**

We consider $\Omega \subset CH \{P_1, P_2, ..., P_N\}$, which after reordering and renumbering the extremal points from $\{P_1, P_2, ..., P_N\}$ is a convex polygon defined by $\{\bar{P}_1, \bar{P}_2, ..., \bar{P}_M\}$: ($\bar{P}_1 \rightarrow \bar{P}_2 \rightarrow ... \rightarrow \bar{P}_M \rightarrow \bar{P}_1$).

It is readily seen that the $N - M$ points in the interior of the convex hull polygon define discs that are covered by the M discs determined by the external points.

Indeed, if $P_k$ is a point in $\{P_1, P_2, ..., P_N\} \setminus \{\bar{P}_1, \bar{P}_2, ..., \bar{P}_M\}$ we have that $S_{P_k}(\Omega) \subset S_{\bar{P}_k}(\Omega)$ where $\bar{P}_k$ is the point where the ray $[\Omega P_k)$ exits the convex hull (see Figure 5).

The point $\bar{P}_k$ is on a boundary segment $[\bar{P}_l \bar{P}_{l+1}]$ of the convex hull and $S_{\bar{P}_l}(\Omega) \cup S_{\bar{P}_{l+1}}(\Omega)$ clearly covers $S_{\bar{P}_k}(\Omega)$, since all three circles intersect at $\Omega$ and at its projection on the line $(\bar{P}_l \bar{P}_{l+1})$, denoted by $Q_l$ (see Figure 6).

Therefore let us define the shape $S \triangleq \bigcup_{i=1}^{M} S_{\bar{P}_i}(\Omega)$ and compute its area explicitly as a function of the location of $\Omega$.

Consider the convex polygon $\bar{P}_1, \bar{P}_2, ..., \bar{P}_M \rightarrow \bar{P}_1$ and the point $\Omega$ inside it (see Figure 7).
Figure 5: Discs defined by internal points are covered by discs defined by external points.

Figure 6: The discs $S_{\bar{P}_l}(\Omega), S_{\bar{P}_{l+1}}(\Omega)$ and $S_{\tilde{P}_k}(\Omega)$ intersect at $\Omega$ and $Q_l$.

Figure 7: The convex polygon $\bar{P}_1, \bar{P}_2, ..., \bar{P}_M \rightarrow \bar{P}_1$ with an internal point $\Omega$.

The diameters $[\Omega \bar{P}_1]$ are segments that form a "star configuration" about $\Omega$, their length being $d_i \triangleq d[\Omega \bar{P}_i]$. Let us denote by $Q_i$ the projections of $\Omega$ on
the lines \( \overline{P_i P_{i+1}} \pmod{M} \). Also define the angles
\[
\begin{align*}
\angle \overline{P_i \Omega Q_i} &= \alpha_i \\
\angle Q_i \Omega \overline{P_{i+1}} &= \beta_{i+1}
\end{align*}
\]
i = 1, 2, ..., M
as illustrated in Figure 8.

Figure 8: Definition of the angles \( \alpha_i, \beta_{i+1} \) for the different possible locations of \( Q_i \)

We recall the following simple facts about areas of segments in a circle (see Figure 9):

Figure 9: Basic properties of triangles and circular segments
1. The area of a circular segment is given by

\[ S_{\text{segment}}(QP) = \frac{1}{4} d^2 \left( \alpha - \frac{1}{2} \sin(2\alpha) \right) \]  

(14)

2. The area of the triangle \( \Omega QP \) is

\[ S(\triangle \Omega QP) = \frac{1}{4} d^2 \sin(2\alpha) \]  

(15)

Plugging (15) into (14), we can express

\[ S_{\text{segment}}(QP) = \frac{1}{4} d^2 \alpha - \frac{1}{2} S(\triangle \Omega QP) \]  

(16)

With these preliminary definitions and basic facts in mind, we can calculate the area of the union of discs \( \bigcup_{i=1}^{M} S_{\bar{P}_i}(\Omega) \) and the area of the convex hull \( CH \{ \bar{P}_1, \bar{P}_2, ..., \bar{P}_M \} \) in terms of the distances \( d_i \) and the angles \( \alpha_i \) and \( \beta_i \) (see Figure 10).

