Minimum weight feedback vertex sets in circle $n$-gon graphs and circle trapezoid graphs

Fanica Gavril
Computer Science Dept., Technion, Haifa 32000, Israel, gavril@cs.technion.ac.il

ABSTRACT: A circle $n$-gon is the region between $n$ non-crossing chords of a circle; the sides of a circle $n$-gon are either chords or arcs of the circle. A circle $n$-gon graph is the intersection graph of a family of circle $n$-gons in a circle. The family of circle trapezoid graphs is exactly the family of circle 2-gon graphs and the family of circle graphs is exactly the family of circle 1-gon graphs. The family of circle $n$-gon graphs contains the circle polygon graphs which have an intersection representation by circle polygons, each polygon with at most $n$ chords.

We describe a polynomial time algorithm to find a minimum weight feedback vertex set, or equivalently, a maximum weight induced forest, in a circle $n$-gon graph.

KEY WORDS: minimum feedback vertex set, maximum induced forest, circle polygon graph, circle trapezoid graph, circle graph

1. Introduction

We consider only finite graphs $G(V,E)$ with no parallel edges and no self-loops, where $V$ is the set of vertices and $E$ the set of edges. Two vertices connected by an edge are called adjacent. For $U \subseteq V$, $G(U)$ is the vertex subgraph induced by $U$. A tree is an acyclic undirected connected graph. A forest is a collection of trees. A tree $t$ is a rooted tree if it is directed and has exactly one vertex $root(t)$ of in-degree zero called root. The sons of a vertex $w$ in a rooted tree $t$ are the vertices having incoming edges from $w$; the successors of $w$ are the vertices $u$ reachable from $w$ by a directed path from $w$ to $u$.

A feedback vertex set of an undirected graph $G$ is a vertex set whose deletion leaves an induced forest. The problem of finding a minimum feedback vertex set, or equivalently a maximum induced forest, of a graph $G$ is NP-complete for bipartite and planar graphs, but polynomial for interval, cocomparability, AT-free, chordal and circle graphs [1,6,12,16,18].
A graph $G$ is an intersection graph of a family $S$ of subsets of a set if there is a one-to-one correspondence between the vertices of $G$ and the subsets in $S$ such that two vertices are adjacent iff their corresponding subsets in $S$ have a non-empty intersection [17].

Interval graphs are intersection graphs of families of intervals on a line. A circle trapezoid is the region between two non-crossing chords of a circle. Circle trapezoid graphs are intersection graphs of circle trapezoids [4,19]. Circle polygon graphs [9,11] are intersection graphs of families of convex polygons inscribed in a circle. The family of circle polygon graphs contains the circular-arc graphs, the circle trapezoid graphs, the circle graphs and the chordal graphs. Circle polygon graphs have polynomial time algorithms for maximum clique and maximum independent set [7] while their recognition, minimum coloring and minimum covering by cliques problems are NP-complete [5,10,13]. These graphs are of interest in computer science, genetics and ecology [14,15].

In the present paper we introduce a slightly more general notion than circle polygons, to include the circle trapezoids. A circle $n$-gon is the region between $n$ non-crossing chords of a circle; the sides of a circle $n$-gon are either chords or arcs of the circle (for example $c$ in Fig. 1a has arcs 3-5, 8-14 and chords 3-14,5-8). A circle $n$-gon graph is the intersection graph of a family of circle $n$-gons in a circle, or equivalently is the intersection graph of their boundaries. Note that $n$ is a constant for a family of circle $n$-gon graphs. The family of circle trapezoid graphs [4] is exactly the family of circle 2-gon graphs and the family of circle graphs [2,3] is exactly the family of circle 1-gon graphs.

The family of circle $n$-gon graphs contains the circle polygon graphs which have an intersection representation by circle polygons, each polygon with at most $n$ chords. The reason for introducing this family of graphs is the following: The algorithm in the present paper is exponential in $n$ and therefore polynomial when $n$ is a constant. Circle trapezoid graphs $G(V,E)$ are also circle polygon graphs but their representation as intersection graphs of circle polygons may require polygons with $O(|V|)$ chords (for example when the neighborhood of a vertex contains an $O(|V|)$ independent set) in which case our algorithm requires exponential time in $|V|$. But, when applied directly to circle trapezoid graphs our algorithm is polynomial in $|V|$.

