Causality, Knowledge and Coordination in Distributed Systems

Research Thesis

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Ido Ben-Zvi

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# Contents

Acknowledgements i  
Table of Contents iii  
Table of Figures v  
Abstract 1  

1 Introduction 3  
1.1 Causal Analysis in Distributed Systems . . . . . . . . . . . . 3  
1.2 Causality, Knowledge, Coordination . . . . . . . . . . . . 6  
1.3 The Interpreted Systems Framework . . . . . . . . . . . . 7  
1.4 Defining Knowledge in a Distributed System . . . . . . . . . 9  
1.5 The Synchronous Model . . . . . . . . . . . . . . . . . . . . 11  
1.5.1 The Synchronous Context $\gamma^b$ . . . . . . . . . . . . 11  
1.5.2 Detailed Specification for $\gamma^b$ . . . . . . . . . . . . 12  
1.6 Road Map . . . . . . . . . . . . . . . . . . . . . . . . . . . 14  
1.7 Related Work . . . . . . . . . . . . . . . . . . . . . . . . . 17  

2 Response Problems and Knowledge in Synchronous Systems 19  
2.1 Studying Coordination via Response Problems . . . . . . . . 19  
2.2 Knowledge Gain in Synchronous Systems . . . . . . . . . . . 22  
2.3 Relating Response Problems and Knowledge Gain . . . . . . 23  
2.3.1 Response Problem to Knowledge Gain . . . . . . . . . . 23  
2.3.2 Knowledge Gain to Response Problem . . . . . . . . . . 26  

3 Gaining Nested Knowledge 29  
3.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . 29  
3.2 Bound Guarantees . . . . . . . . . . . . . . . . . . . . . . . . 33
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3  Syncausality</td>
<td>34</td>
</tr>
<tr>
<td>3.4  Double Response</td>
<td>36</td>
</tr>
<tr>
<td>3.5  Centipedes</td>
<td>38</td>
</tr>
<tr>
<td>3.6  Knowledge Requires Syncausality</td>
<td>40</td>
</tr>
<tr>
<td>3.7  Nested Knowledge Requires Centipedes</td>
<td>44</td>
</tr>
<tr>
<td>3.8  Varying Nondeterminism in Message Trans</td>
<td>49</td>
</tr>
<tr>
<td>3.9  Sufficiency of Centipedes for Knowledge Gain</td>
<td>51</td>
</tr>
<tr>
<td>3.10 Conclusions</td>
<td>56</td>
</tr>
<tr>
<td>4    Gaining Common Knowledge</td>
<td>58</td>
</tr>
<tr>
<td>4.1  Introduction</td>
<td>58</td>
</tr>
<tr>
<td>4.2  Centibrooms</td>
<td>60</td>
</tr>
<tr>
<td>4.3  Common Knowledge Requires Centibrooms</td>
<td>61</td>
</tr>
<tr>
<td>4.4  The Simultaneous Global Snapshot Protocol</td>
<td>64</td>
</tr>
<tr>
<td>4.5  Sufficiency of Centibrooms for Common Knowledge Gain</td>
<td>69</td>
</tr>
<tr>
<td>4.6  Common Knowledge as a Finite Conjunction</td>
<td>70</td>
</tr>
<tr>
<td>4.7  Conclusions</td>
<td>76</td>
</tr>
<tr>
<td>5    Gaining Nested Common Knowledge</td>
<td>77</td>
</tr>
<tr>
<td>5.1  Introduction</td>
<td>77</td>
</tr>
<tr>
<td>5.2  Relating Ordered Group Response and Nested Common Knowledge</td>
<td>78</td>
</tr>
<tr>
<td>5.3  Generalized Centipedes</td>
<td>82</td>
</tr>
<tr>
<td>5.3.1 Nested Common Knowledge Gain Requires Generalized Centipedes</td>
<td>83</td>
</tr>
<tr>
<td>5.4  Conclusions</td>
<td>93</td>
</tr>
<tr>
<td>6    Generalized Ordering of Events</td>
<td>94</td>
</tr>
<tr>
<td>6.1  Introduction</td>
<td>94</td>
</tr>
<tr>
<td>6.2  Condensed Representation of GR</td>
<td>96</td>
</tr>
<tr>
<td>6.3  Generalized Ordering Requires Multiple Generalized Centipedes</td>
<td>98</td>
</tr>
<tr>
<td>6.4  Conclusions</td>
<td>100</td>
</tr>
<tr>
<td>7    Gaining Knowledge of Ignorance</td>
<td>102</td>
</tr>
<tr>
<td>7.1  Introduction</td>
<td>102</td>
</tr>
<tr>
<td>7.2  Bounded Communication and Cones of Influence</td>
<td>103</td>
</tr>
<tr>
<td>7.3  Transmission Guarantees and Knowledge of Ignorance</td>
<td>107</td>
</tr>
<tr>
<td>7.4  Conclusions</td>
<td>114</td>
</tr>
<tr>
<td>8    Discussion</td>
<td>116</td>
</tr>
</tbody>
</table>
List of Figures

2.1 Example 1 .................................................. 21
3.1 Example 2 .................................................. 30
3.2 Examples 3 and 4 .......................................... 31
3.3 Timing guarantees ......................................... 33
3.4 ................................................................. 36
3.5 ................................................................. 37
3.6 ................................................................. 37
3.7 ................................................................. 37
3.8 ................................................................. 37
3.9 Coordination in Double Response ...................... 38
3.10 A centipede ............................................... 39
3.11 A centipede in Example 3 ................................ 40
3.12 Collapsed centipede variations ......................... 50
3.13 The syncausal relation and reflected knowledge state ........................................... 52
3.14 The timing guarantee and reflected knowledge state ........................................... 54
3.15 Syncausal relation and timing guarantee, with induced knowledge state .................. 55
3.16 Centipede for Theorem 7 ............................... 55
4.1 The network of Example 5 .............................. 58
4.2 Example 5 .................................................... 59
4.3 A centibroom .............................................. 60
4.4 The setup for Lemma 22 ................................. 74
5.1 A generalized centipede .................................. 82
6.1 Production process for Munchy Crunchy chocolate bar ....................................... 95
6.2 Generalized centipedes in the Crunchy Munchy production line .............................. 100
7.1 The future causal cones of $\theta$ in asynchronous systems and $\gamma^{\max}$ ................................. 105
Abstract

Effecting coordination across remote sites in a distributed system is an essential part of distributed computing, and also an inherent challenge. Bereft of telepathy and other extrasensory perceptual powers, the processes must rely on message passing in order to achieve it.

In 1978, a fascinating analysis of communication in asynchronous systems was suggested by Leslie Lamport [26]. Lamport takes his cue from the theory of special relativity, where the bounded expansion of light through space and time marks the limits of causal affectability: nothing can travel faster than light, and so causal influence too must be limited by the speed of light. Of course, in typical distributed systems nothing as exotic as traveling at near the speed of light ever comes up. But here, in analogy to light, causal influence cannot travel faster than the messages that traverse the inter-process void do. The import of Lamport’s paper for distributed computing cannot be overestimated. The causal analysis determines a notion of temporal precedence, a sort of weak notion of time, which is otherwise missing in asynchronous systems. This notion has been extensively utilized in various applications.

Yet Lamport’s analysis, and the reliant body of research that has been conducted since, is mostly limited to systems that are asynchronous. In this thesis we go beyond the existing body of literature by investigating causality in synchronous systems. In such systems, the boundaries of causal influence are not charted out exclusively by message passing. Here time itself, passing at a uniform (or almost uniform) rate for all processes, is also a medium by which causal influence may fan out. This thesis studies, and characterizes, the intricate combinations of time and message passing that govern causal influence in synchronous systems.

It turns out that knowledge based analysis [15] provides a well tailored formal framework within which causal notions can be studied. As we show, the formal notion of knowledge is highly appropriate for characterizing causal influence in terms of information flow. The idea of using knowledge in such
circumstance was first brought up by Chandy and Misra in [7]. We broaden their analysis and deepen its methodological infrastructure.

In order to study coordination rigorously, we define several generic classes of coordination problems that pose various temporal ordering requirements on the participating processes. These coordination problems provide natural generalizations of real life requirements. We then analyze the causal conditions that underly suitable solutions to these problems. The analysis is conducted in two stages: first, the temporal ordering requirements are reduced to epistemic conditions. Then, these epistemic conditions are characterized in terms of the causal communication patterns that are necessary and sufficient to bring them about.

Whilst in asynchronous systems causal influence is characterized by a straightforward application of the temporal precedence order defined by Lamport, in synchronous systems the causal communication patterns are more complex. We identify several such patterns, each of them being a minimal requirement in some class of coordination problems: we start with syncausality, an immediate generalization of Lamport’s ordering, and move on to centipedes and centibrooms, structures that combine message passing and timing constraints. These latter two are shown to be special cases of the generalized centipede. These patterns lead us up in an increasingly complex hierarchy of ordering requirements, culminating in a characterization of the minimal communication pattern that is necessary to ensure any specification given as a partial ordering on the temporal precedence of events.
Chapter 1

Introduction

1.1 Causal Analysis in Distributed Systems

In distributed systems, a group of autonomous processes with limited means of communication are typically set to cooperate and coordinate their local actions in order to achieve a system-wide global requirement. In general, the less processes know of actions and of occurrences at remote sites, the more difficult the task of coordination becomes.

Of particular difficulty is achieving coordination in asynchronous systems, where no guarantees are given regarding the rate at which each process proceeds, and message delivery can be indefinitely postponed. In a seminal paper [26], Lamport proposed the happened-before relation $\rightarrow$ between events in asynchronous systems, and based on this relation a mechanism for logical clocks that allow processes to exercise some control over the ordering of events.

Lamport takes his lead from the theory of special relativity, where the way by which light dissipates in space over time determines upper bounds on the spread of information and of causality in general, as nothing can get from source to target faster than light itself. Applying this analogy to distributed systems, Lamport notes that in asynchronous systems, causality and information cannot travel faster than the messages that are sent and received between the processes. This suggests that the following relation on events applies to $e$ and $e'$ whenever the occurrence of $e'$ is causally dependent upon occurrence of $e$.\footnote{The formulation differs slightly from that of [26] as we do not impose irreflexivity, for the sake of a simpler formulation.}

**Definition 1 (Happened-before)** Fix an execution $r$ of the system. The
happened-before relation $→$ over events of $r$ is the smallest relation satisfying the following conditions:

1. If $e$ and $e'$ are events in the same process, and $e$ comes before or with $e'$, then $e \rightarrow e'$.

2. If $e$ is the sending of a message by one process and $e'$ is the receipt of the same message by another process, then $e \rightarrow e'$.

3. If $e \rightarrow e''$ and $e'' \rightarrow e'$ then $e \rightarrow e'$.

As messages are never delivered before they are sent, happened-before implies that whenever $e \rightarrow e'$, the occurrence of event $e$ temporally precedes (or is simultaneous with) the occurrence of $e'$, even if the events occur at distinct sites. Thus, a partial ordering that is implicit in every execution of a distributed system is made explicit.

The event ordering that is determined by the happened-before relation is sometimes referred to as a causal ordering. Causality is an elusive concept whose nature has been widely contested over the centuries. Lamport’s relation circumvents these sticky philosophical issues, in the following sense. Whenever events $e, e'$ do occur in an execution and $e \not\rightarrow e'$, then event $e$ cannot be a cause of event $e'$ under any interpretation of causality, as its effect has not reached the site of $e'$ by the time it occurs. Note that, strictly speaking, the converse scenario where $e \rightarrow e'$ does hold can only mean that $e$ is a potential cause of $e'$, as the occurrence of $e'$ may have been nondeterministic, or based on the occurrence of events other than $e$.

Lamport offers an immediate application for causal ordering. Logical clocks are defined as local counters ($Clock_i$ for each process $i$), that assign a number to each local event. By timestamping each message sent with the current value of the sender’s counter, a simple mechanism is devised to make sure for all events $e, e'$ occurring at sites $i, j$ respectively, that $Clock_i(e) < Clock_j(e')$ whenever $e \rightarrow e'$ and $i \neq j$.

The immense import of Lamport’s paper on the development of theoretical and practical distributed systems cannot be overestimated. For us it is important to summarize by saying that Lamport defined a relation, based on communication patterns (the inter-process message chains that establish the $→$ relation), that traces the dissemination of causal effect in a system. Moreover, he showed how this causal ordering can be used to establish a temporal ordering on events.
Chandy and Misra’s follow up paper [7] explicitly relates Lamport’s relation to knowledge. This paper offers a reinterpretation of Lamport’s ideas in terms of knowledge, rather than of coordination. A more in-depth coverage of knowledge in distributed systems is offered in Section 1.3, while the current discussion will be kept to an intuitive level. In distributed systems each process is immediately acquainted only with its own local state. Thus, facts that pertain to local states of remote sites may be hidden from it. Now consider a run (or execution) of the system where, at the current time, process \( i \)’s local state is \( \ell \). If, when one looks at all possible runs, an arbitrary fact \( \varphi \) holds true of the system whenever \( i \)’s local state is \( \ell \), then \( i \) is said to “know” that \( \varphi \): there is simply no way that, given its local state \( \ell \), fact \( \varphi \) could fail to hold. How would process \( i \) come to know that, say, the value of process \( j \)’s local variable \( X_j \) is 10?

A simple answer can be given if \( X_j = 10 \) is an invariant specified by the protocol. To filter away such “uninteresting” cases, what if at time \( t \) process \( j \) itself does not know that \( X_j = 10 \), and at time \( t' > t \) process \( i \) knows that process \( j \) knows that \( X_j = 10 \)? Chandy and Misra call such a development knowledge gain, where process \( i \) comes to gain new knowledge about the state of process \( j \). They surmise, and then prove, that in such a case it must be that process \( j \) at time \( t \) is happened-before related to process \( i \) at time \( t' \). We will denote such a relation with \( (j, t) \rightarrow (i, t') \). More generally, Chandy and Misra show that if at time \( t' \) process \( i_k \) knows that process \( i_{k-1} \) knows that... process \( i_1 \) knows that process \( j \) knows that \( X_j = 10 \), then it must be that there are times \( t = t_0 < t_1 < \cdots < t_k = t' \) such that \( (j, t_0) \rightarrow (i_1, t_1) \rightarrow \cdots \rightarrow (i_k, t_k) \).

While Lamport relates communication to coordination, Chandy and Misra relate it to knowledge gain. In both cases the happened-before relation can be seen to give as good a characterization as can be achieved of the spread of causal effect in the system. However, once we have formalized the notions of knowledge and of coordination with which the thesis deals, we will show in Section 2.3 that knowledge gain is a necessary condition for coordination, and thus provides a “closer to home” approximation of causality than coordination. As such, we will study it extensively in the thesis, with the aim of giving a precise understanding of causality in synchronous systems.

\(^2\)Since \( X_j \) is a part of \( j \)’s local state, by definition process \( j \) will know its value at all times, so if \( j \) doesn’t know that \( X_j = 10 \), it must be that \( X_j \neq 10 \).

5
1.2 Causality, Knowledge, Coordination

Roughly sketched, the scenery drawn out by Lamport and by Chandy and Misra for asynchronous systems shows that communication is prerequisite for knowledge gain and that, similarly, knowledge gain is necessary for the coordinated ordering of events. These relations justify associating Lamport’s happened-before relation with causality in such settings.

In this thesis we will investigate causality as it manifests itself in synchronous settings. Example 1, presented in the next chapter, will show us that the happened-before relation no longer characterizes causal relations in their entirety under synchrony. Our main goals will be to identify the communication patterns that do characterize causality here.

Our method is to define various scenarios where knowledge gain, as a rigorously defined approximate for causality, takes place. Each of the following chapters is dedicated to such a scenario. In Chapters 3 and 4 we provide the Ordered and Simultaneous Response problems as motivating leads. Given the necessity of nested and common knowledge gain for the OR and SR problems respectively, characterizing solutions to these coordination problems in terms of causality pretty much reduces to an analysis of knowledge gain in such terms.

The study of causal relations leading to knowledge gain is thus relevant in the context two differing research programmes:

- Focussing on the relations between knowledge gain and causality, we hope to make the thesis results instrumental in the widely defined field of epistemic analysis in multi agent systems. The thesis results may be applicable in the linguistic study [28], as well as in game theoretic analysis of interactive epistemics [2, 8], and possibly also in the philosophical analysis of causality [44, 51].

- By encompassing also the relations between knowledge and coordination, we relate coordination directly to communication. Unlike knowledge, coordination and communication are both tangible, and results characterizing one in terms of the other would be easier to apply.

Thus, even in the context of more applicative study of distributed systems, knowledge based analysis can be made to play a subtle, if highly beneficial, role. Knowledge is a powerful tool for extracting underlying generalizations in such systems and is our best approximation for causal phenomena. Once these generalizations have been
properly characterized, direct connections between them can be drawn out, pretty much obsoleting the interpretive epistemic layer.

One final guiding principal for the inquiries made in this thesis needs to be mentioned. It is widely understood that different process protocols lead to widely varying characteristics for the system as a whole. Nevertheless, our key results are not protocol dependent, and in this sense they characterize all synchronous systems. We adhere to the idea of characterizing systems rather than protocols throughout the thesis, and even where protocol specific results are given, they bear significance for all systems (by showing that our definitions are tight). The one exception to this guideline is made in Chapter 7, where gaining knowledge of ignorance is discussed.

1.3 The Interpreted Systems Framework

Results pertaining to knowledge gain in distributed systems provide the main formal backbone of this thesis. For this reason we utilize the interpreted systems framework of Fagin, Halpern, Moses, and Vardi [15]. We shall simplify its exposition somewhat here, and review just enough of the details to support the formal analysis. Essentially all of the definitions in this section are taken from [15].

Informally, we view a multi-process system as consisting of a set \( P = \{1, \ldots, n\} \) of processes connected by a communication network. We assume that, at any given point in time, each process in the system is in some local state. A global state is just a tuple \( g = \langle \ell_e, \ell_1, \ldots, \ell_n \rangle \) consisting of local states of the processes, together with the state \( \ell_e \) of the environment. The environment’s state accounts for everything that is relevant to the system that is not contained in the state of the processes.

A run is a function from time to global states. Intuitively, a run is a complete description of what happens over time in one possible execution of the system. A point is a pair \((r, t)\) consisting of a run \( r \) and a time \( t \). If \( r(t) = \langle \ell_e, \ell_1, \ldots, \ell_n \rangle \), then we use \( r_i(t) \) to denote process \( i \)'s local state \( \ell_i \) at the point \((r, t)\), for \( i = 1, \ldots, n \), and \( r_e(t) \) to denote \( \ell_e \). For simplicity, time here is taken to range over the natural numbers rather than the reals (so that time is viewed as discrete, rather than dense or continuous). Round \( t \) in run \( r \) occurs between time \( t - 1 \) and \( t \).

We identify a protocol for a process \( i \) with a function from local states of \( i \) to nonempty sets of actions. (We mostly consider deterministic protocols, in which each local state is mapped to a singleton set of actions. Such a
A joint protocol is just a sequence of protocols \( P = (P_1, \ldots, P_n) \), one for each process.

We generally study knowledge in runs of a given protocol \( P \) in a particular setting of interest. To do this, we separately describe the setting, or context, in which \( P \) is being executed. Formally, a context \( \gamma \) is a tuple \( (G_0, P_e, \tau) \), where \( G_0 \) is a set of initial global states, \( P_e \) is a protocol for the environment, and \( \tau \) is a transition function.\(^3\) The environment is viewed as running a protocol (denoted by \( P_e \)) just like the processes; its protocol is used to capture nondeterministic aspects of the execution, such as the actual transmission times, external inputs into the system, etc. The transition function \( \tau \) describes how the actions performed by the processes and the environment change the global state. Thus, if \( g \) is a global state and \( \vec{a} = \langle a_e, a_1, \ldots, a_n \rangle \) is a joint action (consisting of an action for the environment and one for each of the processes), then \( \tau(\vec{a}, g) = g' \) specifies that \( g' \) is the state that results when \( \vec{a} \) is performed in state \( g \). When modeling asynchronous systems, we assume that some processes will be executing a NULL action, of which they are not even aware (their local states are left unaltered).

A run \( r \) is consistent with a protocol \( P \) if it could have been generated when running protocol \( P \). Formally, run \( r \) is consistent with joint protocol \( P \) in context \( \gamma \) if

1. \( r(0) \in G_0 \), so that it starts from a \( \gamma \)-legal initial global state, and
2. for all \( t \geq 0 \), the transition from global state \( r(t) \) to \( r(t + 1) \) is the result of performing one of the joint actions specified by \( P \) and the environment protocol \( P_e \) (the latter is specified in \( \gamma \)) in the global state \( r(m) \). That is, if \( P = (P_1, \ldots, P_n) \), \( P_e \) is the environment’s protocol in context \( \gamma \), and \( r(m) = \langle \ell_e, \ell_1, \ldots, \ell_n \rangle \), then there must be a joint action \( \vec{a} = \langle a_e, a_1, \ldots, a_n \rangle \) such that \( a_e \in P_e(\ell_e) \), \( a_i \in P_i(\ell_i) \) for \( i = 1, \ldots, n \), and \( r(m + 1) = \tau(\vec{a}, r(m)) \) (so that \( r(m + 1) \) is the result of applying the joint action \( \vec{a} \) to \( r(m) \)).

We use \( R(P, \gamma) \) to denote the set of all runs of \( P \) in \( \gamma \), and call it the system representing \( P \) in context \( \gamma \).

A description of the specific context \( \gamma \) that we deal with throughout the thesis is found in Section 1.5.

\(^3\) Depending on the application, a context can include additional components, to account for fairness assumptions, probabilistic assumptions, etc. Moreover, additional aspects of a context that are usually suppressed from the notation are nonempty sets \( \text{Int} \) and \( \text{Ext} \) of internal actions for the processes and external inputs, respectively.
1.4 Defining Knowledge in a Distributed System

We aim at a logical analysis of gained knowledge regarding the occurrence of events. The interpreted systems framework [15] provides us with much of the necessary machinery here. We focus on a simple logical language in which the set \( \Phi \) of primitive propositions consists of propositions of the form \( \text{occurred}(e) \), \( \text{ND}(e) \) and \( \text{time} = t \) for all events \( e \) and times \( t \). To obtain the logical language \( \mathcal{L} \), we close \( \Phi \) under propositional connectives and knowledge formulas. Thus, \( \Phi \subset \mathcal{L} \), and if \( \varphi \in \mathcal{L} \), \( i \in \mathbb{P} \) and \( G \subseteq \mathbb{P} \), then \( \{K_i\varphi, E_G\varphi, C_G\varphi\} \subset \mathcal{L} \). The formula \( K_i\varphi \) is read \( \text{process } i \text{ knows } \varphi \), \( E_G\varphi \) is read \( \text{everyone in } G \text{ knows } \varphi \), and \( C_G\varphi \) is read \( \varphi \text{ is common knowledge to } G \). In addition, we add a timestamping operator as well. Thus, if \( \varphi \in \mathcal{L} \) and \( t \in \mathbb{N} \), then \( \text{At}_t\varphi \in \mathcal{L} \).

The truth of a formula is defined with respect to a triple \((R, r, t)\). We write \((R, r, t) \models \varphi\) to state that \( \varphi \) holds at time \( t \) in run \( r \), with respect to system \( R \). It is always assumed that \( r \in R \) in a triple \((R, r, t)\). The precise meaning of nondeterminism in this system is given in Section 1.5 below. Denoting by \( r_i(t) \) process \( i \)'s local state at time \( t \) in \( r \), we inductively define

\[
(R, r, t) \models \text{ND}(e) \quad \text{iff} \quad \text{event } e \text{ occurs in } r \text{ and is nondeterministic there;}
\]

\[
(R, r, t) \models \text{occurred}(e) \quad \text{iff} \quad \text{event } e \text{ occurs at time } t' \text{ in } r \text{ such that } t' \leq t;
\]

\[
(R, r, t) \models \text{time} = t' \quad \text{iff} \quad t' = t;
\]

\[
(R, r, t) \models \text{At}_t\varphi \quad \text{iff} \quad (R, r, t') \models \varphi;
\]

\[
(R, r, t) \models K_i\varphi \quad \text{iff} \quad (R, r', t') \models \varphi \text{ for every run } r' \text{ satisfying } r_i(t) = r_i'(t');
\]

\[
(R, r, t) \models E_G\varphi \quad \text{iff} \quad (R, r, t) \models K_i\varphi \text{ for every } i \in G; \quad \text{and}
\]

\[
(R, r, t) \models C_G\varphi \quad \text{iff} \quad (R, r, t) \models (E_G)^k\varphi \text{ for every } k \geq 1.
\]

Propositional connectives are handled in the standard way, and their clauses are omitted above. In some cases it will be convenient to also syntactically derive the proposition \((R, r, t) \models \text{occurs}(e) \text{ iff } (R, r, t) \models \text{occurred}(e) \land \text{time} =\)

\footnote{In this thesis we do not investigate complexity and decidability issues pertaining to the use of explicitly timestamped formulas. The system’s existing constraints on transmission times require some sort of temporal metric on formulas, and we opt for this choice based on the clarity and conciseness that it offers.}
0 ∨ At_{t-1}\neg\text{occurred}(e)), so \((R, r, t) \vdash \text{occurs}(e)\) holds true if \(e\) occurs exactly at time \(t\) in \(r\).

By definition, \(K_i\varphi\) is satisfied at \(r \in R\) and time \(t\) if \(\varphi\) holds at all points at which \(i\) has the same local state as in \((r, t)\). Thus, given \(R\), the local state determines what processes know. Intuitively, a fact \(\varphi\) is common knowledge to \(G\) if everyone in \(G\) knows \(\varphi\), everyone knows that everyone knows \(\varphi\), and so on \(ad\) \text{infinitum}. In particular, if \((R, r, t) \vdash C_G\varphi\) then \((R, r, t) \vdash K_i K_i K_i \cdots K_i\varphi\), for every string \(K_i K_i K_i \cdots K_i\) and \(h > 0\).

We write \(R \models \varphi\) and say that “\(\varphi\) is valid in \(R\)” if \((R, r, t) \vdash \varphi\) holds for all \(r \in R\) and \(t \geq 0\). A formula is valid, written \(\models \varphi\), if it is valid in all systems \(R\).

It is convenient to treat boundary cases for some of the operators in the following way: for \(|G| = 0\) we have \(R \models E_G\varphi\) for all \(\varphi\), and hence also \(R \models C_G\varphi\). For all \(G\), we say that \((R, r, t) \vdash (E_G)^0\varphi\) iff \((R, r, t) \vdash \varphi\).

In the context of distributed systems, the knowledge operator embodies an important function that is often left unstated. Intuitively, a process “knows” \(\varphi\) if it is in possession of ample evidence that \(\varphi\) is true. Essentially, all such evidence must be based on the local state of the process. The local states of other processes are not immediately available for it to inspect. We consider the local state \(\ell\) of process \(i\) as “ample evidence that \(\varphi\)” if at every possible point in the execution of the distributed system where the local state of \(i\) is \(\ell\), \(\varphi\) holds. Thus knowledge can be seen as a localizing qualifier for the inner formula \(\varphi\). To see this, consider the two statements below.

- \((R, r, t) \vdash \varphi\), and
- \((R, r, t) \vdash K_i \varphi\)

The former statement is straightforward: \(\varphi\) obtains at the distributed system in run \(r\) at time \(t\). The latter statement makes a stronger claim: not only does \(\varphi\) obtain in \(r\) at \(t\), but it also holds in every possible point in which \(i\)'s local state is the same as it is now (at the point \((r, t)\)).

Formulas pertaining to nested knowledge, such as \((R, r, t) \vdash K_j K_i \varphi\), can now also be given a formal interpretation. What it means is that process \(j\)'s local state at \((r, t)\) provides ample evidence to support the claim that process \(i\)'s local state at \((r, t)\) provides ample evidence that \(\varphi\) is true at \((r, t)\).
1.5 The Synchronous Model

1.5.1 The Synchronous Context $\gamma^b$

We are interested in characterizing the effect of synchronous constraints on distributed systems. We therefore define a class of synchronous contexts $\gamma^b$ that ensure the following properties for all systems defined on top of them:

- The set of processes is denoted by $P$. These are connected by a network of weighted channels. For each pair of processes $i, j$ connected by a communication channel, the weights $\min_{ij}$ and $\max_{ij}$ denote the minimal and maximal transmission times for messages over the channel, respectively. In all cases $\min_{ij} \geq 1$. Whenever there is no upper bound on transmission we have that $\max_{ij} = \infty$.

- We assume that processes can receive external inputs from the outside world. These are determined in a genuinely nondeterministic fashion, and are not correlated with anything that comes before in the run, or with external inputs currently received by other processes.

- The scheduler, which we typically call the environment, is in charge of choosing the external inputs, and of determining message transmission times. The latter are also determined in a nondeterministic fashion, subject to the delivery time constraints as detailed by the weights on the channels.

- Time is identified with the natural numbers $\mathbb{N}$, and each process is assumed to take a step at each time $t \in \mathbb{N}$. For simplicity, the processes follow deterministic protocols. Hence, a given protocol $P$ for the processes and a given behavior of the environment completely determine the run.

- Events are sends, receives, external inputs and internal actions. All events in a run are distinct, and we denote a generic event by the letter $e$. For simplicity, events do not take time to be performed. At a given time point a process can perform an arbitrary finite set of actions.

We shall, for the most part, be concerned with contexts that are more restrictive than $\gamma^b$. Thus, any $\gamma^b$ context where $1 = \min_{ij} \leq \max_{ij} < \infty$ for all channels $(i, j)$ will be called a $\gamma^{max}$ context, it is a context whose systems are characterized by the existence of finite upper bounds on delivery time.
Similarly, a $\gamma^b$ context where $1 \leq \min_{ij} < \max_{ij} = \infty$ for every channel $(i, j)$ is a $\gamma^{\text{min}}$ context. This is a context where there are only lower bounds on transmission times.

Whichever superscript $\alpha$ is used to denote a particular context (as in $\gamma^\alpha$), will also be made use of to denote a system $\mathcal{R}^\alpha = \mathcal{R}(P, \gamma^\alpha)$ where $P$ is any arbitrary protocol.

1.5.2 Detailed Specification for $\gamma^b$

A synchronous context $\gamma^b$ is defined as a tuple $(G^0, P^b, \tau)$ where

The environment’s state  Recall that the environment’s state keeps track of relevant aspects of the global state that are not represented in the local states of the processes. We assume that the environment’s state has three components $\ell_e = (\text{Net}_e, \text{t}_e, \text{Hist}_e)$, where

1. $\text{Net}$ is a labelled graph $(\mathbb{P}, E, \text{max}, \text{min})$ describing the network topology and bounds on transmission times. Its nodes are processes, and a directed edge $(i, j) \in E$ captures the fact that there is a channel from $i$ to $j$ in the system. Moreover, the labels $1 \leq \max_{ij} \in \mathbb{N} \cup \{\infty\}$ and $1 \leq \min_{ij} \in \mathbb{N}$ are upper and lower bounds respectively on the time that a message sent on $(i, j)$ can be in transit. The contents of $\text{Net}$ are not affected by $\tau$, and so $\text{Net}$ remains constant throughout the run.

2. The variable $\text{t}$ keeps track of global time. As we shall see its value starts at $\text{t} = 0$, and advances by 1 following each round. Finally,

3. $\text{Hist}_e$ records the sequence of joint actions performed so far. The $\text{Hist}_e$ component uniquely determines the contents of all channels. Indeed, a message $\mu$ is in transit at a given global state $g$ if the $\text{Hist}_e$ component in $g$ records that $\mu$ has been sent, and does not record its delivery.

Process local states  We assume that local states have three components $\ell_i = (\text{Net}_i, \text{t}_i, \text{data}_i)$, where $\text{Net}_i$ and $\text{t}_i$ are copies of the $\text{Net}$ and $\text{t}$ values from the environment’s state. The component $\text{data}_i$ serves as the data segment for the process $i$. Its contents are a function of the protocol $P$ and the transition function $\tau$.

\footnote{This holds true even if we allow message loss by setting $bij = \infty$.}
The set $G_0$ of initial global states  We assume that associated with $\gamma^b$ there is a set $\text{Init}_i$ of possible initial states for each process $i \in P$. We define $G_0$ to be the set of global states $gl = (\ell_e, \ell_1, \ldots, \ell_n)$ satisfying: (1) $t_i = 0$ for all $i \in P$ and $t = 0$; (2) the network components $\text{Net}$ and $\text{Net}_i$ are all identical; (3) for every $i \in P$, $\text{hist}_i = \langle \text{init}_i \rangle$, with $\text{init}_i \in \text{Init}_i$; and (4) $\text{Hist}_e$ is the empty sequence.

Actions and external inputs  Associated with the context $\gamma^b$ are sets $\text{Int}$ of internal actions for the processes and sets $\text{Ext}$ of external inputs, respectively. For ease of exposition we assume that $\bot \in \text{Ext}$, where $\bot$ stands for the empty external input. Moreover, we generally assume that $\text{Ext} \neq \{\bot\}$, so that there is at least one nontrivial possible external input. We assume that processes can perform send actions and internal actions. The local action $a_i(k)$ that $i$ contributes to the joint action in round $k + 1$ consists of a finite sequence of distinct send and internal actions. (Recall that the local action is determined by the protocol, based on the local state.) We use external inputs to model spontaneous events. They are generated by the environment. In addition to external inputs, the environment is in charge of message delivery. Thus, the environment’s action $a_e(k)$ consists of a finite sequence of external inputs to be delivered to various individual processes, a subset $\pi$ of $P$ that are activated in the current round, and a (possibly empty) set of messages that are to be delivered in the current round.

The environment’s protocol $P_e^b$  The environment in $\gamma^b$ is in charge of delivering external inputs to processes and determining message deliveries. For every global state $gl$ we define $P_e^b(gl)$ to be the set of actions $a_e = (\sigma_x, \sigma_d)$ such that

1. $\sigma_x : P \rightarrow \text{Ext}$ is a sequence assigning to each process $i \in P$ an external input (possibly the empty input $\bot$) it receives in the current round, and

2. $\sigma_d$ is a sequence $\langle M_1 \ldots M_{|P|} \rangle$ where (i) for every $i \in \text{Proc}$ the set $M_i$ consists of messages that are in transit in $gl$, (ii) $M_i$ contains all messages in transit to $i$ whose transmission time bounds, as specified in $\text{Net}$, will be violated (expire) if the message is not delivered in the current round, and (iii) none of the messages in $M_i$ are such that if delivered in the current round, will violate the existing minimal transmission time constraints.
Notice that $P^b_e$ is genuinely nondeterministic. Exactly one of the actions in $P^b_e(gl)$ will be performed in global state $gl$ in any given instance. By definition of $R(P, \gamma^b)$, however, if $r(k) = gl$ then the system contains a run extending the prefix $r(0), \ldots, r(k)$ for every possible environment action in $P^b_e(gl)$. Another point to note is that our definition does not enforce (and hence does not assume) FIFO transmission; had we done so, channels would be considered to be queues, and the nondeterministic choices of messages to deliver would have to obey FIFO order. It should also be noted that the scheduler makes sure to comply with all existing constraints: minimal and maximal transmission times, as well as process rate. Finally, the fact that external inputs are delivered in a nondeterministic fashion implies they are not correlated in any way, and they do not depend on anything that happens before they are delivered. This is the sense in which external inputs can be viewed as independent, “spontaneous” events.

The transition function $\tau$ The transition function $\tau$ implements the joint actions in a rather straightforward manner. In every round: (i) the global clock variable $t$ and the local variables $t_i$ of all $i \in \pi$ are advanced by one; (ii) a copy of the joint action is added to the environment’s history log $Hist_e$; and (iii) For every process $i$, a record of all current round message deliveries and external inputs to the process is written in $data_i$. Note that this record is overwritten in every round, so that a protocol must take special measures in order to maintain a persistent copy of these contents.

1.6 Road Map

This chapter has provided an outline of the necessary background upon which it is built: the causal analysis of distributed systems introduced by Lamport, and the knowledge-based framework of Fagin et. al. The rest of the thesis describes novel results obtained as part of our research. We conclude it with a roadmap that offers a general outline of what the thesis is all about.

Chapter 2 defines the formal and conceptual “playground” within which our research is conducted. We start by introducing the Ordered and the Simultaneous Response problems: two generic coordination problems that set constraints on the temporal ordering of events. The problem definitions involve a set of required responses to a spontaneous non-deterministic event. As we argue there, spontaneity is a required ingredient if we want to study those cases of coordination that necessitate information flow in the system.
This rather abstract notion of information flow is also given a formal interpretation, in terms of knowledge gain. Apart from definitions, the chapter also provides initial claims and their proofs. We study and prove the relations between the types of response problems and correlated epistemic states of gained knowledge: nested knowledge is necessary and sufficient for ensuring correct solutions to the Ordered Response problem. Similarly, common knowledge is necessary and sufficient for Simultaneous Response. A discussion on the role of knowledge as an intermediate layer between causality and coordination concludes this chapter.

