A “Thermodynamic” Approach to Multi-Robot Cooperative Localization*

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Abstract
We propose a new approach to the simultaneous cooperative localization of a very large group of simple robots capable of performing dead-reckoning and sensing the relative position of nearby robots. In the last decade, the use of distributed optimal Kalman filters (KF) to address this problem has been studied extensively. In this paper, we propose to use a very simple encounter based averaging process (denoted by EA). The idea behind EA is the following: every time two robots meet, they average their location estimates.

We assume that two robots meet whenever they are close enough to allow relative location estimation and communication. At each meeting event, the robots average their location estimations thus reducing the localization error. Naturally, the frequency of the meetings affects the localization quality. The meetings are determined by the robots’ movement pattern. In this work we consider movement patterns which are “well mixing” i.e. every robot meets other robots and eventually all of the robots frequently. For such a movement pattern, the time course of the expected localization error is derived. We prove that EA is asymptotically optimal and requires significantly less computation and communication resources than KF.

1 Introduction
Localization is the task of estimating a robot’s self location in space and has been identified as one of the key problems in robotics. The localization problem can be studied under two types of assumptions. We may consider that the robots can estimate their location by sensing their surroundings and using the sensor readings along with stored knowledge they possess regarding the environment (e.g. a map). Alternatively, we may assume that the robots are very simple and lack such capabilities. In this case we assume that the robots know their initial location and updates its location estimate based on on-board odometry and compass readings.

There is a vast body of literature discussing self localization under the first set of assumptions where the main challenge is how to best incorporate the large

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quantity of data gathered by the robot (or robots) into a consistent world view. The means to gather data are the so-called exteroceptive sensors which survey the world and the proprioceptive sensors which continuously monitor the robot’s motion in space via e.g., odometry, compass readings, wheel encoders, etc. This paper’s focus is on the second set of assumptions. The reader interested in the more general scenario is referred to the book by Borenstein et al. [1] and the excellent survey and book of Thrun [2, 3].

In the variant of the localization problem that we shall consider, it is assumed that, initially, every robot knows its precise location in a commonly agreed upon coordinate system. Every robot then uses only odometry in order to track its location, by a process which is sometimes called “dead-reckoning”. However, due to noisy sensor readings, in time, the self-location estimate diverges from the robot’s real location. When a group of robots perform localization, the localization error can be reduced by sharing information between them. In order to do so, some simple exteroceptive capabilities are needed. We shall assume that every robot is able to sense the relative location of nearby robots and to communicate with them.

We denote the cooperative localization algorithm proposed in this paper by “Encounter Averaging” (EA). In EA, every robot moves in the area while maintaining an estimate of its location using odometry and whenever two robots are within sensing and communication range, they average their location estimates. In case the two robots’ localization errors are uncorrelated, such an averaging will result in reducing the error for both robots. It is important to note that any odometry-based approach will face difficulties when the odometry errors are correlated. For example, consider a group of robots going uphill on a slippery terrain. The robots’ wheels will tend to slip so the robots will measure forward speed higher than their actual speed. Since all robots’ localization is biased in the same direction, sharing information between the robots clearly cannot compensate for that error.

Two robots “meet” when the distance between them is less than $V$ and they can sense each other and communicate. Upon meeting, the robots average their location estimations thus reducing the localization error, hence the frequency of meetings affects the localization quality. Roughly speaking, the higher the frequency of meeting the lower the localization error. The frequency of meetings is determined by the robots’ movement pattern which is application-dependent. In this work, we consider movement patterns which are “well mixing” in the sense defined below.

**Definition 1 (Well Mixing Movement Pattern (WMMP)).** If the probability of a meeting between any two robots at any given time is constant then the robots follow a well mixing movement pattern.

Let $p(r_i, r_j; t)$ be the probability that robots $r_i$ and $r_j$ meet at time $t$. The movement pattern is WMMP, by definition, if for any two robots $r_i \neq r_j$ and any time $t$, $p(r_i, r_j; t) = p$ where $p$ is a constant. A practical movement pattern which is roughly WMMP is presented in Section 4 as an example. This movement pattern is later used in the simulations.

In this work, a simple “independent error” model (IEM) is considered. In IEM, the odometry errors incurred at each step are independent of the state of the robot. The localization errors accumulate as two-dimensional Gaussian variables with linearly increasing variance. Furthermore, the errors added at
different times are statistically independent. This model is relevant, for example, when considering a small flying platform in an indoor environment. In this case, the odometry errors are due to small air currents which are independent of the flying platform actions. We are currently working toward applying the analysis proposed in this paper (i.e. under the WMMP assumption) on more practical error accumulation models.

The focus of the analysis is on the expected localization error, similarly to thermodynamic physics which address expected values. In a previous preliminary report\cite{4}, we have analyzed EA using different simplifying assumptions. In this work, these assumptions are replaced with the WMMP model. The results of this paper are more precise than what we have reported before. Nevertheless, when considering the limit of a large group of robots, and after a long “stabilization” time, the final results turn out to be identical.

2 The Encounter Averaging Process

In our work time is discrete i.e. \( t = 0, 1, 2 \ldots \). A group of \( M \) identical independent robots is considered. The robots are modeled as points on the plane. A finite flat domain \( \Omega \subset \mathbb{R}^2 \) is considered. The area of \( \Omega \) is denoted by \(|\Omega|\). Every robot can communicate with its neighbors up to a limited distance \( V \). The robots are equipped with sensors which enable them to detect other nearby robots and sense their relative location (up to distance \( V \)). In order to keep the model simple, it is assumed that these sensors are error free i.e. the robots are able to sense the relative location of each other accurately. Our formulation can be extended to include noisy exteroceptive sensors by incorporating uncertainty into the meeting process.

Let \( J \) be the matrix of all ones of size \( M \times M \) and \( I \) - the identity matrix of size \( M \times M \). Note that for any \( k \geq 1 \), \( J^k = M^{k-1}J \). The notation \( z \sim N(0, \sigma^2) \) means that for any \( k \geq 1 \), \( J^k = M^{k-1}J \).

The location of robot \( r_i \) at time \( t \) in respect to a fixed reference frame is denoted by the vector \( X_i(t) = [x_i(t), y_i(t)]^T \) where \( x_i(t), y_i(t) \) are the robot’s coordinates. Let \( v(t) \) be the robot speed and \( \phi_i(t) \) its direction at time \( t \). The robot coordinates are readily updated as follows:

\[
X_i(t+1) = X_i(t) + \begin{bmatrix}
\cos(\phi_i(t)) \\
\sin(\phi_i(t))
\end{bmatrix} v(t)
\]

The location estimate of robot \( r_i \) at time \( t \) is denoted by \( \hat{X}_i(t) = [\hat{x}_i(t), \hat{y}_i(t)]^T \). Initially \( \hat{X}(0) = X(0) \). The estimate error is given by the vector \( \hat{X} = X - X = [\hat{x}_i(t), \hat{y}_i(t)]^T \).