Gavril [7,8] introduced the family of interval-filament graphs which are defined as follows: In a Euclidean plane $PL$, consider a line $L$ defined by $y=0$, drawn from left to right. On $L$, consider a family of closed intervals $I$. For an interval $[l,r] \in I$, we define an interval-curve $c$ in $PL$ as a continuous function $c: [l,r] \rightarrow \mathbb{R}^+$ having $c(l)=c(r)=0$: an interval-curve $c$ starts and ends at the endpoints of $[l,r]$ and is delimited by them. Consider a family
of interval-curves fulfilling that $\cup c \in \mathbb{C}$ is a continuous curve: $a = \cup c$ is called an interval-filament. Two interval-filaments $a_1, a_2$ intersect if $a_1 \cap a_2 \neq \emptyset$. An interval-filament is delimited in $PL$ by its two extreme endpoints in $L$, hence, if two intervals are disjoint, their interval-filaments do not intersect. Clearly, the union of two intersecting interval-filaments is an interval-filament. To each interval in $I$ we assign a distinct vertex $v$ and denote by $i(v)$ the interval to which $v$ is assigned; let $V = \{v \mid i(v) \in I\}$. For each interval $i(v) \in I$ we consider an interval-filament $a(v)$ connecting the left and right endpoints $l_{i(v)}, r_{i(v)}$ of $i(v)$; $FI = \{a(v) \mid i(v) \in I\}$ is a family of interval-filaments and its intersection graph $G(V, E)$ is an interval-filament graph, as introduced by Gavril [7,8]. For an interval-filament $a$ we denote by $i(a)$ the interval delimiting $a$.

In the present paper we describe a polynomial time algorithm to find a maximum weight induced forest, or equivalently a minimum weight feedback vertex set, in a circle $n$-gon graph; the algorithm is a generalization of the algorithm for circle graphs in [6]. The algorithm is exponential in $n$, but is polynomial when $n$ is constant working fairly well for $n=1,2,3$. For $n=1$, the algorithm is very similar to the algorithm in [6] for circle graphs.

In Section 2 we show how to transform a representation of a circle $n$-gon graph by a family of circle $n$-gons into a representation by a family of interval-filaments with special properties. In Section 3 we analyze the structure of a maximum induced forest in a circle $n$-gon graph, using the representation by interval-filaments. In Section 4 we describe a polynomial time algorithm to find a maximum induced forest in a circle $n$-gon graph. The lemmas and algorithms in Sections 3, 4 remain true when the vertices are weighted.

2. Representation of circle $n$-gons as interval-filaments on a line

Gavril [7] proved that every circle polygon graph $G$ is an interval-filament graph. Similarly, every circle $n$-gon graph is an interval-filament graph whose representation on a line can be obtained as follow: Consider a representation by circle $n$-gons of a circle $n$-gon graph $G$ on a circle $CR$ and let $A$ be point on $CR$ (Fig. 1a). For every circle $n$-gon, which does not contain $A$ we delete its chord whose endpoints are closest to $A$; clearly, the intersection relationship does not change. Now (Fig. 1b), we open $CR$ at $A$ and straighten it into a line $L$, while stretching the original chords into arcs and the original arcs into line-segments, without changing the intersection relationship. The remaining boundary of every circle $n$-gon becomes an interval-filament on $L$, composed by a sequence of arcs and line-
segments delimited in $PL$ by its endpoints. In addition, such a family of interval-filaments, to be called a family of $n$-gon-interval-filaments, has the following property:

**Property 1.** Two arcs of interval-filaments on $L$ do not intersect if and only if they have disjoint intervals or one appears between the legs of the other, since the chords in $CR$ corresponding to two non-intersecting arcs are non-crossing (Fig. 1).

Hence, if two $n$-gon-interval-filaments intersect, then their intervals intersect. For a vertex $v$, we denote by $n(v)$ the number of chords of the original circle $n$-gon representing $v$; $n$ is the maximum $n(v)$ among all vertices $v$. Thus, the number of arcs of its corresponding $n$-gon-interval-filaments $a(v)$ is $n(v)$ or $n(v)-1$. We call legs of $a(v)$ the endpoints of $i(v)$ and the points of intersection of the arcs of $a(v)$ with $L$; we number the legs of $a(v)$ from left to right by $r_{v,1}$, $r_{v,2}$, …, $r_{v,m(v)}$. Clearly, the two legs of an arc of $a(v)$ are consecutive among the legs of $a(v)$ and $a(v)$ has at most $2n(v)+2$ legs. Throughout the paper, we assume that both a circle $n$-gon graph and its intersection model by $n$-gon-interval-filaments are given; w.l.o.g. we assume that no two $n$-gon-interval-filaments have a common leg. We denote by $EP$ the set of points on $L$ which are legs of $n$-gon-interval-filaments. The $1$-gon-interval-filaments obtained from the representations of circle graphs [6] have each two legs.

