On the Complexity of Constructing Minimum Reload Cost Path-Trees

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Abstract. The reload cost concept refers to the cost that occurs at a vertex along a path on an edge-colored graph when it traverses an internal vertex between two edges of different colors. The reload cost depends only on the colors of the traversed edges. Previous work on reload costs focuses on the problem of finding a spanning tree that minimizes the total reload cost from a source vertex to all other vertices in a directed graph and proves that this problem is not approximable within any polynomial time computable function of the input size. In this paper we study the complexity and approximability properties of numerous special cases of this problem. We consider both directed and undirected graphs, bounded and unbounded number of colors, bounded and unbounded degree graphs, and bounded and unbounded inter-color reload costs. This problem occurs in various applications such as transportation networks, energy distribution networks, and telecommunications.

Keywords:
Reload cost model, approximation algorithms, network design, network optimization.

1 Introduction

1.1 Background

Various network optimization problems can be modeled using edge-colored graphs. In this work we focus on a network design problem under the reload cost model, associated with the cost incurred while traversing through a vertex via two consecutive edges of different colors. The reload cost associated with the traversal of this vertex depends only on the colors of the incident traversed edges. Although reload costs have important applications in many areas such as transportation networks, energy distribution networks and telecommunications, they have received little attention in the literature.

Each carrier in an intermodal cargo transportation network can be represented by a color. The cost of transferring cargo from one carrier to another
be modeled by reload costs. In energy distribution networks, transfer of energy from one type of carrier to another has a reload cost. In telecommunications, reload costs arise in many different settings. In heterogeneous networks, routing may necessitate switching from one technology (such as a cellular network) to another (such as a wireless local area network) and this switching cost can be modeled by using the reload cost concept. Even within the same technology, switching between networks of different service providers have different costs corresponding to reload costs [2, 12]. Recently, dynamic spectrum access (DSA) networks have been studied in the wireless networking literature [1,11]. Assigned frequency bands in an ad hoc DSA network can be significantly far away from one another. Hence, unlike other wireless networks, switching from one frequency band to another in a DSA network can have a significant cost in terms of delay and power consumption. This frequency switching cost can be modeled using reload costs.

1.2 Related Work

The reload cost concept is introduced in [12] where the focus is on the problem of finding a spanning tree in an edge colored undirected graph having minimum diameter with respect to reload costs. It is proven that, in its most general case, the problem is hard to approximate within any polynomial-time computable function \( f(n) \), and is hard to approximate within any constant factor less than 3 even when the instance is restricted to graphs of maximum degree 5. A polynomial time algorithm for graphs with maximum degree at most 3 is also provided. In [7] integer programming formulation is used to minimize the sum of the reload costs of all paths between pairs of vertices in a spanning tree.

The work [2] presents numerous path, tour, and flow problems concerning reload costs. In particular, it focuses on the Minimum Reload Cost Path-Tree (MinRCPT) problem, which is to find a spanning tree that minimizes the total reload cost from a source vertex to all other vertices. It is shown that, in a directed graph, MinRCPT is inapproximable within any polynomial-time computable function. Given such an inapproximability result, a natural research direction is to investigate the hardness of the problem in specific cases. Such an approach was taken in [5,6,12], although the focus was on other problems.

In [5] the authors study the minimum diameter reload cost spanning tree problem when restricted to graphs with maximum degree 4. In particular, it is proved that if reload costs are unrestricted, the problem cannot be approximated within any constant \( c < 2 \), and it cannot be approximated within any constant \( c < 5/3 \) if reload costs satisfy the triangle inequality. [8] studies problems that find a path/trail/walk by minimizing the total reload cost. It focuses on cases where reload costs are symmetric/asymmetric and do/do not satisfy the triangle inequality. [6] focuses on the minimum reload cost cycle cover problem, which aims to span the vertices of an edge-colored graph by a set of vertex-disjoint cycles.
1.3 Our Contribution