The IEM implies that the localization errors added at each time step are independent of the robot state. It is further assumed that the errors are distributed normally, i.e.
\[ \hat{X}_i(t+1) = \hat{X}_i(t) + \begin{bmatrix} \cos(\phi_i(t)) \\ \sin(\phi_i(t)) \end{bmatrix} v(t) + \begin{bmatrix} n_{i,x}(t) \\ n_{i,y}(t) \end{bmatrix} \]

where \( n_{i,x}(t) \sim N(0,\sigma_0^2) \) (\( n_{i,y}(t) \sim N(0,\sigma_0^2) \)) is the noise added to \( \hat{x}_i(\hat{y}_i) \) at time \( t \) and \( \sigma_0 \) is a constant. The \( x \) and \( y \) errors are independent. Note that in IEM, error is accumulated even when the robot is stationary. This is a reasonable assumption in conjunction with flying or underwater platforms.

Equation 2 can be written in the following manner

\[ \hat{X}_i(t) = X_i(t) + \sum_{t'=0}^{t-1} \begin{bmatrix} n_{i,x}(t') \\ n_{i,y}(t') \end{bmatrix} \]

Hence,

\[ \tilde{X}_i = \sum_{t'=0}^{t-1} \begin{bmatrix} n_{i,x}(t') \\ n_{i,y}(t') \end{bmatrix} \sim \begin{bmatrix} N(0,t \cdot \sigma_0^2) \\ N(0,t \cdot \sigma_0^2) \end{bmatrix} \]

So when no localization correction mechanisms are applied, the variance of the localization error grows linearly in time.

**Lower Bound**

Before describing the proposed protocol, a lower bound on any cooperative localization algorithm is presented. The bound is derived for the full visibility and communication case i.e. at every time step every robot senses the relative position of all other robots and can communicate with them. The bound is obtained by applying the optimal Kalman Filter. The IEM assumption is that \( \hat{x}_i \) and \( \hat{y}_i \) are independent therefore they can be analyzed separately. Hence only one coordinate, the \( x \)-error will be analyzed here. The same results apply to \( y \) as well.

**Theorem 2.** Considering IEM and any cooperative localization algorithm, the expected variance of the localization error of any robot is bounded by

\[ E[\tilde{x}_i(t)^2] \geq \frac{\sigma_0^2}{M} \]

where \( M \) is the number of robots and \( \tilde{x}_i(0) = 0 \).

**Proof.** Using Kalman filtering notations, the system is modeled as

\[ x(t) = x(t-1) + u(t) + w(t) \]

where \( x(t) = [x_1(t) ... x_M(t)]^T \), \( u(t) = [v(t) \cos(\phi_1(t)) ... v(t) \cos(\phi_M(t))]' \) and \( w(t) = [n_{1,x}(t) ... n_{M,y}(t)]' \). Under full visibility and without observation noise, we define the observable vector: \( z(t) = H(t) x(t) \) where

\[ H(t) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 1 & -1 \end{bmatrix} \]
Using the $H(t)$ above, one realizes that $z(t)$ enables the computation of the relative location between all robot pairs, see [5, 6]. Note that $H^{-1}H = H^TH^{-1} = I - \frac{1}{M} J$. Using the Kalman formalism we can derive the evolution of the state covariance matrix recursively under the set of relative location observations as follows:

$$
P(t|t-1) = P(t-1|t-1) + I \sigma^2_0 \quad (7)
$$

$$
S(t) = H(t) P(t|t-1) H^T(t) \quad (8)
$$

$$
K(t) = P(t|t-1) H^T(t) S^{-1}(t) \quad (9)
$$

$$
P(t|t) = (I - K(t) H(t)) P(t|t-1) \quad (10)
$$

Where $P$ is the state covariance matrix and $K$ is the Kalman gain. Since $P$ is symmetric, it is commutative with both $I$ and $J$, hence

$$
P(t|t) = \left( I - \left( I - \frac{1}{M} J \right)^2 \right) \left( P(t-1|t-1) + I \sigma^2_0 \right) \quad (11)
$$

$$
P(t|t) = \frac{J}{M} \left( P(t-1|t-1) + I \sigma^2_0 \right) \quad (12)
$$

Using the initial condition $P(0) = 0$ we get

$$
P(t|t) = \frac{J}{M} \sigma^2_0 t \quad (13)
$$

**Average Upon Meeting**

In our proposed cooperative self-localization scheme, whenever two robots meet, they apply the following meeting protocol. Let $r_i$, $r_j$ be two robots who meet at time $t$. The meeting protocol is described for robot $r_i$; $r_j$ follows the same procedure simultaneously. Upon meeting, $r_i$ asks $r_j$ “what is your estimation of my location?”. $r_j$ replies with

$$
\left[ \hat{x}_j(t^-) \right. \begin{array}{c} \hat{y}_j(t^-) \end{array} \right] + \left[ \begin{array}{c} x_i(t) \\ y_i(t) \end{array} \right] - \left[ \begin{array}{c} x_j(t) \\ y_j(t) \end{array} \right] \quad (14)
$$

Then, $r_i$ sets his location estimate to be the average of his previous estimate and the coordinates received from $r_j$, i.e.

$$
\hat{x}_i(t) = \frac{\hat{x}_i(t^-) + (\hat{x}_j(t^-) + x_i(t) - x_j(t))}{2} \quad (15)
$$

$$
\bar{x}_i(t) = \hat{x}_i(t) - x_i(t) = \frac{\hat{x}_i(t^-) + \bar{x}_j(t^-)}{2} \quad (16)
$$

and the same for $\hat{y}$. Therefore the new error of each of the robots is the average of their old errors.
3 Analysis: The Covariance Evaluation Process

The IEM assumption is that $\tilde{x}_i$ and $\tilde{y}_i$ are independent therefore they can be analyzed separately. Hence only one coordinate, the $x$-error will be analyzed here. The same results will apply to $y$ as well. Let $P_t$ be the covariance matrix of the localization errors in $x$ at time $t$, i.e.

$$P_t = \begin{bmatrix} 
\sigma_{11}(t) & \sigma_{12}(t) & \cdots & \sigma_{1M}(t) \\
\sigma_{21}(t) & \sigma_{22}(t) & \ddots & \sigma_{2M}(t) \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{M1}(t) & \sigma_{M2}(t) & \cdots & \sigma_{MM}(t) 
\end{bmatrix}$$

(17)

where the components of $P_t$ are defined by $\sigma_{ij}(t) = \text{Cov}[\tilde{x}_i(t), \tilde{x}_j(t)]$.