The main properties of families of $n$-gon-interval-filaments are the following:

**Lemma 1.** A family of interval-filaments is a family of $n$-gon-interval-filaments if and only if each interval-filament is a sequence of line-segments and arcs and for every two interval-filaments $a, b$ if an arc $x$ of $a$ appears between the legs of an arc $y$ of $b$, then the arcs $x, y$ do not intersect.
**Proof:** Consider a family of \( n \)-gon-interval-filaments. By the way they are obtained from circle \( n \)-gons, each \( n \)-gon-interval-filament is a sequence of line-segments and arcs on \( L \). Consider two \( n \)-gon-interval-filaments \( a, b \) such that an arc \( x \) of \( a \) appears between the legs of an arc \( y \) of \( b \). Then, by Property 1, the arcs \( a, b \) do not intersect.

Conversely, consider a family of interval-filaments, each interval-filament being a sequence of line-segments and arcs on \( L \), and fulfilling that for every two interval-filaments \( a, b \) if an arc \( x \) of \( a \) appears between the legs of an arc \( y \) of \( b \), then the arcs \( x, y \) do not intersect. If the leftmost and the rightmost legs on \( L \) are in the same interval-filament, we denote them by \( A \). We number all other legs of the interval-filaments in their order from left to right on \( L \) (Fig. 1b).

Now, on a circle \( CR \) (Fig. 1a) we denote a point \( A \) and a set of points numbered anticlockwise by the numbers of the legs on \( L \). Clearly, every line-segment on \( L \) becomes an arc between the points with the same numbers. In addition, we connect by a chord every two points whose numbers correspond on \( L \) to the two legs of an arc. Since two arcs on \( L \) do not intersect iff they have disjoint intervals or one appears between the legs of the other, it follows that two non-intersecting arcs become two non-crossing chords. Thus, the intersection relationship of the interval-filaments is retained on the circle. Now, for every arc of an interval-filament we add a chord on \( CR \) connecting the points corresponding to its leftmost and rightmost legs on \( L \). Clearly, we obtain a family of circle \( n \)-gons. □

**Property 2.** Two \( n \)-gon-interval-filaments \( a, b \) do not intersect iff \( i(a) \cap i(b) = \emptyset \), or the interval of one, say \( i(a) \), is contained between the two legs of an arc of \( b \); this follows from the definition of \( n \)-gon-interval-filaments and from Property 1.

**Lemma 2.** Two \( n \)-gon-interval-filaments intersect, iff the interval of each contains a leg of the other or one is contained in a line-segment of the other.

**Proof:** Consider two intersecting \( n \)-gon-interval-filaments \( a, b \); hence \( i(a) \cap i(b) \neq \emptyset \). If one of them, say \( a \), does not contain a leg of the other \( b \), then either \( i(a) \) is contained in a line-segment of \( b \), or \( i(a) \) is contained between the two legs of an arc of \( b \). But if \( i(a) \) is contained between the two legs of an arc of \( b \), then by Property 2, \( a \) and \( b \) cannot intersect, thus \( i(a) \) is contained in a line-segment of \( b \).

Conversely, consider two \( n \)-gon-interval-filaments \( a, b \). If the interval of one, say \( a \), is contained in a line-segment of the other \( b \), then the two \( n \)-gon-interval-filaments intersect since the endpoints of \( a \) are contained in \( b \). Assume now that none is contained in a line-segment of the other and that the interval of each contains a leg of the other. If \( a, b \) do not
intersect, then by Property 2, the interval of one, say \( i(a) \), is contained between the two legs of an arc of \( b \), contradicting the assumption that \( i(a) \) contains a leg of \( b \). □

**Corollary 3.** A graph is a circle \( n \)-gon graph if and only if it is the intersection graph of a family of \( n \)-gon-interval-filaments.