In this work we investigate some special cases of MinRcpt depending on parameters such as directed graphs (digraphs) and undirected graphs, bounded and unbounded vertex degrees, bounded and unbounded number of colors, and bounded and unbounded reload cost values. We assume without loss of generality that the minimum non-zero reload cost value is 1, and we show the following results:

- There is a constant $\rho > 0$, such that MinRcpt is hard to approximate within $\rho \cdot \log |V|$ in digraphs even when there are only two colors and all reload costs are either 0 or 1.
- There is a constant $\rho > 0$, such that it is hard to approximate MinRcpt within $\rho \cdot \log |V|$ in undirected graphs even when all reload costs are either 0 or 1.
- MinRcpt is in APX-Hard in both directed and undirected graphs, even when there are only two colors, all reload costs are either 0 or 1, and the maximum degree of the graph is bounded by any constant $B \geq 5$.

We summarize our results in Table 1. In Section 2 we introduce notation, definitions and preliminary results. Section 3 presents our inapproximability results for directed graphs, while Section 4 focuses on our results for undirected graphs. Finally in Section 5 we discuss further research directions.

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Table 1. Summary of results

2 Preliminaries

Notations and Problem Statement: Given an undirected graph $G = (V(G), E(G))$ and a vertex $v \in V$, $\delta_G(v)$ denotes the set of edges incident to $v$ in $G$, and $d_G(v) \overset{def}{=} |\delta_G(v)|$ is the degree of $v$ in $G$. The minimum and maximum degrees of $G$ are defined as $\delta(G) \overset{def}{=} \min \{d_G(v) | v \in V(G)\}$ and $\Delta(G) \overset{def}{=} \max \{d_G(v) | v \in V(G)\}$ respectively. When the graph $G$ is clear from the context we omit it from the notations and write $V, E, \delta(v), d(v), \delta$ and $\Delta$. We use similar notation for digraphs.

We consider edge colored graphs $G$, where the edges are colored with colors from a finite set $C$ of colors and $c : E \mapsto C$ is the coloring function. We denote by $E_c$ the set of edges of $E$ colored $c$, and $G_c = (V(G), E(G)_c)$ is the subgraph of $G$ having the same vertex set as $G$, but only the edges colored $c$. 

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The reload costs are given by a real function \( rc : C \times C \mapsto \mathbb{R} \) where \( rc(c, c) = 0 \) for every color \( c \in C \), i.e., reload costs occur only between different colors. We assume without loss of generality that the minimum non-zero reload cost value is 1 and we denote the maximum reload cost value as \( RC_{\max} \). Let \( P \) be a path \((e_1 - e_2 - \ldots - e_\ell)\) where \( \ell \) is the length of \( P \), and \( rc \) a reload cost function. The reload cost incurred by \( P \) is \( rc(P) = \sum_{i=1}^{\ell} rc(c(e_{i-1}), c(e_i)) \).

Given a function \( \text{prev} : E \mapsto E \) that assigns to every edge (resp. arc) of \( G \) some previous edge (resp. arc), such that \( \text{prev}(e) \) is incident to \( e \), we define the reload cost of an edge \( e \in E \) as \( rc(e) = 0 \) for every color \( c \), \( rc(c, c(e)) \) and \( rc(c(e), c(e)) \). We extend this definition to a directed tree \( T \) rooted at \( r \) as follows: Given a directed tree \( T \) rooted at \( r \), let \( \text{prev}(e_1) = e_1 \) and also \( \text{prev}(e_i) = e_{i-1} \) for every \( 1 < i \leq \ell \). Clearly, \( rc(P) = \sum_{i=1}^{\ell} rc(e_i) \) (note that the first term is zero).

We extend this definition to a directed tree \( T \) of \( G \) as follows: Given a directed tree \( T \) rooted at \( r \), let \( \text{prev}(e_1) = e_1 \) and also \( \text{prev}(e_i) = e_{i-1} \) for every \( 1 < i \leq \ell \). Clearly, \( rc(P) = \sum_{i=1}^{\ell} rc(e_i) \) (note that the first term is zero).