We would like to examine the evolution of $P$ in time. For example, when no correction (averaging) mechanisms are applied, the covariance between the localization error of any two robots is zero i.e. for any $i \neq j$, $\sigma_{ij}(t) = 0$, and the components on the main diagonal of $P$ grow linearly in time i.e. for any $i$, $\sigma_{ii}(t) = t \cdot \sigma_0^2$. In other words, $P_t = \sigma_0^2 \cdot I \cdot t$.

When EA is applied, the evolution of $P$ is more complex. Considering a group of robots performing EA, $P_t$ can be derived from $P_{t-1}$ in two stages. In the first, localization errors are added and in the second, meetings are accounted for. Let $P_{t-1}$ be the covariance matrix after the localization errors were added, right before meetings are accounted for. The covariance matrix after considering the meetings is defined by $P_t$. The error accumulation stage is given by

$$P_t = P_{t-1} + I \cdot \sigma_0^2$$

(18)

To account for meetings, consider a time step in which robot $r_i$ and $r_j$ meet. As shown earlier, after the meeting took place, the localization errors are given by

$$\tilde{x}_i(t) = \frac{\tilde{x}_i(t-1) + \tilde{x}_j(t-1)}{2}$$

(19)

$$\sim N \left( 0, \frac{\sigma_{ii}(t-1) + \sigma_{jj}(t-1) + 2\sigma_{ij}(t-1)}{4} \right)$$

(20)

In order to gain insight of Equation 20, assume that $\sigma_{ii}(t-1) = \sigma_{jj}(t-1) = \sigma$ and observe two limit cases:

Case 1. In case $\tilde{x}_i(t-1)$, $\tilde{x}_j(t-1)$ are independent (i.e. $\sigma_{ij}(t-1) = 0$) then $\sigma_{ii}(t) = \frac{1}{2}\sigma$ i.e. the variance of $\tilde{x}_i(t-1)$, $\tilde{x}_j(t-1)$ was halved.

Case 2. In case $\tilde{x}_i(t-1)$, $\tilde{x}_j(t-1)$ are fully correlated (i.e. $\sigma_{ij}(t-1) = \sigma$) then $\sigma_{ii}(t) = \sigma_{ii}(t-1)$ i.e. the localization was not improved.

Note that after the meeting, the robots’ localization errors are identical thus additional averaging of the location estimations will not reduce the error.
Additional elements of $P$ are also affected by the meeting. Consider any robot $r_k$ ($k \neq i, j$),

$$
\sigma_{ik} (t) = \frac{E \left[ \tilde{x}_i (t) \cdot \tilde{x}_k (t) \right]}{2} = \frac{E \left[ \tilde{x}_i (t^-) + \tilde{x}_j (t^-) \right]}{2} = \frac{1}{2} (\sigma_{ik} (t^-) + \sigma_{jk} (t^-)) \tag{21}
$$

So $\sigma_{ik} (t)$ equals the average of $\sigma_{ik} (t^-)$ and $\sigma_{jk} (t^-)$. 

The process of updating $P$ as a result of a meeting between $r_i$ and $r_j$ can be carried out by averaging rows $i$ and $j$ of $P$ and afterward columns $i$ and $j$. It is important to note that EA does not change the sum of all the elements of $P$. Nevertheless, the EA process “spreads the error” from the main diagonal to the rest of the matrix. Since the robots’ localization errors are determined by the values of the main diagonal, spreading the error is desired.

Considering any matrix $X$, Let $[X]_{kl}$ be the component of $X$ located in row $k$ and column $l$. Let $A^j$ be the following matrix:

$$
[A^j]_{kl} = \begin{cases} 
1 & k = l \neq i, j \\
\frac{1}{2} & (k = i \lor k = j) \\
\frac{1}{2} & (l = i \lor l = j) \\
0 & \text{else}
\end{cases} \tag{24}
$$

Note that any $k \geq 1$, $(A^j)^k = A^j$. As $A^j$ is symmetric, it is commutative with $J$. For any matrix $X$, $A^jX$ is the matrix resulting from averaging the $i$'th and $j$'th rows of $X$ and $XA^j^T$ is the result of averaging the $i$'th and $j$'th columns.

Hence, to account for a meeting between robots $r_i$ and $r_j$, $P$ is updated by the operation $A^jPA^j^T$.

Let $A_t$ be the matrix which describes the meetings at time $t$ and let $S_t \subseteq S$ be the set of robot pairs which met at time $t$. $A_t$ is a product of matrices of the form $A^j$. Since $A^j$ and $J$ are commutative, so are $A_t$ and $J$. The expected value $E_{S_t} [A_tA_t^T]$ is derived in the next lemma.

**Lemma 3.** Under the WMPP assumption and for small enough $p$, $E_{S_t} [A_tA_t^T] = a_1 \cdot I + a_j \cdot J$ where $a_1 = 1 - \frac{M^2}{2}$ and $a_j = \frac{M}{2}$.

**Proof.** According to the WMPP assumption, the probability of a meeting between any two robots is $p$. In order to derive $E_{S_t} [A_tA_t^T]$, it is assumed that $p$ is small enough so the probability of two meetings in a time step is negligible. This assumption can be justified by noting that $p$ is proportional to the size of the time step. Hence by shortening the time step we can decrease $p$ as much as required.

There are $\binom{M}{2}$ robot pairs, hence the probability that in a time step there is no a meeting is given by $(1 - p)^{\binom{M}{2}} \approx 1 - \binom{M}{2} p$. In case there is no meeting $A_t = I$ thus $A_tA_t^T = I$. In case there is a meeting, any pair of robots can be the meeting pair with a probability of $1/\binom{M}{2}$. So in case there is a meeting
Denoted by Equation 35 we get

\[ y = \text{Lemma 4.} \]

For any

\[ \begin{align*}
&\text{Using the initial condition } P_0 \equiv E_{S_0} \left[ A_t A_t^T \right] \\
&\text{Substitution of the induction hypothesis } P_t = \alpha (t) \cdot I + \beta (t) \cdot J \text{ into Equation 36 yields}
\end{align*} \]

Following Equation 18, \( P_t \) is given by the recursive equation

\[ P_t = A_{t-1} P_{t-1} A_{t-1}^T \]

We are interested in the expected value of \( P_t \) averaged over the sets \( S_0 \ldots S_{t-1} \) denoted by \( \bar{P}_t \equiv E_{S_0 \ldots S_{t-1}} [P_t] \). By deriving the expected values of both sides of Equation 35 we get

\[ \bar{P}_t = E_{S_{t-1}} \left[ A_{t-1} \left( P_{t-1} + \sigma_0^2 \cdot I \right) A_{t-1}^T \right] \]