Chapter 3 is the first in a series of four chapters that each study and characterize a particular coordination problem. This chapter studies the Ordered Response problem, but it also introduces several key notions that are utilized in the following chapters. We use the set of process-time pairs as the domain in which causal relations are defined. This domain is more suitable than the set of events, or of processes, given the synchronous characteristics of the system. The two most basic causal relations that we use are timing guarantees and syncausality, the latter being a generalization of Lamport’s happened-before.

In asynchronous systems, the correct ordering of more than two events requires the repeated application of the happened-before relation to each pair of subsequent events. A careful analysis of solutions to the most simple cases of Ordered Response reveals that such ordering in synchronous systems requires complex relations between all of the related process-time nodes. We define a causal structure, the centipede, that combines both syncausality and timing guarantees, and prove the Centipede Theorem, showing that the existence of a centipede is necessary for ensuring the correct ordering of a sequence of events. For ease of exposition, the formal results of this chapter (as well as those of Chapters 4 through 6) are given using the $\gamma^{max}$ context, in which only upper bounds are defined. We show however that the theorem also applies in two complementing boundary cases: one, where there are no upper bounds on delivery, and the other where delivery times are fixed. In the former case, the centipede structure is trivialized into a Lamport-style message chain. We conclude this chapter by showing that our definition is tight, in the sense that under some protocols the existence of the centipede is also sufficient for proper event ordering. We suggest the Full Information Protocol (fip) for synchronous systems for this purpose.

Moving on to the next type of response problem, Chapter 4 investigates the causal structures necessary for ensuring the simultaneous happening of events. As such, it constitutes a complete break with existing analysis of asynchronous causality, where no such constraint can be ensured. We
introduce the *centibroom* structure, a variant of the centipede, and prove the Centibroom Theorem, an analog to the Centipede Theorem that shows the existence of a centibroom to be necessary for ensuring simultaneous actions. Sufficiency of the centibroom for such coordinated responses is also proved, under fip. As an application of the Centibroom Theorem we suggest two novel variants of a global snapshot algorithm for synchronous systems, one of which is shown to provide optimal time complexity.

The particular form of the centibroom, and the results of Chapter 4 showing that it is a prerequisite for common knowledge, provide a clear and graphic demonstration that the nature of common knowledge is finitistic, despite its familiar definition being based on an infinite conjunction of facts. Further investigation into the properties of the centibroom and of common knowledge is used to show that, roughly speaking, it takes time to obtain deeply nested knowledge without “collapsing” into common knowledge. Nevertheless, this result is shown to be dependent upon the protocol being followed. A counterexample is suggested where every level of nested knowledge may be achieved without common knowledge ensuing.

Chapters 5 and 6 deal with generalizations of both the Ordered Response and the Simultaneous Response problems. First, the *Ordered Group Response* problem is defined, which can be seen as an immediate “merge” of Ordered and Simultaneous Response requirements. In analogy, the *generalized centipede* structure is defined, and is shown to be necessary in solutions to the Ordered Group Response problem. Then we take the generalization even further and define the *Generalized Response* Problem, where the required temporal ordering of events can be specified using any partial order. Characterization is provided in terms of sets of generalized centipedes. Our understanding of common knowledge is advanced further by showing how such an epistemic state is dependent upon the joint histories of the processes in the group.

Chapter 7 takes a different stance from the one followed to this point. In epistemic terms, each of the response problems we considered thus far is reduced into a rather complex requirement concerning knowledge about knowledge, which is then reduced further into a causal communication condition that is highly dependent upon the existence of upper bounds on message delivery times. In Chapter 7 we consider the complementary approach: we ask what is the causal condition that will ensure knowledge about ignorance rather than knowledge about knowledge. This leads us into a more detailed discussion of causal cones of influence, from which the conditions for such ignorance are then distilled. As it turns out, it is the existence of *lower bounds* on transmission times that makes such knowledge, which is of value
in competitive settings, possible.

Chapter 8 brings the thesis to conclusion and discusses possible further research and various open questions.

### 1.7 Related Work

A great abundance of work pertaining to Lamport’s happened-before relation has been collected over the years, and we shall make no attempt to provide a survey of this work. A thorough report can be found in [48]. Two of the most widely known works that build atop it are Mattern’s generalization of Lamport’s scalar clocks into vector clocks in [30, 17] and Chandy and Lamport’s utilization in the snapshot algorithm [6] which is further discussed in Chapter 4. Another related work is the Chandy and Misra paper [7] discussed in Section 1.1.

Formal study of knowledge, and knowledge about knowledge, touches on many fields, ranging from philosophy [29] and psychology [9], to linguistics [20, 39], economics [2], AI [31], cryptography [12, 46, 19] and distributed systems [23, 7, 40]. The interpreted systems framework, epitomized in [15], stands at the base of related research in to distributed computing [36, 13, 24].

Explicit and implicit use of time bounds, introduced in Chapter 3, for coordination and improved efficiency is ubiquitous in distributed computing. An elegant example of its use is made by Hadzilacos and Halpern in [21]. That knowledge can be gained by way of NULL messages when timing guarantees are available has been part of the folklore for decades. To our knowledge, Lamport [27] was first to explicitly explore the use of NULL messages beyond their customary timeout semantics. In effect Lamport’s state machine protocols in that paper are based on an implicit notion of causality which we will later (see Chapter 3) define as syncausality, yet no attempt at rigorous formalization is made there, and the general role of time bounds is not developed. A tutorial by Moses [34] suggests as a viable topic for future work performing an explicit analysis of the effect of NULL messages on knowledge gain. He also presents an example in which communication can be saved by using timeouts. However, [34] does not suggest modifying Lamport causality to suit synchronous systems, and none of the new notions or technical results in this thesis were suggested in [34]. Krasucki and Ramanujam in [25] study of the interaction between knowledge and the ordering of events in a distributed system. They consider concurrency in a rather abstract setting, where they show that causality is related to the existence of particular partially ordered sets. They do not explicitly study
the synchronous model, however, and do not explicitly consider synchronous
time bounds on channels. Moses and Bloom [35] perform a knowledge-based
analysis of clock synchronization in the presence of bounds on transmission
times. They generalize Lamport’s relation by defining a notion of timed
causality $e \xrightarrow{\alpha} e'$ that corresponds to $e$ taking place at least $\alpha$ time units
before $e'$. It appears that $\xrightarrow{\alpha}$ is a quantitative generalization of Lamport
causality for the purpose of determining relative timing of events. A similar
notion appears in the work of Patt-Shamir and Rajsbaum [43].

Chapter 4 deals extensively with relations between nested knowledge,
common knowledge and time. The growing body of research dealing with
the dynamics of interactive epistemology has brought to light some of the
intricate relations here [23, 15, 50]. Halpern and Moses [23] proved that com-
mon knowledge cannot be gained in the face of unreliable or asynchronous
communication. Parikh and Ramanujam [41] investigate nested knowledge
in connection to formal languages. Common knowledge is typically perceived
in terms of an infinite conjunction of $E^k$, for $k > 0$. There are also definitions
of common knowledge in terms of a fixed point (see, e.g., [29, 15, 5]). Fischer
and Immerman [18] first showed that the level of nested knowledge that can
be achieved without “collapsing” into common knowledge are bounded in
finite state systems. The combinations of nested and common knowledge
that are discussed in Chapter 5 are, interestingly, somewhat similar to those
found in Chwe’s [8] game theoretic analysis of coordination.
Chapter 2

Response Problems and Knowledge in Synchronous Systems

2.1 Studying Coordination via Response Problems

Existing literature about causality in distributed systems deals almost exclusively with asynchronous systems. This is not really surprising, given that a major application for Lamport’s happened-before is in providing some sort of synchronized layer on top of asynchronous communication networks, and that in synchronous systems such a layer is already provided for.

Yet in distributed systems, being able to share a certain global sense of time is, in most cases, only a means to an end, the real purpose being coordinating events across remote sites. Is the coordination of events in a synchronous system as easy as looking at the clock? In some cases yes. Consider a simple system where Zoe and Xerxes operate based on a prearranged protocol that ensures that Zoe will pick Xerxes up for the movies at 7:45. Come 7:45, Xerxes looks at the clock, gets up and goes outside and into Zoe’s car that had just come by.

At other times though, a global clock is not enough to ensure coordination. If Zoe’s arrival hinges upon her getting through all work meetings by 6:50 (an unpredictable occurrence), then Xerxes may find himself alone at 7:45. In synchronous systems that allow for nondeterministic occurrences, a global clock cannot by itself ensure proper coordination across sites, if all events hinge upon some initial nondeterministic occurrence as a trigger. And yet many coordination tasks depend upon external input, whether in
the form of timing or of an assignment to an unknown parameter, as a trigger. Such external input is, for all practical purposes, nondeterministic.

We capture the essence of such coordination tasks in the following manner. We identify a particular spontaneous external input as a *triggering event*, denoted by $e_t$. A run in which the trigger $e_t$ occurs is said to be *triggered*. An intended *response* to such a trigger is specified by a pair $\alpha_h = \langle a_h, i_h \rangle$ with $a_h$ being an action for process $i_h$. A response $\alpha_h$ takes place if process $i_h$ performs the action $a_h$. An instance of the Ordered Response problem is parametrized by a tuple $\langle e_t, \alpha_1, \ldots, \alpha_k \rangle$, consisting of a trigger and a sequence of responses. Formally, we define the following class of problems.

**Definition 2 (Ordered Response)** A protocol $P$ solves the instance $\text{OR} = \langle e_t, \alpha_1, \ldots, \alpha_k \rangle$ of the Ordered Response problem if it guarantees that

1. in a triggered run, every response $\alpha_h$, for $h = 1, \ldots, k$, will occur; moreover, if $h < k$ then $\alpha_h$ will happen before (i.e., no later than) $\alpha_{h+1}$ does. Finally,

2. none of the responses $\alpha_h$ occur in runs that are not triggered.

Consider the following simplified scenario, where such a problem is implied.

**Example 1** Charlie’s bank account is temporarily suspended due to credit problems. Should Charlie make a sufficient deposit at his local branch, Banker Bob at headquarters will re-activate the account. Alice holds a cheque from Charlie, but trying to cash it before the account is re-activated will grant her a fine, rather than cash.

Alice, Bob and Charlie can communicate over a communication network as depicted in Figure 3.1a, where the labels represent maximal transmission times. In particular, messages from Charlie to Bob and Alice take up to 10 and 12 days to be delivered, respectively. This scenario can be viewed as an instance of $\text{OR}$ in which a deposit by Charlie is the triggering event, and the responses are the account re-activation by Bob followed by Alice’s cashing of the cheque.

Intuitively, we expect Alice, Bob and Charlie to communicate in order to ensure a proper ordering of events. Indeed, as shown by Chandy and Misra,

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1For simplicity, we assume that $e_t$ happens at most once in any given run, as do each one of the actions performed in response to it.
if the network were asynchronous there would be no other way to ensure correct ordering except by a message chain linking Charlie to Bob, and then to Alice. The synchronous system of Example 1 offers more freedom. For example, Figure 2.1b shows Charlie sending word to Alice, who sends on the message to Bob and waits for 5 rounds to make sure of its arrival before safely cashing her check. We use a dashed arrow to denote that a message sent by Alice at \( t + 3 \) is sure to arrive at Bob’s by \( t + 8 \).

The Ordered Response problem captures a natural coordination scenario, and its precise specification provides us with a clearly defined scope within which synchronous causality can be investigated. Chapter 3 studies it further and establishes the exact scope of flexibility in communication that is allowed, for establishing ordering such synchronous systems. As we will see in the following sections of this chapter, knowledge plays a key role in providing such characterizations.

We now turn to define another type of coordination problem. In synchronous systems it is often desirable to perform actions simultaneously at different sites, a classic example being the firing squad problem [10]. A natural variant of OR is the Simultaneous Response problem, defined as follows.

**Definition 3 (Simultaneous Response)** Let \( e_t \) be an external input. Then \( \text{SR} = (e_t, \alpha_1, \ldots, \alpha_k) \) defines an instance of the Simultaneous Response problem. A protocol solves the instance \( \text{SR} \) if it guarantees that if the triggering event \( e_t \) occurs, then at some later point all actions \( \alpha_1, \ldots, \alpha_k \) in the response set of \( \text{SR} \) will be performed simultaneously.

A causal analysis of the Simultaneous Response problem will be conducted in Chapter 4. Note that the Simultaneous Response problem can be characterized by means of multiple ordering response problems. Let \( \text{SR} = (e_t, \alpha_1, \ldots, \alpha_k) \). For each pair of required simultaneous responses
α, β ∈ {α₁, ..., αₖ}, we define two ordering response problems: OR^{αβ} = ⟨eₜ, α, β⟩ and OR^{βα} = ⟨eₜ, β, α⟩. A protocol that solves both OR^{αβ} and OR^{βα} will solve the simultaneous subproblem SR^{αβ} = ⟨eₜ, α, β⟩. A protocol that solves both OR^{αβ} and OR^{βα} for every α, β ∈ {α₁, ..., αₖ} will solve SR.

Nevertheless, defining the simultaneous requirement as a separate problem is worthwhile because solutions to such problems give rise to tighter epistemic characterizations, in the form of common knowledge among the set of responding processes. Chapters 5 and 6 will investigate aspects of generalizing the ordering requirements beyond ordered and simultaneous responses.

2.2 Knowledge Gain in Synchronous Systems

As noted in Chapter 1, Chandy and Misra used knowledge gain to refer to a scenario wherein process i “gains knowledge”, or “learns”, that some fact ϕ pertaining to process j holds. In an asynchronous system, as it turns out, the only way for i to gain knowledge about j is by means of a message chain relating the two. Thus, in the asynchronous setting, knowledge gain reflects the way information flows in the system.

Taken at face value, Chandy and Misra’s notion no longer captures information flow when we move to synchronous systems, as process i may learn facts about process j by a mere glance at the clock (for example, by noting that my watch shows 4pm I “learn” that your watch shows 4pm too right now, something that I did not know before, while it was still 3:59). In order to maintain the desirable association with information flow we turn, as we did in the previous section, to nondeterministic occurrences.

Definition 4 (Knowledge gain in synchronous systems) We will say that knowledge gain occurs in the interval [t..t′] of run r whenever a nondeterministic event occurs at some process j no sooner than time t, and process i knows of this occurrence by time t′.

At an intuitive level, we expect that knowledge of an ND (nondeterministic) event is dependent upon communication, and hence knowledge gain of such facts will reflect information flow. The following chapters will pursue the relations between knowledge gain and communication. Natural generalizations of the above notion of knowledge gain are nested knowledge gain and common knowledge gain.
Nested knowledge gain occurs within the interval $[t..t']$ of run $r$ if an ND event $e$ occurs at process $j$ within the interval, and by the end of the interval $K_{i_1}K_{i_{h-1}} \cdots K_{i_1}$ occurred($e$) holds for some sequence of processes $(i_1, i_2, ..., i_h)$. On a similar vein, common knowledge gain occurs when, by the end of the interval of time, $C_G$ occurred($e$) holds for some group of processes $G$.

The suggestive similarity of nested and common knowledge to the Ordered and Simultaneous Response problems respectively, will be examined in Section 2.3.

### 2.3 Relating Response Problems and Knowledge Gain

This section charts out the formal relations between the two previously defined response problems, and knowledge gain.

#### 2.3.1 Response Problem to Knowledge Gain

We start by looking at the Ordered Response problem. Intuitively, the coordination of responses so that they occur in a particular sequence suggests that knowledge gain is involved. In order to ensure that the events occur in sequence, each responder $k$ must know that the previous $k-1$ responses, as well as the trigger event, had already occurred (or are occurring right now).

We now show that this is indeed the case. A caveat concerning knowledge is that we must assume of processes that they do not forget that they had already performed a response. Formally, this property is described as follows.

**Definition 5 (Response recall)** Let OR be defined by $(e_t, \alpha_1, ..., \alpha_k)$ and assume that $R = R(P, \gamma)$ is a system of runs for a protocol $P$ where all of the responses may occur, and $\gamma$ is any arbitrary context. Protocol $P$ recalls responses for OR if for all $1 \leq h \leq k$, $r, r'$ in $R$ and $t' \leq t$, if $\alpha_h$ occurs at $(i_h, t')$ in $r$ and $(r, t) \sim_i (r', t)$, then $\alpha_h$ also occurs at $(i_h, t')$ in $r'$.

As we’ll show in Section 3.6, the response recall assumption is not needed when Ordered Response is related directly to communication, rather than to knowledge. Note that the following theorem, and Theorem 2 as well, are not dependent on the particular context $\gamma$ being used.
**Theorem 1** Let \( \text{OR} = \langle e_\gamma, \alpha_1, \ldots, \alpha_k \rangle \) be an instance of \( \text{OR} \), and assume that protocol \( \text{P} \) solves \( \text{OR} \) in \( \gamma \) and that it recalls responses for it. Let \( r \in \mathcal{R} \) be a run in which \( e_\gamma \) occurs, let \( 1 \leq h \leq k \), and let \( t_h \) be the time at which \( i_h \) performs action \( a_h \) in \( r \). Then

\[
(\mathcal{R}, r, t_h) \not\vDash K_{i_h}K_{i_{h-1}} \cdots K_{i_1}(\text{occurred}(e_\gamma) \land \text{ND}(e_\gamma))
\]

**Proof** We prove the theorem by induction on \( k \).

1. \( k = 1 \): By definition of \( r \), process \( i_1 \) performs \( a_1 \) at \( t_1 \). If \( (\mathcal{R}, r, t_1) \not\vDash K_{i_1}\text{occurred}(e_\gamma) \) then by definition of \( \vDash \) there exists a run \( r' \in \mathcal{R} \) such that \( (r, t_1) \sim_{i_1} (r', t_1) \) and where \( (\mathcal{R}, r', t_1) \not\vDash \text{occurred}(e_\gamma) \). Yet as \( \text{P} \) solves \( \text{OR} \), in \( r' \) action \( a_1 \) is performed only if \( e_\gamma \) has occurred, contradiction. Therefore it must be that \( (\mathcal{R}, r, t_1) \vDash K_{i_1}\text{occurred}(e_\gamma) \).

By definition of \( \text{OR} \), \( e_\gamma \) is an external input event and hence nondeterministic. This is universally true in the system, and hence \( (\mathcal{R}, r, t_1) \vDash K_{i_1}\text{occurred}(e_\gamma) \) implies \( (\mathcal{R}, r, t_1) \vDash K_{i_1}(\text{occurred}(e_\gamma) \land \text{ND}(e_\gamma)) \).

2. \( k > 1 \): Suppose that it is the case that

\[
(\mathcal{R}, r, t_k) \not\vDash K_{i_k}K_{i_{k-1}} \cdots K_{i_1}(\text{occurred}(e_\gamma) \land \text{ND}(e_\gamma))
\]

Then by definition of \( \vDash \) there exists a run \( r' \in \mathcal{R} \) such that \( (r, t_k) \sim_{i_k} (r', t_k) \) and where \( (\mathcal{R}, r', t_k) \not\vDash K_{i_{k-1}}K_{i_{k-2}} \cdots K_{i_1}(\text{occurred}(e_\gamma) \land \text{ND}(e_\gamma)) \). Since the protocol recalls responses, we now obtain that

\[
(\mathcal{R}, r', t_{k-1}) \not\vDash K_{i_{k-1}}K_{i_{k-2}} \cdots K_{i_1}(\text{occurred}(e_\gamma) \land \text{ND}(e_\gamma))
\]

However, again as \( \text{P} \) solves \( \text{OR} \), it also solves the sub-problem \( \text{OR}' \) defined by \( \langle e_\gamma, \alpha_1, \ldots, \alpha_{k-1} \rangle \). As \( \alpha_k \) is also performed by \( i_k \) in \( r' \), it must be that \( \alpha_1, \ldots, \alpha_{k-1} \) too get performed in \( r' \). By the inductive hypothesis we get that for all \( 1 \leq h \leq k - 1 \)

\[
(\mathcal{R}, r', t_h) \vDash K_{i_h}K_{i_{h-1}} \cdots K_{i_1}(\text{occurred}(e_\gamma) \land \text{ND}(e_\gamma))
\]

In particular we get that

\[
(\mathcal{R}, r', t_{k-1}) \vDash K_{i_{k-1}}K_{i_{k-2}} \cdots K_{i_1}(\text{occurred}(e_\gamma) \land \text{ND}(e_\gamma))
\]

This contradicts the previous result, and therefore it must be the case that

\[
(\mathcal{R}, r, t_h) \vDash K_{i_h}K_{i_{h-1}} \cdots K_{i_1}(\text{occurred}(e_\gamma) \land \text{ND}(e_\gamma))
\]

for all \( 1 \leq h \leq k \), as required.
We now turn to consider the Simultaneous Response problem. Here too there is an intuitive connection with knowledge gain. If all responses are always performed simultaneously, then every responding site must know that the other sites are responding too. Yet as the next theorem shows, the simultaneous response requirement implies an even stronger epistemic condition, in the form of common knowledge. As all responses must occur simultaneously, there is actually no need here to assume the response recall property from the arbitrary protocol.

**Theorem 2** Let $SR = (e_1, a_1, \ldots, a_k)$, and assume that protocol $P$ solves $SR$ in $\gamma$. Moreover, let $G = \{i_1, \ldots, i_h\}$ be the set of processes appearing in the response set of $SR$. Finally, let $r \in R$ be a run in which $e_1$ occurs, and let $t$ be the time at which the response actions are performed in $r$. Then $(R, r, t) \models C_G(\text{occurred}(e_1) \land \text{ND}(e_1))$.

**Proof** Fix $h, g \in \{1..k\}$. We first show that $R \models \text{occurred}(a_h) \rightarrow E_G(\text{occurred}(a_h)) \land \text{occurred}(e_1)$. Choose $r', t'$ such that $(R, r', t') \models \text{occurred}(a_h)$. Note that since $P$ solves $SR$ and since response actions are performed only upon the occurrence of the trigger event $e_1$, we get $(R, r', t') \models \text{occurred}(a_h) \leftrightarrow \text{occurred}(a_g)$ and $(R, r', t') \models \text{occurred}(a_h) \rightarrow \text{occurred}(e_1)$. From the former equivalence and $(R, r', t') \models \text{occurred}(a_h)$ we obtain that $(R, r', t') \models \text{occurred}(a_g)$. Since performing a local action is written, at least for the current round, in the process’s local state, we obtain that $(R, r', t') \models K_{i_g} \text{occurred}(a_g)$. Now using the former equivalence again we get that $(R, r', t') \models K_{i_g} \text{occurred}(a_h)$, and using the latter implication we get $(R, r', t') \models K_{i_g} \text{occurred}(e_1)$. Putting these results together we conclude that $(R, r', t') \models K_{i_g} (\text{occurred}(a_h) \land \text{occurred}(e_1))$. Since $g$ is arbitrarily chosen in $G$, we get $(R, r', t') \models E_G(\text{occurred}(a_h) \land \text{occurred}(e_1))$, from which it follows that $(R, r', t') \models \text{occurred}(a_h) \rightarrow E_G(\text{occurred}(a_h) \land \text{occurred}(e_1))$

by our choice of $r', t'$. As false antecedents imply anything, we conclude that $R \models \text{occurred}(a_h) \rightarrow E_G(\text{occurred}(a_h) \land \text{occurred}(e_1))$.

Recall the Knowledge Induction Rule, that derives $R \models \varphi \rightarrow C_G \psi$ from $R \models \varphi \rightarrow E_G(\varphi \land \psi)$. Setting $\varphi = \text{occurred}(a_h)$ and $\psi = \text{occurred}(e_1)$ we apply the rule, and based on the above result obtain $R \models \text{occurred}(a_h) \rightarrow \text{occurred}(e_1)$.
$C_G\text{occurred}(e_t)$. We conclude by noting that $(\mathcal{R}, r, t) \models \text{occurs}(a_h)$ by assumption, and hence also $(\mathcal{R}, r, t) \models C_G\text{occurred}(e_t)$.

By definition of SR, $e_t$ is an external input event and hence nondeterministic. This is universally true in the system, and hence in particular $(\mathcal{R}, r, t) \models C_G\text{occurred}(e_t)$ implies $(\mathcal{R}, r, t) \models C_G(\text{occurred}(e_t) \land \text{ND}(e_t))$. \[\square\text{Theorem 2}\]

Theorems 1 and 2 show that, in a precise sense, knowledge gain is a prerequisite for coordinated response. The next section will show that nested and common knowledge gain indeed characterize ordered and simultaneous responses, in the sense that they define the minimal epistemic prerequisites for such types of coordination.

### 2.3.2 Knowledge Gain to Response Problem

It is immediately apparent that no general law exists showing that knowledge gain implies a solution to a response problem, for processes are not, in general, required to act in any way upon the knowledge that they gain.

In order to bridge the gap between knowledge and action, we would need to add requirements on the protocol being followed by the processes. Since, as stated in Section 1.2, this thesis is focussed on producing protocol-independent results, we do not delve deeply into such additions. \[^2\]

Nevertheless, as we will show, there exist protocols where indeed knowledge gain implies a solution to the response problem. Proving the existence of such a protocol comes to show that the relevant response problem (say ordered response), is indeed characterized by the related type of knowledge gain (in this case, nested knowledge gain). The existence of such a protocol shows that, in general, no epistemic state stronger than nested knowledge can be gained as a result of solving the ordering response problem. \[^3\]

To exemplify the existence of protocols where nested knowledge gain implies a solution to OR, we introduce the following property for protocols.

**Definition 6 (Non-Hesitant Protocol)** Protocol $P$ is non-hesitant with respect to an Ordered Response problem $\text{OR} = \langle e_t, \alpha_1, \ldots, \alpha_k \rangle$ if for each $h \leq k$, process $i_h$ performs $\alpha_h$ as soon as $K_{i_h}K_{i_{h-1}} \ldots K_{i_1}(\text{occurred}(e_t) \land \text{ND}(e_t))$ is established, but no sooner.

---

\[^2\] One could try, for example, to characterize those protocols where knowledge gain *does* imply a solution to the related response problem.

\[^3\] A similar argument is presented in Sections 3.9 and 4.5, in order to show that the yet to be defined centipede and centibroom communication patterns characterize nested and common knowledge gain, respectively.
As the following lemma shows, nested knowledge gain is sufficient for solving OR problems in protocols that are non-hesitant with respect to the problem. As for the existence of actual protocols that comply with the above definition, we can make things easy by assuming the context $\gamma^{\max}$ (as we do in the next two chapters). Here it is easy to find protocols that satisfy non-hesitance, as well as consideracy (defined below) qualifications by insisting that processes timestamp their messages.

**Lemma 1** Let $\text{OR} = \langle e_t, \alpha_1, \ldots, \alpha_k \rangle$ and let $P$ be a non-hesitant protocol with respect to $\text{OR}$. If for every $r \in \mathcal{R}^{\max} = \mathcal{R}(P, \gamma^{\max})$ such that $e_t$ occurs at $(i_0, t)$ in $r$ there exists time $t'$ such that

$$(\mathcal{R}^{\max}, r, t') \models K_{i_k}K_{i_{k-1}} \cdots K_{i_1} (\text{occurred}(e) \land \text{ND}(e)),$$

then $P$ solves $\text{OR}$.

**Proof** Assume $r \in \mathcal{R}^{\max}$ such that $e_t$ occurs at $(i_0, t)$ in $r$. From

$$(\mathcal{R}^{\max}, r, t') \models K_{i_k}K_{i_{k-1}} \cdots K_{i_1} (\text{occurred}(e) \land \text{ND}(e))$$

we obtain the existence of $t_k \leq t'$ such that

(a) $(\mathcal{R}^{\max}, r, t_h) \models K_{i_k}K_{i_{k-1}} \cdots K_{i_1} (\text{occurred}(e) \land \text{ND}(e))$, and

(b) $t_k = 0$ or $(\mathcal{R}^{\max}, r, t_k - 1) \not\models K_{i_k}K_{i_{k-1}} \cdots K_{i_1} (\text{occurred}(e) \land \text{ND}(e))$.

By repeated applications of the Knowledge Axiom we extend this result into a series $t \leq t_1 \leq \cdots \leq t_k \leq t'$ such that for every $h \leq k$

(a) $(\mathcal{R}^{\max}, r, t_h) \models K_{i_h}K_{i_{h-1}} \cdots K_{i_1} (\text{occurred}(e) \land \text{ND}(e))$, and

(b) $t_h = 0$ or $(\mathcal{R}^{\max}, r, t_h - 1) \not\models K_{i_k}K_{i_{h-1}} \cdots K_{i_1} (\text{occurred}(e) \land \text{ND}(e))$.

As $P$ is non hesitant with respect to $\text{OR}$, we get that for every $h \leq k$ process $i_h$ performs $\alpha_h$ at $t_h$, and we are done. ■

Once again switching to the Simultaneous Response problem, we need a protocol where processes are more considerate, in order to ensure a solution to the problem.

**Definition 7 (Considerate protocol)** Protocol $P$ is considerate with respect to a Simultaneous Response problem $\text{SR} = \langle e_t, \alpha_1, \ldots, \alpha_k \rangle$ if for each $h \leq k$, process $i_h$ performs $\alpha_h$ as soon as $C_{i_1, \ldots, i_k} (\text{occurred}(e_t) \land \text{ND}(e_t))$ is established, but no sooner.
Lemma 2 Let $SR = \langle e, \alpha_1, \ldots, \alpha_k \rangle$, let $G = \langle i_1, \ldots, i_k \rangle$ and let $P$ be a considerate protocol with respect to $SR$. If for every $r \in \mathcal{R}^{\max} = \mathcal{R}(P, \gamma)$ such that $e$ occurs at $(i_0, t)$ in $r$ there exists time $t'$ such that $(\mathcal{R}^{\max}, r, t') \models C_G(\text{occurred}(e) \land \text{ND}(e))$, then $P$ solves OR.

The lemma’s proof is immediate if we consider that $C_G \varphi \leftrightarrow K_g C_G \varphi$ for any $g \in G$. 


Chapter 3

Gaining Nested Knowledge

3.1 Introduction

This chapter investigates the minimal communication needed in order to achieve nested knowledge gain in synchronous systems. As such, its results provide an immediate generalization of previous findings pertaining to asynchronous ones.

We have argued elsewhere (see Chapter 2) that knowledge gain can be seen as a close approximation of the spread of causal effect in distributed systems. Yet even though it is more rigorously defined than causality, knowledge is still a rather abstract notion. Thus, we motivate our investigation by studying the more concrete Ordered Response problem. We will show that in order for a protocol to solve the problem, a certain generalization of message chains must relate the trigger and responding sites in every triggered run.

Sections 3.1 through 3.6 will introduce and discuss the new concepts involved in the analysis, and informally sketch out the results in terms of a necessity relation tying in communication to Ordered Response solutions. Sections 3.6 to 3.9 will then retrace our steps and provide the necessary formal underpinnings that uphold these results. The methodology, as discussed in Section 1.2, will be to prove that certain communication patterns are necessary in order for knowledge gain to arise, and then to use Theorem 1 to similarly relate these patterns to an Ordered Response.

For the sake of clear presentation, we assume throughout this chapter and the next one that all examples and proofs take place over a synchronous system in which upper bounds are given for every existing communication channel. We have denoted contexts that generate such systems by $\gamma_{\text{max}}$ (see
Section 1.5 for more). We start by scrutinizing the frozen account example first shown in Section 2.1.

**Example 2** Charlie’s bank account is temporarily suspended due to credit problems. Should Charlie make a sufficient deposit at his local branch, Banker Bob at headquarters will re-activate the account. Alice holds a cheque from Charlie, but trying to cash it before the account is re-activated will grant her a fine, rather than cash. Alice, Bob and Charlie can communicate over a communication network as depicted in Figure 3.1a. In particular, messages from Charlie to Bob and Alice take up to 10 and 12 days to be delivered, respectively. This scenario can be viewed as an instance of OR in which a deposit by Charlie is the event, and the responses are the account re-activation by Bob followed by Alice’s cashing of the cheque.

In a particular instance, depicted in Figure 3.1b, Charlie makes a deposit at time $t$, and immediately broadcasts a message stating this to both Alice and Bob. The message reaches Bob in 2 days and Alice in 4. Bob immediately re-activates Charlie’s account \(^1\) at time $t + 2$. When can Alice deposit the cheque? The cheque would be cashed successfully at any time after $t + 2$. However, Alice only knows about Charlie’s deposit at $t + 4$. But even at that point, she must keep waiting. In the absence of additional information indicating when Bob actually received Charlie’s message, she is only guaranteed that this will happen by time $t + 10$. Knowing Bob’s protocol, she can safely submit the cheque at or after time $t + 10$, but not sooner.

In this example, Alice acts after Bob does. While in an asynchronous setting she would need to obtain explicit notification that Bob acted, in the synchronous setting considered in Example 2 she can base her action on the

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\(^1\)For ease of exposition, we assume throughout the thesis that actions are performed instantaneously; alternative assumptions would not significantly affect the analysis.
Example 3 In a setting similar to Example 2, Susan is Bob’s supervisor at the bank. The network is now as depicted in Figure 3.2a. Suppose that Charlie broadcasts his deposit to all three, and that communication is delivered as in Figure 3.2b. In this case Alice can, as before, submit her cheque at $t + 10$. But she can do even better. Since she receives a message from Susan at $t + 8$ that was sent at $t + 3$, the bound on the Susan-Bob channel ensures her that Bob is to be notified of Charlie’s deposit by time $t + 7$. Thus Charlie’s account will also be solvent as of time $t + 7$, and Alice can safely cash her cheque in this case upon receiving Susan’s message.

In both examples, the timing of Alice’s action depends on the time bounds, but Example 3 shows a more complex interaction between message arrivals and time bounds. In Example 2 Alice combines information gained by means of a message chain from Charlie to her with the known time bound on the Charlie-Bob channel. Charlie’s message to her serves both to notify her about the occurrence of a deposit event, and as a temporal anchor for a timing argument that allows her to properly coordinate her response with Bob’s action.

Example 3 starts out the same for Alice. She is still notified of the deposit event by a message from Charlie at $t + 4$. As before, this message can also be used to coordinate her response to follow Bob’s by clearing the
cheque at \( t + 10 \). But then the message from Susan arrives, and a second message chain between Charlie and Alice is completed. Alice already knows about Charlie’s deposit based on the earlier message. The new message serves her to coordinate a temporally “tighter” response to Bob’s action at \( t + 8 \) rather than \( t + 10 \). For the purpose of coordination, Susan’s message plays a similar role in Example 3 to that played by Charlie’s in Example 2.

Had the triggering event \( e_t \) been unconditionally guaranteed to take place at some time \( t_0 \), then the protocol could directly specify the times \( t_k \geq t_{k-1} \geq \cdots \geq t_1 \geq t_0 \) at which the response actions could be performed with proper coordination. However, since \( e_t \) is a spontaneous event, knowledge about its occurrence must “flow” from \( i_0 \) to the responding sites. With respect to coordination, however, the above examples demonstrate that coordination between the sites does not necessary require explicit communication between successive responses. A site \( i_{h+1} \) may be able to know that \( \alpha_h \) has taken place by combining a priori knowledge regarding timing guarantees, information it has regarding other processes’ protocols, and information it obtains via explicit communication.

Examples 2 and 3 illustrate how a process can come to coordinate its response with another process despite the lack of explicit communication between the sites, based on knowledge of existing upper bounds on communication. But bounds can be used in an additional fashion. Namely, if by time \( t + \max ij \) process \( j \) receives no message sent by \( i \) at time \( t \), then \( j \) can discover that no such message was sent [27]. Depending on \( i \)’s protocol, this can provide \( j \) with information about \( i \)’s state at time \( t \). Consider the following refinement of Example 3.

**Example 4** In the network of Example 3 depicted in Figure 3.2a, suppose that Susan sends Alice a message in every round as long as Susan has not heard from Charlie about an appropriate deposit. In this particular instance, Susan receives a message from Charlie at time \( t+2 \), at which point she stops sending her update messages. At time \( t+7 \) Alice will be able to “time-out” on Susan’s time \( t+2 \) message. She then knows that Susan heard from Charlie at \( t+2 \). Moreover, knowing that Susan relays information to Bob as before, Alice knows that Bob heard about the deposit no later than time \( t+6 \). Hence, Alice can safely cash her cheque at time \( t+7 \) rather than \( t+10 \). \( \square \)

In Example 4 Alice learns of Charlie’s deposit without receiving any message whatsoever. She clearly receives no message chain originating from Charlie. Nevertheless, it seems instructive to think of Susan as sending Alice a “NULL message” in the sense of [27] at time \( t+2 \), carrying relevant
information, by not sending an actual message. Under this interpretation, Example 4 contains a message chain from Charlie to Alice that consists of Charlie’s concrete message to Susan, followed by Susan’s NULL message to Alice.