In the rest of this paper, we omit parameter \( r \) in \( rc(T, r) \) when it is clear from the context and simply write \( rc(T) \).

**Approximation Algorithms.** Let \( \Pi \) be a minimization problem and \( \rho \geq 1 \). A (feasible) solution \( S \) of an instance \( I \) of \( \Pi \) is a \( \rho \)-approximation if its objective function value \( O_\Pi(I, S) \) is at most \( \rho \) times the optimal objective function value \( O_\Pi^* (I) \). An algorithm \( ALG \) is said to be a \( \rho \)-approximation algorithm for an optimization problem \( \Pi \) if \( ALG \) returns a \( \rho \)-approximation \( ALG(I) \) for every instance \( I \) of \( \Pi \). A problem \( \Pi \) is said to be \( \rho \)-approximable if there is a polynomial-time \( \rho \)-approximation algorithm for it. \( \Pi \) is said to be \( \rho \)-inapproximable if there is no polynomial-time \( \rho \)-approximation algorithm for it unless \( P = NP \). Given a real function \( f \), \( \Pi \) is said to be in \( f \)-APX-Hard if there is a constant \( c > 0 \) such that \( \Pi \) is \( (c \cdot f(|I|)) \)-inapproximable where \(|I|\) is the size of the instance \( I \), and \( \Pi \) is said to be in \( f \)-APX if there is a constant \( c > 0 \) such that \( \Pi \) is \( (c \cdot f(|I|)) \)-approximable. If \( \Pi \) is both in \( f \)-APX-Hard and \( f \)-APX then it is said to be in \( f \)-APX-Complete. When \( f \) is a constant these complexity classes are called sim-
ply APX-Hard, APX and APX-Complete, respectively. A polynomial time approximation scheme (PTAS) is an infinite family of algorithms \(\{\text{ALG}_\epsilon \mid \epsilon > 0\}\) such that \(\text{ALG}_\epsilon\) is a \((1 + \epsilon)\)-approximation algorithm with running time \(O(n^{f(\epsilon)})\) for some function \(f\). A problem \(H\) is said to be in PTAS if there is a PTAS for it.

The minimum set cover problems: An instance of the \(\text{MINIMUMWEIGHTEDSETCOVER}\) problem is a triplet \((U, S, w)\) where \(U = \{u_1, u_2, \ldots, u_n\}\) is a finite ground set of elements, \(S = \{S_1, S_2, \ldots, S_m\}\) is a collection of subsets of \(U\), and \(w : S \mapsto \mathbb{R}\) is a real weight function on the sets. Given such an instance, one has to find a subset \(\mathcal{C} \subseteq S\) that covers \(U\), i.e. \(\cup \mathcal{C} \cup S = U\). The goal is to minimize \(w(\mathcal{C}) \defeq \sum_{S_i \in \mathcal{C}} w(S_i)\). The special case in which all the weights are 1 (thus \(w(\mathcal{C}) = |\mathcal{C}|\)) is referred to as the \(\text{MINIMUMSETCOVER}\) problem and we denote its instances as \((U, S)\). The greedy algorithm of [3] is an \(H_n\)-approximation to both problems, where \(H_n = \sum_{i=1}^{n} \frac{1}{i}\) is the \(n\)-th harmonic number. It is also shown in [9] that it is hard to approximate \(\text{MINIMUMSETCOVER}\) within a factor \(o(\log n)\).

3 Lower Bounds for Directed Graphs

In this section we provide hardness results for digraph. Our results hold even for the simplest possible reload cost matrix, i.e. a uniform metric with two colors. For general digraphs we show a lower bound of \(\Omega(\log(\min(V)))\), and for digraphs with bounded degree we show that the problem is APX-Hard.

3.1 General Digraphs

Theorem 1. There is a constant \(\rho > 0\), such that \(\text{MINDIRECTEDRCPT}\) is \(\rho \cdot \log |V|\)-inapproximable even when \(|C| = 2\) and \(RC_{\text{max}} = 1\).