Using the initial condition \( P_0 = 0 \) we have the following result:

**Lemma 4.** For any \( t \geq 0 \), \( \bar{P}_t \) is of the form

\[ \bar{P}_t = \alpha (t) \cdot I + \beta (t) \cdot J. \]

**Proof.** The proof is by induction on time. The induction base is \( \bar{P}_0 = 0 \). Substitution of the induction hypothesis \( \bar{P}_t = \alpha (t) \cdot I + \beta (t) \cdot J \) into Equation 36 yields

\[ \bar{P}_{t+1} = E_{S_t} \left[ A_t \left( \frac{\alpha (t) + \sigma_0^2}{\alpha + \beta} \cdot I \right) A_t^T \right] \]

\[ \begin{align*}
&\text{Proof.} \quad \text{The proof is by induction on time.} \\
&\text{The induction base is } \bar{P}_0 = 0. \text{ Substitution of the induction hypothesis } \bar{P}_t = \alpha (t) \cdot I + \beta (t) \cdot J \text{ into Equation 36 yields}
\end{align*} \]
where we have used the commutativity of $A_t$ and $J$, Lemma 3 and $a_I + a_J \cdot M = 1$.

From Lemma 4 and Equation 39 we find the coupled set of difference equations for $\alpha(t)$ and $\beta(t)$, i.e.

\[
\begin{align*}
\alpha(t + 1) &= a_I \left( \alpha(t) + \sigma_0^2 \right) \\
\beta(t + 1) &= \beta(t) + a_J \left( \alpha(t) + \sigma_0^2 \right)
\end{align*}
\]

Using the initial conditions $\alpha(0) = \beta(0) = 0$ the solution to this set, for $a_I, a_J < 1$, is given by

\[
\begin{align*}
\alpha(t) &= \frac{a_I}{1 - a_I} \left( 1 - a_I^t \right) \sigma_0^2 \\
\beta(t) &= \frac{\sigma_0^2}{M} t - \frac{1}{M} \alpha(t)
\end{align*}
\]

Hence we have

\[
\begin{align*}
\tilde{P}_t &= \frac{2 - M p}{M p} \left( 1 - \left( 1 - \frac{M p}{2} \right)^t \right) \sigma_0^2 \cdot I \\
&\quad + \left( \frac{\sigma_0^2}{M} t - \frac{2 - M p}{M p} \left( 1 - \left( 1 - \frac{M p}{2} \right)^t \right) \sigma_0^2 \right) \cdot J
\end{align*}
\]

Using $M p \ll 1$ and $t \gg 1$ we get

\[
\tilde{P}_t \simeq \frac{\sigma_0^2}{M} J \cdot t + \frac{2\sigma_0^2}{M p} I
\]

Therefore the components of $P_t$ not on the main diagonal grow linearly in time at a rate of $\sigma_0^2 / M$. The values of the main diagonal grow with the same rate. However, there is a gap of $2\sigma_0^2 / M p$ between the values on main diagonal and the rest of the covariance matrix. The evaluation of $\tilde{P}_t$ (Equation 45) is identical to the result we derived in our previous work\cite{4}.

**The effect of a Landmark**

Consider a landmark placed in a fixed point in the environment. The robots know the exact coordinates of the landmark. Every robot that is within the landmark sensing range updates his localization accordingly. As a consequence, every time a robot senses the landmark, it’s localization error is reset to 0. Furthermore, since his new localization is uncorrelated with the other robots, the covariance of that robot with all other robots is also set to 0.

Similarly to the WMMP assumption, it is assumed that the probability that any robot $r$ will hit the landmark at any time $t$ is constant. That probability is denoted by $p_t$.

When robot $r_i$ sense the landmark, the $i$’th column and row of $P$ are zeroed. It is equivalent to applying the operator $L^tP L_i^T$ where

\[
[L^t]_{kl} = \begin{cases} 
1 & k = l \neq i \\
0 & \text{else}
\end{cases}
\]

\[
9
\]

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Let $B_t$ be the set of robots which have sensed the landmark at time $t$. In resemblance to $A_t$, $L_t$ is the matrix which implies sensing the landmark at time $t$. $L_t$ is fully determined by the set $B_t$. $P$ is recursively given by

$$P_{t+1} = A_t L_t (P_t + \sigma_0^2 \cdot I) L_t^T A_t^T$$  \hfill (47)

Performing expectation over the sets $S_t - S_i$ where $S_i$ yields

$$\bar{P}_{t+1} = E_{S_t, B_t} \left[ A_t L_t \left( \bar{P}_t + \sigma_0^2 \cdot I \right) L_t^T A_t^T \right]$$  \hfill (48)

where $\bar{P}_t = E_{S_{t-1}, B_{t-1}} \left[ P_t \right]$. As opposed to $A_t$ and $J$, $L_t$ and $J$ are not commutative, so a slightly different method is required in order to obtain the difference equations for $\alpha(t)$ and $\beta(t)$.

**Lemma 5.** $E_{B_t} \left[ L_t (\alpha \cdot I + \beta \cdot J)L_t^T \right] = b_l I + b_j J$ where $b_l = (1 - p_l)(\alpha + p_l \beta)$ and $b_j = (1 - p_j) \beta$.

**Proof.** Let $X = E_{B_t} \left[ L_t (\alpha \cdot I + \beta \cdot J)L_t^T \right]$. Consider any component on the main diagonal of $X$ i.e. $[X]_{kk}$. In case robot $r_k$ did not sense the landmark then $[X]_{kk} = (\alpha \cdot I + \beta \cdot J)_{kk} = \alpha + \beta$. In case robot $r_k$ did sense the landmark then $[X]_{kk} = 0$. Consider any component not on the main diagonal i.e. $[X]_{kl}$ where $k \neq l$. In case one of the robots $r_k, r_l$ have sensed the landmark then $[X]_{kl} = 0$ otherwise $[X]_{kl} = \beta$. The probability of any robot to sense the landmark is $p_l$ so

$$E_{B_t} \left[ [X]_{kk} \right] = (1 - p_l)(\alpha + \beta)$$  \hfill (49)

$$E_{B_t} \left[ [X]_{kl} \right] = (1 - p_l)^2 \beta$$  \hfill (50)

$$E_{B_t} [X] = (1 - p_l) \left[ \frac{(\alpha + p_l \beta) \cdot I}{1 + (1 - p_l) \beta \cdot J} \right]$$  \hfill (51)

Using Lemmas 4, 5 and Equation 48 we can write the following Lemma. The proof is similar to the proof of Lemma 4.