Lamport utilized NULL messages in [27] for his algorithms implementing the state machine model in the synchronous setting. In the current paper however, NULL messages will be used to define a relation called *syncausality*, extending happened before. A syncausal chain will then be a chain consisting of a sequence of concrete and NULL messages. Syncausal chains are required for information flow regarding nondeterministic events such as spontaneous external inputs.

### 3.2 Bound Guarantees

In the synchronous model we consider, every directed communication link between adjacent processes $i$ and $j$ provides a bound $\max_{ij}$ on the maximal transmission time of messages. These local bounds naturally induce more general bounds, or guarantees as we call them, for any pair of (not necessarily directly) connected sites.

![A variant network graph](image1)

(a) A variant network graph

![Transmission distances](image2)

(b) Transmission distances

Figure 3.3: Timing guarantees

Consider Figure 3.3a, showing a variant of the network graph shown in Figure 3.2a. Assuming that all of the sites involved are fully cooperative in relaying messages, a message sent by Charlie with destination Bob can be guaranteed to arrive after 9 rounds, if it travels from Charlie to Susan and then to Bob, rather than from Charlie to Bob directly. Similarly, a message from Susan to Alice can be guaranteed to arrive after no more than 5 rounds, if it makes the roundabout trip through Bob. Charlie and Alice may also communicate, with a bound of 11 rounds, if they use Susan and Bob as a relays.
Thus, the naturally induced transmission distance between processes \( h \) and \( k \), which we denote by \( D(h, k) \), is the minimal distance between \( h \) and \( k \) in the weighted directed network graph, in which the bounds \( \max_{ij} \) on transmission times are the edge weights. In particular, \( D(i, i) = 0 \) for all \( i \in P \). Figure 3.3b shows the induced transmission distances for the network of Figure 3.3a. Values that differ from the ones in (a) are shown in a gray box.

We find it convenient to represent that time instant \( t \) on process \( i \)'s timeline by the pair \((i, t)\), called a process-time node, or node for short. Based on the transmission time bounds for the channels, we define the following bound guarantee relation among process-time nodes:

**Definition 8 (Bound Guarantee)** We say that \((i, t)\) and \((j, t')\) are related by a bound guarantee, and write \((i, t) \rightarrow (j, t')\), iff \( t + D(i, j) \leq t' \).

Observe that bound guarantees are independent of the speed at which messages actually arrive at a particular run; they depend only on the weighted network topology. If \((i, t) \rightarrow (j, t')\) then it is possible to guarantee that a message sent by \( i \) at time \( t \) will arrive at \( j \) by time \( t' \), assuming that relay is instantaneous. Since the bound-weighted network is assumed to be known to the processes, the passage of time can allow a process to obtain knowledge about remote events that would not be available, say, in an asynchronous setting.

The next sections will explore the ways by which this knowledge can be exploited.

### 3.3 Syncausality

In Example 4 Alice learns of Charlie’s deposit without a message chain from Charlie reaching her. A message chain of a slightly more general type does exist there, however, in which Susan’s not sending a message to Alice at time \( t + 3 \) is a Null message. More formally, consider a network in which \( i \) and \( j \) are directly connected by a communication link with bound \( \max_{ij} \). Then \( i \) can be thought of as “sending” a Null message over this channel at \((i, t)\) if it sends no physical message over the channel at time \( t \). This message is considered as being “delivered” to \( j \) at \((j, t + \max_{ij})\) (see [27]). In the presence of clocks and bound guarantees, Null messages can serve to transfer information between processes. By identifying that no message was sent at \((i, t)\), process \( j \) may be able to draw nontrivial conclusions about \( i \)'s state and \( i \)'s knowledge there.
We now formally define Syncausality, a generalization of Lamport’s happened-before relation that accounts for NULL messages and is thus based on “generalized” message chains that can contain NULL messages as well as normal messages. The relation is defined over process-time nodes rather than events, since defining not sending and not receiving messages as explicit events would be cumbersome.

**Definition 9 (Syncausality)** Fix a run $r$. The syncausality relation $\rightarrow$ over nodes of $r$ is the smallest relation satisfying the following four conditions:

1. If $t \leq t'$, then $(i, t) \rightarrow (i, t')$;

2. If some message is sent at $(i, t)$ and received at $(j, t')$ then $(i, t) \rightarrow (j, t')$;

3. If no message is sent at $(i, t)$ to $i$’s neighbor $j$ then $(i, t) \rightarrow (j, t + \max_{ij})$; and

4. If $(i, t) \rightarrow (h, \hat{t})$ and $(h, \hat{t}) \rightarrow (j, t')$, then $(i, t) \rightarrow (j, t')$.

Clauses (1), (2) and (4) correspond to the local precedence, message precedence and transitivity clauses that define the happened-before relation. Syncausality thus refines (and hence directly generalizes) happened-before. The third clause corresponds to timeout precedence, capturing the case of a NULL message being sent by $(i, t)$ and eventually received at $(j, t + \max_{ij})$. We can thus view syncausality as being based on syncausal chains, consisting of a chain of actual and NULL messages.

Syncausality is also a generalization of bound guarantees. Note that nodes $(i, t)$ and $(j, t + \max_{ij})$ will always be syncausally related: either $(i, t)$ does send $j$ a message, in which case the message is received by $(j, t + \max_{ij})$ and $(i, t) \rightarrow (j, t + \max_{ij})$ will hold by message and local precedence, or no such message is sent, which case $(i, t)$ and $(i, t) \rightarrow (j, t + \max_{ij})$ will hold by timeout precedence. Indeed, a straightforward induction on the number of edges in the shortest path of length $D(i, j)$ between nodes $i$ and $j$ in the network immediately yields:

**Lemma 3** If $(i, t) \rightarrow (j, t')$ then $(i, t) \rightarrow (j, t')$ in every run $r$.

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**Footnote:** In fact, given the natural definition of $\rightarrow$ over process-time nodes, $\rightarrow$ is the coarsest common refinement of $\rightarrow$ and $\rightarrow'$. 

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**Page 35**
Notice that the bound guarantee relation depends only on the network and the transmission bounds \( \max_{ij} \). We view it as being given \textit{a priori} as part of the context. By contrast, clause (2) of the syncausality relation, capturing message precedence, depends on the actually realized message transmission times in a given run. Therefore, syncausality is run-dependent.

We will show in Section 3.6 that knowledge about the occurrence of nondeterministic events can be obtained only by way of syncausal chains. Consequently, for the Ordered Response problem we can show the following:

**Theorem 3** Let \( P \) be a protocol solving \( \text{OR} = \langle e_t, \alpha_1, \ldots, \alpha_k \rangle \). If \( e_t \) occurs at \( (i_0, t) \) in \( r \in \mathbb{R}^{\max} \), and \( \alpha_h \) does at \( (i_h, t_h) \), then \( (i_0, t) \xrightarrow{} (i_h, t_h) \) in \( r \).

The formal proof of Theorem 3 can be found in Section 3.6. It is obtained by first showing that indeed \( i_h \) needs to know that \( e_t \) occurred at \( (i_h, t_h) \), and then showing that without the syncausal connection process \( i_h \) can not know this.

### 3.4 Double Response

The examples in the introduction all involve a simple problem of the form \( \text{OR} = \langle e_t, \alpha_1, \alpha_2 \rangle \). We call this form a \textit{double response}. We can view a double response as incorporating two single responses to the triggering event, with an added coordination requirement to ensure that that \( \alpha_2 \) does not occur before \( \alpha_1 \).

If the first response is performed by \( i_1 \) at time \( t_1 \), and the second by \( i_2 \) at \( t_2 \), then Theorem 3 implies that in a triggered run of any protocol solving \( \text{OR} \), necessarily \( (i_0, t) \xrightarrow{} (i_1, t_1) \) and \( (i_0, t) \xrightarrow{} (i_2, t_2) \) must hold. A number of ways by which the required coordination between \( i_1 \) and \( i_2 \) may be achieved have already been informally considered in Sections 3.1 and 2.1.

First, if \( (i_0, t) \xrightarrow{} (i_1, t_1) \xrightarrow{} (i_2, t') \), as seen in Figure 3.4, then it is easy to see that \( t_1 \leq t' \). Wavy arrows have replaced the straight ones used in Section 3.1, to denote the possibility of non-trivial syncausal chains connecting the nodes. This case echoes the chain structures prevalent in the asynchronous model, with syncausal chains replacing the pure message chains. Yet as our previous examples have shown, there are other possible means for coordination.

![Figure 3.4](image-url)
In Example 2 we had $i_2$ (played out by Alice) waiting until a time $t' = t + 10$ such that $(i_0, t) \rightarrow (i_1, t')$. So by time $t'$ when $a_2$ is performed, $i_1$ must surely have gotten the message about the occurrence of $e_1$ and have performed $a_1$. This possibility is schematically shown in Figure 3.5. The figure shows the syncausal relations that are necessary in order for $i_2$ to coordinate its action with $i_1$. As $i_2$ need not be aware of the realized connection between $i_0$ and $i_1$ and the time $t_1$ when $a_1$ was actually performed, this connection is not shown.

In Examples 3 and 4 there exists some node $(i_3, t_3)$, representing Susan at times $t + 3$ and $t + 2$ respectively, that is tied into the coordination process as depicted in Figure 3.6.

In this case $a_2$ is set by $i_2$ to a time $t'$ such that $(i_3, t_3) \rightarrow (i_1, t')$. Observe that the shadowed lines in Figure 3.2b outline an underlying formation identical to the one shown in Figure 3.6.

Another formation that can be used to ensure coordination had been brought up in Example 1. Consider Figure 3.7. Here $i_2$ serves as a relay for $i_1$, and performs $a_2$ at a time $t'$ such that $(i_2, t_2) \rightarrow (i_1, t')$. In terms of the cheque clearance scenario from Section 3.1, this would be equivalent to Alice getting a message from Charlie at time $t_2$ and then forwarding it on to Bob.

Knowing that her message will not take longer than 5 days to arrive, she waits until day $t_2 + 5$ and then clears her cheque.

What about other formations? Consider the scenario depicted in Figure 3.8. It can be associated with the situation in Example 2, at time $t' = t + 6$. At this stage, both Alice and Bob have received a message from Charlie, but Alice cannot be sure that the message to Bob has indeed arrived, because the transmission distance between Charlie and Bob is 10. Returning to the figure, at $t'$ process $i_2$ cannot perform $a_2$ without risking the possibility that $a_1$ has not yet been performed. A protocol in which $i_2$ does do $a_2$ at $t'$ is one that does not solve the instance of the OR problem in cases where the message to $i_1$ takes longer than 6 time units.
steps to arrive. As we are assuming a protocol that solves the ordering problem in all runs, such a scenario is impossible.

We can show that the formations in Figures 3.4 to 3.7 exhaust the possible causal formations in a protocol that solves the double response OR problem. All of the possible configurations considered above are described by the generalized Figure 3.9, where $\theta$ is a parameterized node on a syncausal path between $(i_0, t)$ and $(i_2, t_2)$. If we set $\theta = (i_1, t_1)$ we get the setting shown in Figure 3.4. Similarly, setting $\theta$ to $(i_0, t)$, $(i_3, t_3)$ and $(i_2, t_2)$ gives us Figures 3.5, 3.6 and 3.7 respectively.\(^3\)

![Figure 3.9: Coordination in Double Response](image)

### 3.5 Centipedes

The double response problem incorporates two aspects required for ordering responses: notification of the responding sites regarding the occurrence of the trigger event, and coordination of the responses between these sites. This section tackles the OR problem in its most general form, where any number of responses may be required to the trigger’s occurrence.

Comparing the individual instances in Figures 3.4 to 3.7 to their generalization in Figure 3.9, we see a pattern emerging, where the actual node standing in for the parameterized $\theta$ gets informed of the occurrence of the trigger event, and serves to split the path and route the new information to both $(i_1, t')$ and $(i_2, t')$. The condition on this splitting node is that it must be able to guarantee the arrival of information at $t_1$ by time $t'$. This promise can then be used by $i_2$ to coordinate its own action with $i_1$’s.

Intuitively, we would expect then to see similar split-and-promise mechanisms crop up further down the line when multiple responses are required. This idea gives rise to the centipede structure, defined below.

\(^3\)Recall that both $\rightarrow$ and $\rightarrow\rightarrow$ are reflexive relations, so for example setting $\theta = (i_0, t)$
Definition 10 (Centipede) Let \( r \in \mathcal{R}^{\text{max}} \), let \( i_h \in \mathcal{P} \) for \( 0 \leq h \leq k \) and let \( t \leq t' \). A centipede for \( \langle i_0, \ldots, i_k \rangle \) in the interval \( (r, t..t') \) is a sequence of nodes \( \theta_0 \leadsto \theta_1 \leadsto \cdots \leadsto \theta_k \) such that \( \theta_0 = (i_0, t) \), \( \theta_k = (i_k, t') \), and \( \theta_h \rightarrow (i_h, t') \) holds for \( h = 1, \ldots, k - 1 \).

A centipede for \( \langle i_0, \ldots, i_k \rangle \) in the interval \( (r, t..t') \) is depicted in Figure 3.10. Extending Figure 3.9, Figure 3.10 shows a syncausal chain extending between \( (i_0, t) \) and \( (i_k, t') \), and along this chain a sequence of “route splitting” nodes \( \theta_1, \theta_2, \ldots \) such that each \( \theta_h \) can guarantee the arrival of a message to \( i_h \) by time \( t' \). Such a message can serve to inform \( i_h \) of the occurrence of the trigger event, and as the set of previously made guarantees gets shuffled on to the next splitting node, each responding site \( i_h \) can be confident that all previous sites \( \langle i_1, \ldots, i_{h-1} \rangle \) had already responded.

We remark that, since both \( \leadsto \) and \( \rightarrow \) are reflexive, it is possible for adjacent \( \theta_j \)'s to coincide. Moreover, it is possible (in fact, probably quite common) that \( \theta_h = (i_h, t_h) \) for some \( t_h \leq t' \). Observe that every simple (Lamport-style) message chain gives rise to a centipede of a simple form in which all body nodes \( \theta_h \) are co-located in this sense with their respective leg nodes. It follows that a centipede is a natural, albeit nontrivial, generalization of a Lamport-causal chain.

While the centipede structure may seem rather intuitive, it is not at all clear that such a structure should exist whenever the OR problem is solved. Nevertheless, Theorem 4 below shows that this is the case, thus providing a concise statement of the communication structures that are required for the ordering of events in every synchronous system.

Theorem 4 (Centipede Theorem) Let \( P \) be a protocol solving the OR = OR\( \langle e_t, \alpha_1, \ldots, \alpha_k \rangle \) in \( \gamma^{\text{max}} \), and assume that \( e_t \) occurs at \( (i_0, t) \) in \( r \in \mathcal{R}^{\text{max}} \).

is not at odds with \( \theta \leadsto (i_0, t) \).
If $i_k$ performs $a_k$ at time $t'$ in $r$ then there is a centipede for $\langle i_0, \ldots, i_k \rangle$ in $(r, t..t')$.

Note however the subtle relation between the Ordered Response requirement and the structures that it necessitates. Suppose that $e_t$ occurs at time $t$ in a run $r \in \mathcal{R}(P, \gamma_{\max})$ where $P$ solves $\text{OR} = \langle e_t, \alpha_1, \ldots, \alpha_k \rangle$. Then there exist times $t_1 \leq t_2 \leq \cdots \leq t_k$ where the actions $\alpha_1, \alpha_2, \ldots, \alpha_k$ are performed respectively. The theorem implies that for each $h \leq k$ there exists a centipede for $\langle i_0, \ldots, i_h \rangle$ in $(r, t..t_h)$. Such a centipede could be used to inform all $\langle i_0, \ldots, i_h \rangle$ processes of the occurrence of $e_t$, and would also serve to coordinate response $\alpha_h$ so as not to occur before any of the responses $\alpha_1, \ldots, \alpha_{h-1}$. However, a centipede for $\langle i_0, \ldots, i_h \rangle$ in $(r, t..t_h)$ need not be a sub-structure in a centipede for $\langle i_0, \ldots, i_{h+1} \rangle$ that occurs in $(r, t..t_{h+1})$.

Recall Example 3 in the introduction, here redrawn in Figure 3.11. The centipede for $\langle \text{Charlie}, \text{Bob} \rangle$ is the sequence $\langle \langle \text{Charlie}, t \rangle, \langle \text{Bob}, t + 2 \rangle \rangle$. The centipede for $\langle \text{Charlie}, \text{Bob}, \text{Alice} \rangle$ that is used by Alice to coordinate her actions with those of Bob is the sequence $\langle \langle \text{Charlie}, t \rangle, \langle \text{Susan}, t + 3 \rangle, \langle \text{Alice}, t + 7 \rangle \rangle$.

\section{Knowledge Requires Syncausality}

This section begins the second part of the chapter, wherein we review the findings described thus far in light of the knowledge-based analysis paradigm. This will provide us with formal proofs to all quoted theorems, as well as a deeper understanding of the forces at play.

A word on terminology. Despite not having proved Theorems 3 and 4 just yet, we will take to referring to syncausality and the centipede as causal structures, anticipating the results of the coming sections.

We start by considering the basic syncausality relation $\rightsquigarrow$. In Section 3.3 we stated that every response event must be syncausally related to the trigger $e_t$. In this section we prove this claim formally.
A simple and useful property of syncausality is a slight extension of the idea that if one node syncausally affects another, then the former must have happened before the latter one:

**Lemma 4** If $(i, t) \leadsto (j, t')$ then $t \leq t'$, with $t = t'$ holding only if $i = j$.

**Proof** All $(i, t) \leadsto (j, t')$ causality instances generated by clause 1 have the property that $i = j$ and $t \leq t'$. By definition of the synchronous context $\gamma_{max}$, messages take at least one time step to be delivered. Moreover, the upper bounds on message transmission times are assumed to satisfy $max_{ij} \geq 1$. Therefore, all $(i, t) \leadsto (j, t')$ causality instances generated by clauses 2 and 3 have the property that $i \neq j$ and $t < t'$. As a result, a straightforward induction on the number of times the transitivity clause 4 is applied in a derivation of $(i, t) \leadsto (j, t')$ yields the desired claim. ■

Lamport relates the happened-before relation to light cones in Minkowski space-time [26]. In the same vein, it is natural to consider past and future causal “cones” induced by syncausality.

**Definition 11** (*past and fut cones*) We define the future causal cone of a node $\alpha$ (in run $r$) to be

$$\text{fut}(r, \alpha) = \left\{ \langle \theta, ND_\theta \rangle : \alpha \leadsto \theta \text{ in } r \text{ and } ND_\theta \text{ is the set of ND events and initial states in } \theta \text{ in } r \right\}.$$  

Similarly, the past causal cone of $\alpha$ is

$$\text{past}(r, \alpha) = \left\{ \langle \theta, ND_\theta \rangle : \theta \leadsto \alpha \text{ in } r \text{ and } ND_\theta \text{ is the set of ND events and initial states in } \theta \text{ in } r \right\}.$$  

We will often treat the sets $\text{past}$ and $\text{fut}$ more simply as sets of nodes, rather than sets of pairs, when the related ND events are immaterial in the context.

Observe that the cones induced by syncausality in synchronous systems are significantly larger than the ones that follow just from Lamport’s happened-before relation. Moreover, just as the future and past cones meet at the current point in space-time for light cones, we can use Lemma 4 to show:

**Lemma 5** For all runs $r \in R_{max}^{\text{max}}$ and nodes $\alpha$ and $\beta$:

1. $\text{fut}(r, \alpha) \cap \text{past}(r, \alpha) = \{\alpha\}$, and  
2. $\alpha \leadsto \beta \iff \text{fut}(\alpha) \cap \text{past}(\beta) \neq \emptyset.$
Proof Let \( \alpha = (i, t) \) be a process-time node. For part (1), observe that \( \alpha \rightsquigarrow \alpha \) since \( \rightsquigarrow \) is reflexive by clause (1) of Definition 9. Hence \( \alpha \in \text{fut}(r, \alpha) \cap \text{past}(r, \alpha) \). By Lemma 4, if \( \alpha \neq \beta = (j, t') \in \text{past}(\alpha) \), then \( t' < t \). Similarly, if \( \alpha \neq \theta = (h, t'') \in \text{fut}(\alpha) \) then \( t < t'' \). It follows that \( \text{fut}(r, \alpha) \cap \text{past}(r, \alpha) = \alpha \).

For part (2), assume that \( \alpha \rightsquigarrow \beta \). Then \( \alpha \in \text{past}(\beta) \) by definition of \( \text{past} \). Since \( \alpha \in \text{fut}(\alpha) \) by part (1), it follows that \( \alpha \in \text{fut}(\alpha) \cap \text{past}(\beta) \neq \emptyset \). For the other direction, suppose that \( \theta \in \text{fut}(\alpha) \cap \text{past}(\beta) \). Then by definition we have that \( \alpha \rightsquigarrow \theta \) and \( \theta \rightsquigarrow \beta \). By transitivity of \( \rightsquigarrow \) (clause 4) we have that \( \alpha \rightsquigarrow \beta \), and we are done.

The next step in relating knowledge to syncausality in synchronous systems comes from the observation that the events that occur in the past (syncausal) cone of a node completely determine the local state at the node. A proof by induction on all nodes \((j, t')\) with \(0 \leq t' \leq t\) shows:

**Lemma 6** Let \( r, r' \in \mathcal{R}_{\max} \).
If \( \text{past}(r, (i, t)) = \text{past}(r', (i, t)) \) then \( r_i(t) = r'_i(t) \).

**Proof** A straightforward proof by induction on \( t' \) in the range \( 0 \leq t' \leq t \) shows that, for all \( j \in \mathbb{P} \), if \((j, t') \in \text{past}(i, t)\) then \( r_j(t') = r'_j(t') \). By assumption, \( r \) and \( r' \) agree on initial states in \((j, 0)\). The induction step is proved based on the fact that each local state which is not initial is determined by the previous local state of the same process and by ND events in that process in the last round. Thus, \( r_j(0) = r'_j(0) \) for all \( j \in \mathbb{P} \). The claim follows from the fact that \((i, t) \in \text{past}(r, (i, t))\).

Since the knowledge of a process in \( \mathcal{R}_{\max} \) is determined by its local state, Lemma 6 implies that this knowledge is determined by the past causal cone. We can now state and prove, using Lemma 6, the following knowledge gain theorem for two processes:

**Theorem 5 (Basic Knowledge Gain)** Assume that \( e \) takes place at \((i_0, t)\) in \( r \in \mathcal{R}_{\max} = \mathcal{R}(P, \gamma_{\max}) \). If \((\mathcal{R}_{\max}, r, t') \models K_{i_1}(\text{occured}(e) \land \text{ND}(e)) \) then \((i_0, t) \rightsquigarrow (i_1, t')\).

**Proof** Let \( e \) be an ND event occurring at \((i_0, t)\) in the run \( r \in \mathcal{R}_{\max} \). We shall prove the contrapositive: If \((i_0, t) \not\rightsquigarrow (i_1, t')\) then \((\mathcal{R}_{\max}, r, t') \not\models K_{i_1}(\text{occured}(e) \land \text{ND}(e)) \). By assumption, \( \mathcal{R}_{\max} = \mathcal{R}(P, \gamma_{\max}) \) for some protocol \( P \). Let \( r' \in \mathcal{R}_{\max} \) be a run identical to \( r \) until (but not including) time \( t \), in which
(i) the environment’s actions at all nodes in $\text{past}(r, i_1, t')$ are identical to those in $r$; and

(ii) the environment’s actions at $i_0$ in the interval $[t, t']$ are identical to those in $r$ with the exception that $e$ does not occur in any of the related nodes. Thus if $t = 0$ then $i_0$’s initial state in $r'$ differs from that in $r$ by not including $e$, and similarly for other local states. Finally,

(iii) all messages delivered to nodes not in $\text{past}(r, i_1, t')$ are delivered at the maximal possible transmission time according to the bounds $\text{max}_{ij}$.

To see that such a run $r'$ indeed exists in $\mathcal{R}^{\text{max}}$, we note that clauses (ii) and (iii) relate to nodes outside $\text{past}(r, (i_1, t'))$ and thus by assumption do not contradict clause (i), and that by definition an external input event is entirely independent of the run’s past history, so its non occurrence in an interval of time is possible. Since $\mathcal{R}^{\text{max}}$ contains all runs of $P$ in $\gamma^{\text{max}}$, it must include $r'$.

Notice that $(\mathcal{R}^{\text{max}}, r', t') \not\models (\text{occurred}(e) \land \text{ND}(e))$: first, since events are distinct in a run and since $e$ occurs at time $t$ in $r$, it does not occur in $r$ at any time previous to $t$. Since $r'$ is identical to $r$ until time $t$ the same applies for $r'$. Next, if $e$ is an external input, then the possibility for $e$ to occur at some $\bar{t} \in [t, t']$ is foiled by (ii), while if it is a message receive, then the following argument applies. Since $(i_0, t) \not\Rightarrow (i_1, t')$ then also $(i_0, \bar{t}) \not\Rightarrow (i_1, t')$ for all $\bar{t} \geq t$, as $(i_0, t) \sim (i_0, \bar{t})$ is true of all $\bar{t} \geq t$. It follows that $(i_0, \bar{t}) \notin \text{past}(r, (i_1, t'))$ for all $\bar{t} \geq t$. So either the message is postponed beyond time $t'$, or else it cannot be thus postponed, in which case $e$ is not an ND event (an early receive) when it occurs in $r'$. For all cases we get by definition of $\models$ that $(\mathcal{R}^{\text{max}}, r', t') \not\models (\text{occurred}(e) \land \text{ND}(e))$. By definition, $r'$ agrees with $r$ on initial states, external inputs, and delivery times on nodes of $\text{past}(r, (i_1, t'))$. Thus by Lemma 6 we have that $(r, t') \sim_{i_1} (r', t')$, and therefore $(\mathcal{R}^{\text{max}}, r, t') \not\models K_{i_1}(\text{occurred}(e) \land \text{ND}(e))$, as desired. $\square$

The proof of Theorem 5 is obtained by constructing a run $r'$ indistinguishable to $i_1$ at $t'$ from $r$ in which no ND events occur outside $\text{past}(r', (i_1, t')) = \text{past}(r, (i_1, t'))$. Theorem 5 captures a natural sense in which syncausality is a notion of potential causality for the synchronous model.

The proof of Theorem 3 can now be derived. For protocols that recall responses, the theorem follows immediately from Theorems 5 and 1. We prove the theorem for arbitrary protocols by relating the general case to that of protocols that recall responses (Definition 5).
Proof of Theorem 3 Let $P$ be a protocol solving $\text{OR} = \text{OR}(e_t, \alpha_1, ..., \alpha_k)$. Define $P'$ to be a protocol that differs from $P$ only in that every process $i$ maintains a list called $\text{Responses}$, to which it adds an item $(a_h, t_h)$ whenever $i$ performs a response $\alpha_h$. (We assume w.l.o.g. that no list with this name is used by $P$.) Notice that this list is an auxiliary variable that does not affect the behavior of the protocol. Indeed, there is an isomorphism between the runs of $P$ and those of $P'$, where the same nondeterministic events and the same actions take place at all nodes of corresponding runs. In particular, since $P$ solves $\text{OR}$, then so does $P'$. By construction, $P'$ recalls responses. Denote $R_{\text{max}}' = R(P', \gamma_{\text{max}})$. Assume that $e_t$ occurs in the run $r$ of $P$ and let $r'$ be the corresponding run of $P'$.

Let $1 \leq h \leq k$. Since $P'$ recalls responses, we have by Theorem 1 that $(R_{\text{max}}', r', t') \models K_{i_h} \text{occurred}(e_t)$. As every external input is, in particular, an ND event, this gives us $(R_{\text{max}}', r', t') \models K_{i_h} (\text{occurred}(e_t) \land \text{ND}(e_t))$. We now use Theorem 5 and the fact that $e_t$ occurs at $(i_0, t)$ to conclude that $(i_0, t) \Rightarrow (i_h, t')$ in $r'$.

Since all actions and communication events in $r$ and in $r'$ are the same, it follows that $(i_0, t) \Rightarrow (i_h, t')$ in $r$ too, as required.

3.7 Nested Knowledge Requires Centipedes

When we move beyond single response problems into the double and $k$ response variants, Theorem 1 provides us with nested knowledge conditions. Showing that nested knowledge implies the existence of a centipede requires a substantial formal theory. This section develops the required theory.

The first relevant notion is captured by the following definition:

Definition 12 (Bridge nodes) Fix $r$ and let $\alpha \Rightarrow \alpha'$. We say that $\beta$ bridges $\alpha$ and $\alpha'$ if

1. $\alpha \Rightarrow \beta \Rightarrow \alpha'$ and
2. $\alpha \Rightarrow \beta' \Rightarrow \beta$ implies $\beta' = \beta$, for all nodes $\beta'$.

Intuitively, a bridge is an earliest node that is syncausally affected by $\alpha$ and precedes $\alpha'$ by way of a timing guarantee. Interestingly, bridges are guaranteed to exist:

Lemma 7 If $\alpha \Rightarrow \alpha'$ then there is a node $\beta$ bridging $\alpha$ and $\alpha'$.
Proof Let \( \alpha \sim \alpha' \), where \( \alpha = (i, t) \) and \( \alpha' = (j, t') \). By Lemma 4 we have that \( t \leq t' \). We prove the claim by induction on \( d = t' - t \). The base case is \( d = 0 \), in which case \( t = t' \) and by Lemma 4 we have that \( \alpha = \alpha' \). Since \( \alpha \sim \alpha \rightarrow \rightarrow \alpha \) holds, and \( \alpha \sim \beta' \rightarrow \rightarrow \alpha \) holds only for \( \beta' = \alpha \), it follows that \( \beta = \alpha \) is a bridge as required. For the inductive step, let \( d > 0 \) and assume that the claim holds for all pairs of causally related nodes with time differences strictly smaller than \( d \). Since \( \alpha' \rightarrow \alpha' \) by definition, the assumption that \( \alpha \sim \alpha' \) clearly implies that \( \alpha \rightarrow \alpha' \rightarrow \alpha' \). We consider two cases. If there is no node \( \beta' \neq \alpha' \) such that \( \alpha \sim \beta' \rightarrow \alpha' \) then \( \beta = \alpha' \) is the desired bridge node. Otherwise, such a \( \beta' = (i_1, t_1) \) exists. As before, we obtain by Lemma 4 that \( t_1 < t' \). In particular, \( d_1 = t_1 - t < t' - t = d \). Thus, since \( \alpha \sim \beta' \) we have by the inductive assumption for \( d_1 \) that there is a node \( \beta \) bridging \( \alpha \) and \( \beta' \). It follows that \( \alpha \sim \beta \rightarrow \beta' \rightarrow \alpha' \) and \( \beta \) satisfies the minimality clause (2) of Definition 12 with respect to \( \alpha \). As \( \rightarrow \rightarrow \) is a transitive relation, we obtain that \( \alpha \sim \beta \rightarrow \alpha' \) and the claim follows.

Bridges are closely related to early message receives:

Lemma 8 If \( \alpha \neq \beta \) and \( \beta \) bridges \( \alpha \) and \( \alpha' \), then there exists some \( \beta' \) such that \( \alpha \sim \beta' \sim \beta \) and the syncausal chain \( \beta' \sim \beta \) consists of a single early receive.

Proof Denote \( \alpha = (i_1, t_1) \) and \( \beta = (i_2, t_2) \). If \( \beta \) bridges \( \alpha \) and \( \alpha' \) then, in particular, \( \alpha \sim \beta \). If, in addition, \( \alpha \neq \beta \) then \( t_1 < t_2 \) by Lemma 4. It follows that \( \alpha \sim \beta' \sim \beta \) where \( \beta' \sim \beta \) is derivable by clause (1), (2), or (3), and \( \beta' = (i', t') \) for some \( t' < t_2 \). If \( \beta' \sim \beta \) is derivable by (1) or (3), then \( \beta' \rightarrow \alpha' \) and \( \beta \) does not bridge \( \alpha \) and \( \alpha' \), contradicting the assumption. The alternative is that \( \beta' \sim \beta \) is derivable from (2) but not from (3), and hence \( \beta' \sim \beta \) must be an early receive, as claimed.

The existence of bridge nodes as a special kind of node motivates an alternative approach to defining centipedes, based on bridge nodes. We start with the notion of centinodes:

Definition 13 (Centinode) We inductively define node \( \theta \) to be a \( \langle i_0, \ldots, i_k \rangle \) centinode in \( (r, t..t') \) as follows.

\[
\begin{align*}
\text{k} = 0: & \quad \theta \text{ is a } \langle i_0 \rangle \text{ centinode iff } \theta = (i_0, t); \text{ while} \\
\text{k} > 0: & \quad \theta \text{ is a } \langle i_0, \ldots, i_k \rangle \text{ centinode iff there exists a } \langle i_0, \ldots, i_{k-1} \rangle \text{ centinode } \\
& \quad \theta' \text{ in } (r, t..t'), \text{ such that } \theta \text{ bridges between } \theta' \text{ and } (i_k, t') \text{ in } r.
\end{align*}
\]
As a straightforward conclusion of Lemma 7 we can show:

**Lemma 9**  The following three are equivalent:

1. A centipede for \( \langle i_0, \ldots, i_k \rangle \) in \( (r, t..t') \) exists;
2. A \( \langle i_0, \ldots, i_k \rangle \) centinode in \( (r, t..t') \) exists; and
3. A centipede \( \langle \theta_0, \ldots, \theta_k \rangle \) for \( \langle i_0, \ldots, i_k \rangle \) in \( (r, t..t') \) exists, in which every node \( \theta_j \) is a \( \langle i_0, \ldots, i_j \rangle \) centinode in \( (r, t..t') \), for \( j = 0, \ldots, k \).

**Proof**  The truth of 3 \( \Rightarrow \) 2 is immediate. A straightforward induction on \( k \) shows that 2 \( \Rightarrow \) 1, as the current centinode \( \theta_k \) is syncausally related to \( \theta_{k-1} \) and timing guarantee related to \( (i_k, t') \). We now prove that 1 \( \Rightarrow \) 3. Assume that \( C = \langle \theta_0, \ldots, \theta_k \rangle \) is a centipede for \( \langle i_0, \ldots, i_k \rangle \) in \( (r, t..t') \). We define by induction on \( h \leq k \) centipedes \( C_h = \langle \theta'_0, \ldots, \theta'_h, \theta_{h+1}, \ldots, \theta_k \rangle \) in \( (r, t..t') \), in which the nodes \( \theta'_0 \) to \( \theta'_h \) are centinodes. The final centipede \( C_k \) in the construction satisfies the conditions of 3.

- \( h = 0 \) : By definition, the initial node \( \theta_0 \) in \( C \) is a \( \langle i_0 \rangle \) centinode in \( (r, t..t') \). Defining \( \theta'_0 = \theta_0 \) we have \( C_0 = C \).
- \( h > 0 \) : Assume that a centipede \( C_{h-1} = \langle \theta'_0, \ldots, \theta'_{h-1}, \theta_h, \ldots, \theta_k \rangle \) as described above has been constructed. By Lemma 7 there exists a node \( \theta'_h \) bridging \( \theta'_{h-1} \) and \( \theta_h \). Since \( \theta_h \rightarrow (i_h, t') \) we get that \( \theta'_h \) is a centinode for \( \langle i_0, \ldots, i_h \rangle \) in \( (r, t..t') \). Define \( C_h = \langle \theta'_0, \ldots, \theta'_h, \theta_{h+1}, \ldots, \theta_k \rangle \). If \( h = k \) then we are done. Otherwise, since \( \theta_h \rightarrow \theta_{h+1} \) and \( C_{h-1} = \langle \theta'_0, \ldots, \theta'_{h-1}, \theta_h, \ldots, \theta_k \rangle \) is a centipede for \( \langle i_0, \ldots, i_k \rangle \) in \( (r, t..t') \), we obtain that \( C_h \) is also such a centipede, as required.