Proof. We provide an approximation preserving reduction from \(\text{MINIMUMSETCOVER}\) to \(\text{MINDIRECTEDRCPT}\). It is known (see [10]) that there is a constant \(\rho' > 0\) such that \(\text{MINIMUMSETCOVER}\) is \(\rho' \log |V|\)-inapproximable even if we restrict the problem to instances in which the number of sets is polynomial in the number of elements\(^4\), i.e. \(|S| \leq |U|^k\) for some fixed \(k > 0\). Given an instance \((U, S)\) of \(\text{MINIMUMSETCOVER}\) we construct an instance \((G, C, c, r, rc)\) of \(\text{MINDIRECTEDRCPT}\) as follows (see Figure 1):

\[ G = (V, E) \text{ is a digraph where } V = \{r\} \cup S \cup S' \cup U, S' = \{S'_i | S_i \in S\}. \]

\[ E = E_1 \cup E_2 \cup E_3 \cup E_4 \text{ where } E_1 = \{(r, S_i) | S_i \in S\}, E_2 = \{(r, S'_i) | S_i \in S\}, E_3 = \{(S_i, S'_i) | S_i \in S\}, E_4 = \{(S'_i, u_j) | u_j \in S_i, S_i \in S\}. \]

\[ C = \{c_1, c_2\} \text{ and the reload cost function } rc \text{ is such that } rc(c_1, c_2) = rc(c_2, c_1) = 1. \]

Recall that \(rc(c_1, c_1) = rc(c_2, c_2) = 0\), by definition. Also

\[ c(e) = \begin{cases} c_1 \text{ if } e \in E_1 \cup E_2 \\ c_2 \text{ otherwise.} \end{cases} \]

\(^4\) This is slight abuse of notation, because the result presented in [10] is valid under the assumption \(NP \neq \text{DTIME}(n^{O(\log \log n)})\), an assumption stronger than \(P \neq NP\).

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Let $C^*$ be an optimal set cover, and without loss of generality assume $C^* = \{S_1, S_2, \ldots, S_k\}$. Consider the spanning tree $T(C^*)$ of $G$ rooted at $r$ with the arcs $E(T(C^*)) = E_1 \cup E_2 \cup E_3 \cup E_4^1$ where $E_2 = \{(r, S_i') \in E_2 | i > k\}$, $E_3^1 = \{(S_i, S_i') \in E_3 | 1 \leq i \leq k\}$, and $E_4^1 \subseteq E_4$ is such that each $u_j$ has exactly one incoming arc $(S_i', u_j)$ in $E_4^1$ and also $1 \leq i \leq k$. Note that there is always such an arc because there is at least one set in $S_i \in C^*$ containing $u_j$ as $C^*$ is a set cover. The reload costs of $E_1$ and $E_2$ are zero because $\text{prev}(e) = e$ for every edge $e \in E_1 \cup E_2$. The reload cost of an edge $e \in E_3^1$ is 1 and the reload cost of an edge $e \in E_4^1$ is zero because $c(e) = c(\text{prev}(e)) = c_2$. Therefore $\text{rc}(T(C^*)) = |E_2| = k = |C^*|$. We conclude that the value $\text{rc}^*$ of an optimal solution of $(G, C, r, rc)$ is at most $|C^*|$. 

Let $T = (V, E_T)$ be a spanning tree of $G$ rooted at $r$ constituting a $\rho$-approximation to the constructed instance $(G, C, c, r, rc)$ of $\text{MinDirectedRcpt}$. Note that $E_1 \subseteq E_T$, because otherwise some vertex $S_i \in S$ is not reachable from $r$ in $T$. Also for every $i$ exactly one of $(r, S_i'), (S_i, S_i')$ is in $E_T$ because otherwise either $S_i'$ is not reachable from $r$ or there is an undirected cycle in $T$.