**Lemma 6.** For any $t \geq 0$, $\bar{P}_t$ is of the form $\bar{P}_t = \alpha(t) \cdot I + \beta(t) \cdot J$.

**Proof.** The proof is by induction on $t$. The induction base is $\bar{P}_0 = 0$. Substitution of the induction hypothesis $\bar{P}_t = \alpha(t) \cdot I + \beta(t) \cdot J$ into Equation 48 yields

$$\bar{P}_{t+1} = E_{S_t, B_t} \left[ A_t L_t \left( \bar{P}_t + \sigma_0^2 \cdot I \right) L_t^T A_t^T \right]$$  \hfill (52)

$$= (1 - p_l)$$

$$\cdot E_{S_t} \left[ A_t \left[ \frac{(\alpha(t) + \sigma_0^2 + p_l \beta(t)) \cdot I}{1 + (1 - p_l) \beta(t) \cdot J} \right] A_t^T \right]$$  \hfill (53)

$$= (1 - p_l) \left[ \frac{(\alpha(t) + \sigma_0^2 + p_l \beta(t)) \cdot I}{1 + (1 - p_l) \beta(t) \cdot J} \right]$$  \hfill (54)

$$= (1 - p_l) \left[ \frac{a_l (\alpha(t) + \sigma_0^2 + p_l \beta(t))}{1 + (a_j p_l + 1 - p_l) \beta(t)} \cdot J \right]$$  \hfill (55)

\hfill \square
Using Lemma 6 and Equation 55 we can find the set of difference equations for \( \alpha(t) \) and \( \beta(t) \), i.e.

\[
\alpha(t + 1) = (1 - p_l) a_I \left( \alpha(t) + \sigma_0^2 + p_l \beta(t) \right)
\]

\[
\beta(t + 1) = (1 - p_l) \left( a_J \alpha(t) + a_J \sigma_0^2 + (a_J p_l + 1 - p_l) \beta(t) \right)
\]

We could not solve this set. However, the steady state solution is given by

\[
\alpha(t) = a_I \sigma_0^2 \frac{2 + p_l^2 - 3 p_l}{2 - 2 a_I - a_J - p_l + 3 a_I p_l + a_J p_l - a_I p_l^2}
\]

\[
\beta(t) = a_J \sigma_0^2 \frac{1 - p_l}{p_l(2 + 3 a_I p_l + a_J p_l - 2 a_I - a_J - a_I p_l^2 - p_l)}
\]

Note that the steady state solution is constant in time thus a single landmark is sufficient to make the localization error bounded.

By substituting \( a_I \) and \( a_J \) from Lemma 3 and using \( p_l \ll 1 \), \( p_l \ll Mp \ll 1 \) and \( M \gg 1 \) we get

\[
\varphi_t \simeq \frac{\sigma_0^2}{2 \sigma_0^2 p_l} \cdot J + \frac{2 \sigma_0^2}{M p \cdot I}
\]

These steady state results are identical to the result obtained in the report[4].

4 The Random Billiard Walk Example

As an example of a movement pattern which is well mixing, we propose the “random billiard walk” (RBW). In RBW, all robots travel at a constant speed \( v_0 \) and the heading of every robot is generally fixed. Upon hitting an obstacle, the robot randomly selects a new heading. In the experiments without obstacles, in order to avoid static patterns, every robot randomizes a new heading once in a long while.

Consider a robot performing RBW in a domain \( \Omega \). Denote by \( r_i(t) \) the position of robot \( r_i \) at time \( t \). For any \( a \subseteq \Omega \) and time \( t \), let \( q_a(t) \) be the probability that \( r_i(t) \in a \) and

\[
q_a = \lim_{t \to \infty} q_a(t)
\]

Deriving \( q_a \) for RBW is, to the best of our knowledge, an open problem. However, since RBW can be modeled as a Markov chain, the limit in Equation 61 exists1. Somewhat similar geometrical random walks were studied, see a survey in [7]. In our experiments, the simple approximation of \( q_a \simeq \frac{|a|}{|\Omega|} \) was used and have produced good results.

Assumption 7 below states that RBW is very fast mixing i.e. the convergence of the limit in Equation 61 is infinitely fast. Clearly, the assumption is false. However, in our experiments, the mixing time was quite fast so Assumption 7 is a good approximation, see the experimental results in the end of this section.

1RBW can be modeled as a Markov chain by including the heading in the configuration.
Assumption 7. For any robot \( r \), \( a \subseteq \Omega \), and two times \( t, t' \) such that \( t' > t \),

\[
Pr \{ r(t') \in a \mid r(t) \} = q_a
\]

Using Assumption 7, we evaluate \( p(r_i, r_j; t) \) - the probability that robots \( r_i \) and \( r_j \) meet at time \( t \). Our method of evaluating \( p(r_i, r_j; t) \) resembles the method used for discrete domains\[8\]. Fix two robots \( r_i \) and \( r_j \). Let \( V_t \) be the circle of radius \( v \) around \( r_i(t-1) \) and \( V_{t-1} \) the circle around \( r_i(t) \), see Figure 1a. So robot \( r_i \) meets robot \( r_j \) at time \( t \) if and only if \( r_j(t) \in V_t \) and \( r_j(t-1) \notin V_{t-1} \).

Note that in case \( r_j(t) \in V_t \) and \( r_j(t-1) \in V_{t-1} \) the distance between \( r_i \) and \( r_j \) at time \( t \) is less than \( v \). However, in this case, further exchange of information between the robots will not improve their localization (as shown in the previous Section), so it is not considered as a meeting.

Let

\[
p(r_i, r_j; t) = Pr \left[ r_j(t) \in V_t \land r_j(t-1) \notin V_{t-1} \right]
\]

\[
= \int_{V_t} Pr \left[ r_j(t) \in da \land r_j(t-1) \notin V_{t-1} \right] da
\]

\[
= \int_{V_t} Pr \left[ r_j(t) \notin V_t \mid r_j(t) \in da \right] Pr \left[ r_j(t) \in da \right] da
\]

\[
= \int_{V_t} Pr \left[ r_j(t-1) \notin V_{t-1} \mid r_j(t) \in da \right] q_{da} da
\]

Due to Assumption 7, in case \( r_j(t) \in da \), \( r_j(t-1) \notin V_{t-1} \) is distributed uniformly over the circle of radius \( v_0 \) with a center at \( r_j(t) \). So \( Pr \left[ r_j(t-1) \notin V_{t-1} \mid r_j(t) \in da \right] \) equals the part of this circle which is not inside \( V_{t-1} \) (see the bold arc in Figure 1a). Let \( \Delta x = x_j(t) - x_i(t), \Delta y = y_j(t) - y_i(t) \). We face the problem of calculating the intersection of two circles at distance \( d = \sqrt{(\Delta x + v_0)^2 + \Delta y^2} \),
see Figure 1b. \( \theta \) is given by:

\[
\theta = \begin{cases} 
2 \arcsin \left( \frac{\sqrt{4d^2V^2-(d^2-v_0^2+V^2)^2}}{2dV_0} \right) 
\end{cases} 
\]

if \( r_j(t) \notin V_{t-1} \)

\[ \cdots \] (66)

so

\[
Pr \left[ r_j(t-1) \notin V_{t-1} \mid r_j(t) \in da \right] = \frac{2\pi - \theta}{2\pi} \] (67)

Substitution of Equation 67 into 65 yields,

\[
p(r_i, r_j; t) = \frac{2\pi - \theta}{2\pi} q_{da} da \] (68)

which can be calculated numerically. Recall that when the robots follow a WMMP, by definition, for any \( r_i \neq r_j \) and \( t \), \( p(r_i, r_j; t) = p \) where \( p \) is a constant. As required, \( p(r_i, r_j; t) \) given in Equation 68 is independent of \( r_i, r_j \) and \( t \) hence - is constant.