![Figure 10: Computing the excess area \( \Delta S \) and the area of the convex hull \( S_{CH} \) (note that \( \theta_{i+1} = \angle Q_k \Omega Q_{k+1} = \angle Q_k \Omega P_{k+1} + \angle P_{k+1} \Omega Q_{k+1} = \beta_{k+1} + \alpha_{k+1} \))](image)

Let us express the excess area \( \Delta S \) defined in (13) as a sum of circular segments. It can be seen from Figure 8 that the excess area over the CH edge \([\bar{P}_i \bar{P}_{i+1}]\) in the three possible scenarios is either the sum or difference of the circular segments lying on the chords \([Q_i \bar{P}_i], [Q_i \bar{P}_{i+1}]\).

If \( Q \in [\bar{P}_i \bar{P}_{i+1}] \) (see Figure 8a), the excess area over \([\bar{P}_i \bar{P}_{i+1}]\), denoted \( \Delta S_i \), is

\[ \Delta S_i = S_{(\text{segment})} (Q_i \bar{P}_i) + S_{(\text{segment})} (Q_i \bar{P}_{i+1}) \]

and

\[ S(\triangle \bar{P}_i \Omega \bar{P}_{i+1}) = S(\triangle \bar{P}_i \Omega Q_i) + S(\triangle Q_i \Omega \bar{P}_{i+1}) \]
Thus using (16),

\[ \Delta S_i = \left[ \frac{1}{4} d_i^2 \alpha_i - \frac{1}{2} S(\triangle \bar{P}_i \Omega Q_i) \right] + \left[ \frac{1}{4} d_{i+1}^2 \beta_{i+1} - \frac{1}{2} S(\triangle Q_i \Omega \bar{P}_{i+1}) \right] \]

\[ = \frac{1}{4} d_i^2 (\alpha_i + \beta_{i+1}) - \frac{1}{2} \left[ S(\triangle \bar{P}_i \Omega Q_i) + S(\triangle Q_i \Omega \bar{P}_{i+1}) \right] \]  

(17)

\[ = \frac{1}{4} d_i^2 (\alpha_i + \beta_{i+1}) - \frac{1}{2} S(\triangle \bar{P}_i \Omega \bar{P}_{i+1}) \]

Observing that \( \angle \bar{P}_i \Omega \bar{P}_{i+1} = \alpha_i + \beta_{i+1} \), this yields

\[ \Delta S_i = \frac{1}{4} d_i^2 (\angle \bar{P}_i \Omega \bar{P}_{i+1}) - \frac{1}{2} S(\triangle \bar{P}_i \Omega \bar{P}_{i+1}) \]  

(18)

If \( Q \not\in [\bar{P}_i \bar{P}_{i+1}] \) and is on the continuation of the line determined by \([\bar{P}_i \bar{P}_{i+1}] \) beyond \( \bar{P}_{i+1} \) (see Figure 8b).

\[ \Delta S_i = S_{(\text{segment})}(Q, \bar{P}_i) - S_{(\text{segment}})(Q, \bar{P}_{i+1}) \]

and

\[ S(\triangle \bar{P}_i \Omega \bar{P}_{i+1}) = S(\triangle \bar{P}_i \Omega Q_i) - S(\triangle Q_i \Omega \bar{P}_{i+1}) \]

with \( \angle \bar{P}_i \Omega \bar{P}_{i+1} = \alpha_i - \beta_{i+1} \), resulting in

\[ \Delta S_i = \left[ \frac{1}{4} d_i^2 \alpha_i - \frac{1}{2} S(\triangle \bar{P}_i \Omega Q_i) \right] - \left[ \frac{1}{4} d_{i+1}^2 \beta_{i+1} - \frac{1}{2} S(\triangle Q_i \Omega \bar{P}_{i+1}) \right] \]

\[ = \frac{1}{4} d_i^2 (\alpha_i - \beta_{i+1}) - \frac{1}{2} \left[ S(\triangle \bar{P}_i \Omega Q_i) - S(\triangle Q_i \Omega \bar{P}_{i+1}) \right] \]  

(19)

\[ = \frac{1}{4} d_i^2 (\angle \bar{P}_i \Omega \bar{P}_{i+1}) - \frac{1}{2} S(\triangle \bar{P}_i \Omega \bar{P}_{i+1}) \]

Finally, if \( Q_i \) is on the line determined by \([\bar{P}_{i+1} \bar{P}_i] \) beyond \( \bar{P}_i \) (see Figure 8c).

