Algorithms for phase retrieval with a (rough) phase estimate available: a comparison

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In this work we consider a variation of the phase retrieval problem where the phase is not lost completely but is known to lie within a certain interval. For this problem, we develop two new algorithms and evaluate their performance compared with the classical Hybrid Input-Output (HIO) method. One of the algorithms is a modification of the Hybrid Input-Output method which uses the additional information about the phase, and the second is based on convex optimization techniques. Extensive tests demonstrate that the latter algorithm outperforms significantly both the original HIO and its phase-aware modification. © 2010 Optical Society of America

OCIS codes: 000.0000, 999.9999.

1. Introduction

The classical phase retrieval problem amounts to the reconstruction of an image from the magnitude of its Fourier Transform (FT). For many years the problem has been playing a central role in certain branches of optics, astronomy, and crystallography. Recent advent of a high-resolution imaging technique called Coherent Diffraction Imaging (CDI) extends the list of applications directly relying upon our ability to recover the phase of an image's Fourier transform [1].

The currently prevailing method of reconstruction is the Hybrid Input-Output (HIO) method [2], which is based on the alternating projections method due to Gerchberg and Saxton (GS) [3]. However, unlike the GS method, HIO is much less prone to stagnation and, in fact, provides good results in case of noiseless measurements and a good estimation of the actual image support. Despite the great success and wide usage of the HIO algorithm, the quest for a faster and more noise robust methods persists. A natural candidate would
be convex optimization methods\(^1\). Unfortunately, the phase retrieval problems turns out to be particularly tough for such methods. A study in [4] demonstrates that Newton-type algorithms fail miserably for anything but tiny problems. To the best of our knowledge, no convex optimization based algorithm exists that is capable of solving the classical phase retrieval problem. Nevertheless, it has been shown recently that the situation changes dramatically if some phase information is available: in [5] a convex optimization method was presented for the case where a rough estimate of the phase (up to \(\pi\) radians) is available. In this case, the algorithm reported there provides a reconstruction rate that is orders of magnitude faster than the HIO method. Moreover, this algorithm yields much better reconstruction quality when the measurements contain noise.

Another advantage of optimization based algorithms is their ability to incorporate additional information into the computational scheme. For example, in addition to the Fourier magnitude one might have available a low-resolution (blurred) version of the sought image. In this case, current methods can only deal with the deconvolution and phase retrieval problems separately. For example, in [6] at the first stage a deconvolution is performed and then, at the second stage, the classical HIO method is applied starting with the result of the deconvolution as its initial guess. Methods that can combine additional information directly can provide better and more robust results. In [7] an optimization base method was reported for simultaneous phase retrieval and de-blurring. In [8] the method was extended to simultaneous phase retrieval combined with deconvolution, focusing, and holography.

In this work we concentrate on a more detailed performance comparison of the algorithms for phase retrieval with partial phase information. Specifically, we compare the standard HIO algorithm with the convex optimization method described in [5] and a new algorithm which is a generalization of the HIO method that uses the phase uncertainty interval information. The latter algorithm is called Phase-Aware Hybrid Input-Output (PA-HIO). While this generalization is significantly faster than the original HIO method it is still much slower than the method introduced in [5]. Our results confirm that much of the improvement comes from the algorithm and not only from the additional available data.

The rest of the paper is organized as follows. In Section 2 we present a modification of the HIO algorithm that uses the additional phase information. A short description of our convex optimization method is presented in Section 3. Section 4 is devoted to quantitative results of our simulations, followed by a discussion and conclusions in Section 5.

\(^1\)Here, by convex optimization methods we refer to classical optimization algorithms such as gradient descent and Newton-type methods.
2. A ”phase aware” modification of the Hybrid Input-Output method

Let us start with the notation used throughout this paper. The sought signal and its Fourier transform are denoted by \( x \) and \( \hat{x} \) respectively. In this work \( x \) is a function of two-dimensional space coordinates denoted by \( t \), however, these are usually omitted for the sake of brevity. The Fourier transform operator is denoted by \( F \) and we use the matrix-vector product notation to represent an application of the transform:

\[
\hat{x} \equiv Fx. \tag{1}
\]

In iterative methods we use a subscript, e.g., \( x_k \), to designate the value of \( x \) at the \( k \)-th iteration.