Unfortunately, \( p \) can not be measured directly in an experiment. Instead, the prediction of \( p \) is validated by measuring the histograms of \( \delta \) and \( n(k; \Delta t) \) as defined below. Let \( \delta \) be the time elapsed between two successive meetings of a specific robot with any other robot i.e. \( \delta \) is the “free time” between meetings. According to the WMMP model, \( \delta \) is distributed geometrically with a mean of \( 1/p_r \), i.e. \( Pr[\delta = k] = p_r \) \( (1 - p_r)^{k-1} \) where \( p_r \) is the probability that robot \( r \) meets any other robot at time \( t \) \( (p_r = 1 - (1 - p)^M-1) \). The histograms of \( \delta \) for three experiments with varying number of robots are presented in Figure 2 for a torus environment and a 9-rooms environment. A sketch of the 9-rooms environment can be found in Figure 9. The experiments show that the estimation of \( \delta \) (hence \( p_r \)) is accurate for both the torus and the 9-rooms.

For any two robots \( r_i \neq r_j \) let \( N_{i,j}(\Delta t) \) be the number of meetings between \( r_i \) and \( r_j \) in a time period of length \( \Delta t \). Hence for any \( 0 \leq k \leq \Delta t \):

\[
Pr \left[ N_{i,j}(\Delta t) = k \right] = \binom{\Delta t}{k} p^k (1-p)^{\Delta t-k} \] (69)

Let \( n(k; \Delta t) \) be the number of robot pairs which have met exactly \( k \) times in the time period \( \Delta t \). The expected value of \( n(k; \Delta t) \) is given by:

\[
E \left[ n(k; \Delta t) \right] = \frac{M}{2} \cdot Pr \left[ N_{i,j}(\Delta t) = k \right] \] (70)

Simulation results of a single run of RBW in a box environment are presented in Figure 3. The experiment result agree with the theoretical estimations for both the short time scale \( (\Delta t = 500) \) and the long time scale \( (\Delta t = 5000) \). This strong agreement supports Assumption 7 and our claim that RBW is WMMP.
Figure 2: The histogram of $\delta$. The solid lines are the theoretical estimations and the markers are the simulation results. Every line in the figure is a result of a single run of 5000 time cycles.

Simulation results for a 9-rooms environment are presented in Figure 4. For the 9-rooms environment, the experiment result are somewhat far of the predictions. The probability of a short time period between consecutive meetings was found to be higher than expected. It is well understood. For example, in case robots $r_i$ and $r_j$ have recently met, they are probably in the same room so there is high probability that they will meet again soon, in contradiction to the WMMP assumption in which the probability of meeting any other robot is the same. Furthermore, the obstacles enlarge the mixing time hence making Assumption 7 less valid. So, for the 9-rooms environment, RBW is only roughly WMMP. As a result, we would expect that the localization error predictions for the 9-rooms environment will be less accurate.

To conclude:

- $p$ was derived and was found accurate for both the torus and the 9-rooms environment.
- For environments with few obstacles (e.g. torus, box etc.), RBW is WMMP.
- When the environment include many obstacles, RBW is only roughly WMMP. As a result, the localization error predictions will be less accurate.

Derivation of $P_l$

In this section we derive $p_l$ i.e. the probability of robot $r$ sensing the landmark at time $t$. It is derived in a manner similar to $p_r$. Consider any specific robot $r$. Let $B$ be a circle of radius $V$ around the landmark, see Figure 5. $r$ collides with the landmark at time $t$ if $r(t) \in B$ and $r(t - 1) \not\in B$ (see the derivation of $p$).
Figure 3: Results of a single run of RBW in a box environment where $M = 100$. The solid lines are the theoretical estimations of $n(k; \Delta t)$ and the dots are the simulation results.

Figure 4: Results of a single run of RBW in a 9-rooms environment where $M = 100$. The solid lines are the theoretical estimations of $n(k; \Delta t)$ and the dots are the simulation results.

Figure 5: Illustration of the beacon meeting probability derivation process.
To derive $p_l$, consider an infinitesimal area of size $da \subseteq B$,

$$p_l = Pr [r(t) \in B \land r(t-1) \notin B]$$

$$= \int_B Pr [r(t) \in da \land r(t-1) \notin B] da$$

$$= \int_B q_{da} \cdot Pr [r(t-1) \notin B \mid r(t) \in da] da$$

(71)

(72)

(73)

In case $r(t) \in da$, we assume that $r(t-1)$ is uniformly spread over the circle of radius $v_0$ (the step length) with a center at $r(t)$. So $Pr [r(t-1) \notin B \mid r(t) \in da]$ equals the part of this circle which is not inside $B$ (the bold arc in Figure 5). Let $r(t) = (x_r(t), y_r(t))$ and WLOG assume that the landmark location is $(0,0)$. We face the problem of calculating the intersection of two circles at distance $d = \sqrt{x_r^2 + y_r^2}$. $\theta$ is given by:

$$\theta = \begin{cases} 
2\pi - 2\sin \left( \frac{\sqrt{4d^2V^2-(d^2-v^2+V^2)^2}}{2d} \right) & \text{if } d \geq V - v \\
0 & \text{else} 
\end{cases}$$

(74)

so

$$Pr [r(t-1) \notin B \mid r(t) \in da] = \frac{2\pi - \theta}{2\pi}$$

(75)

$$p_l = \int_B \frac{2\pi - \theta}{2\pi} q_{da} da$$

(76)

$p_l$ can be derived numerically. Note that $p_l$ is independent of $r$ and $t$, hence - is a constant.