Lemma 9 allows using centinodes and centipedes interchangeably. Indeed, clause 3 suggests that we can without loss of generality think of centipedes as consisting of a sequence of centinodes. We are now ready to prove our main theorem, stating that the existence of a centipede is a necessary condition for attaining nested knowledge of an ND event:

**Theorem 6 (Knowledge Gain)**  Let \( P \) be a deterministic protocol, and let \( r \in \mathcal{R}_{max} = \mathcal{R}(P, \gamma_{max}) \). Assume that \( e \) is an ND event at \( (i_0, t) \) in \( r \). If \( (\mathcal{R}_{max}, r, t') \models K_{i_k} K_{i_{k-1}} \cdots K_{i_1} (\text{occurred}(e) \land \text{ND}(e)) \), then there is a centipede for \( \langle i_0, \ldots, i_k \rangle \) in \( (r, t..t') \).
Proof We shall prove the contrapositive form: if no centinode \( \langle i_0, \ldots, i_{k-1}, i_k \rangle \) exists in \((r, t..t')\), then \((R_{max}^r, r, t') \nRightarrow K_{i_k}K_{i_k-1} \cdots K_{i_1}(\text{occurred}(e) \land \text{ND}(e))\).

We reason by induction on \( k \geq 1 \):

\( k = 1 \) This case is a rephrasing of Theorem 5: By assumption, there is no \( \langle i_0, i_1 \rangle \) centinode in \((r, t..t')\). In other words, there is no node \( \theta \) bridging \((i_0, t)\) and \((i_1, t')\). By Lemma 7 it follows that \((i_0, t) \nrightarrow (i_1, t')\). Thus, by Theorem 5 we have that \((R_{max}^r, r, t') \nRightarrow K_{i_1}(\text{occurred}(e) \land \text{ND}(e))\), as claimed.

\( k \geq 2 \) Assume inductively that the claim holds for \( k-1 \). Moreover, assume that no \( \langle i_0, \ldots, i_k \rangle \) centinode exists in \((r, t..t')\). For every \( r' \in R_{max}^r \) let \( C^r = \{ \theta^1, \ldots, \theta^d \} \) be the set of \( \langle i_0, \ldots, i_{k-1} \rangle \) centinodes in \((r', t..t')\). Observe that \( \theta' \nrightarrow (i_k, t') \) in \( r \) for all \( \theta' \in C^r \), since otherwise we would have by Lemma 7 that there is a bridging node \( \theta' \rightarrow \beta \rightarrow (i_k, t') \). But \( \beta \) would then be a \( \langle i_0, \ldots, i_{k-1}, i_k \rangle \) centinode in \((r, t..t')\), contradicting our assumption.

We consider two cases. First suppose that \((i_0, t) \in C^r\). Given that \((i_0, t) \nrightarrow (i_k, t')\) we have that \((R_{max}^r, r, t') \nRightarrow K_{i_k}(\text{occurred}(e) \land \text{ND}(e))\) by Theorem 5 above, and since \( K_{i_k}K_{i_k-1} \cdots K_{i_1}(\text{occurred}(e) \land \text{ND}(e)) \) validly implies \( K_{i_k}(\text{occurred}(e) \land \text{ND}(e)) \) is valid, we obtain that \((R_{max}^r, r, t') \nRightarrow K_{i_k}K_{i_k-1} \cdots K_{i_1}(\text{occurred}(e) \land \text{ND}(e))\), as claimed.

Next suppose that \((i_0, t) \notin C^r\). Let \( r' \in R_{max}^r \) be a run such that \( r' \) is identical to \( r \) until (but not including) time \( t \), and where

(i) the environment’s actions at all nodes in \( \text{past}(r, (i_k, t')) \) are identical to those in \( r \); and

(ii) all messages delivered to nodes not in \( \text{past}(r, (i_k, t')) \) are delivered at the maximal possible transmission time according to the bounds \( \text{max}\gamma_{ij} \).

To see that such a run \( r' \) indeed exists in \( R_{max}^r \), we note that clause \( (ii) \) relates to nodes outside \( \text{past}(r, (i_k, t')) \) and thus by assumption does not contradict clause \( (i) \), and that by definition all early message receives can be delayed, independent of the run’s past or concurrent events. Since \( R_{max}^r \) contains all runs of \( P \) in \( \gamma_{max} \), it must include \( r' \).

Showing that such a run exists repeats the arguments in Theorem 5. From \( r' \) being identical to \( r \) until time \( t \), from clause \( (i) \) above and from Lemma 6 it follows that \( r'_{i_k}(t') = r_{i_k}(t') \). Notice that by construction of \( r' \) we have that \( \alpha \rightarrow \alpha' \) holds in \( r' \) only if \( \alpha \rightarrow \alpha' \) in \( r \), and that every
early receive in $r'$ is an early receive in $r$. Considering that bounds are universal in all runs, we obtain that every bridge node in $r'$ is also a bridge node in $r$, and hence that $C'' \subseteq C'$. By definition of $r'$, none of the nodes in the set $C'$, and hence also in $C''$, experiences an early receive in $r'$. Yet from Lemma 8 and from $(i_0, t) \notin C''$ it follows that every node $\theta' \in C''$ must be a nontrivial bridge node in $r'$, thus experiencing an early receive. We therefore conclude that $C'' = \emptyset$.

Based on the inductive hypothesis and the definition of $C''$ we get

$$(\mathcal{R}'_{max}, r', t') \not\models K_{i_{k-1}} \cdots K_{i_1} (\text{occurred}(e) \land \text{ND}(e)).$$

As $r_{ik}(t) = r'_{ik}(t)$, we obtain using the definition of Knowledge operator that

$$(\mathcal{R}'_{max}, r, t') \not\models K_{i_k} K_{i_{k-1}} \cdots K_{i_1} (\text{occurred}(e) \land \text{ND}(e)),$$

and we are done.

\[\square \text{Theorem 6}\]

Based on the Knowledge Gain Theorem, we can proceed to prove Theorem 4, the Centipede Theorem. For protocols that recall responses, the Centipede theorem follows immediately from Theorems 6 and 1. We prove the Centipede Theorem for arbitrary protocols by relating the general case to that of protocols that recall responses, as we did in Theorem 3.

**Proof of Theorem 4** Let $P$ be a protocol solving $\text{OR} = \text{OR}\langle e_t, \alpha_1, \ldots, \alpha_k \rangle$. Define $P'$ to be a protocol that differs from $P$ only in that every process $i$ maintains a list called Responses, to which it adds an item $(a_h, t_h)$ whenever $i$ performs a response $\alpha_h$. (We assume w.l.o.g. that no list with this name is used by $P$.) Notice that this list is an auxiliary variable that does not affect the behavior of the protocol. Indeed, there is an isomorphism between the runs of $P$ and those of $P'$, where the same nondeterministic events and the same actions take place at all nodes of corresponding runs. In particular, since $P$ solves $\text{OR}$, then so does $P'$. By construction, $P'$ recalls responses. Denote $\mathcal{R}' = \mathcal{R}(P', \gamma_{\text{max}})$. Assume that $e_t$ occurs in the run $r$ of $P$ and let $r'$ be the corresponding run of $P'$. Since $P'$ recalls responses, we have by Theorem 1 that

$$(\mathcal{R}', r, t_k) \models K_{i_k} K_{i_{k-1}} \cdots K_{i_1} \text{occurred}(e_t).$$

Since $e_t$ is an external input in all runs of $\mathcal{R}'$, we have that $\mathcal{R}' \models \text{occurred}(e_t) \Rightarrow (\text{occurred}(e_t) \land \text{ND}(e_t))$. Hence,

$$(\mathcal{R}', r, t_k) \models K_{i_k} K_{i_{k-1}} \cdots K_{i_1} (\text{occurred}(e_t) \land \text{ND}(e_t)).$$
By Theorem 6 we thus obtain that there is a centipede for $\langle i_0, \ldots, i_k \rangle$ in $(r', t..t')$. Since all actions and communication events in $r$ and in $r'$ are the same, it follows that there is a centipede for $\langle i_0, \ldots, i_k \rangle$ in $(r, t..t')$, and we are done. ■

3.8 Varying Nondeterminism in Message Transmission

A better grasp of the dynamics and flexibility of the centipede structure, and of the scope of the related Knowledge Gain Theorem, is afforded by considering two particular models, with very different characteristics. On one extreme, we define the Asynch-delivery model to be one in which $\max_{ij} = \infty$ for all channels $i \rightarrow j$. On the other extreme, we consider the Fixed-delivery model to be one in which every message on a channel $i \rightarrow j$ spends exactly $\max_{ij} < \infty$ time units in transit.

We define the Asynch-delivery model as a context $\gamma^{as}$ which is a $\gamma^b$ context where $\max_{ij} = \infty$ for all existing channels. This is a model where processes share a global clock but communication is asynchronous. In this case, clause (3) in the definition of syncausality cannot be used to infer syncausality of any pair of nodes, and thus $\Rightarrow$ coincides with $\rightarrow$.

In this model, the Knowledge Gain then reduces to a theorem equivalent to Chandy and Misra’s Knowledge Gain Theorem for totally asynchronous contexts [7].

**Lemma 10** Let $P$ be an arbitrary protocol, let $R^{as} = R(P, \gamma^{as})$. Assume $e$ is an ND event occurring at $(i_0, t)$ in $r$. If $(R^{as}, r, t') \models K_{i_0} K_{i_1} \cdots K_{i_k} (\text{occurred}(e) \land \text{ND}(e))$, then there is a chain $(i_0, t) \rightarrow (i_1, t_1) \rightarrow \cdots \rightarrow (i_k, t_k)$ in $(r, t..t')$.

**Proof** By the theorem’s assumptions and by applying the Knowledge Gain Theorem, we obtain that there must exist a centipede $(\langle i_0, t, \theta_1, \ldots, \theta_{k-1}, (i_k, t') \rangle)$ for $\langle i_0, i_1, \ldots, i_k \rangle$ in $(r, t..t')$.

Yet when $b = \infty$ for all channels, we get that if $(i, t) \sim (j, t')$ and $t \leq t' < \infty$ then the syncausal relation cannot be based on applications of clause (3) of the definition of syncausality on page 35. Thus, it must be that $(i, t) \rightarrow (j, t')$. Moreover, if $t \leq t' < \infty$ and $(i, t) \rightarrow (j, t')$ then it must be that $i = j$.

Thus in the existing centipede it must be that for all $h < k$, $\theta_h = (i_h, t_h)$ and $\theta_h \rightarrow \theta_{h+1}$. Thus providing us with a message chain linking $i_0, i_1, \ldots, i_k$
As seen in Figure 5.1, under the Asynch-delivery model the centipede’s legs are shortened to length 0, and the syncausal relations between it’s body nodes collapse into Lamport’s happened-before.

The Fixed-delivery model runs opposite to the Asynch-delivery one in removing the nondeterministic aspect in message delivery. Thus, not only do processes share a global clock, but they also share knowledge of the exact time it takes each message to be delivered.

We define this model as a $\gamma^b$ context where $\min_{ij} = \max_{ij}$ for all channels, and denote it $\gamma^f$. Under this model, every message sent arrives exactly at its related channel’s bound guarantee. The syncausal relation then acquires the same extension as that of the timing guarantee, and centipede body nodes are collapsed into a single node, as shown in Figure 5.1.

**Lemma 11** Let $P$ be an arbitrary protocol, and let $\mathcal{R}^f = \mathcal{R}(P, \gamma^f)$. Fix $r$ and assume $e$ is an ND event occurring in $(i_0, t)$ in $r$.

If $(\mathcal{R}^f, r, t') \models K_{i_0}K_{i_{k-1}} \ldots K_{i_1}(\text{occurred}(e) \land \text{ND}(e))$, then $(i_0, t) \rightarrow (i_h, t')$ for all $h \leq k$.

**Proof** Note that in the context $\gamma^f$ we have that $(i, s) \rightarrow (j, s')$ iff $(i, s) \rightarrow (j, s')$, for all processes $i, j$ and times $s, s'$. By the Knowledge Gain Theorem there must exist a centipede for $(i_0, i_1, \ldots, i_k)$ in $(r, t..t')$. In other words, for each $h \leq k$ there exists some $\theta_h$ such that $(i_0, t) \rightarrow \theta_h \rightarrow (i_h, t')$. Thus we get that $(i_0, t) \rightarrow \theta_h \rightarrow (i_h, t')$ and hence $(i_0, t) \rightarrow (i_h, t')$ as required. □
3.9 Sufficiency of Centipedes for Knowledge Gain

The Knowledge Gain and Centipede Theorems (Theorems 6 and 4 respectively) state that the centipede structure is necessary for gaining nested knowledge occurrence of nondeterministic events and for solving the OR problem. These results hold in a strong sense, regardless of the protocol used by the processes. Our goal in this section is to show that these results are tight.

We cannot prove that centipedes are sufficient means to achieve these ends for all protocols, because the knowledge actually transferred by messages depends on the protocol, and may be insufficient. The most we can do in order to prove the tightness of our definitions is to show that there exist specific protocols under which centipedes are sufficient for knowledge gain. We will do so for the following version of the full information protocol.

**Definition 14 (Full-information Protocol)** In the full information protocol for synchronous systems, denoted \( fip \), every process \( i \in P \) sends its local state on each of its outgoing channels at every time step. Moreover, each process retains a history of every event that has taken place locally, and every message received, along with their times of occurrence.

We will denote with \( R^{\text{fip}}(\gamma_{\text{max}}) \) the system \( R(\text{fip}, \gamma_{\text{max}}) \). In \( fip \) the processes convey all of their knowledge as fast as they can. Roughly speaking, knowledge is spread in the system as fast as possible, given the transmission times allowed by the environment in the given run. While our stated goal is to prove that under \( fip \) the Knowledge Gain and Centipede Theorems are tight, our results will be somewhat stronger. These theorems show that centipedes are necessary for knowledge gain regarding nondeterministic events. As we shall see, in the context of \( fip \), there is no need to restrict attention to nondeterministic events. The causal structures in question are sufficient for knowledge gain regarding general events (and more general facts).

Three simple but very useful properties of the timestamping operator \( A_t \) are captured by the following immediate lemma:

**Lemma 12** For every formula \( \varphi \) and times \( t, t' \), the following formulas are valid in \( R^{\text{max}} \):

\[
\text{TS1} \quad \models \quad A_{t'}(\varphi \leftrightarrow A_t \varphi)
\]

\(^4\)See [42] and [34] for some interesting observations on the connections between protocols and message meanings.
\[ \text{TS2} \vdash \text{At}_t \text{At}_t \varphi \leftrightarrow \text{At}_t \varphi \]
\[ \text{TS3} \vdash \text{At}_t (K_i \varphi \leftrightarrow K_i \text{At}_t \varphi) \]

Notice that, as described, processes following fip have perfect recall. Since they maintain their local histories, they do not forget what they knew. Of course, the truth of a transient fact can change over time. Knowing that the time is 3 is possible at time 3 but not at time 4. But whether \( \varphi \) held at time \( t \) does not change. More formally, perfect recall and the presence of clocks give us the following knowledge-preservation property, which states that if at time \( t \) process \( i \) knows that \( \varphi \), then at every future time point the process will know (or remember) that it knew \( \varphi \) at \( t \).

**Lemma 13** If \( t' \geq t \), then the following formula is valid in \( \mathcal{R}^{\text{fip}} \):
\[ \text{TS4} \vdash \text{At}_{t'} K_i \varphi \rightarrow \text{At}_{t'} K_i \text{At}_t K_i \varphi. \]

The proof of the following lemma will make use of the \( \sim^t \) relation, which results from a single application of one of the clauses (1), (2), or (3) of syncausality.

We are now ready to show that syncausality alone is sufficient to ensure knowledge transfer under fip. Figure 3.13 below provides a graphical visualization of what Lemma 14 shows. Namely, that if \( K_i \varphi \) holds at time \( t \) and \((i, t) \sim (j, t')\), then at \( t' \) process \( j \) knows that at time \( t \) process \( i \) knew that \( \varphi \).

**Lemma 14** If \((\mathcal{R}^{\text{fip}}, r, t) \models K_i \varphi \) and \((i, t) \sim (j, t')\) in \( r \), then \((\mathcal{R}^{\text{fip}}, r, t') \models K_j (\text{At}_t K_i \varphi) \).

\[ \text{Figure 3.13: The syncausal relation and reflected knowledge state} \]

**Proof** Given that \((i, t) \sim (j, t')\), we have that
\[ (i, t) = (j_0, s_0) \sim^{nt} (j_1, s_1) \sim^{nt} \ldots \sim^{nt} (j_n, s_n) = (j, t') \]

We prove the claim by induction on \( n \).
\( n = 0 \): In this case, \( j = i \) and \( t = t' \). Positive introspection gives us \((\mathcal{R}^{\text{fip}}, r, t) \models K_i K_i \varphi \). Let \( \varphi' = K_i \varphi \), and apply TS3 based on \((\mathcal{R}^{\text{fip}}, r, t) \models K_i \varphi' \). This gives us \((\mathcal{R}^{\text{fip}}, r, t') \models K_i K_i \varphi \) as required.

\( n > 0 \): Assume inductively that \((\mathcal{R}^{\text{fip}}, r, s_{n-1}) \models K_{j_{n-1}} (At_{r} K_i \varphi) \). By definition of the sequence we have that \((j_{n-1}, s_{n-1}) \sim^{\text{ax}} (j_n, s_n) \). By definition of \( \sim^{\text{ax}} \) there are three options to consider, corresponding to clauses (1)–(3) of syncausality:

1. \( j_{n-1} = j_n \) and \( s_{n-1} \leq s_n \): In this case by TS4 we have that \((\mathcal{R}^{\text{fip}}, r, s_{n-1}) \models At_{s_n} K_{j_n} At_{s_{n-1}} K_{j_{n-1}} (At_{r} K_i \varphi) \), which is reduced to

\[
(\mathcal{R}^{\text{fip}}, r, s_n) \models K_{j_n} At_{s_{n-1}} K_{j_{n-1}} (At_{r} K_i \varphi) .
\]

Applying the Knowledge Axiom and TS2 to every \((r', s_n) \sim_{j_n} (r, s_n) \) we obtain \((\mathcal{R}^{\text{fip}}, r, s_n) \models K_{j_n} (At_{r} K_i \varphi) \).

2. process \( j_{n-1} \) sends a message in round \( s_{n-1} \), which is received by \( j_n \) in round \( s_n \): Since the protocol used is \( \text{fip} \), message contents consist of the local state of sender. Based on the inductive assumption we get that \((\mathcal{R}^{\text{fip}}, r, s_n) \models K_{j_n} At_{s_{n-1}} K_{j_{n-1}} (At_{r} K_i \varphi) \).

This implies \((\mathcal{R}^{\text{fip}}, r, s_n) \models K_{j_n} (At_{r} K_i \varphi) \) as in case (1).

3. \((j_{n-1}, j_n) \) is a network channel and no message is sent by \( j_{n-1} \) to \( j_n \) at time \( s_n \): Since in \( \text{fip} \) every process sends its local state to all neighbors in every round, this option is not viable in \( r \).

The lemma is proved based on the fact that in \( \text{fip} \) processes constantly send explicit messages on all outgoing channels, so a syncausal chain in \( \text{fip} \) never contains a link that is based on clause (3) of syncausality. Hence, if \((i, t) \Rightarrow (j, t') \) in \( r \), then there must exist a chain of “real” messages linking the two nodes. Messages in \( \text{fip} \) contain the local state of the sender. Hence, \( i \)’s local state at time \( t \) is propagated through the message chain until it reaches \( j \).

Lemma 15 below makes use of the further guarantees made by the \( \Rightarrow \) relation. Recall that \((i, t) \Rightarrow (j, t') \) implies that \((i, t) \sim (j, t') \), by Lemma 3. Moreover, The \( \Rightarrow \) relation is determined by the context \( \gamma^{\text{max}} \) alone. So that if \((i, t) \Rightarrow (j, t') \) holds in a run \( r \) of the system, it will do so in all runs of the system. Thus, process \( i \) knows already at time \( t \) that its current knowledge
will be available to \( j \) at \( t + D(i, j) \). This situation is depicted in Figure 3.14. Translated into English, the figure shows that if at time \( t \) process \( i \) knows that \( \varphi \), and if \((i, t) \rightarrow (j, t')\), then at time \( t' \) process \( i \) also knows that at time \( t' \) process \( j \) will know that at time \( t \) process \( i \) knew that \( \varphi \).

**Lemma 15** If \((\mathcal{R}_{\text{fip}}, r, t) \models K_i \varphi \) and \((i, t) \rightarrow (j, t')\), then \((\mathcal{R}_{\text{fip}}, r, t) \models K_i (\text{At}_{t'} K_j (\text{At}_t K_i \varphi))\).

![Figure 3.14: The timing guarantee and reflected knowledge state](image)

**Proof** Since \((i, t) \rightarrow (j, t')\), and since this property is determined by the network independently of the particular run \( r \), we have that \((i, t) \rightarrow (j, t')\) in every run \( r' \in \mathcal{R}_{\text{fip}} \). Moreover, by Lemma 3 we have that \((i, t) \sim (j, t')\) in every such run. Applying Lemma 14 to every run \( r' \) such that \((r', t) \sim_i (r, t)\) we obtain that \((\mathcal{R}_{\text{fip}}, r', t') \models K_j (\text{At}_t K_i \varphi)\). By \(TS1\) we obtain \((\mathcal{R}_{\text{fip}}, r', t') \models \text{At}_{t'} K_j (\text{At}_t K_i \varphi)\). By choice of runs \( r' \) we now conclude that \((\mathcal{R}_{\text{fip}}, r, t) \models K_i (\text{At}_{t'} K_j (\text{At}_t K_i \varphi))\). 

Lemmas 14 and 15 capture essential epistemic aspects of the fip in the synchronous context \( \gamma^{\max} \), based in part on perfect recall. Composing them gives us Lemma 16, which is at the heart of the proof of the sufficiency Theorem 7 below. The causal and epistemic states described by the lemma are shown in Figure 3.15.

**Lemma 16** If \((\mathcal{R}_{\text{fip}}, r, t) \models K_i \varphi \) and \((i, t) \sim (j, t_j) \rightarrow (\ell, t_\ell)\) then \((\mathcal{R}_{\text{fip}}, r, t) \models K_j \text{At}_{t_j} K_i K_{t_i} K_i \varphi\).

**Proof** Since \((\mathcal{R}_{\text{fip}}, r, t_i) \models K_i \varphi \) and \((i, t_i) \sim (j, t_j)\), Lemma 14 gives us \((\mathcal{R}_{\text{fip}}, r, t_j) \models K_j \text{At}_{t_j} K_i \varphi\). Now as \((j, t_j) \rightarrow (\ell, t_\ell)\), using Lemma 15 we get \((\mathcal{R}_{\text{fip}}, r, t_j) \models K_j \text{At}_{t_j} K_j \text{At}_{t_\ell} K_i \varphi\). Finally, applying validity \(TS2\) reduces the result to \((\mathcal{R}_{\text{fip}}, r, t) \models K_j \text{At}_{t_i} K_i \text{At}_{t_i} K_i \varphi\) as required.

We are now ready to prove that in fip, the existence of a centipede is sufficient for nested knowledge gain.

54
Theorem 7 If \((R^{ip}, r, t) \models K_{i_0} \varphi\) and there is a centipede for \(\langle i_0, \ldots, i_k \rangle\) in \((r, t..t')\), then \((R^{ip}, r, t') \models K_{i_k} K_{i_{k-1}} \cdots K_{i_1} K_{i_0} (At_t K_{i_0} \varphi)\).

Proof Let \(\langle (j_0, t_0), \ldots, (j_k, t_k) \rangle\) be a centipede for \(\langle i_0, \ldots, i_k \rangle\) in \((r, t..t')\), such that \((i_0, t) = (j_0, t_0)\) and \((j_k, t_k) = (i_k, t')\), as seen in Figure 3.16.

We show by proceeding inductively on each “body” node \((j_h, t_h)\) for \(0 \leq h \leq k\) that \((R^{ip}, r, t_h) \models K_{j_h} (At_t K_{i_h} \cdots K_{i_0} (At_t K_{i_0} \varphi))\). Recall that the global time \(t\) appears as a component of all local states. Thus, \((r^1, t^1) \sim_{i_0} (r^2, t^2)\) is possible only if \(t^2 = t^1\).

\(h = 0\): As \((i_0, t) \rightarrow (i_0, t')\), applying Lemma 15 to assumption \((R^{ip}, r, t) \models K_{i_0} \varphi\) gives us \((R^{ip}, r, t) \models K_{i_0} At_{t'} K_{i_0} K_{i_0} (At_t K_{i_0} \varphi)\). Since \(i_0 = j_0\) we get \((R^{ip}, r, t) \models K_{j_0} At_{t'} K_{i_0} K_{i_0} \varphi\).

\(h > 0\): Assume for \(h - 1\) and show for \(h\). For clarity, define

\[\Psi_h = K_{i_h} \cdots K_{i_0} (At_t K_{i_0} \varphi)\]

The inductive assumption gives us that

\[(R^{ip}, r, t_{h-1}) \models K_{j_{h-1}} (At_{t'} \Psi_{h-1})\].

Figure 3.15: Syncausal relation and timing guarantee, with induced knowledge state

Figure 3.16: Centipede for Theorem 7
By definition of the centipede we have that \((j_{h-1}, t_{h-1}) \rightsquigarrow (j_h, t_h)\) and that \((j_h, t_h) \rightarrow (i_h, t')\). By Lemma 16 we have that
\[
(R^{\text{fip}}, r, t_h) \models K_{j_h} (\text{At}_{t'} \text{K}_{i_h} \text{At}_{t'} \Psi_{h-1}).
\]

Using validity TS3 we reduce this to
\[
(R^{\text{fip}}, r, t_h) \models K_{j_h} (\text{At}_{t'} \text{K}_{i_h} \Psi_{h-1}),
\]
which is the required \((R^{\text{fip}}, r, t_h) \models K_{j_h} (\text{At}_{t'} \Psi_h)\). This concludes the inductive proof.

In particular, we obtain \((R^{\text{fip}}, r, t') \models K_{i_k} (\text{At}_{t'} \text{K}_{i_k} \text{K}_{i_{k-1}} \ldots \text{K}_{i_0} (\text{At}_{t} \text{K}_{i_0} \varphi))\) since \(j_k = i_k\) and \(t_k = t'\). Using the Knowledge Axiom we obtain that
\[
(R^{\text{fip}}, r, t') \models \text{At}_{t'} \text{K}_{i_k} \text{K}_{i_{k-1}} \ldots \text{K}_{i_0} (\text{At}_{t} \text{K}_{i_0} \varphi).
\]
Finally, using TS1, \((R^{\text{fip}}, r, t') \models K_{i_k} \text{K}_{i_{k-1}} \ldots \text{K}_{i_0} (\text{At}_{t} \text{K}_{i_0} \varphi)\) as desired. \(\Box\) Theorem 7

Theorem 7 proceeds by tracing the knowledge states of the centipede’s “body” nodes.\(^5\) These nodes provide a communication path that “feeds” the endpoints \(i_1, i_2, \text{ etc.}\). A subtle point is that each body node \(\theta_h\) already knows that its related endpoint \(i_h\) will know by \(t'\) what it (i.e. \(\theta_h\)) knows. This is information that \(\theta_h\) can also pass on to the next body node \(\theta_{h+1}\).

We obtain the following result by immediate application of Lemma 1 to Theorem 7.

**Theorem 8 (Nested Knowledge Sufficiency)** Let \(P\) be an fip protocol that is also non-hesitant for OR = \(\langle e_\tau, \alpha_1, \ldots, \alpha_k \rangle\). If for every \(r \in R^{\text{fip}} = R(P, \gamma^{\text{max}})\) in which \(e\) is an ND event at \((i_0, t)\) there exists time \(t'\) such that a centipede for \((i_0, \ldots, i_k)\) exists in \((r, t..t')\), then \(P\) solves OR.

### 3.10 Conclusions

This chapter starts out by introducing and discussing several new concepts related to causality in synchronous systems. Thus, the bound guarantee and syncausality relations lead up to the centipede structure. Then the formal theory is developed that results with the Knowledge Gain Theorem, and thence the Centipede Theorem. Finally, it is shown that the causal structures defined are tight, in the sense that there exists a protocol where

---

\(^5\)The knowledge state of a node is a convenient abuse of language, that refers to the knowledge state of the process related to the node, at the time related to the node.
the existence of a centipede is sufficient for knowledge gain and for solving the OR problem.

Our results all hold in particular in the case in which \( \max_{ij} = \infty \) for all channels, so that communication is asynchronous (although processes share the global clock and can move at each step). Because communication is asynchronous, bound guarantees are useless in this setting. Syncausality reduces to Lamport’s happened-before, and all possible centipedes collapse to message chains. Thus, our results also apply to such contexts, reproving Chandy and Misra’s Knowledge-gain theorem in a slightly more general setting. Asynchrony of communication alone suffices for this type of implosion.

How knowledge actually evolves in a system will depend on the particular protocol used. As a first study of the role that protocols play in determining information flow in the synchronous contexts \( \gamma^{\max} \), we have analyzed the full-information protocol and have shown that the definitions for syncausality and centipede are not only necessary but also sufficient for nested knowledge under such protocols. If one adds the non-hesitance property then the causal structures also suffice for solving the OR problem. It follows that our characterization of coordination in terms syncausality and centipedes is, in a precise sense, tight.
Chapter 4
Gaining Common Knowledge

4.1 Introduction

This chapter analyzes the causal relations that lead to common knowledge gain and to simultaneous coordination. A well-known result [23] shows that common knowledge cannot be gained in asynchronous systems. Common knowledge can, however, be gained in synchronous ones. As such, the results in this chapter have no counterpart in asynchronous systems. The state of common knowledge has been shown to play an important role in agreements and in coordinating simultaneous actions [23, 14, 15].

As before, we provide a more concrete motivation for our investigation by considering the Simultaneous Response problem, defined in Section 2.1. Consider the scenario depicted in the following example.

Example 5 The Wikileaks whistle blowing site is about to uncover yet another state secret. It strikes a bargain with El País and The New York
As soon the secret becomes available to Wikileaks (the exact timing depends upon an external source and is thus unknown), it will pass on the information to the papers using time stamped messages. The contract with Wikileaks states that both papers are to publish the scoop simultaneously, or not at all. The parties involved communicate over the network shown in Figure 4.1. Note that the scenario sketches out an instance of SR where a spontaneous event at Wikileaks is to be followed by a pair of simultaneous publication events.

Figure 4.2: Example 5

Suppose that the secret becomes available at time $t$ and that Wikileaks sends messages to the NYT and El País right after (let’s keep the Middleman out of it for now).

In Figure 4.2a Wikileak’s messages to the NYT and El País arrive at times $t + 3$ and $t + 5$ respectively. The editors both wait until $t + 10$ before simultaneously publishing the secret.

Figure 4.2b offers an alternative scenario. Here the Middleman is also notified by Wikileaks, and it sends on messages to both papers. Despite the fact that the messages sent by Wikileaks to the papers both arrive by $t + 3$, and that the Middleman’s messages arrive by $t + 8$, the papers must wait until $t + 9$ in order to ensure simultaneous publication of the scoop.

Recall that, given Theorem 2, the simultaneous response requirement is reduced to a requirement for common knowledge of the occurrence of the ND event. Example 5 is thus best analyzed in terms of knowledge gain. In Figure 4.2a, as soon as the message to El País arrives, we have that $K_E \text{Secret} \land K_N \text{Secret}$. But we also have $K_E K_N \text{Secret}$, since the Spanish editor can work out that send time was $t$ and that a message to the NYT will have arrived by $t + 3$ at the latest. $K_N K_E \text{Secret}$ does not hold however, because the message to El País may take longer than 5. By waiting until
t + 10 we also have $K_N K_E \text{Secret}$. Since both bounds have been reached, and since the bounds are common knowledge, we also get that $K_E K_N K_E \text{Secret}$, $K_N K_E K_N K_E \text{Secret}$, etc. As this ever lengthening nesting of knowledge points out, at $t + 10$ the group of papers \{NYT, El País\} has gained common knowledge of the secret, $C_{E,N} \text{Secret}$.

In Figure 4.2b similar calculations will convince the reader that, based on the Middleman’s messages, common knowledge arises already at $t + 9$. Note that mutual knowledge ($K_E \text{Secret} \land K_N \text{Secret}$, established at time $t + 3$) and even mutual nested knowledge ($K_N K_E \text{Secret} \land K_E K_N \text{Secret}$, established at time $t + 7$) do not necessarily lead to common knowledge. For example, at $t + 7$ $K_E K_N K_E \text{Secret}$ does not hold: the Spanish editor is thinking that as far as the editors in New-York are concerned, a message from Wikileaks to El País could arrive as late as $t + 10$, and that the message from the Middleman may not have arrived in New-York as yet.

4.2 Centibrooms

As illustrated above the existence of a centipede, even under the best of terms where messages contain all relevant information, may not suffice for ensuring common knowledge gain. The analysis suggests that it is only when a node exists from which messages are guaranteed to have arrived at the sites of all parties concerned, that common knowledge may arise.

We now define a communication structure that echoes this intuition.

**Definition 15 (Centibroom)** Let $t \leq t'$ and $G \subseteq \mathbb{P}$. Node $\theta$ is a centibroom for $(i_0, G)$ in $(r, t..t')$ if $(i_0, t) \leadsto \theta$ and $\theta \rightarrow (i_h, t')$ holds for all $i_h \in G$.

The centibroom node $\theta$ is syncausally connected to the originating node of the nondeterministic event, which enables it to be informed of the event’s
occurrence. Node $\theta$ is also connected by bound guarantees to the time $t'$ nodes of all processes in $G$. Intuitively, this makes it possible for $\theta$ to guarantee that a message sent to any $i_h \in G$ will have arrived by $t'$. Note that, once again, Figures 4.2a and 4.2b contain centibroom structures (in both figures all communication that is not a part of the centibroom is dimmed out).

The Centibroom Theorem, formulated below and proved in the next section, shows that indeed in order to coordinate a simultaneous response, a centibroom must exist that connects the responding sites to the triggering one. The Centibroom Theorem can be seen as an extension of the Centipede Theorem that applies to the SR problem.

**Theorem 9 (Centibroom Theorem)** Let $P$ be a protocol solving $SR = \langle e_t, \alpha_1, \ldots, \alpha_k \rangle$ in $\gamma^{\max}$, and assume that $e_t$ occurs at $(i_0, t)$ in $r \in \mathcal{R}^{\max}$. If the response actions are performed at time $t'$ in $r$, then there is a centibroom $\langle i_0, G \rangle$ in $(r, t..t')$.

### 4.3 Common Knowledge Requires Centibrooms

Clearly, centibrooms are simpler structures than general centipedes. Notice, however, that a centibroom for $G = \{j_1, \ldots, j_\ell\}$ can be considered as a condensed representation of infinitely many centipedes, each of which can support knowledge gain of a particular formula. More concretely, we have the following.

**Lemma 17** Let $G \subseteq \mathcal{P}$, and let $\theta$ be a centibroom for $\langle i_0, G \rangle$ in $(r, t..t')$. Then for every sequence $\langle i_1, \ldots, i_k \rangle \in G^k$ of processes in $G$, the sequence $(i_0, t) \cdot \theta^k$ (where $\theta$ repeats $k$ times) is a centipede for $\langle i_0, \ldots, i_k \rangle$ in $(r, t..t')$.

**Proof** Fix a sequence $\mathcal{C} = \langle i_1, \ldots, i_k \rangle \in G^k$. The sequence $\langle (i_0, t), \theta \cdots \theta, (i_k, t') \rangle$, with $k - 1$ repetitions of $\theta$, is a centipede for $\mathcal{C}$, since:

- $\theta$ is a centibroom for $\langle i_0, G \rangle$, so $(i_0, t) \leadsto \theta$, and
- $\theta \cdot (i_k, t')$ implies $\theta \leadsto (i_k, t')$, and
- $\theta \leadsto \theta$ due to reflexivity of $\leadsto$, finally
- $\theta$ is a centibroom for $\langle i_0, G \rangle$ so $\theta \cdot (i_h, t')$ for all $1 \leq h \leq k - 1$. 

61
Notice that Lemma 17 does not bound the value of $k$, nor does it restrict the possibility of repetitions in the sequence $\langle i_1, \ldots, i_k \rangle$ in question. We are now ready to show that the centibroom serves as the structure underlying common knowledge.

**Theorem 10 (Common Knowledge Gain)** Let $P$ be an arbitrary protocol, let $G \subseteq \mathbb{P}$, and let $r \in R^{\max}$. Assume that $e$ is an ND event at $(i_0, t)$ in $r$. If $t' > t$ and $(R^{\max}, r, t') \models C_G((\text{occurred}(e) \land \text{ND}(e)))$, then there is a centibroom $\hat{\theta}$ for $\langle i_0, G \rangle$ in $(r, t..t')$.