Let $Q \subseteq S$ be the set of vertices $S_i$ such that $(r, S_i') \notin E_T, (S_i, S_i') \notin E_T$. Let also $\overline{Q} = S \setminus Q$ be the set of vertices of $S_i$ such that $(r, S_i') \in E_T, (S_i, S_i') \notin E_T$. Let $U_Q$ be the set of vertices of $U$ that are reachable from $Q$ in $T$, and $\overline{U_Q} = U \setminus U_Q$ the set of vertices of $U$ that are not reachable from $Q$ in $T$. Then the vertices of $\overline{U_Q}$ are reachable from $\overline{Q}$ in $T$.

We now construct a solution $T' = (V, E_{T'})$ with $\text{rc}(T') \leq \text{rc}(T)$ and $\overline{U_Q} = \emptyset$. Consider a vertex $u_j \in \overline{U_Q}$, let $S_i' \in \overline{Q}$ its parent vertex in $T$. Now consider the subtree of $T$ containing the arcs $(r, S_i), (r, S_i')$ and all the arcs going from $S_i'$ to its children. The reload cost of this subtree is equal to the number of children of $S_i'$, thus at least 1. On the other hand the subtree obtained by removing the arc $(r, S_i')$ and adding the arc $(S_i, S_i')$ has reload cost exactly 1. In the new subtree all the children of $S_i'$ are in $U_Q$. We can proceed with this transformation until $\overline{U_Q} = \emptyset$ to get a solution $T'$ as claimed. Now $U_Q = U$ and thus $Q$ is a set cover. It can be easily verified that $\text{rc}(T') = |Q|$. We conclude

$$|Q| = \text{rc}(T') \leq \text{rc}(T) \leq \rho \cdot \text{rc}^* \leq \rho \cdot |C^*|.$$ 

Thus $Q$ constitutes a $\rho$-approximation to $\text{MINIMUMSETCOVER}$. Finally, we note that $|V| = |U| + 2|S| + 1 \leq 2|U|^{k+1}$ and $\log |V| \leq (k + 1)\log |U| + 1$. Let $\rho'$ be a constant for which $\text{MINIMUMSETCOVER}$ is $(\rho' \log |U|)$-inapproximable, and let $\rho < \rho'/(k + 1)$. If $\text{MinDirectedRcpt}$ is $\rho \log |V|$-approximable then we can use it to find a $(\rho \log |U| + O(1))$-approximation to $\text{MINIMUMSETCOVER}$, a contradiction. \hfill \Box

### 3.2 Digraphs with Bounded Degree

**Theorem 2.** For any integer constant $B \geq 5$, $\text{MinDirectedRcpt}$ is $\text{APX}$-Hard even when $\Delta(G) = B$, $|C| = 2$ and $RC_{\text{max}} = 1$.

**Sketch of Proof:** For every $k \geq 3$, $\text{MINIMUMSETCOVER}$ is in $\text{APX}$-Hard even for instances where the sets are of cardinality at most $k$ and each element appears
in at most 2 sets [4]. Given such an instance \((U, S)\) of \textsc{MinimumSetCover} we construct an instance \((G, C, r, rc)\) of \textsc{MinDirectedRcpt} in two steps. In the first step, we construct an instance of \textsc{MinDirectedRcpt} as we did in the proof of Theorem 1 (see Figure 1). In the second step we replace the tree induced by \(\{r\} \cup S\) with a directed binary tree rooted at \(r\), having \(S\) as its leaves and all its edges colored \(c_1\). We do the same transformation to the tree induced by \(\{r\} \cup S'\). Note that this does not change the reload costs of the original edges and the reload costs of the new edges are zero, i.e. same as the reload costs of the removed edges. By the choice of the instance \((U, S)\), every vertex in \(U\) has (in-)degree at most 2. Moreover, the degree of every vertex in \(S'\) is at most \(k + 2\), the degree of \(r\) is at most 4, and the degree of all the other vertices is at most 3. Therefore \(\Delta(G) \leq k + 2\). As the reload costs did not change, using the same proof as in Theorem 1 we can show that a \(\rho\)-approximation to \textsc{MinDirectedRcpt} implies a \(\rho\)-approximation to the instance under consideration of \textsc{MinimumSetCover}. Assume, by way of contradiction, that for some \(B \geq 5\), there exists a PTAS for the \textsc{MinDirectedRcpt} problem on graphs with \(\Delta(G) \leq B\). Using our reduction, this implies a PTAS for \textsc{MinimumSetCover} instances for which the sets have at most \(B - 2 \geq 3\) elements and every element appears in at most 2 sets, a contradiction.