5 Discussion and Simulations

In the scenario where EA is applied without a landmark, the expected localization error of a robot is given by

$$\dot{x}(t) \sim \dot{y}(t) \sim N (0, \sigma_{\text{diag}}^2 (t))$$

$$\sigma_{\text{diag}}^2 (t) = \frac{\sigma_o^2}{M} \cdot t + \frac{2\sigma_o^2}{M_p}$$

(77)

(78)

i.e. the variance of the localization error ($\sigma_{\text{diag}}^2 (t)$) comprises a time dependent component and a constant component. When no error correction mechanisms are applied, $\sigma_{\text{diag}}^2 (t) = \sigma_o^2 \cdot t$. Hence by applying EA, the error growth rate is reduced by a factor of $M$. However, a constant component is added. This constant component is a result of the time the odometry errors require to average over the robots and is inversely proportional to the frequency of meetings ($M_p$). By Theorem 2, the optimal cooperative localization algorithm employing a Kalman filter based on all possible relative location observations yields $\sigma_{\text{diag}}^2 (t) = \frac{\sigma_o^2}{M} \cdot t$. Hence, rather surprisingly, the localization estimates provided by EA are optimal up to a constant i.e. asymptotically optimal.
Note that, with EA, the variance of the error increases linearly in time. The error can be made bounded by introducing a landmark. With a landmark, the localization error is given by Equation 77 with,

\[ \sigma_{\text{diag}}^2(t) = \frac{2\sigma_0^2}{M_p} + \frac{\sigma_0^2}{2M_p} \tag{79} \]

In any of the cases above, the predicted localization error is given by

\[ E[e(t)] = \sqrt{\frac{\pi}{2} \sigma_{\text{diag}}^2(t)} \tag{80} \]

Simulations were used in order to validate the analytical results. The following parameters were computed from experimental data:

\[ \sigma_{\text{diag}}^2(t) = \frac{1}{M} \sum_{i=1}^{M} |P_t|_{ii} \tag{81} \]

\[ \sigma_{\text{cov}}^2(t) = \frac{1}{M(M-1)} \sum_{i=1}^{M} \sum_{j \neq i} |P_t|_{ij} \tag{82} \]

\[ E[e(t)] = \frac{1}{M} \sum_{i=1}^{M} \sqrt{\tilde{x}_i^2(t) + \tilde{y}_i^2(t)} \tag{83} \]

\( \sigma_{\text{diag}}^2 \) is the variance of the localization error averaged over the robots and \( \sigma_{\text{cov}}^2 \) is the covariance averaged over all robot pairs. \( E[e] \) is the localization error averaged over the robots. In all figures displaying experimental results, the predicted values are presented using solid lines. The simulation results are presented using error bars or markers.

The values of \( \sigma_{\text{cov}}^2 \), \( \sigma_{\text{diag}}^2 \) and \( E[e] \) for a single simulation run are presented in Figure 6. The environment for this run was a torus of size 100 × 100, the group comprised \( M = 100 \) robots where \( V = 3 \), \( v_0 = 1 \) and \( \sigma_0^2 = 0.01 \). The experiments show that the predictions of \( \sigma_{\text{diag}}^2 \) and \( \sigma_{\text{cov}}^2 \) are accurate. This is expected since RBW is indeed WMMP on the torus, as shown in Section 4. The average error was found to be very noisy for a single run. Hence the mean of the average error over 50 runs is presented in Figure 7. The average is less noisy. Therefore, it can be observed that the error is also predicted well.

The values of \( \sigma_{\text{cov}}^2 \), \( \sigma_{\text{diag}}^2 \) for a single run on a torus with a landmark are presented in Figure 8a. The average of the mean error over 50 runs is presented in Figure 8b. The simulation results are in agreement with the analysis.

The simulations have shown that the predictions are accurate when the environment is a torus. We have experimented with more environments. Sketches of the environments used can be found in Figure 9. The values of \( \sigma_{\text{cov}}^2 \) and \( \sigma_{\text{diag}}^2 \) for a single run in a 9-rooms environment are presented in Figure 10. For a 9-rooms environment, \( \sigma_{\text{cov}}^2 \) and the growth rate of \( \sigma_{\text{diag}}^2 \) are predicted well but the gap between \( \sigma_{\text{cov}}^2 \) and \( \sigma_{\text{diag}}^2 \) is higher than expected. Recall that this gap is the result of the time required to average the error over the robots and is given by \( 2\sigma_0^2/M_p \). On the torus, the robots travel freely hence at every time step there is a probability of \( M_p \) to meet a “fresh” robot i.e. a robot with a shared covariance of \( \sigma_{\text{cov}}^2 \). In other words, on the torus, RBW is WMMP. On the contrary, in the 9-rooms environment, there is a higher probability to meet
Figure 6: $\sigma^2_{cov}$, $\sigma^2_{diag}$ and $E[e]$ for a single run on a torus where $M = 100$, $|\Omega| = 100^2$, $V = 3$, $v_0 = 1$ and $\sigma_0^2 = 0.01$. The solid lines are the theoretical predictions and the error bars are the simulation results.

Figure 7: Average of the mean localization error over 50 runs.
2000 4000 6000 8000 10000
0 0.1 0.2 0.3 0.4
σ 2
(a) The lower line is \( \sigma^2_{\text{cov}} \) and the upper is \( \sigma^2_{\text{diag}} \). (b) Average of the mean error over 50 runs.

Figure 8: (a) \( \sigma^2_{\text{cov}}, \sigma^2_{\text{diag}} \) for a single run and (b) \( E[e] \) averaged over 50 runs. The environment was a torus with a single landmark. Parameters: \( M = 100, |\Omega| = 100^2, V = 3, v_0 = 1 \) and \( \sigma_0^2 = 0.01 \). The solid lines are the theoretical predictions and the error bars are the simulation results.

0 5000 10000
0.2 0.4 0.6 0.8
E[e]
t
Figure 9: Sketches of the environments used in the simulations. The blue dots are the robots.

(a) Box. (b) 9-rooms. (c) Ring.

Putting it another way, RBW is less WMMP. Meeting a “fresh” robot reduces the localization error much more efficiently than meeting a “dirty” one. Hence the error spreads less efficiently and the gap between \( \sigma^2_{\text{cov}} \) and \( \sigma^2_{\text{diag}} \) is larger than predicted.

Simulation results on more environments can be found in Figure 11. The results are similar to the ones for the 9-rooms environment i.e. the gap is larger then expected.

6 Relation to Previous Work

The process discussed here resembles other averaging processes that have been discussed in the literature. For example, Tanny and Wellner[9]; Proshcan and Shaked[10]; and Xiao and Boyd [11] discuss three different processes of random
Figure 10: $\sigma^2_{\text{cov}}$ and $\sigma^2_{\text{diag}}$ for a single run in 9-rooms where $M = 100$, $V = 3$, $v_0 = 1$ and $\sigma^2_0 = 0.01$. The lower line is $\sigma^2_{\text{cov}}$ and the upper is $\sigma^2_{\text{diag}}$.