**Proof** Assume the notations and conditions of the theorem. Denote $G = \{i_1, \ldots, i_k\}$ and $d = t' - t$. Since $(R^{\max}, r, t') \models C_G(\text{occurred}(e) \land \text{ND}(e))$ we have by definition of common knowledge that $(R^{\max}, r, t') \models E_G^{k(d+1)}(\text{occurred}(e) \land \text{ND}(e))$. In particular, this implies that $(R^{\max}, r, t') \models (K_{ii_1} \cdots K_{ii_k})^{d+1}(\text{occurred}(e) \land \text{ND}(e)),$

where $(K_{ii_1} \cdots K_{ii_k})^{d+1}$ stands for $d + 1$ consecutive copies of $K_{ii_1} \cdots K_{ii_k}$. By the Knowledge Gain Theorem 6, there is a corresponding centipede $\sigma = \langle \theta_0, \theta_1, \ldots, \theta_k(d+1) \rangle$ in $(r, t..t')$. Denote $\theta_h = (i_h, t_h)$ for all $0 \leq h \leq k \cdot (d + 1)$. Recall that, by definition, $\theta_h \asymp \theta_{h+1}$ holds for all $h < k \cdot (d + 1)$. By Lemma 4 we obtain that if $\theta_h \neq \theta_{h+1}$ then $t_h < t_{h+1}$. It follows that there can be at most $d + 1$ distinct nodes $\alpha_1 \sim \alpha_2 \sim \cdots \sim \alpha_\ell$ in $\sigma$. Every $\alpha_h$ represents a segment $\theta_{x_1}, \ldots, \theta_{x_h}$ of the nodes in $\sigma$. By the pigeonhole principle, one of the $\alpha$’s must represent a segment consisting of at least $k$ of the $\theta$’s in $\sigma$. Denoting this node by $\hat{\alpha}$, we obtain that $\hat{\alpha} \rightarrow (i_h, t')$ for every $i_h \in G$. Moreover, by definition of the centipede and transitivity of $\sim$ we have that $(i_0, t) \sim \hat{\alpha}$. It follows that $\hat{\alpha}$ is a centibroom for $\langle i_0, G \rangle$ in $(r, t..t')$.

\[ \Box \text{Theorem 10} \]

The proof of Theorem 10 is based on the Knowledge Gain Theorem 6. Recall that $C_G \varphi$ implies arbitrarily deeply nested knowledge of $\varphi$. Every such nested knowledge formula implies the existence of a centipede. A nested knowledge formula is constructed whose centipede has sufficiently many nodes that at least one of them must be a centibroom for $G$ at $t'$.

In Chapter 3 we defined the centinode, which is an instance of the centipede whose every “body” node is a bridge to the related “leg” node. We now similarly identify and prove the existence of a bridging centibroom.

**Definition 16 (Bridging centibroom)** Let $t \leq t'$ and $G \subseteq \mathbb{P}$. Node $\theta$ is a bridging centibroom for $\langle i_0, G \rangle$ in $(r, t..t')$ if
• \( \theta \) is a centibroom for \( \langle i_0, G \rangle \) in \( (r, t..t') \); and

• \( \theta \) bridges \( (i_0, t) \) and \( g \) for every \( g \in G \).

**Lemma 18** Fix \( r \in R^{max} \) and assume that \( \theta \) is a centibroom for \( \langle i_0, G \rangle \) in \( (r, t..t') \). Then there exists a node \( \theta' \) that is a bridging centibroom for \( \langle i_0, G \rangle \) in \( (r, t..t') \).

**Proof** By Lemma 7 there exists a node \( \psi \) such that \( \psi \) bridges \( (i_0, t) \) and \( \theta \). Node \( \psi \) is a bridging centibroom since

- \( (i_0, t) \sim \psi \rightarrow \theta \rightarrow g \) implies \( (i_0, t) \sim \psi \rightarrow g \) for all \( g \in G \)

- \( \alpha \sim \psi' \rightarrow \psi \) implies \( \psi' = \psi \) by definition of bridge.

Theorem 10 shows that common knowledge can arise in synchronous systems *only* when there exists a centibroom structure, centered about the centibroom node. The above Lemma 18, together with Lemma 8, points out that there must exist a bridging centibroom for the group, in which a nondeterministic *pivotal event*, either an early receive or possibly an external input when \( \psi = (i_0, t) \), occurs. This demonstrates that the nature of common knowledge is finitistic, despite its familiar definition being based on an infinite conjunction of facts. This phenomenon is consistent with the analysis of common knowledge in the work on fault-tolerance [13, 36, 32]. There, too, common knowledge arises at some time \( t' \) exactly if there is some property \( S \) of the correct nodes that ensures that all processes will know by time \( t' \) that the property \( S \) held in the run.

We remark that Theorem 10 relates to a familiar situation involving the evolution of knowledge in broadcasts. In a flooding protocol or a radio broadcast, for example, the contents being broadcast become common knowledge to a growing set of participants with time. Typically, after a time interval equivalent to the diameter of the system, the contents can become common knowledge to *all* processes in the system.

The proof of Theorem 9 is now immediate: we show that the existence of a centibroom is a necessary condition for solving the Simultaneous Response problem by applying the Common Knowledge Theorem 10 to Theorem 2.
4.4 The Simultaneous Global Snapshot Protocol

Before exploring further the theoretical implications for the centibroom structure, we pause to consider a possible application.\(^1\)

A well known application for Lamport’s causal relation is the global snapshot algorithm, proposed by Chandy and Lamport in [6]. This algorithm is used to record a consistent global state in asynchronous systems. A global snapshot of the system at a given run \(r\) and time \(t\), which we will denote with \(\text{Snap}(r, t)\), consists of records of the local states of all processes in the system, and of the communication channels, at that point in the run. Technically, communication channels do not possess a memory, so their state must be reconstructed by the processes. Interestingly, the Chandy-Lamport algorithm cannot ensure that the global snapshot that it actually records is in fact a global state in the current run. No protocol can grant such assurances in an asynchronous system. Rather, the algorithm ensures that the recorded snapshot is consistent with the current run in the following sense:

**Definition 17 (Snapshot consistency)** Fix \(r \in \mathcal{R}(P, \gamma)\) for arbitrary protocol \(P\) and context \(\gamma\). Snapshot \(S^*\) is consistent with the interval \([t_s, t_e]\) of \(r\) if there exists \(r' \in \mathcal{R}(P, \gamma)\) and times \(t'_s \leq t'_* \leq t'_e\) such that

1. \(\text{Snap}(r, t_s) = \text{Snap}(r', t'_s)\),
2. \(S^* = \text{Snap}(r', t'_*)\), and
3. \(\text{Snap}(r, t_e) = \text{Snap}(r', t'_e)\).

Mechanisms for recording global states come in useful, for example, in association with recovery from system failure. In fact, many applications use such algorithms in order to retain “checkpoints”: global states that can be “rolled back” into, when failure occurs [38]. The Centibroom Theorem suggests a synchronous variation for Chandy and Lamport’s original algorithm. We will actually consider two variants: the first being message optimal, and the second providing time optimization.

When activated, the Simultaneous Global Snapshot Protocol results with all processes simultaneously recording their local states at a time \(t\), and all messages that are in transit on inbound communication channels at that

\(^1\)We thank Gadi Taubenfeld for suggesting this application to us.
time. Observe that given the synchronous nature of the system, simultaneity is a necessary requirement for achieving a consistent global state. Allowing two processes \( i \) and \( j \) to record their local states at \( t_i \) and \( t_j \) respectively, where \(|t_i - t_j| > 0\) may, in the general case, result in an inconsistency: it may be the case that there are no possible global states that includes the local states defined by \((i, t_i)\) and \((j, t_j)\) both, due to simultaneous actions that are always performed by \( i \) and \( j \) together at some time \( t_i < t' \leq t_j \).

Summing up, if snapshot \( S^* \) is consistent with the interval \([t_s, t_e]\) of run \( r \), then there exists some time \( t_\ast \in [t_s, t_e] \) such that \( S^* = \text{Snap}(r, t_\ast) \).

The algorithm is quite simple. We mark with \( \text{Diameter}_i \) the distance of the process \( j \) furthest from \( i \), when measuring based on \( D_{ij} \). We assume that the protocol may be initiated (from the outside) at any process in the system, or even in several places in the system. Algorithm 1 shows the protocol’s pseudo code. The (arbitrary) initiator node \((i, t)\) floods the system with \textit{initiate} messages, that indicate time \( t' = t + \text{Diameter}_i \) as the time at which the “snapshot” must be taken. By definition of \( \text{Diameter}_i \), these messages arrive at all sites by time \( t' \). At \( t' \) every process \( j \) records its own local state, and starts recording incoming communications on each of its inbound channels. Recording the channel \( h \mapsto j \) takes place from time \( t' \), until \( t' + \max_{ij}_h \), but only messages that are not marked with an extra “ignore” bit are recorded. Apart from carrying on these recordings, the processes are free to carry on with their (non snapshot related) tasks. However, if these tasks demand that a process \( j \) send a message on some \textit{outbound} channel \( j \mapsto h \) prior to time \( t' + \max_{j_h} \), then this message is marked “ignore” by appending an extra bit set to 1 to the message.

Note that a different mechanism could be employed for the purpose of recording the contents of communication channels. Rather than starting to record upon snapshot, the alternative mechanism would have each process constantly keeping a long-enough tail on its history so that when snapshot occurs at time \( t \), for each channel \( i \mapsto j \), process \( i \) can recount all messages sent on the channel which may, potentially still be en route. Those would be all messages sent after \( t - \max_{ij} \). At the price of greater stress on memory resources, the algorithm would complete the snapshot recording faster. Although in order to gain a complete picture of the state of the channel \( i \mapsto j \), we would have to further compare the local states of \( i \) and \( j \) at the snapshot time. For this reason we opt for the version presented below, its simplicity being better suited for our explanatory purposes.

The following lemma proves the protocol’s correctness.

\textbf{Lemma 19} Choose \( r \in \mathcal{R}(P^{\text{snap}^{-1}}, \gamma_{\text{max}}) \) where snapshot initiation occurs

\[ \]
Algorithm 1 Simultaneous Global Snapshot - \(P^{\text{snap-1}}\)

1: \textbf{procedure} \textit{Initiator node} \((i, t)\):
2: \hspace{1em} snapshot \(\leftarrow t + \text{Diameter}_i\)
3: \hspace{1em} for all outgoing channels \(i \mapsto h\) do
4: \hspace{2em} send\_h(\text{initiate}(\text{snapshot}))
5: \textbf{procedure} \textit{Arbitrary node} \((j, t')\):
6: \hspace{1em} if receive \text{initiate}(S) then
7: \hspace{2em} snapshot \(\leftarrow S\)
8: \hspace{2em} for all outgoing channels \(i \mapsto h\) do
9: \hspace{3em} send\_h(\text{initiate}(\text{snapshot}))
10: \hspace{1em} if \(t' = \text{snapshot}\) then record local state
11: \hspace{1em} for all incoming channels \(h \mapsto j\) do
12: \hspace{2em} receive \text{msg} on channel
13: \hspace{2em} if \(\text{snapshot} \leq t' < \text{snapshot} + D(h, j) \land \text{msg}.\text{transparent} \neq 0\)
14: \hspace{3em} record \text{msg}

at \((i, t)\). Then there exists a time \(t' \geq t + \text{Diameter}_i\) where each process \(j\) contains

1. record of its local state at time \(t + \text{Diameter}_i\), and
2. record of incoming messages en route at time \(t + \text{Diameter}_i\).

\textbf{Proof} That all process local states are simultaneously recorded at \(t'\) is straightforward from the definitions. That exactly those messages that were in transit at time \(t'\) are recorded can be seen by noting first that all messages in transit on channel \(h \mapsto j\) at \(t'\) are guaranteed to arrive by time \(t' + \text{max}_{hj}\), at which point recording on that channel stops. Moreover, messages sent after \(t'\) but which arrive at \(j\) before \(t' + \text{max}_{hj}\) will be marked transparent and will not be recorded. Thus, the algorithm is correct in recording the global state at time \(t'\).

The algorithm is straightforward. A revised version of the algorithm can ensure time optimality. The protocol starts the same, with the initiating node \((i, t)\) flooding the system with \textit{initiate} messages bearing the value \(t + \text{Diameter}_i\). However, in this version, every process \(j\) that gets such a message at time \(t'\) checks to see whether it can ensure an even quicker simultaneous recording response, i.e. whether \(t' + \text{Diameter}_j < t + \text{Diameter}_i\). If so, it will start to flood the system with \textit{initiate} messages bearing \(t' + \text{Diameter}_j\).
Algorithm 2 Simultaneous Global Snapshot - $P_{\text{snap}}^{-2}$

1: procedure Initiator node $(i, t)$:
2:    $\text{snapshot} \leftarrow t + \text{Diameter}_i$
3:    for all outgoing channels $i \mapsto h$ do
4:       $\text{send}_h(\text{initiate}(\text{snapshot}))$
5: procedure Arbitrary node $(j, t')$:
6:    if receive initiate(S) then
7:       if $S \leq t' + \text{Diameter}_j$ then
8:          $\text{snapshot} \leftarrow S$
9:       else
10:          $\text{snapshot} \leftarrow t' + \text{Diameter}_j$
11:    for all outgoing channels $i \mapsto h$ do
12:       $\text{send}_h(\text{initiate}(\text{snapshot}))$
13:    if $t' = \text{snapshot}$ then record local state
14:    for all incoming channels $h \mapsto j$ do
15:       receive msg on channel
16:       if $\text{snapshot} \leq t' < \text{snapshot} + D(h, j) \wedge \text{msg.transparent} \neq 0$
17:          record msg
Lemma 20  Protocol $P_{\text{snap}-2}$ has the following two properties:

**Correctness:** It is correct.

**Optimality:** No other protocol can ensure a shorter delay between initiation and time of snapshot.

**Proof**

**Correctness:** Fix a run $r \in \mathcal{R}(P_{\text{snap}-2}, \gamma_{\text{max}})$ where initiation of snapshot algorithm occurs at $\theta_0 = (i_0, t_0)$, setting snapshot time for $t'_0 = t_0 + \text{Diameter}_{i_0}$. If no shorter term initiate messages are issued within the interval $[t_0, t'_0]$ then $r$ is also a $P_{\text{snap}-1}$ run, and is thus correct by Lemma 19.

Otherwise, let $t'_1 < t'_0$ be the earliest snapshot time suggested after initiation, and let $\theta_1 = (i_1, t_1)$ be the issuing node. As $t'_1 = t_1 + \text{Diameter}_{i_1}$ and no process issues a shorter term initiate message, $\text{initiate}(t'_1)$ is guaranteed to arrive at all processes no later than $t'_1$. Again, as no process issues a shorter term snapshot suggestion, the local variable $\text{snapshot}$ is equal to $t'_1$ at time $t'_1$ in all processes. Now, based on Lemma 19, the run is shown to be correct.

**Optimality:** By the Centibroom Theorem, any protocol in which a simultaneous action on the part of all processes is dependent upon snapshot initiation must contain a centibroom for $\langle \theta_0, \mathcal{P} \rangle$ where $\theta_0 = (i_0, t_0)$ is the initiation node. Choose a run $r \in \mathcal{R} = \mathcal{R}(P_{\text{snap}-2}, \gamma_{\text{max}})$ initiation occurs at $\theta_0$ and snapshot at $t'_0$.

Suppose that there exists a centibroom node $(i_1, t_1)$ for $\langle \theta, \mathcal{P} \rangle$ in $\langle r, t..t'_1 \rangle$, where $t'_1 < t'_0$. Assume without loss of generality that for every $t'' < t'_1$ there are no centibrooms for $\langle \theta, \mathcal{P} \rangle$ in $\langle r, t..t'' \rangle$. By definition of centibroom, $\theta_0 \rightsquigarrow \theta_1$ and $\theta_1 \rightsquigarrow (h, t'_1)$ for all $h \in \mathcal{P}$. At $t_1$ or sooner, $i_1$ receives an $\text{initiate}(S)$ message with some suggested snapshot time $S$. Since $t_1 + \text{Diameter}_{i_1} = t'_1 < t'_0 \leq S$, and as $i_1$ is following $P_{\text{snap}-2}$, it immediately starts to flood the system with $\text{initiate}(t'_1)$ messages. As no shorter term suggestion is made, by the above proof of the correctness of $P_{\text{snap}-2}$, snapshot occurs at $t'_1 < t'_0$, in contradiction to the assumption that snapshot occurs at $t'_0$.

We thus obtain that for every run $r \in \mathcal{R}$ in which initiation occurs at $\theta_0$, the shortest interval within which a centibroom can be established is $[t_0, t'_0]$, where $t'_0$ is the time at which snapshot actually occurs.

68
As all nodes in \( \text{fut}(\theta_0) \) flood the initiate messages, there cannot be a protocol \( P' \) where information about initiation decimates any faster than in \( P_{\text{snap}^{-2}} \), and hence in particular a centibroom cannot be established any faster than in \( P_{\text{snap}^{-2}} \), and so delay between initiation and snapshot is at least as long as it is in \( P_{\text{snap}^{-2}} \).

\[ \blacksquare \]

### 4.5 Sufficiency of Centibrooms for Common Knowledge Gain

We proceed to show that the centibroom indeed characterizes common knowledge gain in synchronous systems, in the same way nested knowledge gain is characterized by centipedes. We will show that the existence of a centibroom is sufficient for common knowledge gain in every \( \mathcal{R}^{\text{fp}} \) system by using the Induction Rule for Common Knowledge, which states that from \( \mathcal{R}^{\text{fp}} \models \alpha \rightarrow E_G(\alpha \land \beta) \), we can infer \( \mathcal{R}^{\text{fp}} \models \alpha ightarrow C_G \beta \). Importantly, processes must now make explicit use of their capability to discern global time in order to gain common knowledge, due to the essential part played by bound guarantees.

**Theorem 11** If \( (\mathcal{R}^{\text{fp}}, r, t) \models K_{i_0} \varphi \) and there is a centibroom node \( \theta \) for \( \langle i_0, G \rangle \) in \( (r, t, t') \), then \( (\mathcal{R}^{\text{fp}}, r, t') \models C_G(\text{At}(\varphi, t)) \).

**Proof** Assume that the conditions of the theorem hold, and let \( \theta = (j, t_j) \). In particular, \( (j, t_j) \rightarrow (i, t') \) for every \( i \in G \). From \( (\mathcal{R}^{\text{fp}}, r, t) \models K_{i_0} \varphi \) and \( (i_0, t) \sim (j, t_j) \) in \( r \) we have by Lemma 14 that \( (\mathcal{R}^{\text{fp}}, r, t_j) \models K_j(\text{At}(\varphi, t)) \). We now use the induction rule with \( \alpha \) set to \( (\text{time} = t') \land \text{At}(K_j(\text{At}(\varphi, t))), t_j \), and \( \beta \) being \( \text{At}(\varphi, t) \). Since \( \mathcal{R}^{\text{fp}} \models \alpha \rightarrow \beta \) in this case, it suffices to show that \( \mathcal{R}^{\text{fp}} \models \alpha \rightarrow E_G \alpha \). Thus, let \( r' \in \mathcal{R}^{\text{fp}} \) and fix time \( \hat{t} \). If \( (\mathcal{R}^{\text{fp}}, r', \hat{t}) \models \alpha \) then \( \alpha \rightarrow E_G \alpha \) is trivially satisfied in \( (r, \hat{t}) \). Now suppose that \( (\mathcal{R}^{\text{fp}}, r', \hat{t}) \models \alpha \), giving us that \( \hat{t} = t' \) and thus \( (\mathcal{R}^{\text{fp}}, r', t') \models \text{At}_j, K_j(\text{At}_i \varphi) \). This, in turn, gives us \( (\mathcal{R}^{\text{fp}}, r', t_j) \models K_j(\text{At}_i \varphi) \) by application of \( TS1 \) (Lemma 12). Fix \( i \in G \). Since \( (j, t_j) \) is a centibroom node, we have \( (j, t_j) \rightarrow (i, t') \). By Lemma 3 it is also the case that \( (j, t_j) \sim (i, t') \) in \( r' \). Using Lemma 14 we now obtain \( (\mathcal{R}^{\text{fp}}, r', t') \models K_i, K_j(\text{At}_i \varphi) \). Moreover, the fact that the time is part of the local state in \( \gamma^{\text{max}} \) implies that \( (\mathcal{R}^{\text{fp}}, r', t') \models K_i(\text{time} = t') \). It follows that \( (\mathcal{R}^{\text{fp}}, r', t') \models K_i \alpha \), and since \( i \) was an arbitrarily chosen member of \( G \) then \( (\mathcal{R}^{\text{fp}}, r', t') \models E_G \alpha \). It follows
that $\mathcal{R}_{\text{fip}} \models \alpha \rightarrow E_G \alpha$. Since $\beta = \text{At}(\varphi, t)$ we obtain by the Induction Rule that $\mathcal{R}_{\text{fip}} \models \alpha \rightarrow C_G(\text{At}(\varphi, t))$. Finally, since $(\mathcal{R}_{\text{fip}}, r, t') \models \alpha$ we obtain that $(\mathcal{R}_{\text{fip}}, r, t') \models C_G(\text{At}(\varphi, t))$, as desired. $\Box_{\text{Theorem 11}}$

In order to relate the centibroom in a fip system to a solution to the simultaneous response problem $\text{SR}$, we must tie in common knowledge to action. Such a connection is established if we assume that the protocol is also considerate with respect to $\text{SR}$ (see Definition 21). We obtain the following result by immediate application of Lemma 2 to Theorem 11.

**Theorem 12 (Nested Knowledge Sufficiency)** Let $P$ be an fip that is also considerate with respect to $\text{SR} = \langle e, \alpha_1, \ldots, \alpha_k \rangle$. If for every $r \in \mathcal{R}_{\text{fip}} = \mathcal{R}(P, \gamma_{\text{max}})$ in which $e$ is an ND event at $(i_0, t)$ there exists time $t'$ such that a centibroom for $\langle i_0, \ldots, i_k \rangle$ exists in $(r, t..t')$, then $P$ solves $\text{SR}$.

### 4.6 Common Knowledge as a Finite Conjunction

Common knowledge is typically perceived in terms of an infinite conjunction of $E^k$, for $k > 0$. There are also definitions of common knowledge in terms of a fixed point (see, e.g., [29, 15, 5]). The centibroom structure and the necessity of centibrooms for common knowledge supports the fixed-point view: the only way in which a new fact can become common knowledge is if there is a singular point, represented by the centibroom node $\theta$, which carries the information that $\theta$ is a centibroom for all processes in $G$ at time $t'$. At time $t'$, everyone can become aware of its existence, and the fixed-point yields common knowledge. This is also consistent with the view advocated by [9, 29], that a shared environment is required for common knowledge to arise.

Even though the fixed point definition implies the infinite conjunction, Fischer and Immerman [18] showed that in finite-state systems, where the set of all global states in a system $R$ is finite, there is a power $m$ such that $C_G \varphi$ is equivalent to $E_m^G \varphi$. The fip protocol, with its perfect recall property in the synchronous context $\gamma_{\text{max}}$, produces a state space whose size is unbounded. Nevertheless, given the role of the centipede and centibroom structures in $\gamma_{\text{max}}$, we now show that there are cases in which common knowledge is a finite conjunction under fip in $\gamma_{\text{max}}$ as well.

Roughly speaking, when running fip it takes time to obtain deep knowledge without having common knowledge. Indeed, we obtain a sharp bound on the depth of $E^k_G$ that can be obtained $d$ time units after the occurrence of
a nondeterministic event. Given a group of size $|G| = g$ and natural number $d > 0$, we denote by $M_{dg} = (d - 1)(g - 1) + 2$. We prove

**Theorem 13** Let $e$ be an ND event occurring at $(i_0, t)$ in $r \in R_\text{fp}$, let $d > 0$, and $|G| = g$. If $(R_\text{fp}, r, t + d) \models E_{M_{dg}} \text{occurred}(e)$ then $(R_\text{fp}, r, t + d) \models C_G(\text{occurred}(e) \land \text{ND}(e))$.

Note that although a centipede’s “body” nodes $(\theta_0, \ldots, \theta_k)$ are naturally conceived of as distinct, they need not be such. Yet recall that by Lemma 4, when two body nodes are distinct, their time components must also be distinct.

Theorem 13 follows directly by Theorem 11 from the following lemma:

**Lemma 21** Let $r \in R^{\text{max}}$, $d > 0$, $G \subseteq \Pi$ with $g = |G|$, and assume that $e$ is an ND event at $(i_0, t)$ in $r$. If $(R^{\text{max}}, r, t + d) \models E_{M_{dg}} \text{occurred}(e)$ then there exists a centibroom node for $(i_0, G)$ in $(r, t \ldots t + d)$.

**Proof** Assume that $(R^{\text{max}}, r, t + d) \models E_{M_{dg}} \text{occurred}(e)$. If $(i_0, t)$ is a centibroom for $(i_0, G)$ in $(r, t \ldots t + d)$ then we are done. Otherwise $|G| > 1$, and moreover there is some $j \in G$ such that $(i_0, t) \not\rightarrow (j, t + d)$. For notational convenience, let us denote the processes of $G$ by $\{j_0, \ldots, j_{g-1}\}$, where $(i_0, t) \not\rightarrow (j_0, t + d)$. Denote $M = M_{dg} - 1$ and let $f(h) = j_{(h \mod g)}$ for all $h \leq M$. Thus, $f$ maps natural numbers into members of $G$, every interval of $g$ adjacent numbers are mapped to the full set $\{j_0, \ldots, j_{g-1}\} = G$, and $f(0) = j_0$. We focus on a knowledge formula of the form

$$\Psi(e) = K_{f(M)}K_{f(M-1)} \cdots K_{f(1)}K_{f(0)} \text{occurred}(e).$$

Observe that there are $M + 1 = (|G| - 1) \cdot (d - 1) + 2$ knowledge operators in $\Psi(e)$, all of which belong to processes in $G$. By assumption, $(R^{\text{max}}, r, t + d) \models E_{G^{M+1}} \text{occurred}(e)$, and hence in particular $(R^{\text{max}}, r, t + d) \models \Psi(e)$. The Knowledge Gain Theorem implies that there exists a centipede for $(i_0, f(0), f(1), \ldots, f(M))$ in $(r, t \ldots t + d)$. Let

$$\langle (i_0, t), \Omega^0, \Omega^1, \ldots, \Omega^{M-1}, (f(M), t + d) \rangle$$

be such a centipede. By definition of a centipede we have that $(i_0, t) \rightsquigarrow \Omega^0$ and $\Omega^0 \rightsquigarrow (j_0, t + d)$. Since ‘‘$\rightsquigarrow$’’ is transitive, the fact that $(i_0, t) \not\rightarrow (j_0, t + d)$ implies that $(i_0, t) \not\rightarrow \Omega^0$. Since ‘‘$\rightsquigarrow$’’ is reflexive we have that $(i_0, t) \neq \Omega^0$. Recall by definition of $f$ that $f(M) = f(M - 1) + 1 \mod g$. 

71
Since \( g > 1 \), clearly \( f(M) \neq f(M - 1) \). Hence, by Lemma 4 we have that \((f(M), t + d) \rightarrow (f(M - 1), t + d)\). It follows that \( \Omega^{M-1} \neq (f(M), t + d) \).

By Lemma 4, if \( \Omega^h = \Omega^{h'} \) then \( \Omega^h = \Omega^{h''} = \Omega^{h'} \) for every \( h'' \) in the range \( h \leq h'' \leq h' \). Let \( \Phi_1, \ldots, \Phi_D \) denote the maximal sub-sequence of distinct nodes in the sequence \( \Omega^0, \ldots, \Omega^{M-1} \). Lemma 4 implies that the times at which the nodes \((i_0, t), \Phi_1, \ldots, \Phi_D, (f(M), t + d)\) occur form a strictly increasing sequence, and so \( D \leq d - 1 \). For all \( b \) in the range \( 1 \leq b \leq D \) define 

\[
\text{Lemma 22} \quad \text{For every } t \geq 0, \ d > 0 \text{ and } g > 1 \text{ there exists a run } r \in \mathcal{R}^{\text{fip}} \text{, an ND event } e \text{ at } (i_0, t) \text{ in } r \text{ and a set of processes } G \subseteq \mathcal{P} \text{ of size } |G| = g, \text{ such that}
\]

\[
(\mathcal{R}^{\text{fip}}, r, t + d) \models E_G^{M_d g - 1} \text{ occurred}(e) \land \neg C_G \text{ occurred}(e).
\]

**Proof** Fix \( d, g \). Define \( \gamma_{d,g}^{\text{max}} \) to be a synchronous context with the following properties:

- Let \( G = \{j_0, \ldots, j_{g-1}\} \). For every \( m < g \), denote by \( G_{-m} \) the set \( G \setminus \{j_m\} \).

- Let \( \mathcal{P} = G \cup \{i_0\} \cup \{h_{k,m}\}_{1 \leq k < d, 0 \leq m < g} \). The set of processes is seen in Figure 4.4a.

- The network graph is complete, and the bounds on transmission times are as follows

1. for every \( k < d \) and \( m < g \), \( b(h_{k,m}, j) = 1 \) for all \( j \in G_{-m} \)
2. for every other \( i, j \in \mathcal{P} \), \( b(i, j) = d + 1 \)

For every \( 1 \leq k < d \), use \( H_k \) to denote the set \( \{h_{k,m}\}_{0 \leq m < g} \). Note that for every \( r \in \mathcal{R}^{\text{fip}, d} = \mathcal{R}(\text{fip}, \gamma_{d,g}^{\text{max}}) \), as the processes are running the fip, every process sends every other process a message at every time unit. Note also that there can be no centibroom node for \( (i_0, G) \) in \( (r, t \ldots t + d) \), because
for every process \( i \) there exists at least one \( j \in G \) such that \( b(i, j) > d \). Hence, by Theorem 10, \((R_{\text{fip}}, r, t + d) \models \neg G \text{occurred}(e)\).

Choose \( r \in R_{\text{fip}, d,g} \) such that an ND event \( e \) occurs at \((i_0, t)\) and such that all sent messages arrive at the maximally allowed transmission time, except for the following ones:

1. For every \( h \in H_1 \), the message sent from \( i_0 \) to \( h \) at time 0 arrives at time 1.

2. For every \( 1 \leq k < d - 1 \), for every pair of processes \( h_1 \in H_k \) and \( h_2 \in H_{k+1} \), the message sent by \( h_1 \) to \( h_2 \) at time \( k \) arrives at \( k + 1 \).

3. For every \( h \in H_{d-1} \) and \( j \in G \), the message sent from \( h \) to \( j \) at time \( d - 1 \) arrives at time \( d \).

The existence of \( r \) is guaranteed by definition of \( R_{\text{fip}, d,g} \): the run is a legal possible execution of \( \text{fip} \) in the defined context.

Use \( f(k) \) to denote the value \((g - 1) \cdot k\) for every \( k > 0 \). Fix a sequence \( S = \langle i_0, i_1, \ldots, i_{f(d-1)+1} \rangle \) such that \( \{i_1, \ldots, i_{f(d-1)+1}\} \subseteq G \). Observe that for every \( 1 \leq k < d \), the subsequence \( s_k = \langle i_{f(k-1)+1}, \ldots, i_{f(k)} \rangle \) contains exactly \( g - 1 \) elements, and so there must exist some \( j(k) \in G \) such that \( j(k) \neq i \) for every \( i \in s_k \).

We now define a node sequence \( \langle \langle i_0, t \rangle, \theta_1, \ldots, \theta_{f(d-1)}, (i_{f(d-1)+1}, d) \rangle \) and show that it is a centipede for \( S \) in \((r, 0\ldots d)\). For every \( l = 1..f(d-1) \), let \( k = \lceil \frac{l}{g-1} \rceil \), and define \( \theta_l = (h_{k,j(k)}, k) \). Observe that \( f(k-1) < l \leq f(k) \), and hence by choice of \( j(k) \) that \( b(h_{k,j(k)}, i_l) = 1 \). Since \( k \leq d - 1 \) we obtain that \( \theta_l \rightarrow (i_l, d) \). Moreover, if \( l < f(d-1) \) then \( \theta_l \rightarrow \theta_{l+1} \). For if \( k > \frac{l}{g-1} \) then \( \theta_l = \theta_{l+1} \) and the result stems from the reflexivity of \( \rightarrow \), while if \( k = \frac{l}{g-1} \) then, noting that \( \theta_l \in H_k \) and \( \theta_{l+1} \in H_{k+1} \), we get the result from clause (2) above. Finally, we note that \( \langle i_0, t \rangle \rightarrow \theta_1 = (h_{1,j(1)}, 1) \) since \( h_{1,j(1)} \in H_1 \) and using clause (1), and similarly that \( \theta_{f(d-1)} = (h_{d-1,j(d-1)}, d-1) \rightarrow (i_{f(d-1)+1}, d) \) since \( h_{d-1,j(d-1)} \in H_{d-1} \) and from clause (3) above. Figure 4.4b shows a fragment of the described centipede.

We have shown that there exists a centipede in \((r, 0\ldots d)\) for every sequence \( \langle i_0, i_1, \ldots, i_{f(d-1)+1} \rangle \) such that \( \{i_1, \ldots, i_{f(d-1)+1}\} \subseteq G \). By Theorem 7 we get that \((R_{\text{fip}, d,g}, r, t') \models K_{i_{f(d-1)+1}} K_{i_{f(d-1)}} \cdots K_{i_1} \text{occurred}(e)\) for every such sequence. We thus obtain, considering that \( f(d-1) + 1 = (g - 1)(d - 1) + 1 = M_{d_{g}} - 1 \), that \((R_{\text{fip}, d,g}, r, t') \models E_{G}^{M_{d_{g}}-1} \text{occurred}(e)\) by definition of \( E \) operator.

\[\blacksquare\]
Theorem 13 and Lemma 22 tightly bound the levels of $E_k^G$ that can hold without common knowledge necessarily arising. They draw an essential connection between this bound, the size of the set of processes $G$ in question, and the time that elapses since the ND event of interest occurs. It is natural to ask whether this property is restricted to $f_{ip}$, or perhaps may be true in general. We now show that it is not true for all protocols. In fact, there is a protocol that can attain arbitrary levels of nested knowledge quickly, without giving rise to common knowledge.

**Example 6** Let $\gamma^{max'}$ be a context with $P = \{s,0,1\}$, where the network is V-shaped with $s$ at the base, and the communication bounds are $b_{s,0} = b_{s,1} = 1$. The initial state of process $s$ contains an initial value consisting of a natural number $k \geq 0$. We assume that the protocol $P$ that $s$ is following prescribes the following actions upon receiving an external input (an event that we denote by $e$): If $k$ is odd, then $s$ sends the message $\langle \text{occurred}(e), k \rangle$ to process 1, and the message $\langle \text{occurred}(e), k - 1 \rangle$ to process 0. If $k$ is even, then $s$ sends the message $\langle \text{occurred}(e), k \rangle$ to process 0 and, in case $k > 0$ it also sends the message $\langle \text{occurred}(e), k - 1 \rangle$ to process 1. Moreover, $s$ never sends a message of the form $\langle \text{occurred}(e), d \rangle$ if $e$ does not occur.

Thus, if $k = 0$ then only one process will receive a message, and in all other cases both of them will. Whenever an process receives the message $\langle \text{occurred}(e), h \rangle$, it knows that $e$ occurred but does not know whether $k = h$ or $h + 1$. In particular, upon receiving $\langle \text{occurred}(e), 0 \rangle$, process 0 considers it possible that 1 received nothing and does not know that $e$ occurred.

We now show that arbitrarily deeply nested knowledge can be obtained in this setting within a single time step, without common knowledge arising:

**Lemma 23** In the context of Example 6, let $r \in R = R(P, \gamma^{max'})$, let $G = \{0,1\}$ and assume that the event $e$, consisting of the receipt of an external
input by $s$ at time $t$, in $r$. If the initial value of $s$ in $r$ is $k$ then

$$(R, r, t + 1) \models E_k^G \text{occurred}(e) \land \neg C_G \text{occurred}(e).$$

**Proof** We split the proof into two parts, handled by Lemmas 24 and 25. Assume that $e$ occurs in $r$ at time $t$ as stated, and that the initial value is $k$. By Lemma 24 we have that $(R, r, t + 1) \models E_k^G \text{occurred}(e)$ and by Lemma 24 that $(R, r, t + 1) \models \neg E_{k+1}^G \text{occurred}(e)$. Since $\models \neg E_{k+1}^G \varphi \rightarrow \neg C_G \varphi$ is a validity, the latter implies that $(R, r, t + 1) \models \neg C_G \text{occurred}(e)$, and the claim holds.