\[\Box\]

### 4 Undirected Graphs with Bounded Degree

#### 4.1 Fixed Number of Colors

**Theorem 3.** \textsc{MinRcpt} is \textsc{Apx}-Hard even when \(\Delta(G) \leq 5\), \(|C| = 2\) and \(RC_{\max} = 1\).

**Proof.** We recall that \textsc{MinimumSetCover} is \textsc{Apx}-Hard even for instances where the sets are of cardinality at most 3 and each element appears in at most 2 sets [4]. Given such an instance \((U, S)\) of \textsc{MinimumSetCover} we construct an instance \((G, C, r, rc)\) of \textsc{MinRcpt} as follows (see Figure 2):

\(G = (V, E)\) is a graph where \(V = \{r\} \cup R_1 \cup R_2 \cup S \cup S' \cup X \cup U\), \(S' = \{S_i' \mid i \in S\}\), and \(X = \{x_{ij} \mid i \in S, u_j \in S_i\}\). \(E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5\). \(R_1\) and \(E_1\) are such that the subgraph \((\{r\} \cup R_1 \cup S, E_1)\) of \(G\) is a binary tree with root \(r\) and leaves \(S\). \(R_2\) and \(E_2\) are such that the subgraph \((\{r\} \cup R_2 \cup S', E_2)\) of \(G\) is a binary tree with root \(r\) and leaves \(S'\). \(E_3 = \{\{S_i', S_i\} \mid i \in S\}\), \(E_4 = \{\{S_i', x_{ij}\} \mid u_j \in S_i, S_i \in S\}\), and \(E_5 = \{\{x_{ij}, u_j\} \mid u_j \in S_i, S_i \in S\}\). Note that \(\Delta(G) \leq 5\) and this maximum is attained for nodes of \(S'\) corresponding to sets with 3 elements. \(C = \{c_1, c_2\}\) and the reload cost function \(rc\) is such that \(rc(c_1, c_2) = rc(c_2, c_1) = 1\). Also

\[c(e) = \begin{cases} c_1 & \text{if } e \in E_1 \cup E_2 \cup E_5 \\ c_2 & \text{otherwise.} \end{cases}\]

Let \(C^*\) be an optimal set cover, and without loss of generality assume \(C^* = \{S_1, S_2, \ldots, S_k\}\). Consider the spanning tree \(T(C^*)\) of \(G\) with the edges \(E(T(C^*)) = \)
Every solution of (ii) and (i) every subtree of that this transformation does not violate (i). Let \( S_i \) and \( T_j \) be some node in \( S_i \) closest to \( T_j \). Assume without loss of generality that \( S_i \), \( T_j \) are the closest and second closest to \( r \) among these nodes, respectively. The path between these two nodes in \( T \) does not use \( r \) because they are in the same subtree, therefore it must use some node \( u \in U \). We conclude that there is a path \((r-S_i-x_{ij}-u_j-x_{2j}-S'_2)\) in \( T \) (with possibly the node \( S_1 \) between \( r \) and \( S'_2 \)). We consider two cases: (a) \( \{S_2,S'_2\} \notin E(T) \). In this case we replace \( \{x_{2j},S'_2\} \) by \( \{S_2,S'_2\} \) to obtain a new tree with the same cost and less \( S' \) nodes sharing a subtree. (b) \( \{S_2,S'_2\} \in E(T) \). In this case we replace \( \{x_{2j},S'_2\} \) by \( \{r,S_2\} \) to obtain a tree with the same cost and less \( S' \) nodes sharing a subtree. By applying this procedure as long as there are \( S' \) nodes sharing a subtree, we get a tree having the same reload cost or less, and satisfying (i).