Now, to describe the PA-HIO method let us start with its progenitor: the original HIO. The algorithm starts with an initial (say, random) estimate \( x_0 \) and performs iterative Fourier transforms back and forth between the object and Fourier domain. In the Fourier domain it applies the known magnitude and this new value is used to update the previous iterate. A schematic description of the method and its update rule are given in Figure 1 and Equation (2) respectively.

\[
x_{k+1}(t) = \begin{cases} 
  x'_k(t), & t \notin \vee \\
  x_k(t) - \gamma x'_k(t), & t \in \vee
\end{cases} \tag{2}
\]

Here \( \vee \) denotes the set of coordinates where \( x'_k \) violates the image domain constraints and \( \gamma \) is a damping factor that is set to 0.75 in all our experiments. The modification that converts the HIO method into its phase-aware counterpart PA-HIO is straightforward. At the stage
where the Fourier domain constraints are imposed on the current $\hat{x}$, we not only impose the correct FT magnitude, but also force the phase to lie within the interval $[\alpha, \beta]$ by projecting onto the nearest point. That is, if the current phase already lies within $[\alpha, \beta]$ then only the FT magnitude is imposed. However, if the phase is outside the interval $[\alpha, \beta]$ then it is replaced by either $\alpha$ or $\beta$, depending on which one is closer. Graphically, these manipulations can be represented by projecting the complex-valued entries of $\hat{x}$ onto the arc defined by the Fourier magnitude and phase uncertainty interval $[\alpha, \beta]$ as shown in Figure 2a. As is demonstrated in Section 4 this method is significantly faster than the original HIO algorithm. Of course, we attribute this improvement to the additional phase information available.

3. Convex optimization method

First, we must clarify that the problem is not convex. The term “convex optimization” refers to the fact that we perform a convex relaxation at some stage, and this turns out to be crucial for successful reconstruction. Nevertheless, the problem remains non-convex.

There have been many attempts in the past to solve the phase retrieval problem by the means of convex optimization methods. The most common formulation is as follows

$$\min \| |Fx| - r \|^2 \quad \text{s.t.} \quad x \geq 0$$

where

$$\min \| |Fx| - r \|^2 \quad \text{s.t.} \quad x \geq 0$$

Unfortunately, the phase retrieval problem turns out to be particularly difficult for convex optimization methods. Nevertheless, the modified problem in which the phase is known up to uncertainty of as much as $\pi$ radians can be solved every efficiently as was shown in [5]. Here we give a short description of the method. The main procedure is divided into two stages. At the first stage we perform a convex relaxation by allowing the complex numbers
in the Fourier domain to lie anywhere in the convex region $\mathcal{C}$ as shown in Figure 2b. Hence, we solve the convex problem:

$$\min_{x \geq 0} \|d(Fx, \mathcal{C})\|^2,$$

where $d(a, \mathcal{C})$ denotes the Euclidean distance from a point $a$ to the convex set $\mathcal{C}$. In our experience, several dozen iterations are sufficient to solve this convex problem (see Figures 4a, 5a, 6a, and 7a). In practice, it is even unnecessary to solve this problem completely, it is sufficient to perform about 20 iterations and proceed to the second stage (see Section 4 for details). At the second stage we combine the Fourier magnitude constraints with the convex problem of the first stage and solve the following minimization problem:

$$\min_{x \geq 0} \|d(Fx, \mathcal{C})\|^2 + |||Fx| - r||^2,$$

where $r$ denotes the known Fourier magnitude. Of course, the solution of the first stage is used as the starting point of the second stage. We stress that this partitioning into two stages is crucial: attempts to skip the first stage and start by trying to solve Equation (5) do not usually yield good results.