**Lemma 24** The conditions of Lemma 23 imply that $(R, r, t + 1) \models E_k^G \text{occurred}(e)$.

**Proof** Observe that by the structure of the protocol, $(\text{occurred}(e), d)$ messages are sent only if $e$ indeed takes place. Thus, for both processes $i \in \{0, 1\}$ it is the case that if $i$ receives a message of the form $(\text{occurred}(e), d)$ at time $t + 1$ in $r$ with any value $d \geq 0$, then $(R, r, t + 1) \models K_i \text{occurred}(e)$.

By convention, we define $E_0^G \varphi = \varphi$. We prove by induction on $k \geq 0$ that if the initial value of $s$ in $r$ is $h \geq k$ and $e$ occurs at $(r, t)$, then $(R, r, t + 1) \models E_k^G \text{occurred}(e)$. In particular, this implies that $(R, r, t + 1) \models E_k^G \text{occurred}(e)$ in the case $h = k$, establishing the claim. We consider two case.

$k = 0$ : By assumption, $e$ occurs at time $t$ in $r$, and thus $(R, r, t + 1) \models \text{occurred}(e)$, and by definition of $E_0^G$ also $(R, r, t + 1) \models E_0^G \text{occurred}(e)$.

$k > 0$ : In this case, process $i = \text{parity}(h)$ receives the message $(\text{occurred}(e), h)$, and the other process $j = 1 - i$ receives $(\text{occurred}(e), h - 1)$. According to the protocol, a message $(\text{occurred}(e), d)$ is received if the initial value is either $d$ or $d + 1$, and hence at least as large as $d$. Both processes thus know that the initial value is at least as large as $h - 1$. Since $h \geq k$ by assumption, and by the inductive hypothesis we have that $(R, r, t + 1) \models E_{k-1}^G \text{occurred}(e)$ whenever $h \geq k - 1$, it follows that both $(R, r, t + 1) \models K_i E_{k}^G \text{occurred}(e)$ and $(R, r, t + 1) \models K_j E_{k-1}^G \text{occurred}(e)$. Hence, $(R, r, t + 1) \models E_k^G \text{occurred}(e)$ and we are done.

**Lemma 25** The conditions of Lemma 23 imply $(R, r, t + 1) \models \neg E_{k+1}^G \text{occurred}(e)$.

**Proof** First notice that, in every run $r' \in R$, a process that does not receive a message of the form $(\text{occurred}(e), d)$ does not know that $e$ occurred,
since there is another run $r'' \in R$ in which its local history is identical to then one in $r'$, and where the event does not occur. We can now prove the claim by induction of $k$. If $k = 0$ then process 1 does not receive a $\langle \text{occurred}(e), d \rangle$ by time $t + 1$. Thus, $(R, r, t + 1) \models \neg K_1 \text{occurred}(e)$ and so $(R, r, t + 1) \models \neg E^{1}_G \text{occurred}(e)$, as claimed.

Let $k > 0$ and assume inductively that the claim holds for $k - 1$. By definition of the protocol, process $i = \text{parity}(k)$ receives the message $\langle \text{occurred}(e), k \rangle$, and the other process $j = 1 - i$ receives $\langle \text{occurred}(e), k - 1 \rangle$. There is a run $r' \in R$ in which $r'_j(t + 1) = r_j(t + 1)$ and the initial value is $k - 1$. It follows that $(R, r, t + 1) \models \neg K_j E^{k}_G \text{occurred}(e)$, and thus $(R, r, t + 1) \models \neg E^{k+1}_G \text{occurred}(e)$, and we are done. ■

We note that the epistemic structure obtained here is similar to that which arises in the electronic mail game of Rubinstein [47], and in the co-ordinated attack problem [23]. One distinguishing feature is that in our example here the high degree of nested knowledge is obtained in one step, with two messages, whereas a long interactive exchange of $k$ messages is required to achieve $k$ levels of nesting in the other cases. A similar epistemic structure also arises in the analysis of the initial states of the muddy children puzzle [15], or of the Conway paradox [11].

### 4.7 Conclusions

Taking a step beyond nested knowledge, this chapter develops the theory needed in order to characterize common knowledge gain, an epistemic state that is only possible in synchronous systems [23]. We define the centibroom, a simpler, tighter, communication structure than the centipede, and prove the Common Knowledge Gain Theorem that validates the centibroom’s causal nature. We then show that the centibroom is also necessary in solution to the Simultaneous Response problem.

Based on the fip, first introduced in Chapter 3, it is shown that centibrooms are also sufficient for common knowledge gain. We then utilize this result to determine sharp thresholds regarding when nested knowledge becomes common knowledge under fip. Finally, Example 6 shows that this phenomenon is not universal to all protocols. A protocol exists in which no depth of nested knowledge must imply common knowledge.
Chapter 5

Gaining Nested Common Knowledge

5.1 Introduction

The Ordered Response problem deals with a totally ordered sequence of response, and the Simultaneous Response problem with groups of responses that must be enacted in unison. As seen in Section 2.1, we can look at the required time ordering in an instance \( SR = \langle e_t, \alpha_1, \ldots, \alpha_k \rangle \) as the set of requirements \( \{ Time(e_t) \leq Time(\alpha_g) \leq Time(\alpha_h) | g, h \in 1, \ldots, k \} \).

Two possible extensions of problem specifications come to mind. The first extension, that we call the Ordered Group Response problem, is an immediate generalization of the OR and SR problems.

**Definition 18 (Ordered Group Response)** Let \( e_t \) be an external input and let \( A^h = \langle \alpha_1^h, \ldots, \alpha_{\ell_h}^h \rangle \) be a set of responses of length \( \ell_h \), for every \( h = 1, \ldots, k \). A protocol \( P \) solves the instance \( OGR = \langle e_t, A^1, \ldots, A^k \rangle \) of the Ordered Group Response problem if it guarantees that

1. in a triggered run, for every \( h = 1, \ldots, k \) all of the actions in the response set \( A^h \) are performed simultaneously; moreover, if \( h < k \) then the actions in \( A^h \) will happen before (i.e., no later than) those of \( A^{h+1} \). Finally,

2. none of the responses, in any of the sets \( A^1, \ldots, A^k \), occurs in runs that are not triggered.

We will use \( I^h \) to denote the set of processes \( \{ i \in P | \langle i, a \rangle \in A^h \} \) for every \( h < k \).
It is easy to see that every instance of OR can be rewritten as an instance of OGR where all sets of responses are singletons. Similarly, every instance of SR can be rewritten as an instance of \( \text{OGR} = \langle e_t, A^1 \rangle \) where all simultaneous responses are members of \( A^1 \).

The second extension, which is even wider scoped than OGR, is a problem specification where the required event ordering is given by any arbitrary partial order. The Generalized Ordering problem will be defined and studied in the next chapter. In this chapter we will focus on the OGR problem. We defer giving a leading example until the next chapter, which will make use of this chapter’s results. Apart from providing the foundational results necessary for the next chapter, this chapter can also be seen as providing a unifying account that merges the thus-far separately treated theories that surround OR and SR.

### 5.2 Relating Ordered Group Response and Nested Common Knowledge

In order to relate the new ordering problem to a causal structure, we will first identify an epistemic condition that is implied by protocols solving the problem. As solutions to OR require nested knowledge and those of SR imply common knowledge, we expect that solutions to OGR will necessitate a little of both kinds of epistemic states.

We will say that nested common knowledge of \( \varphi \) obtains at run \( r \in R \) and time \( t \) with respect to groups of processes \( G_1, \ldots, G_k \) if

\[
(R, r, t) \models C_{G_k} C_{G_{k-1}} \cdots C_{G_1} \varphi
\]

holds. As we will show, nested common knowledge is a necessary requirement in protocols that solve the OGR problem. In fact, we show that nested common knowledge is necessitated even if the protocol only weakly solves OGR, according to the following definition.

**Definition 19 (Weakly solving Ordered Group Response)**

Let \( \text{OGR} = \langle e_t, A^1, \ldots, A^k \rangle \) be an instance of the ordered group response problem. A protocol \( P \) weakly solves \( \text{OGR} \) if it guarantees that for every \( h = 1, \ldots, k \) and \( \alpha \in A^h \)

1. for every \( r \in R^{\text{max}} \), time \( t \) and \( \alpha' \in A^h \), \( \alpha \) occurs at \( (r, t) \) iff \( \alpha' \) occurs at \( (r, t) \)
2. for every \( r \in \mathcal{R}^{\text{max}} \), time \( t \) and \( \alpha' \in A^{h'} \) where \( h' \leq h \), if \( \alpha \) occurs at \((r, t)\) then \( \alpha' \) occurs at \((r, t')\) where \( t' \leq t \).

Note that every protocol that solves \( \text{OGR} \) also weakly solves it, but that the opposite implication does not hold. A protocol that weakly solves \( \text{OGR} \) does not necessitate that any of the responses occur in a triggered run.

Recall that in order to prove the relation between solutions to \( \text{OR} \) and nested knowledge in Theorem 1, we had to assume that processes can recall responses that they had performed. We now define a stronger recall requirement for nested common knowledge.

**Definition 20 (Group response recall)** Let \( \text{OGR} = \langle e_t, A^1, \ldots, A^{k} \rangle \) and assume that \( \mathcal{R} = \mathcal{R}(P, \gamma) \) is a system of runs for a protocol \( P \) where all of the responses may occur, and \( \gamma \) is any arbitrary context. Protocol \( P \) recalls group responses for \( \text{OGR} \) if for all \( \alpha \in A^h \) where \( 1 \leq h \leq k \), \( r \in \mathcal{R} \), \( t' \leq t \) and \( i \in \mathbb{P} \), if \((\mathcal{R}, r, t) \models K_i \text{occurred}(\alpha) \) then \((\mathcal{R}, r, t') \models K_i \text{occurred}(\alpha) \).

A protocol recalls group responses if processes, once they know that a response has taken place, never forget this fact. We are now ready to prove that \( \text{OGR} \) requires nested common knowledge.

**Theorem 14** Let \( \text{OGR} = \langle e_t, A^1, \ldots, A^{k} \rangle \), and assume that protocol \( P \) weakly solves \( \text{OGR} \) in \( \gamma \) and that it recalls group responses for \( \text{OGR} \). Let \( r \in \mathcal{R} \) be a run in which \( e_t \) occurs at time \( t_0 \), and where the processes in \( P^h \) perform the actions \( a^h_1 \) to \( a^h_{\ell_h} \) simultaneously at \( t_h \geq t_{h-1} \), for every \( 1 \leq h \leq k \).

Then \((\mathcal{R}, r, t_h) \models C_{P+} C_{P-1} \cdots C_{P+1} (\text{occurred}(e_t) \land \text{ND}(e_t)) \) for every \( 1 \leq h \leq k \).

**Proof** We proceed by induction on \( k \).

\( k = 0 \) : \((\mathcal{R}, r, t_0) \models (\text{occurred}(e_t) \land \text{ND}(e_t)) \) by definition of \( r \).

\( k > 0 \) : We will use the Induction Rule for common knowledge to prove the inductive step. Recall that \( A^k = \langle \alpha^{k}_1, \ldots, \alpha^{k}_{\ell_k} \rangle \).

Fix \( h, g \in \{1, \ldots, \ell_k\} \). We first show that \( \mathcal{R} \models \text{occurs}(a^k_h) \rightarrow E_{G_1} (\text{occurs}(a^k_h) \land C_{P+1} \cdots C_{P+1} (\text{occurred}(e_t) \land \text{ND}(e_t)))) \). Note that as all responses in \( A^k \) are performed simultaneously, we get \( \mathcal{R} \models \text{occurs}(a^k_h) \leftrightarrow \text{occurs}(a^k_g) \). Since whether the (deterministic) protocol \( P \) performs the action \( a^k_h \)
is a function of $i_g$’s local state, we have that $\mathcal{R} \vdash \text{occurs}(a^k_g) \rightarrow K_{i_g}\text{occurs}(a^k_g)$. Now using the former equivalence we get that

\[(*) \quad \mathcal{R} \vdash \text{occurs}(a^k_h) \rightarrow K_{i_g}\text{occurs}(a^k_h).\]

Now choose arbitrary $r',t'$. Suppose that $(\mathcal{R},r',t') \vdash \text{occurs}(a^k_h)$. Note that since $P$ weakly solves OGR, we have both

(i) when $a^k_h$ is performed, the responses in $A^{k-1}$ have already been performed (or are being performed). Say that these have been performed at a time $t'_{k-1} \leq t'$. And,

(ii) protocol $P$ also weakly solves the sub-problem $\text{OGR}' = \langle e_t, A^1, \ldots, A^{k-1} \rangle$. By the inductive hypothesis, we have

\[ (\mathcal{R},r',t'_{k-1}) \vdash C_{I_{k-1}} \cdots C_{I_1}(\text{occurred}(e_t) \land \text{ND}(e_t)) \]

for all $h < k$.

As processes recall group responses and $t'_{k-1} \leq t'$, we obtain from $(ii)$ that $(\mathcal{R},r',t') \vdash C_{I_{k-1}} \cdots C_{I_1}(\text{occurred}(e_t) \land \text{ND}(e_t))$. Given this and the fact that if $(\mathcal{R},r',t') \not\vdash \text{occurs}(a^k_h)$ then $(\mathcal{R},r',t') \vdash \text{occurs}(a^k_h) \rightarrow \alpha$ for any $\alpha$, we conclude that

\[ \mathcal{R} \vdash \text{occurs}(a^k_h) \rightarrow C_{I_{k-1}} \cdots C_{I_1}(\text{occurred}(e_t) \land \text{ND}(e_t)). \]

Combined with $(*)$, we obtain

\[ \mathcal{R} \vdash \text{occurs}(a^k_h) \rightarrow K_{i_g}C_{I_{k-1}} \cdots C_{I_1}(\text{occurred}(e_t) \land \text{ND}(e_t)). \]

Since $g$ was arbitrarily chosen, we get $\mathcal{R} \vdash \text{occurs}(a^k_h) \rightarrow E_{I_k}\text{occurs}(a^k_h)$, and also $\mathcal{R} \vdash \text{occurs}(a^k_h) \rightarrow E_{I_k}C_{I_{k-1}} \cdots C_{I_1}(\text{occurred}(e_t) \land \text{ND}(e_t))$, ending up with

\[ \mathcal{R} \vdash \text{occurs}(a^k_h) \rightarrow E_{I_k}(\text{occurs}(a^k_h) \land C_{I_{k-1}} \cdots C_{I_1}(\text{occurred}(e_t) \land \text{ND}(e_t))), \]

as required.

Let $\Phi = \text{occurs}(a^k_h)$ and $\Psi = C_{I_{k-1}} \cdots C_{I_1}(\text{occurred}(e_t) \land \text{ND}(e_t))$. Applying the Knowledge Induction Rule we get $\mathcal{R} \vdash \Phi \rightarrow E_{I_k}(\Phi \land \Psi)$ from $\mathcal{R} \vdash \Phi \rightarrow E_{I_k}(\Phi \land \Psi)$, giving us that

\[ \mathcal{R} \vdash \text{occurs}(a^k_h) \rightarrow E_{I_k}C_{I_{k-1}} \cdots C_{I_1}(\text{occurred}(e_t) \land \text{ND}(e_t)). \]

Recalling that $(\mathcal{R},r,t) \vdash \text{occurs}(a^k_h)$ by assumption, we obtain that $\mathcal{R} \vdash C_{I_k}C_{I_{k-1}} \cdots C_{I_1}(\text{occurred}(e_t) \land \text{ND}(e_t))$.  

80
An immediate corollary is that nested common knowledge is also necessitated in protocols that solve (not weakly solve) OGR.

**Corollary 1** Let $OGR = \langle e_t, A^1, \ldots, A^k \rangle$, and assume that protocol $P$ solves OGR in $\gamma$ and that it recalls group responses for OGR. Let $r \in R$ be a run in which $e_t$ occurs at time $t_0$, and where the processes in $I^h$ perform the actions $a^h_1$ to $a^h_{\ell_h}$ simultaneously at $t_h \geq t_{h-1}$, for every $1 \leq h \leq k$. Then for every $1 \leq h \leq k$

$$(R, r, t_h) \models C_{I^h} C_{I^{h-1}} \cdots C_{I^1} (\text{occurred}(e_t) \land \text{ND}(e_t)).$$

As mentioned above, both nested knowledge and common knowledge are specific cases of nested common knowledge. As such, Theorems 1 and 2 can be derived as further corollaries from the above one.

To complete the picture, we briefly point out that there are protocols for which nested common knowledge gain implies a solution to OGR.

**Definition 21 (Group considerate protocol)** Let $OGR = \langle e_t, A^1, \ldots, A^k \rangle$, where $A^h = \langle \alpha^h_1, \ldots, \alpha^h_{\ell_h} \rangle$ for every $h = 1, \ldots, k$. Protocol $P$ is group considerate with respect to $OGR$ if for each $h \leq k$ and $m \leq \ell_h$, response $a^h_m$ is carried out by its respective process $i^h_m$ as soon as $C_{I^h} (\text{occurred}(e_t) \land \text{ND}(e_t))$ is established, but no sooner.

**Lemma 26** Let $OGR = \langle e_t, A^1, \ldots, A^k \rangle$, and let $P$ be a group considerate protocol with respect to OGR. If for every $r \in R^{\max} = R(P, \gamma)$ such that $e_t$ occurs at $(i_0, t)$ in $r$ there exists time $t'$ such that

$$(R^{\max}, r, t') \models C_{I^h} C_{I^{h-1}} \cdots C_{I^1} (\text{occurred}(e) \land \text{ND}(e)),$$

then $P$ solves OGR.

The proof repeats the one of Lemma 1, with nested common knowledge replacing nested knowledge. That a process will know immediately that common knowledge has been achieved is given by the validity $C_{G} \varphi \leftrightarrow K_g C_{G} \varphi$ for any $g \in G$. 

81
5.3 Generalized Centipedes

We expect that just as OGR generalizes both OR and SR, a characterizing causal structure will generalize both the centipede and the centibroom. The generalized centipede, defined below, offers just this kind of generalization.

**Definition 22 (Generalized Centipede)** Let \( r \in \mathcal{R}^{\text{max}} \), let \( I^h \subseteq \mathcal{P} \) for \( 1 \leq h \leq k \) and let and \( t \leq t_1 \leq \cdots \leq t_k \). A generalized centipede for \( \langle \theta_0, I^1, \ldots, I^k \rangle \) in \( (r, t..t') \) is a sequence of nodes \( \theta_0 \leadsto \theta_1 \leadsto \cdots \leadsto \theta_k \) such that \( \theta_0 = (i_0, t) \), and \( \theta_h \leadsto (i^h_m, t') \) holds for all \( h = 1, \ldots, k \) and \( i^h_m \in I^h \).

![Figure 5.1: A generalized centipede](image)

In chapters 3 and 4 we found it convenient to consider special kinds of centipedes and centibrooms, namely centinodes and bridging centibrooms, respectively. Once again, the following definition extends both of these special kinds.

**Definition 23 (Bridging Generalized Centipede)** A generalized centipede \( \langle \theta_0, \ldots, \theta_k \rangle \) for \( \langle (i_0, t), I^1, \ldots, I^k \rangle \) in \( (r, t..t') \) is bridging if \( \theta_h \) is a bridging centibroom for \( \langle (i_0, t), I^h \rangle \) in \( (r, t..t') \) for all \( h = 1 \ldots k \).

As the following lemma shows, generalized centipedes and their bridged sub-kind may be freely interchanged.

**Lemma 27** A generalized centipede for \( \langle \theta_0, I^1, \ldots, I^k \rangle \) exists in \( (r, t..t') \) iff a bridging generalized centipede for \( \langle \theta_0, I^1, \ldots, I^k \rangle \) exists in \( (r, t..t') \).
Proof That the existence of a bridging generalized centipede implies that of a generalized centipede is immediate. We now prove the other direction. Assume that \( C = \langle \theta_0, \ldots, \theta_k \rangle \) is a generalized centipede for \( \langle I_0, \ldots, I_k \rangle \) in \( (r, t, \ldots, t') \). We define by induction on \( h \leq k \) generalized centipedes \( C_h = \langle \theta'_0, \ldots, \theta'_h, \theta_{h+1}, \ldots, \theta_k \rangle \) in \( (r, t, \ldots, t') \), in which the nodes \( \theta'_0 \) to \( \theta'_h \) are bridging centibrooms for \( \langle I_0, \ldots, I_h \rangle \), respectively.

\( h = 0 \) : By definition, \( \theta_0 = (i_0, t) \). As \( I_0 = \{ i_0 \} \), \( \theta_0 \) is a trivial bridging centibroom for \( \langle I_0 \rangle \) in \( (r, t, \ldots, t') \), and hence a bridging generalized centipede for \( \langle I_0 \rangle \).

\( h > 0 \) : Assume that a bridging generalized centipede \( C_{h-1} = \langle \theta'_0, \ldots, \theta'_{h-1}, \theta_h, \ldots, \theta_k \rangle \) as described above has been constructed. By Lemma 7 there exists a node \( \theta'_h \) bridging \( \theta'_{h-1} \) and \( \theta_h \). Since \( \theta_h \rightarrow (i^h, t') \) for all \( i^h \in I^h \) we get that \( \theta'_h \) is a bridging centibroom for \( \langle I^0, \ldots, I^h \rangle \) in \( (r, t, \ldots, t') \). Define \( C_h = \langle \theta'_0, \ldots, \theta'_h, \theta_{h+1}, \ldots, \theta_k \rangle \). If \( h = k \) then we are done. Otherwise, since \( \theta'_{h'} \rightarrow \theta_h \sim \theta_{h+1} \) and \( C_{h-1} = \langle \theta'_0, \ldots, \theta'_{h-1}, \theta_h, \ldots, \theta_k \rangle \) is a generalized centipede for \( \langle i_0, \ldots, i_k \rangle \) in \( (r, t, \ldots, t') \), we obtain that \( C_h \) is also such a generalized centipede, as required.

We now formulate the Generalized Centipede Theorem, which will be proved in the next section. The theorem is proved for protocols weakly solving the OGR problem. An immediate corollary gives us that the same conditions hold for protocols that (non-weakly) solve the OGR. This later corollary can be used to derive Theorems 4 and 9 as immediate further corollaries.

**Theorem 15** Let \( OGR = \langle e_t, A^1, \ldots, A^k \rangle \), and assume that protocol \( P \) weakly solves \( OGR \) in \( \gamma \). Let \( r \in R(P, \gamma^{max}) \) be a run in which \( e_t \) occurs at time \( t = t_0 \), and where the processes in \( I^h \) perform the actions \( a^h_1 \) to \( a^h_{t_h} \) simultaneously at \( t_h \geq t_{h-1} \), for every \( 1 \leq h \leq k \), with \( t' = t_k \).

Then there is a generalized centipede for \( \langle I^1, \ldots, I^k \rangle \) in \( (r, t, t') \).

**5.3.1 Nested Common Knowledge Gain Requires Generalized Centipedes**

We start by revisiting the relation between past, the past causal cone, and knowledge. We repeat here the definition of past and fut cones, as we now
wish to make use of the complete data that is encoded in the definitions.

**Definition 11 (reprinted)** We define the *future causal cone* of a node \( \alpha \) (in run \( r \)) to be

\[
\text{fut}(r, \alpha) = \{ \langle \theta, ND\theta \rangle : \alpha \rightarrow \theta \text{ in } r \text{ and } ND\theta \text{ is the set of ND events and initial states in } \theta \text{ in } r \}.
\]

Similarly, the *past causal cone* of \( \alpha \) is

\[
\text{past}(r, \alpha) = \{ \langle \theta, ND\theta \rangle : \theta \rightarrow \alpha \text{ in } r \text{ and } ND\theta \text{ is the set of ND events and initial states in } \theta \text{ in } r \}.
\]

Although \( \text{past} \) and \( \text{fut} \) are sets that contain pairs of node and event-set, we will frequently treat them simply as sets of nodes, when the second component of the pair is irrelevant in the context.

Lemma 6 showed that the local state of a process, and hence also its knowledge state, is determined by its past causal cone. A straightforward extension shows that common knowledge of a group of processes is determined by the union of their past cones.

**Definition 24 (Group past cones)** For every \( G \subseteq P \) and time \( t \), we will write

1. \( \bigcup \text{Past}(r, G, t) \) to denote the set \( \bigcup_{g \in G} \text{past}(r, (g, t)) \), and
2. \( \bigcap \text{Past}(r, G, t) \) to denote the set \( \bigcap_{g \in G} \text{past}(r, (g, t)) \).

**Lemma 28** Fix \( r, r' \in R^{max} \), \( G \subseteq P \) and time \( t \), such that \( \bigcup \text{Past}(r, G, t) = \bigcup \text{Past}(r', G, t) \).

For every \( \varphi \in L \), if \( (R^{max}, r, t) \models C_{G\varphi} \) then \( (R^{max}, r', t) \models C_{G\varphi} \).

**Proof** Suppose that \( (R^{max}, r', t) \not\models C_{G\varphi} \). Then there exists some sequence \((g_1, g_2, \ldots, g_k)\) such that \( (R^{max}, r', t) \not\models K_{g_1}K_{g_2}\cdots K_{g_k}\varphi \). Write \( \varphi' = K_{g_1}\cdots K_{g_k}\varphi \). We get \( (R^{max}, r', t) \not\models K_{g_1}\varphi' \). From \( \bigcup \text{Past}(r, G, t) = \bigcup \text{Past}(r', G, t) \) we obtain, in particular, that \( \text{past}(r, (g_1, t)) = \text{past}(r', (g_1, t)) \). By Lemma 6 we get that \( (R^{max}, r, t) \not\models K_{g_1}\varphi' \), which gives us \( (R^{max}, r, t) \not\models K_{g_1}K_{g_2}\cdots K_{g_k}\varphi \), contradicting the assumption that \( (R^{max}, r, t) \models C_{G\varphi} \). \( \blacksquare \)
The next two lemmas give us an even greater focus on the effect of the causal past upon the current epistemic state. The first lemma points out that the causal past itself is fully determined by those nondeterministic events that occur in it. We add the following definition.

**Definition 25 (Nondeterministic past)** For every \( r \in \mathcal{R}^{max} \), node \( \theta \) and \( G \subseteq \mathcal{P} \),

1. Define \( \text{NDpast}(r, \theta) = \{ \langle \psi, ND_{\psi} \rangle \in \text{past}(r, \theta) : ND_{\psi} \neq \emptyset \} \). This is the set of nodes \( \psi \) such that \( \psi \in \text{past}(r, \theta) \) and either an ND event occurs at \( \psi \) in \( r \) or \( \psi \) is an initial state.

2. Define \( \bigcup \text{NDPast}(r, G, t) = \bigcup_{g \in G} \text{NDpast}(r, (g, t)) \).

3. Define \( \bigcap \text{NDPast}(r, G, t) = \bigcap_{g \in G} \text{NDpast}(r, (g, t)) \).

**Lemma 29** Fix \( r, r' \in \mathcal{R}^{max} \) and node \( (i, t) \).

If \( \text{NDpast}(r, (i, t)) = \text{NDpast}(r', (i, t)) \) then \( \text{past}(r, (i, t)) = \text{past}(r', (i, t)) \).

**Proof** We prove the claim by induction on \( t \).

\( t = 0 \) \quad \text{past}(r, (i, 0)) = \{ \langle s_i, ND_{s_i} \rangle \} \), the singleton initial local state of \( i \) in \( r \). By assumption it is the same state as in \( r' \), and hence \( \text{past}(r, (i, 0)) = \text{past}(r', (i, 0)) \).

\( t > 0 \) Suppose that \( r_i(t) \neq r'_i(t) \). Then, wlog, by Lemma 6 there exists some \( \langle \theta, ND_{\theta} \rangle \in \text{past}(r, (i, t)) \) such that \( \langle \theta, ND_{\theta} \rangle \notin \text{past}(r', (i, t)) \). If \( \theta \sim (i, t) \) in \( r' \) then it must be that \( ND_{\theta} \) is different in \( r \) and \( r' \), in which case we get that \( \text{NDpast}(r, (i, t)) \neq \text{NDpast}(r', (i, t)) \), contrary to the lemma’s assumptions. Hence there must exist some \( \theta \) such that \( \theta \sim (i, t) \) in \( r \) but \( \theta \nRightarrow (i, t) \) in \( r' \).

By Lemma 7 there exists a bridge \( \psi \) such that \( \theta \sim \psi \rightarrow (i, t) \). As \( \theta \notin \text{past}(r', (i, t)) \), it must be that \( \theta \nrightarrow (i, t) \), and hence that \( \psi \neq \theta \).

We now consider two cases:

\( \psi = (i, t) \): In this case, as \( \text{NDpast}(r, (i, t)) = \text{NDpast}(r', (i, t)) \), \( \theta \sim \psi \) and an ND event occurs at \( \theta \) in \( r \), it must be that \( \theta \in \text{past}(r', (i, t)) \), contradicting the assumption that \( \theta \notin \text{past}(r', (i, t)) \).
ψ ≠ (i, t): In this case from Lemma 4, it must be that ψ = (j, t') for some t' < t. From $\text{ND}_{\text{past}}(r, (i, t)) = \text{ND}_{\text{past}}(r', (i, t))$ and ψ ∈ $\text{past}(r, (i, t))$ we obtain that $\text{ND}_{\text{past}}(r, (j, t')) = \text{ND}_{\text{past}}(r', (j, t'))$. By the inductive hypothesis $\text{past}(r, (j, t')) = \text{past}(r', (j, t'))$ and hence, as $\theta \in \text{past}(r, (j, t'))$ it must also be that $\theta \not\in \text{past}(r', (i, t))$, again contra the assumption that $\theta \not\in \text{past}(r', (i, t))$.

Proving the next lemma is done by composing the two previous lemmas.

**Lemma 30** Fix $r, r' \in R^{\text{max}}$, time $t$ and $G \subseteq \mathcal{P}$. If $\bigcup \text{ND}_{\text{past}}(r, G, t) = \bigcup \text{ND}_{\text{past}}(r', G, t)$, then if $(R^{\text{max}}, r, t) \vdash C_G \varphi$ then $(R^{\text{max}}, r', t) \vdash C_G \varphi$.

**Definition 26 (Centibroom past)** Let $r \in R^{\text{max}}$, $G \subseteq \mathcal{P}$ and fix time $t$. The centibroom past of $G$ in $r$ at time $t$ is the set

$$\text{BroomPast}(r, t, G) = \left\{ (\theta, ND_0) \in \bigcup \text{ND}_{\text{past}}(r, G, t) \mid \text{there exists a centibroom for } (\theta, G) \text{ in } (r, 0..t) \right\}.$$

As before, in the case of past and fut, we will often treat $\text{BroomPast}(r, t, G)$ as a set of nodes, rather than a set of pairs. Note that $\text{BroomPast}(r, t, G) \subseteq \bigcup \text{ND}_{\text{past}}(r, G, t) \subseteq \bigcup \text{ND}_{\text{past}}(r, G, t)$.

When $G = \{i, j\}$, we partition the nodes in $\bigcup \text{ND}_{\text{past}}(r, G, t)$ based on the existence of bridging nodes. Recall that a node $b$ bridges $\theta$ and $\psi$ if $\theta \rightsquigarrow b \rightsquigarrow \psi$ and there is no node $b'$ such that $b' \neq b$ and $\theta \rightsquigarrow b' \rightsquigarrow b$. Note that there may exist more than one bridge node connecting $\theta$ and $\psi$, if there is more than one syncausal path between the nodes. In the following partition, for each node $\theta \in \bigcup \text{ND}_{\text{past}}(r, G, t)$ we look at the nodes bridging $\theta$ and $(i, t)$, as well as those nodes bridging $\theta$ to $(j, t)$. Moreover, we focus on those bridging nodes that are earliest: $(i, t)$ is an earliest node bridging $\theta$ and $\psi$ if it bridges the two nodes and if there is no alternate bridging node $(j, t')$ for $\theta$ and $\varphi$ such that $t' < t$.

**Definition 27 (Partitioning $\bigcup \text{ND}_{\text{past}}(r, G, t)$)** Given a run $r \in R^{\text{max}}$, a time $t$ and $G = \{i, j\}$, for each $(\theta, ND_0) \in \bigcup \text{ND}_{\text{past}}(r, G, t)$ we have $\theta \rightsquigarrow (i, t)$ or $\theta \rightsquigarrow (j, t)$. Let $t^\theta_i$ denote the time of the earliest bridge nodes between $\theta$ and $(i, t)$, with $t^\theta_i = \infty$ if there are no bridges between $\theta$ and $(i, t)$ (i.e. $\theta \not\rightsquigarrow (i, t)$). Similarly define $t^\theta_j$. The set $\bigcup \text{ND}_{\text{past}}(r, G, t)$ can be partitioned into the following subsets:
1. BroomPast\(r, t, G\)

2. \(Br_i^r = \{(\theta, ND\theta) \in \bigcup \text{NDPast}(r, G, t) - \text{BroomPast}(r, t, G) | t_i^\theta < t_j^\theta\}\)

3. \(Br_j^r = \{(\theta, ND\theta) \in \bigcup \text{NDPast}(r, G, t) - \text{BroomPast}(r, t, G) | t_j^\theta < t_i^\theta\}\)

4. \(Br_{\text{same}}^r = \{(\theta, ND\theta) \in \bigcup \text{NDPast}(r, G, t) - \text{BroomPast}(r, t, G) | t_i^\theta = t_j^\theta < \infty\}\)

Notice that Definition 27 first constructs the cell for all pairs \((\theta, ND\theta) \in \bigcup \text{NDPast}(r, \{i, j\}, t)\) for which there exists a centibroom for \((\theta, \{i, j\})\) in \(r\), and then takes all the remaining nodes in \(\bigcup \text{NDPast}(r, \{i, j\}, t)\) and partitions them further into \(Br_i^r, Br_j^r\) and \(Br_{\text{same}}^r\). In particular, this means that in case of some \(\theta \in Br_{\text{same}}^r\), where \(t_i = t_j\), the nodes bridging \(\theta\) to \((i, t)\) will be distinct from those bridging \(\theta\) to \((j, t)\), or otherwise we would have that \(\theta \in \text{BroomPast}(r, t, G)\).

Once again using \(t_i^\theta\) to denote the time of the earliest bridge nodes between \(\theta\) and \((i, t)\), with \(t_i^\theta = \infty\) if there are no bridges, we divide the cells of Definition 27 further, by “slicing” each cell according to the time associated with the earliest bridging nodes.

**Definition 28 (slicing the partition cells of \(\bigcup \text{NDPast}(r, \{i, j\}, t)\))**

Given a run \(r \in R^{\text{max}}\), a time \(t\) and \(G = \{i, j\}\), let \(\text{BroomPast}(r, t, G), Br_i^r, Br_j^r\) and \(Br_{\text{same}}^r\) form the partition in Definition 27.

We divide some of the partition cells further into time-slices in the following way

1. \(Br_i^r = \bigcup_{t' \leq t} Br_i^r(t')\), where \(Br_i^r(t') = \{(\theta, ND\theta) \in Br_i^r | t_i^\theta = t'\}\),

2. \(Br_j^r = \bigcup_{t' \leq t} Br_j^r(t')\), where \(Br_j^r(t') = \{(\theta, ND\theta) \in Br_j^r | t_j^\theta = t'\}\), and

3. \(Br_{\text{same}}^r = \bigcup_{t' \leq t} Br_{\text{same}}^r(t')\), where
   \(Br_{\text{same}}^r(t') = \{(\theta, ND\theta) \in Br_{\text{same}} | t_j^\theta = t_i^\theta = t'\}\).