\[ \Delta S_i = -S_{(\text{segment})}(Q_i, \bar{P}_i) + S_{(\text{segment})}(Q_i, \bar{P}_{i+1}) \]

and

\[ S(\triangle \bar{P}_i \Omega \bar{P}_{i+1}) = -S(\triangle \bar{P}_i \Omega Q_i) + S(\triangle Q_i \Omega \bar{P}_{i+1}) \]

with \( \angle \bar{P}_i \Omega \bar{P}_{i+1} = -\alpha_i + \beta_{i+1} \), we similarly get

\[ \Delta S_i = \frac{1}{4} d_i^2 (-\alpha_i + \beta_{i+1}) - \frac{1}{2} \left[ -S(\triangle \bar{P}_i \Omega Q_i) + S(\triangle Q_i \Omega \bar{P}_{i+1}) \right] \]  

(20)

\[ = \frac{1}{4} d_i^2 (\angle \bar{P}_i \Omega \bar{P}_{i+1}) - \frac{1}{2} S(\triangle \bar{P}_i \Omega \bar{P}_{i+1}) \]

Therefore, summing the excess area over all the M discs, we can write:

\[ \Delta S = \sum_{i=1}^{M} \Delta S_i = \sum_{i=1}^{M} \frac{d_i^2}{4} (\angle \bar{P}_i \Omega \bar{P}_{i+1}) - \frac{1}{2} S(\triangle \bar{P}_i \Omega \bar{P}_{i+1}) \]  

(21)
Observe that the convex hull area can similarly be expressed as a sum of triangles:

\[ S_{CH} = \sum_{i=1}^{M} S(\triangle \bar{P}_i \bar{\Omega} \bar{P}_{i+1}) \]  

(22)

Given that observation, we can rewrite (21) as:

\[ \Delta S = \sum_{i=1}^{M} \frac{d_i^2}{4} (\angle \bar{P}_i \bar{\Omega} \bar{P}_{i+1}) - \frac{1}{2} S_{CH} \]  

(23)

Now we wish to find the optimal center that yields the minimal excess area, and since \( S_{CH} \) is independent of \( \bar{\Omega} \), we need to solve:

\[ \bar{\Omega}^* = \arg \min_{\bar{\Omega}} \Delta S = \arg \min_{\bar{\Omega}} \sum_{i=1}^{M} \frac{d_i^2}{4} (\angle \bar{P}_i \bar{\Omega} \bar{P}_{i+1}) \]  

(24)

By extending each edge \([ \bar{P}_{i-1} \bar{P}_i ]\) outside the polygon, an exterior angle is formed at the vertex \( \bar{P}_i \) whose size is exactly \( \theta_i = \angle \bar{P}_i \bar{\Omega} \bar{P}_{i+1} \) (see Figure 10). As \( \theta_i \) is also independent of \( \bar{\Omega} \), (24) becomes:

\[ \bar{\Omega}^* = \arg \min_{\bar{\Omega}} \sum_{i=1}^{M} \theta_i d_i^2 \]  

(25)

This is simply a weighted sum of the square distances of the vertices from \( \bar{\Omega} \), with given constant weights that measure the exterior angles of the convex polygon. Noting that the \( M \) exterior angles sum to \( 2\pi \), the optimizer of (25) is explicitly given by:

\[ \bar{\Omega}^*(x, y) = \left( \frac{\sum_{i=1}^{M} \theta_i x_i}{\sum_{i=1}^{M} \theta_i}, \frac{\sum_{i=1}^{M} \theta_i y_i}{\sum_{i=1}^{M} \theta_i} \right) \]  

(26)

We see that a relatively straightforward calculation provides the optimal location \( \bar{\Omega}^* \) as a weighted average of the points \( \bar{P}_1, \bar{P}_2, ..., \bar{P}_M \), the weights being proportional to the turn angles at \( \bar{P}_i \).

However this is exactly the Steiner center point of the convex polygon defined by the points \( P_1, P_2, ..., P_N \).

\[ \blacksquare \]

Hence Theorem 2 is proved, and also, we have justified the search for the optimal location only within the convex hull of the data points.

4 The Steiner center: some alternative characterizations

The Steiner point (also known as the Steiner curvature centroid) of a convex polygon in \( \mathbb{R}^2 \), is commonly defined as the geometric centroid of the system
obtained by placing a mass equal to the magnitude of the exterior angle at each vertex [4].

For a set of points \( P \subseteq \mathbb{R}^2 \), let \( V_P \) be the set of extreme points of \( P \), i.e. the vertices of the convex hull \( CH(P) \). For every \( p \in V_P \) let \( \alpha_p \) be the interior angle formed on the convex hull boundary at \( p \), and set the weights accordingly:

\[
w(p) = \begin{cases} 
\pi - \alpha_p, & \text{if } p \in V_P \\
0, & \text{if } p \in P - V_P 
\end{cases}
\]  

(27)
such that for \( p \in V_P \) the weight \( w(p) \) is the turn angle at \( p \) on \( CH(P) \).