4. Simulation results

We tested the three methods on a variety of images and results were always consistent with those presented below. To demonstrate the efficacy of our methods we show the results for the three images shown in Figure 3. The images were chosen to be of distinct nature, each representing a different field of phase retrieval applications. “Lena” is a natural image, and as such it possesses many features and a great variety of tones. The second image is a fragment of a VLSI circuit. This image was provided by the KLA Tencor corporation for a joint project on numerical solutions for optical inspection and metrology methods in the VLSI industry. The third depicts a fragment of a Saturn image taken by the Hubble space telescope [9].

The general setup in our experiments is as follows: all images in simulations are of size 128 × 128 pixels and are padded by zeros to the size of 256 × 256 pixels, i.e., two-fold “oversampling” in the Fourier domain; values of the image magnitude (and intensity) lie between zero (black) and one (white); in all experiments the true phase is distributed uniformly at random inside the phase uncertainty interval; the starting point was always chosen as a random image whose values are uniformly distributed in the $[0, 1]$ interval.

For practical purposes the constrained formulation of Equations (4) and (5) was replaced by an unconstrained one by introducing a penalty function for negative values of $x$. Hence, these two problems were solved by minimizing the following functionals:

$$E_c(x) = \|d(Fx, \mathcal{C})\|^2 + ||[x]_-||^2,$$
and

\[ E(x) = \| |Fx| - r\|^2 + \|d(Fx,C)\|^2 + \| [x]_- \|^2. \]  

(7)

Here \([x]_-\) denotes the negative part of \(x\):

\[ [x]_- = \begin{cases} 0, & x \geq 0 \\ x, & x < 0 \end{cases} \]  

(8)

As a solver we used our implementation of the SESOP method [10]. However, very similar results were obtained with other optimization solvers, e.g., L-BFGS [11]. Probably the main difference is that SESOP guarantees that the number of Fourier transforms per iteration is always equal two (as in the HIO and PA-HIO methods), while L-BFGS can perform more transforms. However, in practice the difference is very small.

Besides applicability to different types of images we also show that our method works well for a wide range of phase uncertainties. In Figures 4, 5, 6, and 7 we demonstrate the results for phase uncertainty intervals of 3, 0.3, 0.03, and 0.003 radians respectively. The graphs depict the behavior of the objective functions versus the number of iteration. Specifically, the first sub-figure (a) in these figures depicts the behavior of the convex functional (6) that we minimize at the first stage. Note that 60 iterations are shown only for demonstration purposes since we use only 20 in practice. Sub-figures (b), (c), and (d) depict the progress of minimizing the non-convex functional (7) for the three test images. Note that the second stage was limited by 130 iterations and in many cases our optimization routine stops before reaching this limit because it is tuned to terminate once the objective function reaches values of order 1E-16. As is evident from the figures our convex optimization method significantly outperforms both the HIO and PA-HIO algorithms.
Fig. 4: Reconstruction results: 3 radians phase uncertainty.
Fig. 5: Reconstruction results: 0.3 radians phase uncertainty.
Fig. 6: Reconstruction results: 0.03 radians phase uncertainty.
Fig. 7: Reconstruction results: 0.003 radians phase uncertainty.
5. Conclusions

We presented two methods for phase retrieval when a phase estimate is available. One method is based on the current reconstruction algorithm HIO and another one is a convex optimization method. The latter algorithm demonstrates a significantly faster reconstruction rate over a wide range of phase uncertainties: from 3 to 0.003 radians. Moreover, the convex optimization framework allows great deal of flexibility to incorporate additional information about the sought image. For example, it can be a low-resolution version of it. Or, maybe some prior information such as image being known to be piece-wise constant. Or, sparsity of the image either directly (like in a photograph of stars) or in some basis, e.g., discrete cosine transform. This additional information can be used to speed-up the reconstruction and improve its quality in case of noisy measurements as was shown in [5]. Furthermore, it can allow to drop the phase information at all, thus giving us a convex optimization method for the phase retrieval [7,8].

References