Suppose that there is an edge \( \{x_{ij},u_j\} \in E_5 \setminus E(T) \). Then \( \{S'_1,x_{ij}\} \in E(T) \), because otherwise \( x_{ij} \) is isolated. By removing this edge and adding the edge \( \{x_{ij},u_j\} \) we obtain a tree with at most the same cost and containing one more edge of \( E_5 \). By applying this transformation as long as there are edges in \( E_5 \setminus E(T) \) we obtain a tree with at most the same reload cost, satisfying (ii). Note that this transformation does not violate (i).

Suppose that there is an edge \( \{r,S'_1\} \in E(T) \) and \( S'_1 \) is not a leaf. If \( S'_1 \) has \( k \geq 1 \) adjacent \( X \) nodes in \( T \), then by removing \( \{r,S'_1\} \) from \( E(T) \) and adding the missing edge among \( \{|r,S_1|,\{S_i,S'_1\}\} \) we can reduce the reload cost by at least \( k-1 \geq 0 \) and get a tree with one less \( \{r,S'_1\} \) edge violating (iii). If \( S'_1 \) has no adjacent \( X \) nodes in \( T \), then it is adjacent to \( S_i \), because otherwise it is a leaf of \( T \). By removing \( \{S_i,S'_1\} \) and adding \( \{r,S_i\} \) we get a tree with smaller cost and one less \( \{r,S'_1\} \) edge violating (iii). Clearly these transformations violate neither (i) nor (ii), and applying them as long as \( T \) contains violating edges we get a tree \( T' \) as claimed. \( \square \)
We proceed with the proof of the theorem. Let $T = (V, E_T)$ be a spanning tree of $G$ rooted at $r$ constituting a $\rho$-approximation to the constructed instance $(G, C, c, r, rc)$ of MINRcpt. By the above claim, it can be transformed in polynomial time into a $\rho$-approximation $T' = (V, E_{T'})$ satisfying the conditions of the claim. In such a solution all the $E_2$ edges connect $r$ to a leaf $S'_i$, thus do not incur any reload cost. $E_3$ edges connect $r$ to nodes $S_i$ some of which are leaves of $T'$. Let $Q \subseteq S$ be the set of vertices $S_i$ such that $(r, S_i) \in E_{T'}$ and $S_i$ is not a leaf of $T'$. Every node $S_i \in Q$ is connected to $S'_i$ in $T'$, by an edge with reload cost 1. Thus, the reload cost of these edges is $|Q|$. All the nodes $u_j \in U$ are distributed among the subtrees rooted at a node of $Q$. By (i), every $S'_i$ is unique in its subtree, thus the $u_j$ nodes in each subtree are connected to $S'_i$ by a path $(S'_i \rightarrow x_{ij} \rightarrow u_j)$ of length 2. Therefore $u_j \in S_i$. We conclude that $Q$ constitutes a set cover of $(U, S)$. The reload cost of each such path is 1, incurring a total reload cost of $|U|$. The remaining $E_5$ edges do not incur a reload cost, because their previous edges are also in $E_5$. Therefore $rc(T') = |Q| + |U|$. We conclude

\[
|Q| = rc(T') - |U| \leq rc(T) - |U| \leq \rho \cdot rc^* - |U| \\
\leq \rho(|C^*| + |U|) - |U| = (\rho - 1)|U| + \rho \cdot |C^*| \\
\leq (\rho - 1)3|C^*| + \rho \cdot |C^*| = (4\rho - 3)|C^*|.
\]

(in the last line we used the fact that each set contains at most 3 elements). Therefore $Q$ constitutes a $(4\rho - 3)$-approximation to the MINIMUMSETCOVER problem which is known to be in APX-Hard. Let $\rho' > 1$ be a constant such that MINIMUMSETCOVER is $\rho'$-inapproximable. Then $4\rho - 3 > \rho'$ and $\rho > (\rho' + 3)/4$, implying that for $\rho = (\rho' + 3)/4 > 1$ MINRcpt is $\rho$-inapproximable.