The Steiner point is defined as the normalized weighted center of mass of \( P \) using the weights \( w(p) \). Since \( \sum_{p \in P} w(p) = 2\pi \), this center is:

\[
\Omega_s(P) = \frac{1}{2\pi} \sum_{p \in P} w(p)p
\]  

(28)

An interesting property of the Steiner point arising from this definition is that it is defined solely by the geometry of the convex hull, i.e. given a discrete set of points \( P \), their Steiner center takes into consideration only the subset of these points that serve as vertices of \( CH(P) \). A desirable consequence is the "stability" of this center, in the sense that a continuous displacement of any point \( p \in P \) results in a continuous change of the weights and therefore also of the Steiner center [3].

Another characterization of the Steiner center is by projections [3]: Denote the unit vector at angle \( \theta \in [0, \pi] \) by \( u_\theta = (\cos \theta, \sin \theta) \). Let \( P_\theta \) denote the projection of the set of points \( P \subseteq \mathbb{R}^2 \) on \( u_\theta \), then:

\[
P_\theta = \{u_\theta < p, u_\theta > | p \in P\}
\]  

(29)

and the Steiner center is defined as:

\[
\Omega_s(P) = \frac{2}{\pi} \int_0^\pi mid(P_\theta) d\theta
\]  

(30)

where:

\[
mid(P_\theta) = \frac{u_\theta}{2} (\min_{p \in P} <p, u_\theta> + \max_{p \in P} <p, u_\theta>)
\]  

(31)
is simply the Euclidean center of \( P_\theta \).

Additionally, the Steiner center \( \Omega_s \) of a convex shape has some very interesting properties, among which is its linearity with respect to Minkowski addition. That is, if \( K_1 \) and \( K_2 \) are two convex sets in \( \mathbb{R}^d \), we have that

\[
\Omega_s(K_1 \oplus K_2) = \Omega_s(K_1) + \Omega_s(K_2)
\]  

(32)

where \( \oplus \) stands for vector addition, i.e.

\[
K_1 \oplus K_2 = \{x + y | x \in K_1, y \in K_2\}
\]  

(33)
It is also true that the map $K \rightarrow \Omega_s(K)$ is similarity invariant, i.e.
\[ \Omega_s(tK) = t\Omega_s(K) \] (34)
where
\[ tK = \{ tx | x \in K \subset \mathbb{R}^d, t > 0 \} \] (35)

It is well known (see Shephard [11, 13], Sallee [8] and Schneider [9]) that the above two properties and continuity of a mapping characterize the Steiner point, i.e. if we have a mapping from the set of convex shapes to $\mathbb{R}^d$, i.e. $\Phi : \mathcal{P}^d \rightarrow \mathbb{R}^d$ (where $\mathcal{P}^d$ is the set of convex shapes in $\mathbb{R}^d$) that satisfies:

1. $\Phi(K_1 \oplus K_2) = \Phi(K_1) + \Phi(K_2)$
2. $\Phi(tK) = t\Phi(K)$
3. $\Phi()$ is continuous

then necessarily $\Phi(K) = \Omega_s(K)$.

We hope that this last characterization will enable an elegant proof of the result discussed in this paper for the higher dimensional case.

5 Concluding remarks

This paper presented a novel characterization of the Steiner center as the point that provides the tightest disc coverage for the convex hull of the set of points in the plane.

We first proved that the convex hull of N points in $\mathbb{R}^d$ is covered by the union of d-dimensional discs formed such that their diameters are the segments connecting some point $\Omega$ inside the convex hull with each of the N vertices (this is true for any $\Omega$ chosen inside the convex hull, regardless of its exact location).

Next it was shown for $\mathbb{R}^2$ that the theoretically optimal location of $\Omega$ in the sense of minimizing the area difference between the union of discs and the convex hull is the well known Steiner curvature center. This interesting property is a nice addition to the existing characterizations of the Steiner center.

For higher dimensions, we conjecture that the optimal point is the $\mathbb{R}^d$ Steiner center, but a theoretical proof is yet to be found. Some numerical simulations that were performed in 3D seem to confirm this conjecture.

Acknowledgements

This research was supported by the Israel Science Foundation grant no. 1551/09.
References