(a) Box environment.  
(b) Ring environment.

Figure 11: $\sigma^2_{\text{cov}}$ and $\sigma^2_{\text{diag}}$ for a single run where $M = 100$, $V = 3$, $v_0 = 1$ and $\sigma^2_0 = 0.01$. The lower line is $\sigma^2_{\text{cov}}$ and the upper is $\sigma^2_{\text{diag}}$. 
averaging of vector elements. In [9], at every step of the process, two elements of the vector are chosen randomly and then averaged. In [10], at every step $0 < \lambda < 1$ is chosen randomly and then the largest and smallest elements of the vector are averaged with weights $\lambda$ and $(1 - \lambda)$. Xiao and Boyd [11] derived the averaging probabilities that minimize the convergence time assuming the elements reside in the vertices of a connected graph and every two elements can be averaged only if there is an edge connecting them. For the three processes above, convergences to the uniform vector was proved. Note that the process discussed in this paper and the three averaging processes above can be analyzed using products of random matrices. The convergence of products of random matrices have been studied, see for example [12, 13].

About ten years ago, Sanderson[5, 6] proposed a cooperative localization mechanism based on a central (non-distributed) Kalman Filter. He also presented a distributed algorithm for the fully symmetric case: homogeneous group and a complete relative position measurement graph (RPMG) i.e. at every time step all robots meet all robots. Roumeliotis and colleagues[14, 15, 16, 17, 18] presented a distributed version of KF in which the computation required to maintain the covariance matrix is distributed between the robots. However, every meeting between two robots implies an update of at least $2M$ components of the covariance matrix. Furthermore, all robots must be aware of every update of the covariance matrix. In the distributed KF of Roumeliotis et al., every meeting implies a computation complexity of $\Theta(M^2)$ and communication between all robots, so their algorithm does not scale well. In a later work, Mourikis and Roumeliotis proposed to reduce the computation and communicating loads by lowering the frequency of relative observations[19]. Martinelli proposed to use a hierarchical structure of Kalman filters[20] i.e. the robots are divided into groups, relative observations and corrections are performed within each group, inter-group corrections are performed only between the group leaders hence reducing the computation and communication complexity.

In distinction from the previous work, in EA, the computation complexity implied by a meeting is $\Theta(1)$ and the only communication required is between the two meeting robots. Furthermore, since EA is asymptotically optimal, the benefits of using variants of KF are limited to reducing the constant gap between EA and the optimum achievable. These benefits diminish when considering long time scales. It is important to note that in the KF based approaches every robot maintains, to some extent, an evaluation of the covariance matrix thus has an estimation of its localization accuracy. In EA, the covariance matrix is not shared by the robots hence the robots' accuracy estimation is lower.

Roumeliotis and Rekleitis were the first to analyze the performance of KF[21, 22]. They considered homogeneous robots with complete relative position measurement graph (RPMG). Later, Mourikis and Roumeliotis extended the analysis to include heterogeneous groups and general RPMG[23, 18]. Mourikis and Roumeliotis analyzed KF assuming a fixed RPMG i.e. every robot averages its location with a fixed set of other robots. By fixing the RPMG, they have been able to obtain an exact analysis of the localization process. They also considered changes of the RPMG, but discuss the system state after stabilization.

The model used by Mourikis and Roumeliotis is more suitable to ground robots than IEM. In their model, every robot senses its orientation using a compass and updates its localization based on the distance and direction traveled. The localization errors result from the wheel encoders and compass noises.
So the localization errors added at each time step are independent but are affected by the robot state (heading, speed). We intend to apply our formalism on more realistic models like the one considered in [23] in the future. Even though the model of Mourikis and Roumeliotis is more general than IEM, the analysis of both models produces similar results. The main similarities are:

- The error comprises a time dependent term and a constant term. The time dependent term is monotonically increasing (in time) and is dependent solely on the number of robots and the quality of the odometry. In particular, it is not dependent on the RPMG. Observe Equation 45. The time dependent part is dependent of $\sigma_0^2$ (odometry noise) and $M$ but is independent of $p$ (a characteristic of the RPMG).

- When a single robot (or more) have access to absolute position measurement, the error of all robots become bounded. In our work, this happens when a landmark is introduced.

Considering underwater UAVs, Bahr et al. [24] proposed a variant of KF that uses only range measurement for the update process. With resemblance to KF, Fox et al. proposed to average the location estimations between robots[25]. In their work, every robot estimates its location using Monte Carlo localization[26, 27] i.e. every robot maintains a cloud of points in space with a probability attached to every point. The robot location estimate is the probability function implied by the cloud. When two robots meet, their clouds are averaged.

Kurazume and colleagues[28, 29, 30] proposed a strategy based on “portable landmarks”. In this scheme, every time a robot moves, other robots are holding still while following the robot movement with their sensors. The viewing robots provide the moving one a localization better then given by his own odometry. Several other works were carried out using this scheme, see [31, 32]. In contrast to IEM, in the “portable landmarks” paradigm, it is assumed that robots which do not move do not accumulate error. Since this strategy is not solely odometry-based, it is more resilient to correlation between odometry errors. On the downside, when applying this scheme, the robots’ movements are limited. In EA and KF, no special movement pattern is required and the robots are free to go wherever the task they perform requires.

7 Conclusion

We presented the error averaging (EA) localization scheme inspired by the optimal Kalman filter (KF) proposed by Sanderson[6] and Roumeliotis et al.[17]. The idea behind EA is simple: Whenever two robots meet, they average their location estimates. While being asymptotically optimal, EA requires considerably less communication and computation then KF.

In case the robots have no access to absolute localization information, EA’s localization error is composed of two components. A constant component and a monotonically increasing time dependent component. The constant component results from the time required to average the error between the robots and is a function of the odometry quality and the frequency of meetings. The time dependent component results from the error accumulated by the robots and is a function of the odometry quality and the number of robots i.e. it is independent
of the frequency of meetings. In some cases, robots have access to absolute localization (e.g., a landmark), the localization errors of all robots become bounded.

We analyzed the expected localization quality of EA assuming the movement pattern of the robots is random and well mixing (WMMP). i.e., the probability of a meeting between any two robots at any time is constant. As an example of such a movement pattern, we presented the random billiard walk (RBW). Simulations have shown that the analysis is accurate when the environment is a torus. Hence, RBW is indeed well mixing on the torus. When the environment includes obstacles, RBW is less WMMP. In that case, the time dependent component of the error propagation is predicted well; however, the constant is somewhat higher than expected.

We are currently working toward applying the analysis proposed in this paper (under the WMMP assumption) to more realistic error accumulation models.

References