For every \(G \subseteq P\), we will say that runs \(r\) and \(r'\) are \(G\)-reachable at time \(t\) if there exists a sequence \(\langle r_0, \ldots, r_k \rangle\) such that \(r = r_0, r' = r_k\), and for each \(h < k\) there exists \(i_h \in G\) such that \(r_h \sim (i_h, t)\) \(r_{h+1}\). The next lemma shows that the nodes in cells \(Br_i^r, Br_j^r\) and \(Br_{\text{same}}^r\) are not essential for determining \(G\)-reachability for groups of size 2.
Lemma 31 Fix \( r, r' \in \mathcal{R}^{\max}, \) time \( t, G = \{i, j\} \subseteq \mathcal{P}. \) There exists \( C \geq 0 \) and a sequence of runs \( \langle r_1, r_2, \ldots, r_{2C} \rangle \) such that \( r \) and \( r_{2C} \) are \( G \)-reachable, \( \text{BroomPast}(r_{2C}, t, G) = \text{BroomPast}(r, t, G), \) and \( \text{Br}^{r_{2C}}_i = \text{Br}^{r_{2C}}_j = \text{Br}^{r_{2C}}_{\text{same}} = \emptyset. \)

Proof We prove by induction that for each \( d = 0, \ldots, t \) that there exists a run \( r_{2d} \) such that

1. \( r \) and \( r_{2d} \) are \( G \)-reachable,
2. \( \text{BroomPast}(r_{2d}, t, G) = \text{BroomPast}(r, t, G), \)
3. \( \text{Br}^{r_{2d}}_i = \bigcup_{1 \leq h \leq t-d} \text{Br}^r_i(h), \)
4. \( \text{Br}^{r_{2d}}_j = \bigcup_{1 \leq h \leq t-d} \text{Br}^r_j(h), \) and
5. \( \text{Br}^{r_{2d}}_{\text{same}} = \bigcup_{1 \leq h \leq t-d} \text{Br}^r_{\text{same}}(h). \)

\( d = 0 \) In this case \( r_{2d} = r_0 = r, \) and all requirements trivially hold (for example, \( \text{Br}^{r_0}_i = \text{Br}^r_i = \bigcup_{1 \leq h \leq t} \text{Br}^r_i(h) \)).

\( d > 0 \) Inductively assume the existence of a run \( r_{2d-2} \) satisfying all requirements. Note that all \( \theta \) in \( \text{Br}^{r_{2d-2}}_i, \) in \( \text{Br}^{r_{2d-2}}_j \) and in \( \text{Br}^{r_{2d-2}}_{\text{same}} \) occur no later than at \( t - d + 1. \) Let \( r_{2d-1} \) be a run identical to \( r_{2d-2}, \) except that for every \( \theta \in \text{Br}^{r_{2d-2}}_i(t - d + 1) \cup \text{Br}^{r_{2d-2}}_{\text{same}}(t - d + 1), \theta \leadsto (i, t) \) but \( \theta \not\leadsto (j, t). \)

We now show that such a run exists. Iterating over every node \( \theta \in \text{Br}^{r_{2d-2}}_i(t - d + 1) \cup \text{Br}^{r_{2d-2}}_{\text{same}}(t - d + 1), \) we examine two possible cases:

\( \theta \in \text{Br}^{r_{2d-2}}_{\text{same}}(t - d + 1) \) For every node \( b^\theta_j \) bridging \( \theta \) and \( (j, t), \) arbitrarily choose enough early events on the path \( \theta \leadsto b^\theta_j \) that occur in \( r_{2d-2}, \) and cancel their occurrence in \( r_{2d-1}, \) so as to make sure that \( \theta \not\leadsto b^\theta_j \) in \( r_{2d-1}. \) This is possible since \( \theta \not\leadsto b^\theta_j, \) or we would have that \( \theta \in \text{Br}^r_j \) or \( \theta \in \text{BroomPast}(r, t, G). \)

Since by assumption \( \theta \notin \text{BroomPast}(r_{2d-2}, t, G) \), changing occurrences in \( \text{fut}(r_{2d-1}, \theta) \) does not alter the set \( \text{BroomPast}(r_{2d-1}, t, G). \) Moreover, as there exists a bridge \( b^\theta_i \neq b^\theta_j \) for all \( b^\theta_j, \) making changes in the nodes of \( \text{fut}(r_{2d-1}, \theta) \cap \text{past}(r_{2d-1}, (j, t)) \) does not affect \( \text{NDpast}(r_{2d-2}, (i, t)). \)
\( \theta \in \text{Br}_{t}^{r_{2d-2}}(t - d + 1) \) If \( \theta \not \rightarrow (j, t) \) then \( \theta \) does not require that we alter \( r_{2d-1} \) with respect to the current \( r_{2d-2} \). Otherwise, \( \theta \sim (j, t) \), but for every node \((b_j, t_j)\) bridging \( \theta \) and \((j, t)\) in \( r_{2d-2} \), there exists some \((b_i, t - d + 1)\) bridging \( \theta \) and \((i, t)\) in \( r_{2d-2} \) such that \( t - d + 1 < t_j \). It could be that \((b_i, t - d + 1) \sim (b_j, t_j)\) for some such bridges, or it could be that there is no causal relation between the bridges.

In either case, \((b_i, t - d + 1) \not \rightarrow (b_j, t_j)\) or we would have that \( \theta \in \text{BroomPast}(r, t, G) \). In the first case, we choose \( r_{2d-1} \) such that enough early receive events are canceled along every path from \((b_i, t - d + 1)\) to \((b_j, t_j)\), so that \( \theta \not \rightarrow (b_j, t_j) \) for every such bridge. In the second case we choose \( r_{2d-1} \) such that early receive events can be cancelled anywhere along the path from \( \theta \) to \((b_j, t_j)\), once again resulting in \( \theta \not \rightarrow (b_j, t_j) \). Having gone over all nodes bridging \( \theta \) and \((j, t)\) and removed bridges from \( r_{2d-1} \), we end up with \( \theta \not \rightarrow (j, t) \) in \( r_{2d-1} \).

We get that \( \text{BroomPast}(r_{2d-1}, t, G) = \text{BroomPast}(r_{2d-2}, t, G) \), for the same reasons as in the previous case. Moreover, the early receives cancelled in \( r_{2d-1} \) with respect to \( r_{2d-2} \) are either not in \( \text{past}(r_{2d-1}, (i, t)) \), or are in nodes \( \psi \) such that \( \psi \not \rightarrow (i, t) \). In either case then, canceling early receives does not alter the set \( \text{NDpast}(r_{2d-2}, (i, t)) \) and we have \( \text{NDpast}(r_{2d-1}, (i, t)) = \text{NDpast}(r_{2d-2}, (i, t)) \).

In both cases examined we get that

\[
\text{NDpast}(r_{2d-1}, (i, t)) = \text{NDpast}(r_{2d-2}, (i, t))
\]

and hence that \( r_{2d-1,t}(t) = r_{2d-2,t}(t) \). We also get that

\[
\text{BroomPast}(r_{2d-1}, t, G) = \text{BroomPast}(r_{2d-2}, t, G).
\]

Therefore, based on the inductive hypothesis we obtain that

- \( \{r_{2d-1}\}_i(t) = \{r\}_i(t) \) - the local states of \( i \) at time \( t \) in runs \( r_{2d-1} \) and \( r \) are identical, and

- \( \text{BroomPast}(r_{2d-1}, t, G) = \text{BroomPast}(r, t, G) \).

Moreover, \( \text{Br}_{t}^{r_{2d-1}}(t - d + 1) = \emptyset \), and again based on induction that \( \text{Br}_{t}^{r_{2d-1}} = \bigcup_{1 \leq h \leq t-d} \text{Br}_{t}^{r}(h) \). Finally we have for every \( \psi \in \text{Br}_{\text{same}}^{r_{2d-1}}(t - d + 1) \) that \( \psi \not \rightarrow (j, t) \) and that, as we only remove early receives, \( \text{Br}_{\text{same}}^{r_{2d-1}}(t - d + 1) \subseteq \text{Br}_{\text{same}}^{r_{2d-2}}(t - d + 1) \).
We now apply the same arguments in order to choose the run $r_{2d}$, replacing $j$ with $i$ whenever possible. For every $\theta \in Br_{\text{same}}^{t_d}(t - d + 1)$, we cancel enough early receives so as to make sure that $\theta \not\sim (i, t)$. As we also had that $\theta \not\sim (j, t)$ for every $\theta \in Br_{\text{same}}^{t_d}(t - d + 1)$, we end up with $Br_{\text{same}}^{t_d}(t - d) = \emptyset$ Summing up, we get that

$$r_{2d - 1} \sim (i, t) r_{2d - 2} \text{ and } r_{2d} \sim (j, t) r_{2d - 1},$$

$$Br_{i}^{r_{2d}} = \bigcup_{1 \leq h \leq t - d} Br_{i}^{r}(h),$$

$$Br_{j}^{r_{2d}} = \bigcup_{1 \leq h \leq t - d} Br_{j}^{r}(h),$$

and finally that

$$Br_{\text{same}}^{r_{2d}} = \bigcup_{1 \leq h \leq t - d} Br_{\text{same}}^{r}(h).$$

and the induction is complete.

In particular, for $C = t$ we get the lemma’s required result. ■

We are now ready to prove the following theorem, showing that the state of common knowledge in a group of processes $G$ is characterized in a precise sense by those nodes of their pasts in which an ND event occurs and which are centibroom-related to all of the processes in $G$.

**Theorem 16** Fix $r, r' \in R_{\text{max}}$, time $t$ and $G \subseteq P$.

If $\text{BroomPast}(r, t, G) = \text{BroomPast}(r', t, G)$ then $(R_{\text{max}}, r, t) \vdash C_{G'} \varphi$, iff $(R_{\text{max}}, r', t) \vdash C_{G'} \varphi$.

**Proof** Cases where $|G| = 0$ are trivial, and those where $|G| = 1$ are solved using Lemma 29. So assume that $|G| \geq 2$. Suppose that $(R_{\text{max}}, r', t) \nvdash C_{G'} \varphi$. Then there exists some formula $\varphi' = K_{i_1} K_{i_2} \cdots K_{i_k} \varphi$ and some $G' = \{i, j\} \subseteq G$ such that $(R_{\text{max}}, r, t) \vdash C_{G'} \varphi'$ but $(R_{\text{max}}, r', t) \nvdash C_{G'} \varphi'$.

From Lemma 31 we get that there exists a sequence of runs $\langle r_1, ..., r_k \rangle$ such that

(i) $r \sim (i, t) r_1 \sim (j, t) r_2 \sim (i, t) \cdots \sim (i, t) r_{k - 1} \sim (j, t) r_k$, and

(ii) $\text{BroomPast}(r_k, t, G') = \text{BroomPast}(r, t, G')$, and

(iii) $Br_{i}^{r_k} = Br_{j}^{r_k} = Br_{\text{same}}^{r_k} = \emptyset$.

From (i) above and from $(R_{\text{max}}, r, t) \vdash C_{G'} \varphi$ we get that $(R_{\text{max}}, r_k, t) \vdash C_{G'} \varphi$.

Again applying Lemma 31 we obtain a sequence of runs $\langle r'_1, ..., r'_m \rangle$ such that
(i') \( r' \sim_{(i,t)} r'_1 \sim_{(j,t)} r'_2 \sim_{(i,t)} \cdots \sim_{(i,t)} r'_{m-1} \sim_{(j,t)} r'_m \), and

(ii') \( \text{BroomPast}(r'_m, t, G') = \text{BroomPast}(r', t, G') \), and

(iii') \( Br^r_m = Br^r_m = Br^r_m = \emptyset \).

From (i') above and from \((R_{\text{max}}, r', t) \not\models C_G \varphi\) we get that \((R_{\text{max}}, r'_m, t) \not\models C_G \varphi'\).

Note that given (ii), (iii), (ii'), (iii') and from \(\text{BroomPast}(r,t,G') = \text{BroomPast}(r',t,G') \) we get that \(\bigcup \text{NDPast}(r_k,G',t) = \bigcup \text{NDPast}(r'_m,G',t) \). By using the Theorem’s assumptions regarding ND events and initial states and Lemma 30, we obtain that \((R_{\text{max}}, r_k, t) \models C_G \varphi' \) implies \((R_{\text{max}}, r'_m, t) \models C_G \varphi' \), contradicting the above result \((R_{\text{max}}, r'_m, t) \not\models C_G \varphi' \). \(\square\) Theorem 16

Theorem 16 can be weakened into the following useful corollary. Considering that \(\text{BroomPast}(r,t,G) \subseteq \bigcap \text{NDPast}(r,G,t) \), we get

**Corollary 2** Fix \( r, r' \in R_{\text{max}} \), time \( t \) and \( G \subseteq P \). If \( \bigcap \text{NDPast}(r,G,t) = \bigcap \text{NDPast}(r',G,t) \) then if \((R_{\text{max}}, r, t) \models C_G \varphi \) then \((R_{\text{max}}, r', t) \models C_G \varphi \).

At long last we are ready to prove that nested common knowledge gain necessitates the existence of a generalized centipede that relates the process-groups with the triggering node. The proof proceeds much in the same fashion as that of Theorem 6, but process groups have replaced individual processes. Thus, Theorem 17 generalizes both Theorems 6 and 10.

**Theorem 17 (Nested Common Knowledge Gain)** Let \( P \) be a deterministic protocol, \( I^h \subseteq P \) for \( h = 1 \ldots k \), and let \( r \in R_{\text{max}} = R(P, \gamma_{\text{max}}) \).

Assume that \( e \) is an ND event at \((i_0, t)\) in \( r \). If

\[
(R_{\text{max}}, r, t') \models C_{I_h} C_{I_{h-1}} \cdots C_{I_1} (\text{occurred}(e) \land \text{ND}(e)),
\]

then there is a generalized centipede for \( \langle I^1, \ldots, I^k \rangle \) in \( (r, t..t') \).

**Proof** We shall prove the contrapositive form: if no bridging generalized centipede for \( \langle I^1, \ldots, I^k \rangle \) exists in \( (r, t..t') \), then

\[
(R_{\text{max}}, r, t') \not\models C_{I_h} C_{I_{h-1}} \cdots C_{I_1} (\text{occurred}(e) \land \text{ND}(e)).
\]

We reason by induction on \( k \geq 1 \):
\(k = 1\) By assumption, there is no generalized centipede for \(\langle I^1 \rangle\) in \((r, t.t')\). Hence, by definition of generalized centipede there is no centibroom for \(\langle (i_0, t), I^1 \rangle\) in \((r, t.t')\). By Theorem 10 it follows that \(\langle R^{max}, r, t' \rangle \not\in C_{f1}(\text{occurred}(e) \land \text{ND}(e))\), as claimed.

\(k \geq 2\) Assume inductively that the claim holds for \(k-1\). Moreover, assume that no bridging generalized centipede for \(\langle I^1, \ldots, I^k \rangle\) exists in \((r, t.t')\).

For every \(r' \in R^{max}\) let

\[ C^r' = \left\{ \psi_{k-1} \mid \langle \psi_0, \ldots, \psi_{k-1} \rangle \text{ is a bridging generalized centipede for } \langle I^1, \ldots, I^{k-1} \rangle \text{ in } (r', t.t') \right\}. \]

Observe that for every \(\psi_{k-1}' \in C^r\), there is no centibroom node \(\varphi\) for \(\langle \psi_{k-1}', I^k \rangle\) in \((r, t..t')\). Otherwise, by Lemma 18, there would exist a bridging centibroom \(\varphi'\) for \(\langle \psi_{k-1}', I^k \rangle\), and \(\langle \psi_0', \ldots, \psi_{k-1}', \varphi' \rangle\) would be a bridging generalized centipede for \(\langle I^1, \ldots, I^k \rangle\), contradicting our assumption. Thus, \(C^r \cap \text{BroomPast}(r', t, G) = \emptyset\).

Choose \(r' \in R^{max}\) such that

(i) the environment’s actions at all nodes in \(\text{past}(r, \theta)\) for every \(\theta \in \text{BroomPast}(r, t', G)\) are identical to those in \(r\); and

(ii) all messages delivered to nodes not in \(\text{past}(r, \theta)\) for any of the \(\theta \in \text{BroomPast}(r, t', G)\), are delivered at the maximal possible transmission time according to the bounds \(max_{ij}\).

To see that such a run \(r'\) indeed exists in \(R^{max}\), we note that clauses (i) and (ii) relate to different sets of nodes, that it is impossible by definition of \(\text{BroomPast}(r, t', G)\) that there exists some \(\theta \notin \text{BroomPast}(r, t', G)\) such that \(\theta \in \text{past}(r, \psi)\) for some \(\psi \in \text{BroomPast}(r, t', G)\), and that by definition all early message receives can be delayed, independent of the run’s past or concurrent events. Since \(R^{max}\) contains all runs of \(P\) in \(\gamma^{max}\), it must include \(r'\).

Notice that by construction of \(r'\) we have that \(\alpha \rightsquigarrow \alpha'\) holds in \(r'\) only if \(\alpha \rightsquigarrow \alpha'\) in \(r\), and that every early receive in \(r'\) is an early receive in \(r\). Considering that bounds are universal in all runs, we obtain that every bridge node in \(r'\) is also a bridge node in \(r\), and hence that \(C^r' \subseteq C^r\). By definition of \(r'\), and since \(C^r \cap \text{BroomPast}(r, t', G) = \emptyset\), none of the nodes in the set \(C^r\), and hence also in \(C^r'\), experiences an early receive in \(r'\). Yet from Lemma 8 and from \((i_0, t) \notin C^r'\) it follows
that every node $\theta' \in C^{\prime'}$ must be a nontrivial bridge node in $r'$, thus experiencing an early receive. We therefore conclude that $C^{\prime'} = \emptyset$.

By on the inductive hypothesis we obtain from this that $(R_{\text{max}}, r', t') \not\models C_{I_k} \cdots C_{I_1} (\text{occurred}(e) \land \text{ND}(e))$, and using the Knowledge Axiom we get that $(R_{\text{max}}, r', t') \not\models C_{I_k} C_{I_{k-1}} \cdots C_{I_1} (\text{occurred}(e) \land \text{ND}(e))$. By applying Theorem 16 we get that

$$(R_{\text{max}}, r, t') \not\models C_{I_k} C_{I_{k-1}} \cdots C_{I_1} (\text{occurred}(e) \land \text{ND}(e)),$$

and we are done.

\[\square\text{Theorem 17}\]

Using Theorems 14 and 17, we can now prove Theorem 15. The proof repeats that of Theorem 4 almost to the letter so it will not be repeated here.

\section{5.4 Conclusions}

This chapter introduced the OGR problem, along with nested common knowledge and the generalized centipede. These provide a unifying theory for the concepts and results presented in Chapters 2, 3 and 4.

But these concepts also provide important infrastructure for solutions to the Generalized Ordering problem, which will be discussed in the next chapter. In particular, the notion of \textit{weakly solving} OGR will play a central part.

An interesting result presented in this chapter is Theorem 16, that characterizes common knowledge among a group $G$, based on a subset of nodes in their shared pasts. Thus, neither complete information about the local states of these processes nor about their causal past are needed, in order to settle the scope of common knowledge among the group’s members.
Chapter 6

Generalized Ordering of Events

6.1 Introduction

Previous chapters have studied coordination under various restrictions: linear ordering of responses, simultaneous responses, or a linear ordering of sets of simultaneous responses. In this chapter we remove all structural restrictions on ordering and consider systems where the required ordering of the responses is given by any non-particularized partial order.

As we will see, this ultimate generalization does not spawn yet more intricate causal structures and epistemic states. Rather, solutions require multiple instances of the (already defined) generalized centipedes to exist in triggered runs.

We start with a concrete, if simple, example. Consider the following case, describing the production process for the *Munchy Crunchy* chocolate bar.

**Example 7** Charlie’s Chocolate Factory produces all kinds of chocolate, based on distributed processes that control various machines and manufacturing stages. The Munchy Crunchy is one of the chocolate bars manufactured in the plant. Those processes involved in its production are visualized in Figure 6.1. There are 10 different distributed processes involved in the manufacturing, arranged into 2 initiating singleton clusters and 3 multi-process clusters. Assume that the underlying network graph relating all of the processes is full, and that it contains many other processes besides those shown in the figure.
Figure 6.1: Production process for Munchy Crunchy chocolate bar

The figure describes the required manufacturing process, with arrows signifying activation order. Two processes, each a singleton cluster, initiate manufacturing by sending chocolate and crunchies into the system. Each of these processes is controlled by a human operator. Cluster 1 contains processes that control the input and output valves of a mixing bowl that mixes together chocolate and crunchies. These valves must all be opened simultaneously (this is visualized by a cycle in the ordering graph), but only if both chocolate and crunchies are being streamed into the system. Cluster 2 controls another mixing bowl, with only one input and one output valve. Here pure chocolate for the bar’s coating is blended with unhealthy chemicals. Again, both valves must operate together, but only if chocolate is being streamed in. Finally, the processes in Cluster 3 control the coating, tempering and wrapping machines. These too must start to work simultaneously, but only if the mixing bowls are sending out their blends.

The required manufacturing process described above goes beyond the problem formulations we have thus far seen. We are seeing not only requirements for linear ordering and for simultaneity, but also multiple triggers, and events that are causally dependent upon multiple independent causes.

We now define a class of problems for which the requirements graph shown in Figure 6.1 would be an instance. Note that the graph can also be expressed as a partial order \( (V_{\text{crunchy}}, \preceq_{\text{crunchy}}) \) defined on a set of events, where

\[
V_{\text{crunchy}} = \{ \text{chocolate, crunchies, } \text{choc−in}_1, \text{crunch−in}_1, \text{out}_1, \text{choc−}
\]

95
in\textsubscript{2}, out\textsubscript{2}, coat, temper, wrap}, and

the partial order \( \preceq^{\text{crunchy}} \) is defined for every pair edge-connected pair of processes.

In order to fully express the graph, we need to add to the partial order a distinction between those triggering events that are spontaneous external inputs and those that are responses to such triggers. In Example 7, the set of triggers is \( T^{\text{crunchy}} = \{ \text{chocolate}, \text{crunchies} \} \). We formalize requirements such as the one given above, in the following way.

**Definition 29 (Generalized Response Problem)** An instance of the generalized response problem is defined by a tuple \( \text{GR} = \langle V, T, \preceq \rangle \) where

1. \( V \) is a set of events,
2. \( T \subseteq V \) is the set of ND external inputs in \( V \), and
3. \( \preceq \) is a partial order on \( V \), such that every \( \tau \in T \) is \( \preceq \) minimal (i.e. for every \( \tau \in T \) and \( e \in V \), if \( e \preceq \tau \) then \( e = \tau \)).

A protocol \( P \) solves the instance \( \text{GR} = \langle V, T, \preceq \rangle \) of the Generalized Response problem if it guarantees that in every run \( r \),

1. if \( e \preceq e' \) occur at \( t \) and \( t' \) respectively, then \( t \leq t' \). Moreover,
2. for every \( e' \in V \), \( e' \) occurs in \( r \) iff for every \( e \preceq e' \), \( e \) occurs in \( r \).

Consider a protocol \( P^{\text{crunchy}} \), that solves the instance of \( \text{GR} \) defined by the tuple \( \langle V^{\text{crunchy}}, T^{\text{crunchy}}, \preceq^{\text{crunchy}} \rangle \). Our concern in this chapter is to characterize the necessary causal structures that must obtain in executions of protocols solving instances of \( \text{GR} \), such as the protocol \( P^{\text{crunchy}} \).

### 6.2 Condensed Representation of GR

Each of the clusters in Example 7 contains a cycle, while the manufacturing requirements specify that the events in each cluster must occur simultaneously. The next lemma shows the existence of a cycle does indeed guarantee simultaneity in protocols that solve \( \text{GR} \).
Lemma 32 Let $P$ be a protocol solving $\text{GR} = \langle V, T, \preceq \rangle$. Fix $e, e' \in V$ such that $e \preceq e'$ and $e' \preceq e$.
If $e$ occurs at $(r, t)$ then $e'$ occurs at $(r, t)$ too.

Proof From $e' \preceq e$ and definition of GR, there must be some $t' \leq t$ such that $e'$ occurs at $(r, t')$. Again from definition and from $e \preceq e'$, we also get that $t \leq t'$.

Hence $t' = t$. $\blacksquare$

The GR formalization can be used to designate any required order of events, but often, as in the case of Example 7, it is more sensible to consider the strongly connected components in the graph as single units. Thus for every instance $\langle V, T, \preceq \rangle$ of GR, we consider the condensed form $\langle V', T', \preceq' \rangle$, which is derived by collapsing every strongly connected component in $\langle V, \preceq \rangle$ into a single vertex, or component, $C \in V'$.

The partial order $\preceq'$ holds between $C$ and $C'$ iff there exist $e \in C$ and $e' \in C'$ such that $e \preceq e'$. Since for every $\tau \in T$ if $e \preceq \tau$ for some $e \in V$ then $e = \tau$, the subset of triggering components is $T' = \bigcup_{\tau \in T} \{t\} \subseteq V'$.

We will blur the distinction between $\{\tau\} \in T'$ and $\tau \in T$ freely and freely interchange between the two forms.

Thus the condensed form $\langle V', T', \preceq' \rangle$ may be considered as an instance of GR in its own right. In fact, we will even speak of a component $C \in G'$ as “occurring” at time $t$, if all of its member events are simultaneously occurring at that time. Condensed forms of directed graphs enjoy the desirable property of containing no cycles. This property makes it easier for us to work with them than with the original instance of the problem, and following lemma shows that there is no harm done, as the two forms are equivalent with respect to protocol solutions.

Lemma 33 Let $\text{GR} = \langle V, T, \preceq \rangle$ and let $\langle V', T', \preceq' \rangle$ be the condensed form of GR. Protocol $P$ solves GR iff it solves the condensed form.

Proof

$\Rightarrow$ Assume that $P$ solves GR. Fix run $r$.

1. Fix $C \in V'$ and $e, e' \in C$ such that $e$ occurs at $(r, t)$. By definition of $C$ we have that $e' \preceq e \preceq e'$. Hence it must be that $e'$ occurs at $(r, t)$ too.

2. Fix $C, C' \in V'$ such that $C \preceq' C'$ and $C$ and $C'$ occur at $t$ and $t'$ respectively. Then there exist $e \in C$ and $e' \in C'$ that occur at $(r, t)$ and $(r, t')$ respectively. By definition of $\preceq'$ we get that $t \leq t'$. $\blacksquare$
3. Fix $C' \in V'$ and $\{C_h\}_{h=1..k}$, the set of components such that $C_h \preceq C'$ for all $h$. Let $e' \in C'$ and $e_h \in C_h$ for all $h$. By definition of $\preceq'$ $e_h \preceq e'$ for all $h$. Since $P$ solves $GR$ we get that $e'$ occurs iff $e_h$ occurs for all $h$, and hence $C'$ occurs iff $C_h$ occurs for all $h$.

$\Leftarrow$ The arguments pretty much repeat those in the other direction.

6.3 Generalized Ordering Requires Multiple Generalized Centipedes

The ordering requirement formalized by a generalized ordering problem $\langle V, T, \preceq \rangle$ expresses the idea that an event $e \in V$ should occur iff a set of prerequisite actions had already been performed. These in turn will have their own set of prerequisites, etc. The next lemma reformulates the prerequisites for the occurrence of such an event in terms of chains of linear orderings. Focussing on condensed forms, we are assured that there are no cycles in the graph, and hence no infinite chains to reckon with. Of particular interest for us when considering solutions to $GR$ are component chains, defined below.

**Definition 30 (Component chains)** Let $\langle V, T, \preceq \rangle$ be the condensed form of $GR$. A component chain for $C \in V'$ is a sequence $\langle C_0, C_1, \ldots, C_k \rangle$ of alternating members of $V'$ such that $C_0 \in T'$, $C_h \preceq C_{h+1}$ for all $h < k$, and $C_k = C$.

Given $GR = \langle V, T, \preceq \rangle$, we say that $\langle C_0, C_1, \ldots, C_k \rangle$ is a component chain for $e \in V$ if $e \in C$ for some $C \in V'$ where $\langle V', T', \preceq' \rangle$ is the condensed form of $GR$, and $\langle C_0, C_1, \ldots, C_k \rangle$ is a component chain for $C$.

**Lemma 34** Suppose that protocol $P$ solves $\langle V', T', \preceq' \rangle$, the condensed form of $GR$. Fix $r \in R^{\max}(P, T^{\max})$ and let $C \in V'$ be a component that occurs at $(r, t)$. Then for every component chain $\langle C_0, C_1, \ldots, C_k \rangle$ for $C$ there exist $t_0 \leq t_1 \leq \cdots \leq t_k = t$ such that for every $h \leq k$, $C_h$ occurs at $t_h$.

**Proof** Suppose that there exists a component chain $\langle C_0, C_1, \ldots, C_k \rangle$ for $C$ in which the condition does not hold. Then there exists some $h \leq k$ where one of the following hold
\begin{itemize}

\item \textbf{C} does not occur in \(r\): In this case, as \(P\) solves \(\langle V', T', \preceq' \rangle\), component \(C\) also does not occur in \(r\), contrary to the assumption.

\item \textbf{there exists some} \(h' > h\) \textbf{such that} \(t_h > t_{h'}\): In this case, as \(C_h \preceq' C_{h'}\) it must be that \(C_{h'}\) does not occur at all in \(r\), and hence also that \(C_h\) does not occur, and the case is reduced to the previous one, and thus to a contradiction.

\end{itemize}

Lemma 34 reduces the necessary requirements for the occurrence of a component \(C\) into a set of linearly ordered requirements - one for each component chain leading back from \(C\). Recalling the Ordered Group Response problem from chapter 5 and the notion of weakly solving, we note that each of the linearly ordered requirements just mentioned is in fact a requirement of the protocol that it weakly solve an instance of the OGR problem that is specified by the component chain.

The following theorem formalizes this insight, giving us the necessary condition, in causal terms, for correct solutions to the GR problem.

\textbf{Theorem 18} Let \( GR = \langle V, T, \preceq \rangle \) and let \(P\) be a protocol that solves \(GR\). Fix \( r \in R^\max(P, \gamma^\max) \), and suppose that event \(e \in V\) occurs at \((r, t)\). Then for each component chain \(\langle \{\tau\}, C_1, \ldots, C_k \rangle\) of \(e\), there exists a generalized centipede \(\langle \theta_0, I^1, \ldots, I^k \rangle\) in \((r, t_0 \ldots t_k)\) where \(I^h\) and \(t_h\) are the set of processes where events of \(C_h\) occur and their time of occurrence respectively, and \(\theta_0\) is the node where \(\tau\) occurs.

\textbf{Proof} By Lemma 33 \(P\) solves \(GR\) iff it solves the condensed form \(\langle V', T', \preceq' \rangle\). By Lemma 34 in every run where \(e\) occurs, for every initial component chain \(\langle C_0 = \{\tau\}, C_1, \ldots, C_k \rangle\) for \(C\), there exist \(t_0 \leq t_1 \leq \cdots \leq t_k = t\) such that \(C_h\) occurs at \(t_h\) for all \(h \leq k\). As the occurrences of \(C_h\) for all \(h \leq k\) are necessary whenever component \(C\) occurs, we have by definition that \(P\) weakly solves the OGR instance defined by \(\langle \tau, C_1, \ldots, C_k \rangle\). By Theorem 15, there must exist a generalized centipede for \(\langle I^1, \ldots, I^k \rangle\) in \((r, t_0 \ldots t)\). \(\square_{\text{Theorem 18}}\)

In a protocol that solves an instance \(\langle V, T, \preceq \rangle\) of \(GR\) then, whenever an event \(e \in V\) occurs, there exists a set of generalized centipedes - one for each component chain of \(e\). Returning to Example 7 this means that in a protocol that properly controls the production process for \textit{Crunchy Munchies}, whenever both chocolate and crunchies are being pushed into...
the production system, the minimal communication between the distributed processes must contain all three communication structures see in Figure 6.2.

![Diagram](image)

Figure 6.2: Generalized centipedes in the *Crunchy Munchy* production line

### 6.4 Conclusions

This chapter brings our investigation into the causal structures that underly coordination for purpose of event ordering to an end. Using the generalized centipede and the notion of weakly solving, we have shown that for any prescribed partial ordering on events, the communication requirements in a protocol that solves the generalized ordering can be characterized as a set
of generalized centipedes.

We use the *condensed* representation of the ordering graph, in order to avoid the loops that are created whenever the ordering prescribes simultaneous occurrences. Under this representation, vertices represent the strongly connected components of the original graph.

As we proved, each maximal path in the condensed graph represents a requirement for the existence of a generalized centipede in every run where the path’s ultimate component, or cluster of simultaneous events, occurs. The generalization of the ordering requirements to allow for multiple triggering events is translated into communication requirements that include sets of such centipedes.
Chapter 7

Gaining Knowledge of Ignorance

7.1 Introduction

This chapter takes a different, complementing, look at the way transmission bounds affect knowledge and causality in distributed systems. In place of studying the effects of upper bounds in such systems, we will now focus upon lower bounds on transmission times, and how these affect knowledge gain. Intuitively, the existence of lower bounds makes it possible for one process to gain knowledge of another process’s ignorance respecting a recent event.

Such considerations seem to make more sense in an environment that is motivated by competition, rather than cooperation. In his book The Rothschilds [33], Frederic Morton gives an account of the events in the London Stock exchange at the time of the Battle of Waterloo, in which knowledge about lower bounds on transmission times supposedly played a major role. Morton’s (disputed) account can be summarized as follows: on the night of June 15, 1815. Nathan Rothschild, one of London’s most prominent financiers at the time, was informed by his special private couriers that the Battle of Waterloo was won by the British. Official word by Wellington’s men could only arrive on the next day. On the next morning, Rothschild went to the London Stock exchange, and signaled his agents to furiously sell consuls (government bonds). “He knows...[who won]” was the word among traders. The market crashed, and just before Wellington’s men arrived with the news of victory, Rothschild signaled his agents to buy all available consuls, at a fraction of their original price. He is said to have made a fortune on that day.
For the course of events described by Morton to be plausible, not only was it necessary for Rothschild to know about the outcome before everyone else. He also had to know that the others were ignorant of the outcome. Otherwise, he would fear that one of his rivals could out-smart him, gradually buy his shares and make out with a huge gain at Rothschild’s expense. The epistemic circumstances in this example are based on Rothschild’s courier system being known to have lower minimal transmission times than that of Wellington’s communication lines.

The Battle of Waterloo example above illustrates the importance of knowledge about other’s ignorance in particular circumstances. For another example, consider a sealed-bid first-price auction for mining rights. Suppose that near the auction closing a potential bidder learns of a relevant event $e$, say that gold was found in an adjacent site. The bidder’s valuation of the auctioned rights may have changed. But the decision regarding if, and by what amount, to alter her bid would depend on her knowledge about whether her competitor knows about $e$. In particular, if she knows that he is ignorant of $e$, then she should not increase her bid by a significant amount. The analysis presented will serve to show how our favored bidder can use her information about transmission times to figure out whether her competitor is ignorant of $e$.

Our analysis will start by presenting a novel view of how bounds on message transmission times in a communication network induce causal cones of information flow among events in the system, in analogy with the light cones in Einstein-Minkowski spacetime considered in physics [16, 37].

Based on this probing into causal cones, we will develop the formal theory of knowledge of ignorance. As mentioned above, such considerations are more naturally understood in the context of a competitive environment. For this reason this chapter will not introduce any kind of cooperative ordering task to motivate the analysis.

### 7.2 Bounded Communication and Cones of Influence

Consider a fixed inertial system in which all sites are at rest with each other. In such a setting, light rays carry information at a constant speed $c$ in Euclidean space. In terms of Einstein-Minkowski spacetime, the light rays

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1Our setting can be thought of as consisting of a single inertial system, in which there is a single, non-relativistic, notion of time for all sites.
outgoing from an event (or a 4-dimensional point \( p \)) form a surface in spacetime called the event’s future light cone. The light rays converging on an event form a surface called the event’s past light cone. The spacetime points within \( p \)'s future light cone make up its absolute future and those within its past light cone make up its absolute past: the former are spacetime points that events at \( p \) can influence and the latter are the points that can influence \( p \). Events at points outside both light cones of \( p \) can neither influence nor be influenced by events at \( p \). Such events are considered independent of, or sometimes called concurrent with events at \( p \). Observe that the absolute future and absolute past cones of a point \( p \) are fixed and depend only on the coordinates of \( p \).

In analogy, consider a computer network based on a specific context \( \gamma^b \) where for every channel \( i \rightarrow j \), there is a fixed transmission time: \( \min_{ij} = \max_{ij} \). Moreover, assume that the processes follow the full-information protocol \( \text{fip} \) in which, at every instant, they send a message describing their whole history to all neighbors. With fixed transmission rates and constant message sending, we would get that in every run \( \theta \leadsto \theta' \) iff \( \theta \rightarrow \theta' \). Just as in the case of light traveling in Einstein-Minkowski spacetime, in this setting every node \( \theta \) would define a future cone \( \text{fut}(\theta) = \{ \theta' | \theta \rightarrow \theta' \} \) and a past cone \( \text{past}(\theta) = \{ \theta' | \theta' \rightarrow \theta \} \), as well as nodes that are causally concurrent with respect to \( \theta \). In this section we focus primarily upon future causality, where an intricate dynamics transforms potentiality into necessity, as we shall soon see.