4.2 Any Number of Colors

**Theorem 4.** There is a constant $\rho > 0$, such that MINRcpt is $\rho \cdot \log(|V|)$-inapproximable even when $RC_{\max} = 1$.

**Sketch of Proof:** Similar to Theorem 1, we provide an approximation preserving reduction from MINIMUMSETCOVER to MINRcpt. We show the reduction for the case where all nonzero reload costs are equal to 1.

Given an instance $(U, S)$ of MINIMUMSETCOVER we build an instance $(G, C, c, r, rc)$ of MINRcpt. Consult Figure 3 for the construction. The construction is the same as in Theorem 1 except that each edge $(S'_i, u_j) \in E_4$ has color $c_{i+2}$. The reload cost function is

\[
rc(c_i, c_j) = \begin{cases} 
0 & \text{if } i = j \\
0 & \text{if } \min(i, j) = 2 \text{ and } \max(i, j) > 2 \\
1 & \text{otherwise.}
\end{cases}
\]

The impact of having directed edges between $\{S'_i, u_j\}$ in Figure 1 is achieved in Figure 3 by having undirected edges with different colors between which the reload cost is 1. Hence, the same reduction in Theorem 1 can be applied here. □
5 Conclusion

We have studied special cases of the minimum reload cost path-tree problem in directed and undirected graphs. We developed hardness results on cases such as bounded and unbounded vertex degrees, bounded and unbounded number of colors and bounded reload cost values. Our results indicate that MinRcpt is inherently difficult to approximate even in special cases.

Two main research directions are subject of work in progress: a) to develop approximation algorithms for the special cases considered in this work, b) to get hardness results for other cases not considered in this work. In particular, it will be interesting to find approximation algorithms as well as stronger inapproximability results for directed graphs with an unbounded number of colors.

References

A Figures

Fig. 1. Digraph $G$ corresponding to an instance of \textsc{MinimumSetCover} having $S_1 = \{u_3, u_5\}$, $S_2 = \{u_1, u_3, u_4\}$, $S_3 = \{u_2, u_6\}$, $S_4 = \{u_2, u_3, u_5, u_6\}$. Bold arcs indicate the spanning tree $T(C^*)$ corresponding to the optimum set cover $C^* = \{S_2, S_4\}$. 

$U=\{u_1, u_2, u_3, u_4, u_5, u_6\}$
$S=\{S_1, S_2, S_3, S_4\}$
$S_1=\{u_3, u_5\}$
$S_2=\{u_1, u_3, u_4\}$
$S_3=\{u_2, u_6\}$
$S_4=\{u_2, u_3, u_5, u_6\}$
\(U = \{u_1, u_2, u_3, u_4, u_5, u_6\}\)
\(S = \{S_1, S_2, S_3, S_4\}\)
\(S_1 = \{u_3, u_4\}\)
\(S_2 = \{u_1, u_3, u_4\}\)
\(S_3 = \{u_2, u_6\}\)
\(S_4 = \{u_3, u_5\}\)

**Fig. 2.** Graph G corresponding to an instance of MINIMUMSETCOVER having \(S_1 = \{u_3, u_4\}\), \(S_2 = \{u_1, u_3, u_4\}\), \(S_3 = \{u_2, u_6\}\), \(S_4 = \{u_3, u_5\}\). Bold arcs indicate the spanning tree \(T(C^*)\) corresponding to the optimum set cover \(C^* = \{S_2, S_3, S_4\}\).
Fig. 3. Graph G corresponding to an instance of MinimumSetCover having $S_1 = \{u_3, u_5\}$, $S_2 = \{u_1, u_3, u_4\}$, $S_3 = \{u_2, u_6\}$, $S_4 = \{u_2, u_3, u_5, u_6\}$. Bold arcs indicate the spanning tree $T(C^*)$ corresponding to the optimum set cover $C^* = \{S_2, S_4\}$. 