What happens when transmission times are not fixed? In purely asynchronous settings, where \( \max_{ij} = \infty \) and thus messages can take arbitrarily long to be delivered, a node \( \theta' \) can be influenced by \( \theta = (i, t) \) only if \( \theta \rightarrow \theta' \). Thus, Lamport’s \( \rightarrow \) relation defines a future cone (and a past cone) for every given node. As opposed to the fixed-transmission system described above, however, here the cone may differ significantly between different runs due to the varying transmission times. Figure 7.1 shows the future cone of node \( \theta \) in a specific run, for an observer with complete information about the future. The alternative futures that remain unrealized in the current run are shown in outline. Observe that a “core” cone can be made out in the center of \( \text{fut}(\theta) \), of nodes that are guaranteed a priori to be within \( \text{fut}(\theta) \), and will thus necessarily be affected by \( \theta \). We denote this cone by \( \Box \text{aff}(\theta) \).

In an asynchronous context, this core consists of the set of nodes \((i, t')\) such that \( t' \geq t \).

The picture becomes more interesting in the presence of upper bounds \( \max_{ij} \) on message transmission times. Recall that we denote by \( D_{ih} \) the shortest distance between vertices \( i \) and \( h \) in the \( \max \)-weighted network.
possible causal futures of \( \theta \) in other runs

\[ \text{fut}(\theta) \supseteq \Box \text{aff}(\theta) \]

\( \Box \text{aff}(\theta) \) region is now defined as the set \( \{(j, t') : t' < t + d_{ij}\} \). Figure 7.2 shows the causal cones of \( \theta \) in \( \gamma^b \).

We have considered the sets \( \text{fut}(\theta), \Box \text{aff}(\theta) \) and \( \Box \text{unaff}(\theta) \), which are all easily determined given complete information regarding the run’s infinite

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2 We assume that messages between processes are not instantaneous and spend at least one time unit in transmission.

3 In analogy to the definition of the \( D_{ij} \) values, \( d_{ij} \) is defined as the shortest distance between \( i \) and \( j \) in the \( \min \)-weighted network graph.
execution. To be of practical use however, we should consider whatever it is that can be made known about causal influence, given a specific “present” point in time \( t' \) and assuming that future events in the run are as yet undetermined. We define \( \text{fut}(\theta, t') \) as the set \( \{(j, t'') | \theta \rightarrow (j, t''), t'' \leq t'\} \), the set of nodes that have, by time \( t' \), already been realized as a part of \( \theta \)'s future.

The portion of the run realized by time \( t' \) determines the sets of necessarily affected and unaffected nodes relative to the current time, in a way that extends them beyond \( \boxdot \text{aff}(\theta) \) and \( \boxdot \text{unaff}(\theta) \), respectively. The set \( \boxdot \text{aff}(\theta, t') \) of all nodes that are guaranteed to be causally affected by \( \theta \) given \( \text{fut}(\theta, t') \), is the union of the \( \boxdot \text{aff}(\theta') \) cones of all \( \theta' \in \text{aff}(\theta, t') \). As we already have \( \theta \rightarrow (k, \bar{t}) \), this suffices to ensure that \( (j, t'') \in \text{fut}(\theta) \). We denote with \( \text{Unaff}(\theta, t') \) the set of nodes that are potentially unaffected by \( \theta \) relative to current time \( t' \). This set is the complement of the set \( \boxdot \text{aff}(\theta, t') \).

A more challenging definition is that of the set \( \text{aff}(\theta, t', \bar{t}) \) of nodes that, at time \( t' \), are potentially affected by \( \theta \). A node \( \theta' \) is potentially affected if it is possible, given \( \text{fut}(\theta, t') \), that the current run will evolve so as to include \( \theta' \) in \( \text{fut}(\theta) \). This set is inductively defined: If \( \theta = (i, t) \) then \( \text{aff}(\theta, t') = \{(j, t'') | t'' \geq t + d_{ij}\} \), and \( \text{aff}(\theta, t') = \bigcup_{(k, \bar{t}) \in \text{aff}(\theta, t'-1)} \{(j, t'') | t'' \geq \bar{t} + d_{kj}\} \) for \( t' > t \). The set of necessarily unaffected nodes \( \text{unaff}(\theta, t') \) is the complement of \( \text{aff}(\theta, t') \).

Observe that with time, as larger portions of the run get realized, the set of nodes that are neither necessarily affected by \( \theta \) nor necessarily unaffected by it, given by \( \text{aff}(\theta, t') \cap \text{unaff}(\theta, t') \), monotonically shrinks. This can be visualized by comparing the state of the cones in Figure 7.2 with that of Figure 7.3, that displays the same run at a later point in time. It is the case.
that at time \( t' \) every node in the time interval \([t, t']\) is either in \( \square \text{aff}(\theta, t') \) or in \( \square \text{unaff}(\theta, t') \). Moreover, the \( \square \text{aff}(\theta, t') \) cone and \( \square \text{unaff}(\theta, t') \) region grow monotonically grow with \( t' \).

Figure 7.3: The necessarily affected and unaffected regions by \( \theta \) in \( \gamma^b \), w.r.t. time \( t' > t \)

In summary, while light cones define fixed regions of influence and concurrency, communication dynamically determines the cones of influence and their complements.

### 7.3 Transmission Guarantees and Knowledge of Ignorance

Cones of influence and information flow as discussed in the previous section are clearly closely related to knowledge about knowledge and to knowledge about ignorance. In this section we build on the cones interpretation to analyze the dynamics of what would probably be best termed as “knowledge gain about ignorance”.

For the following analysis we introduce some variations in the formal language that is used in the proofs. In place of logical operators whose validity is dependent upon system, run and time (the \((R, r, t)\) at the left hand side of the \textit{satisfies} operator \( \models \)), this chapter utilizes \textit{timestamped} operators that are dependent only upon a system and a specific run.

The set \( \Phi \) of primitive propositions consists of the propositions \( \text{occurred}_t(e) \) for all events \( e \) and times \( t \), and the propositions \( \theta \models \theta' \) for all pairs of process-time nodes. The logical language \( L \) is obtained by closing \( \Phi \) under propositional connectives and knowledge formulas. We write \( \theta \not\models \theta' \) instead
of \( \neg(\theta \Rightarrow \theta') \). Our knowledge operators are indexed by a node \( \theta = (i, t) \), and so are time stamped. Thus, \( \Phi \subset \mathcal{L} \), and if \( \varphi \in \mathcal{L}, i \in \mathcal{P} \) and \( t \) is a time, then \( K_{(i,t)}\varphi \in \mathcal{L} \). The formula \( K_{(i,t)} \) is read *process i at time t knows \( \varphi *).

We write \( (R, r) \models \varphi \) to state that \( \varphi * holds in the run \( r, with respect to system \( R \). We write \( r \sim_{(i,t)} r' \) whenever process \( i \)'s local state at time \( t \) in \( r \) is identical to it's local state at time \( t \) in run \( r' \), and inductively define

\[
(R, r) \models \theta \Rightarrow \theta' \quad \text{iff} \quad \theta \Rightarrow \theta' \text{ in the run } r;
\]
\[
(R, r) \models \text{occurred}_t(e) \quad \text{iff the event } e \text{ occurs in } r \text{ by time } t; \text{ and}
\]
\[
(R, r) \models K_{(i,t)}\varphi \quad \text{iff } (R, r') \models \varphi \text{ for every run } r' \text{ satisfying } r \sim_{(i,t)} r';
\]

Propositional connectives are handled in the standard way, and their clauses are omitted above. Despite the slight variance in nomenclature, \( K_{(i,t)}\varphi \) still follows [15] in being satisfied if \( \varphi * holds at all points at which i has the same local state as it does at time \( t \). Thus, given \( R, the local state determines what processes know. Note that \( (R, r) \models \text{occurred}_t(e) \) holds iff \( (R, r, t') \models K_{(i,t)}\text{occurred}(e) \text{ for any time } t' \). Similarly, \( (R, r) \models K_{(i,t)}\varphi \) iff \( (R, r, t') \models K_{(i,t)}\varphi \text{ for any time } t' \). So for the most part, this chapter's formal semantics is but an adaptation of those introduced in Chapter 1.

The motivation here is twofold. First, the timestamped language allows us to simplify the presentation. Second, The use of timestamped epistemic operators and the introduction of the Lamport relation into the formal language implies greater expressivity that we hope will ferment new insights into the study of causation in distributed systems.

Our analysis here will be performed within the context \( \gamma_{\text{min}} \), in which lower bounds on message transmission times are available. Recall that in \( \gamma_{\text{min}} \) there are no upper bounds on message transmission times; messages can take an arbitrarily long amount of time to be delivered. In these settings, Lemma 10 tells us that nested knowledge implies a message chain linking the processes. The converse, shown below in Lemma 35, states that under \( \text{fip} \), such a message chain implies nested knowledge.

**Lemma 35** Let \( \mathcal{R} = \mathcal{R}(\text{fip}, \gamma_{\text{min}}) \). Assume \( e \) is an ND event occurring at \( (i_0, t) \) in \( r \). If there is a chain \( (i_0, t) \rightarrow (i_1, t_1) \rightarrow \cdots \rightarrow (i_k, t_k) \in (r, t..t') \), then \( (\mathcal{R}, r, t') \models K_{i_0}K_{i_{k-1}} \cdots K_{i_1} \text{occurred}(e) \land \text{ND}(e) \).

**Proof** The proof is arrived at by first noting that Lemma 14 implies the following for \( \gamma_{\text{min}} \): if \( (\mathcal{R}, r, t) \models K_{i_0}\varphi \) and \( (i, t) \rightarrow (j, t') \) in \( r \), then \( (\mathcal{R}, r, t') \models K_{j}K_{i_0}\varphi \). Then, repeated applications of this result give us the required outcome. ■
In the rest of this section, we will focus on how different cones of influence combine to determine when an process knows that another process is ignorant about an event of interest. We will give a complete characterization of this question for the \textit{fip} and draw implications from this to the general case of arbitrary protocols.

Recall the sealed-bid first-price auction described in the Introduction. Our bidder is named \(i_2\), her competitor is \(i_1\), and the bids need to be in by time \(t_1\). Moreover, \(i_2\) must decide on her bid at time \(t_2\). Finally, the event \(e\) in which information about a newly found gold mine was disclosed occurred at \(\theta^0 = (i_0, t_0)\). The goal, then, is to determine whether \(K_{\theta_2} \neg K_{\theta_1} \text{occurred}_{i_0}(e)\).

Given that processes following \textit{fip} have the perfect recall property, the following lemma shows that the knowledge state of a process determines its causal past.

**Lemma 36** Fix \(r \in \mathcal{R}(\text{fip}, \gamma^{min})\) and nodes \(\theta_0, \theta_1, \theta_2\) such that \(i_0 \neq i_1\).

\[(\mathcal{R}, r) \models K_{\theta_2} \equiv (\theta_0, \theta_1) \iff \theta_0 \rightarrow \theta_1 \rightarrow \theta_2 \text{ in } r.\]

**Proof**

\(\Rightarrow\) If \(\theta_0 \not\rightarrow \theta_1\) then \((\mathcal{R}, r) \not\models K_{\theta_2} \equiv (\theta_0, \theta_1)\) by the Knowledge Axiom, immediately contradicting the lemma’s assumptions.

Suppose now that \(\theta_0 \rightarrow \theta_1\) but that \(\theta_1 \not\rightarrow \theta_2\). Since \(i_0 \neq i_1\), agent \(i_1\) must receive a message at some point \(t'_1 \in (t_0, t_1]\). Since in \(\gamma^{min}\) every message receive is a nondeterministic event, using Lemma 10 we get that \((\mathcal{R}, r) \models K_{\theta_2} \equiv (\theta_0, \theta_1)\). By definition of \(\models\) there exists a run \(r' \sim_{\theta_2} r\) such that \((\mathcal{R}, r') \models \rightarrow (\theta_0, \theta_1)\), and hence \(\theta_0 \not\in \text{past}(r', \rightarrow)\), again leading to contradiction.

\(\Leftarrow\) \((\theta_0, \theta_1)\) is an edge in \(\text{past}(r, \rightarrow, \theta_2)\). From perfect recall we get that \((\theta_0, \theta_1)\) is an edge in \(\text{past}(r', \rightarrow, \theta_2)\) for all \(r' \sim_{\theta_2} r'\). By definition of \(\text{past}\), \(\theta_0 \rightarrow \theta_1\) in every \(r'\), and therefore \((\mathcal{R}, r) \not\models K_{\theta_2} \equiv (\theta_0, \theta_1)\).

As discussed in Chapter 1, the meaning of the lower bounds \(\min_{ij}\) in the \textit{Net} labeled graph component of the context is that certain message chains, in which messages travel faster than the lower bounds specify, are impossible. We say that a sequence of nodes \(\theta_0, \theta_1, \ldots, \theta_m\) is a \textit{legal message chain} with respect to \textit{Net} if for every \(h < m\) we have (i) \(t_h < t_{h+1}\) and (ii)

\textit{Technion - Computer Science Department - Ph.D. Thesis PHD-2011-09 - 2011}
if $i_h \neq i_{h+1}$ then $i_h \mapsto i_{h+1}$ is a channel in $\text{Net}$, and $(t_{h+1} - t_h) \geq \min_{i_h,i_{h+1}}$. Clearly, for every legal message chain, there is a run of $\gamma^\text{min}$ with network $\text{Net}$ in which this message chain is realized, and $\theta_0 \mapsto \theta_1 \mapsto \cdots \mapsto \theta_m$. Conversely, if $\theta \mapsto \theta'$ in a run $r$, then there is a legal message chain starting at $\theta$ and ending at $\theta'$, that is a causal chain in $r$.

By Lemma 10 we have that $K_{\theta_2} \rightarrow K_{\theta_1} \text{ occurred}_{i_0}(e)$ will hold if $\theta_2$ knows that $\theta_0 \not\Rightarrow \theta_1$ in the current run. We now formalize the required conditions, based on causal cones and legal message chains.

**Definition 31 (Set of legal paths)** We denote by $\text{Legal}_{\theta_0, \theta_1}$ the set of legal message chains starting at $\theta_0$ and ending at $\theta_1$.

$\text{Legal}_{\theta_0, \theta_1}$ consists of all message chains that are both within $\Diamond \text{aff}(\theta_0)$ of nodes that can possibly be affected by $\theta_0$, and in the analogous region of nodes that can possibly affect $\theta_1$. See Figure 7.4.

**Definition 32 (Cut)** A $\theta_0\theta_1$-cut is a set of nodes $C$ that appear in the paths of $\text{Legal}_{\theta_0, \theta_1}$ that intersects every path in $\text{Legal}_{\theta_0, \theta_1}$.

The cut $C$ is called $\theta_0$-clean in run $r$ if $\theta_0 \not\Rightarrow c$ in $r$, for every $c \in C$.

Figure 7.4 depicts three different $\theta_0\theta_1$ cuts.

![Figure 7.4: The set $\text{Legal}_{\theta_0, \theta_1}$ of possible chains from $\theta_0$ to $\theta_1$](image)

Lemma 36 tells us that in order for an process to be informed of any communication link between two distinct sites, the site on the receiving side must be in the process’s past causal cone. The above discussion suggests that the existence of a clean cut, in this cone, on the paths between these sites is of importance. Moreover, we should be looking for cuts that are somehow more “recent”. The following definition picks up on this intuition.
Definition 33 (Causal front) Fix nodes $\theta_0, \theta_1, \theta_2$. The causal front of $\theta_0 \theta_1$ with respect to $\theta_2$ in run $r$, denoted by $\text{Front}_{\theta_2}(r, \theta_0, \theta_1)$, is the set of nodes
\[ \{ \phi \mid \phi \text{ is on some chain in } \text{Legal}_{\theta_0 \theta_1} \text{ and } \exists \Psi \in \text{Legal}_{\phi \theta_1} \text{ s.t. } \Psi \cap \text{past}(\theta_2) = \{\phi\} \} \]

Let $\Psi$ be a legal message chain connecting between $\theta_0$ and $\theta_1$ that is also, at least in part, within the scope of $\text{past}(\theta_2)$. By definition of $\text{Front}_{\theta_2}(r, \theta_0, \theta_1)$, it will contain a “latest contact point” of $\theta_2$ with the nodes of $\Psi$. So, as far as $i_2$ knows at time $t_2$, it is possible that $\psi \rightarrow \theta_1$. Now if it is also the case that $\theta_0 \rightarrow \psi$, then a communication path between $\theta_0$ and $\theta_1$ has been established. There is a certain subtlety involved in the definition. The fact that $(i, t)$ is in $\text{Front}_{\theta_2}(r, \theta_0, \theta_1)$ does not mean that $(i, t')$ is not in the front for $t' > t$. We can still have $(i, t') \in \text{Front}_{\theta_2}(r, \theta_0, \theta_1)$ for some $t' > t$, if each of the nodes $(i, t)$ and $(i, t')$ constitutes a latest contact point for some potential path to $\theta_1$.

![Figure 7.5: Thick marking gives a schematic view of $\text{Front}_{\theta_2}(r, \theta_0, \theta_1)$](image)

We are now ready to characterize knowledge of ignorance in $\gamma_{\min}$, by showing that it reduces to existence of a “$\theta_0$-clean” cut in the causal front:

**Theorem 19** Let $r \in \mathcal{R}(\text{fip}, \gamma_{\min})$ and denote $F = \text{Front}_{\theta_2}(r, \theta_0, \theta_1)$. Then $(\mathcal{R}, r) \models K_{\theta_2}(\theta_0 \nleftrightarrow \theta_1)$ iff both (a) $F$ is $\theta_0$-clean, and (b) $F$ is a $\theta_0 \theta_1$-cut.

**Proof**

$\Rightarrow$: Suppose, to the contrary, that $F$ is either not $\theta_0$-clean, or is not a $\theta_0 \theta_1$-cut. Choose a run $r' \in \mathcal{R}$ such that

- $\text{past}(r', \theta_2) = \text{past}(r, \theta_2)$, and where
• all messages sent and delivered outside \( \text{past}(r, \theta_2) \) have minimal transmission times.

That such a run exists is given by \( \mathcal{R} \) being a representing system and by the non-dependence of nondeterministic events on the past of the run in which they occur. By Lemma 6 we get that \( r \sim_{\theta_2} r' \), and hence that \((\mathcal{R}, r') \models \theta_0 \not\models \theta_1 \). Moreover, as \( \text{past}(r', \theta_2) = \text{past}(r, \theta_2) \) we also have that \( \text{Front}_{r'}(r, \theta_2, \theta_0) = \text{Front}_{\theta_2}(r, \theta_0, 1) = F \). We now have two choices:

**F is not a \( \theta_0\theta_1 \)-cut:** Then there exists

\[
\Psi = \langle \theta_0 = \psi_0 \rightarrow^{nt} \psi_1 \cdots \rightarrow^{nt} \psi_n = \theta_1 \rangle \in \text{Legal}_{\theta_0\theta_1}
\]

such that \( \Psi \cap \text{past}(r', \theta_2) = \emptyset \). By definition of \( r' \) we get that \( \psi_0 \rightarrow \psi_1 \cdots \rightarrow \psi_n \) in \( r' \), and hence that \((\mathcal{R}, r') \models \theta_0 \not\models \theta_1 \), contradiction.

**F is not \( \theta_0 \)-clean:** In this case there exists

\[
\Psi = \langle \theta_0 = \psi_0 \rightarrow^{nt} \psi_1 \cdots \rightarrow^{nt} \psi_n = \theta_1 \rangle \in \text{Legal}_{\theta_0\theta_1}
\]

and some \( k < n \) such that \( \psi_0 \rightarrow \psi_k \) and \( \langle \psi_{k+1} \cdots \psi_n \rangle \cap \text{past}(r', \theta_2) = \emptyset \). Again by definition of \( r' \) we get that \( \psi_{k+1} \rightarrow \cdots \rightarrow \psi_n \) in \( r' \).

We obtain that \( \psi_0 \rightarrow \psi_n \) and hence that \((\mathcal{R}, r') \models \theta_0 \not\models \theta_1 \), again contradicting the assumption.

\(\Leftarrow\): Suppose that \((\mathcal{R}, r) \not\models K_{\theta_2} \theta_0 \not\models \theta_1 \). Then there exists a run \( r' \) such that \( r \sim_{\theta_2} r' \), where \((\mathcal{R}, r') \models \theta_0 \models \theta_1 \). Let \( \Psi = \langle \psi_0, \psi_1, \ldots, \psi_n \rangle \) be a sequence such that \( \theta_0 = \psi_0 \rightarrow^{nt} \psi_1 \cdots \rightarrow^{nt} \psi_n = \theta_1 \) in \( r' \).

It follows that \( \Psi \in \text{Legal}_{\theta_0\theta_1} \). Since \( F \) is a \( \theta_0\theta_1 \)-cut, there must exist some \( \phi \in F \cap \Psi \). Since \( \phi \in \Psi \) we get that \( \theta_0 \in \text{past}(r', \phi) \). Since \( F \) is a \( \theta_2 \) causal front we have that \( \phi \in \text{past}(r, \theta_2) \), and as \( r' \sim_{\theta_2} r \) and \( \gamma^{min} \) is causally traced we also obtain that \( \text{past}(r', \phi) = \text{past}(r, \phi) \). This gives us that \( \theta_0 \rightarrow \phi \) in \( r \) too, contradicting the assumption that \( F \) is \( \theta_0 \)-clean in \( r \).

\(\Box\) Theorem 19

Theorem 19 characterizes knowledge of non-causality under fip in a system with lower bounds on transmission times. Based on the knowledge gain
theorem, we can translate this into conditions on when an process will know
that another process is ignorant of the occurrence of an event of interest.
Consider an event $e_0$ that can occur only at $i_0$. We are interested in when
$K_{\theta_2} \rightarrow K_{\theta_1} \text{ occurred}_{i_0}(e_0)$ holds. Clearly, if $i_2$ knows that $e_0$ did not take place,
then it would know that $i_1$ does not know that $e_0$ took place.

Theorem 19 provides a condition enabling knowledge at $\theta_2$ that $\theta_0 \nrightarrow \theta_1$.
Suppose that $\theta_0 = (i_0, t_0)$. Since $\rightarrow$ is transitive, however, $\theta_0 \nrightarrow \theta_1$ implies
that $\theta' \nrightarrow \theta_1$ for all $\theta' = (i_0, t')$ with $t' > t_0$. So, by Lemma 10, $\theta_1$ could not
have knowledge that $e_0$ happened at any time after $t_0$ too. Combining these
observations, we are able to obtain a tight characterization of knowledge
about ignorance regarding the occurrence of a nondeterministic event:

**Theorem 20 (Knowledge of Ignorance Theorem)** Let $r \in \mathcal{R}(\text{fip}, \gamma_{\text{min}})$,
fix a node $\theta_2$, and let $e_0$ be a nondeterministic $i_0$-event. Let $t'$ be the latest
time for which $(\mathcal{R}, r) \models K_{\theta_2} \rightarrow \text{ occurred}_{i_0}(e_0)$ holds, and denote $\theta_0 = (i_0, t' + 1)$.
Then
$(\mathcal{R}, r) \models K_{\theta_2} \rightarrow K_{\theta_1} \text{ occurred}_{i_0}(e_0)$ iff $\text{ Front}_{\theta_2}(r, \theta_0, \theta_1)$ is a $\theta_0$-clean, $\theta_0 \theta_1$-cut.

**Proof**

$\Rightarrow$: We will prove the counter-position. Suppose that $F$ is not a $\theta_0 \theta_1$-cut or
it is not is $\theta_0$-clean. Note that in particular, this means that $\theta_0 \leadsto \theta_1$ and hence that $t_0 \leq t_1$. As $t_0$ is the latest time for which $(\mathcal{R}, r) \models K_{\theta_2} \rightarrow \text{ occurred}_{i_0-1}(e_0)$ holds, there must exist a run $r' \leadsto_{\theta_2} r$ where $e_0$ occurs at $(i_0, t_0)$. As $\gamma_{\text{min}}$ is causally traced we get that $\text{ past}(r', \theta_2) = \text{ past}(r, \theta_2)$ and hence that $\text{ Front}_{r'}(r, \theta_2, \theta_0) \theta_1 = \text{ Front}_{\theta_2}(r, \theta_0, \theta_1) = F$.

Theorem 19 now shows that $(\mathcal{R}, r') \nvdash K_{\theta_2} \theta_0 \nrightarrow \theta_1$. So there must
exist a run $r'' \in \mathcal{R}$ such that $r'' \leadsto_{\theta_2} r'$, where $(\mathcal{R}, r'') \vdash \theta_0 \Rightarrow \theta_1$. As
the processes are following fip we get, using Lemma 35, that $(\mathcal{R}, r') \nvdash K_{\theta_2} \text{ occurred}_{i_0}(e_0)$. Since $t_0 \leq t_1$ we get $(\mathcal{R}, r') \models K_{\theta_1} \text{ occurred}_{i_0}(e_0)$. Finally, since $r'' \leadsto_{\theta_2} r' \leadsto_{\theta_2} r$, we get that $(\mathcal{R}, r) \nvdash K_{\theta_2} \rightarrow K_{\theta_1} \text{ occurred}_{i_0}(e_0)$, contradicting our assumptions.

$\Rightarrow$: We will prove the counter-position. Suppose that $F$ is not a $\theta_0 \theta_1$-cut or
it is not is $\theta_0$-clean. Note that in particular, this means that $\theta_0 \leadsto \theta_1$ and hence that $t_0 \leq t_1$. As $t_0$ is the latest time for which $(\mathcal{R}, r) \vdash K_{\theta_2} \rightarrow \text{ occurred}_{i_0-1}(e_0)$ holds, there must exist a run $r' \leadsto_{\theta_2} r$ where $e_0$ occurs at $(i_0, t_0)$. As $\gamma_{\text{min}}$ is causally traced we get that $\text{ past}(r', \theta_2) = \text{ past}(r, \theta_2)$ and hence that $\text{ Front}_{r'}(r, \theta_2, \theta_0) \theta_1 = \text{ Front}_{\theta_2}(r, \theta_0, \theta_1) = F$.
Theorem 19 now shows that \((\mathcal{R}, r') \not\models K_{\theta_2} \theta_0 \nRightarrow \theta_1\). So there must exist a run \(r'' \in \mathcal{R}\) such that \(r'' \sim_{\theta_2} r'\), where \((\mathcal{R}, r'') \models \theta_0 \Rightarrow \theta_1\). As the processes are following \(\text{fip}\) we get, using Lemma 35, that \((\mathcal{R}, r'') \models K_{\theta_1} \text{ occurred}_{t_0}(e_0)\). Since \(t_0 \leq t_1\) we get \((\mathcal{R}, r') \models K_{\theta_2} \text{ occurred}_{t_1}(e_0)\). Finally, since \(r'' \sim_{\theta_2} r' \sim_{\theta_2} r\), we get that \((\mathcal{R}, r') \not\models K_{\theta_2} \lnot K_{\theta_1} \text{ occurred}_{t_1}(e_0)\), contradicting our assumptions.

\(\Leftarrow\): Choose an arbitrary \(r' \in \mathcal{R}\) such that \(r \sim_{\theta_2} r'\). We consider three options for the occurrence of event \(e_0\):

- \(e_0\) does not occur in run \(r'\): in this case we get, in particular, that \((\mathcal{R}, r') \models \lnot K_{\theta_1} \text{ occurred}_{t_1}(e_0)\).
- \(e_0\) occurs before time \(t_0\): in this case we obtain a contradiction to the theorem’s assumption that \((\mathcal{R}, r) \not\models K_{\theta_2} \lnot \text{ occurred}_{t_0-1}(e_0)\).
- \(e_0\) occurs at some time \(t' \geq t_0\): from \(r \sim_{\theta_2} r'\) and Lemma 36 we get that \(\text{Front}_{r'}(r, \theta_2, \theta_0)_{t_1} = \text{Front}_{\theta_2}(r, \theta_0, \theta_1) = F\). Theorem 19 is now used to show that \((\mathcal{R}, r') \models K_{\theta_2} \lnot \text{ occurred}_{t_1}(e_0)\), and thus that \((\mathcal{R}, r') \models \theta_0 \nRightarrow \theta_1\). By definition of \(\nRightarrow\) we also get that \((i_0, t') \not\models \theta_1\). Using Lemma 10 we conclude that \((\mathcal{R}, r') \models \lnot K_{\theta_1} \text{ occurred}_{t_1}(e_0)\).

We showed that \((\mathcal{R}, r') \models \lnot K_{\theta_1} \text{ occurred}_{t_1}(e_0)\) for all \(r \sim_{\theta_2} r'\). By definition of \(\models\) we get that \((\mathcal{R}, r) \models K_{\theta_2} \lnot K_{\theta_1} \text{ occurred}_{t_1}(e_0)\), as required.

\(\square\) Theorem 20

7.4 Conclusions

While in timing-based algorithms such as clock-synchronization algorithms [22, 1] lower bounds on transmission times are typically of limited impact, our thesis in the current chapter is that lower bounds play a crucial role in determining knowledge about ignorance. This, in turn, can be of value in player’s considerations in non-cooperative settings. In this chapter we characterized when knowledge about ignorance is obtained in runs of the full-information protocol, in the presence of lower bounds. A natural question involves characterizing knowledge of ignorance for general protocols, or in strategic settings in which a player has uncertainty concerning other players’ strategies. Our results have natural implications about more general
settings: if Alice knows that even under the full-information protocol Bob cannot know about Charlie, then she may be able to conclude the same even under lesser communication. But the analysis required for the general question is more subtle, since Alice could hear from intermediate points without having full knowledge of what information they have. There is considerable room for further exploration of this point.

This chapter also draws an analogy between the causal cones that are formed by information in synchronous systems with bounds, and the notion of causal light-cones in physics. The invariance of the speed of light causes the causal cone of a given point in 4-dimensional Einstein-Minkowski space-time to be fixed \textit{a priori} and not change as time proceeds. In contrast, in the digital space of communication networks, upper bounds induce a region of points that are definitely affected by a spontaneous event occurring at a given point, while lower bounds define a region of points guaranteed to \textit{not} be affected. For a given point $\theta = (i, t)$, these regions grow with time, converging at the end of time to form the set of points actually affected by $\theta$. We used this view to motivate our analysis of knowledge of ignorance. We believe that further study of the causal cones and their evolution over time will provide insights into the fundamental properties of synchronous environments.
Chapter 8

Discussion

This thesis investigates causality and coordination in distributed systems. It extends Lamport’s original paper on causality [26] by looking into causality in synchronous systems. As we saw in Chapter 3, our results provide a generalization of Lamport’s work, in the sense that asynchronous networks are modeled as systems with infinite upper transmission bounds.

Our results show that while the dissemination pattern of asynchronous causality is a fairly straightforward extension of the happened-before relation into message chains, synchronous causality spreads in rather complex patterns that combine message deliveries and timing guarantees.

Formally, our study is based on knowledge-based analysis [15]. As it turns out, the notion of knowledge provides a very close formal approximation of causal influence, which also underlies temporal precedence. In fact, the various forms of temporal event orderings that we examine are each reduced to corresponding epistemic conditions. The two basic kinds being linear ordering, extended to nested knowledge, and simultaneous ordering, which is reduced to a common knowledge requirement. In Chapter 6 these two basic ordering types are combined, providing a characterization of the causal pattern requirement for any given partial ordering on events. Chapter 7 introduces a complementing approach, studying the minimal requirements needed for a process to an ensure that an event at a remote site has not yet transpired.

The current study opens up many possible venues for further research. An immediate concern would be to further extend the study of causality into weaker systems, such as systems with clock drifts, or systems where the processes have only partial knowledge of the bounds on communication. In such networks two learning dynamics intertwine. First, the dynamics
covered in this thesis, that explain how information about events in the
current run are spread in the system. A second process of information flow
concerns the gradual learning of the processes about the communication
characteristics of the underlying network. Relevant information to be gained
here for a process is not only what the actual bounds are, but also what other
processes may have learned about these bounds.

More generally, the model suggested here makes pretty strong synchrony
assumptions. Weakening the model by removing the global clock, reducing
the available knowledge about network characteristics, and allowing failures,
is necessary in order to bring our results closer to real world applications.

Another salient extension to system characteristics would be to consider
mobile networks. Here, given the graveness of energy considerations and the
multi-hop nature of communication, our characterizations of minimal com-
munication requirements may be highly relevant. In such systems though,
we expect not only the bounds to vary throughout a run, but also the num-
ber and identity of participating processes. Some causal analyses have been
suggested [45, 4], but they follow the classical asynchronous paradigm.

Another direction for further investigation would be to consider the finer-
gained patterns that are introduced if we consider protocol-specific knowl-
edge. The thesis follows the approach of Chandy and Misra [7], in charting
out the communication that is universally necessary for knowledge gain. It
is clear that once a specific protocol is considered, the set of communication
patterns that lead to knowledge gain is reduced. First, it may be that pro-
cess \( i \) does not gain any information regarding an occurrence at \( j \) despite
the existence of a causal connection, simply because the messages relating
the two processes do not convey this fact (a typical case is a NULL mes-
sage, that carries very limited information content). Moreover, processes
may delay message relaying, or may be committed to a specific communica-
tion channel despite the existence of several alternatives. A protocol-specific
characterization of knowledge gain, and hence of temporal ordering, would
take all such protocol-specified limitations on communication into account.
We would expect tighter necessity conditions for knowledge gain here.

Lamport’s definition for the concurrency of two events \( e \) and \( e' \) in [26] is
that \( e \not\rightarrow e' \) and \( e' \not\rightarrow e \). Further inquiry about concurrency in synchronous
systems is also in place. Of course, given the global clock, the obvious
candidate for concurrency is that both \( e \) and \( e' \) occur at exactly the same
time. However, a more subtle approach may be appropriate here. Despite
the possibility of actually confirming the exact time of occurrence of events,
in many cases the concurrency of two events is actually accidental. What
we are really after is a notion of “temporal independence”: that events \( e \)
and $e'$ may occur at any temporal ordering in relation to each other. Such an inquiry may serve to extend the measure of parallelism in a protocol by identifying and uncoupling simultaneous occurrences that should really be temporally independent.

Several possible extensions of the thesis may be of less relevance to the distributed systems community, referring instead to the study of multi-agent systems under other disciplines. As we mentioned in Chapter 7, one such case is the study of “knowledge of ignorance”. For one process to be able to tell that another process is unaware of a certain occurrence seems to be highly relevant for systems which are of a competitive nature, as studied in game theory. The existing results presented here are by and large incomplete, as we do not know what more complex epistemic states involving ignorance, such as $K_{i_3} \neg K_{i_2} \neg K_{i_1} (\text{occurred}(e_0) \land \text{ND}(e_0))$, would require in terms of (dis)communication. Another possible research direction is the relaxation of knowledge into belief. Formal systems involving belief are ubiquitous in game theory and also in the general study of multi-agent systems [3, 49]. Do the causal patterns that characterize knowledge gain also characterize the spread of belief? We do not currently know.

Finally, our results as presented give a “snapshot” of the communication pattern that must exist, if knowledge has been gained. They insinuate that the dynamics of information flow is such that information moves “outward” from the site of occurrence of an ND event. Yet we do not prove that such is the case. We have began to extend our results in this direction too. Here some subtle considerations must be made in order to accommodate information “flow”. Also, as it turns out, one must consider the possibility that certain information, say fact $\varphi$, is actually spread as separate packets, each containing a fraction of the required information.

All in all, we believe (and hope) that synchronous causality, along with the new concepts and methodology presented in this thesis, will turn out to be a fruitful advancement in the field of distributed computing, as well as for multi-agent systems in general.
Bibliography


פרק 3יועד להבאתו של מושג "סנטיפיד" (מרבה רגליים). הפרק הוא על תיאור ההתקפה של הנסתרים ואיתם החיזוק של הדפוס תקשורת שהמוכנה."שאותו מגדיר לדפוס תקשורת ומפיקה את כל התפריטים לביעת.

כנזרה לעיל, הקויור בין קיימם של דפוס תקשורת ושידור יכול לשמה הפעולה מוחלטת נוכח של המוארעת מוכחת ב铳מצע התייחסות למצב הדוב הרנדני מתוכנות השונות. בכל מרחיצי התהליך שסידור נכון לשימה מורים את התהליך המוחלט והعباد בידינו של השטפים. הפרק 4 עד 6 שמסב של יד מ קופקאר מ_fence מתוכנה של השטפים.

פרקים 7 ו-8 הם על(Container ביבי תיאור ממושג שונם, בעד פרק 7 מוחבב ביבי המוחות. הפרקה: כדי לבלוק לודא שליא מקבוע הפוקס בטרם מספר מבצע פעולות אחורות על ידי

ה hiểu? נבח תוני התיאור בביבת ביבי ממושג ביבי תיאורัน לעי.
(a) An example network

(b) Cheque cleared at \( t + 10 \)

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