**Proof:** A family of circle \( n \)-gons on a circle \( CR \) can be transformed into a family of \( n \)-gon-interval-filaments on a line and conversely, without changing the intersection relationship, by the processes described above. Thus, a graph is a circle \( n \)-gon graph if and only if it is the intersection graph of a family of \( n \)-gon-interval-filaments. □

**Lemma 4.** Consider two \( n \)-gon-interval-filaments such that \( i(a) \subset i(b) \). Then, every \( n \)-gon-interval-filament \( c \) which intersects \( a \) and fulfills \( i(c) \nsubseteq i(b) \) must also intersect \( b \).

**Proof:** Since \( a \) intersects \( c \) and \( i(a) \subset i(b) \), it follows by Lemma 2 that \( i(a) \) and hence \( i(b) \) contains a leg of \( c \) and \( i(c) \not\subset i(b) \). Since \( i(c) \not\subset i(b) \), it follows that \( i(c) \) contains a leg of \( b \). Therefore, by Lemma 2, the \( n \)-gon-interval-filaments \( b, c \), intersect. □

There is a generalization of circle \( n \)-gon graphs defined as follows: Consider a circle \( CR \) which is the base of a cylinder. Assume that an interval-filament on the cylinder has its endpoints on \( CR \), is delimited by its endpoints and is a sequence of \( CR \)-arcs and cylinder-arcs above \( CR \), on the cylinder, with the legs on \( CR \). Consider a family of such interval-filaments fulfilling: (a) no two arcs of \( CR \) delimiting two interval-filaments cover \( CR \); (b) if a cylinder-arc \( x \) of an interval-filament \( a \) appears between the (consecutive) legs of a cylinder-arc \( y \) of an interval-filament \( b \), then these arcs do not intersect (equivalent to Property 1 for \( n \)-gon-interval-filaments). We call cylinder \( n \)-gon graph the intersection graph of such a family of interval-filaments. Clearly, the union of the arcs on \( CR \) delimiting the trees of an induced forest cannot cover \( CR \), otherwise there would be a cycle in a tree. The maximum induced forest problem for cylinder \( n \)-gon graphs \( G(V,E) \) can be reduced to the same problem for circle \( n \)-gon graphs: In every minimal \( CR \)-arc we take a point \( X \), delete from \( G \) the vertices whose corresponding delimiting arcs on \( CR \) contains \( X \), and find a maximum induced forest in the remaining graph, which by Corollary 2 is a circle \( n \)-gon graph. The maximum induced forest of \( G \) is the maximum among these induced forests. Note that we can also define interval or subtree-filaments fulfilling Property 1, on trees.

### 3. Analysis of maximum induced forests in circle \( n \)-gon graphs

Let \( G(V,E) \) be a circle \( n \)-gon graph represented as an intersection graph of \( n \)-gon-interval-filaments.
Consider a maximum induced forest $F_G$ of $G$ (Fig. 2b, without the dashed edges). Let $t_1, t_2, \ldots, t_r$ denote the trees of $F_G$ in order from left to right of the left endpoints of the intervals $i(t_j)$ delimiting the interval-filaments $\cup \{a(u)\mid u \in t_j\}$. For every tree $t_j$ of $F_G$ we take as root its vertex $v_j$ whose interval has leftmost left endpoint. The forest $F_G$ in Fig. 2b has two trees, rooted at $z$ and $u'$. The interval-filaments corresponding to two trees $t_i, t_j$ of $F_G$ do not intersect and their intervals are either disjoint or are contained one into another.

**Lemma 5.** In a rooted tree $t_j$ of $F$, if a vertex $u$ is a successor of a vertex $w$, then $i(w) \subset i(u)$, thus $i(u)$ cannot contain both endpoints of $i(w)$.

**Proof.** Consider in $t_j$ a successor $u$ of a vertex $w$ and assume that $i(w) \subset i(u)$.

Assume that $u$ is not a son of $w$. Let $p$ be the unique path from $\text{root}(t_j)$ to $w$ and let $f=\cup_{v \in p} a(v)$. The continuous curve $f$ does not intersect $a(u)$, otherwise $t_j$ would contain an edge between $u$ and a vertex in $p$. Hence, $i(w) \subset i(u)$ implies $i(f) \subset i(u)$, contradicting the fact that $i(\text{root}(t_j))$ has leftmost left endpoint in $t_j$.

Assume now that $u$ is a son of $w$. Since $a(w)$ intersects its father $z$, $i(w) \subset i(u)$ and $i(z) \subset i(u)$, it follows by Lemma 4 that $z$ must also intersect $u$, which contradicts the fact that $u$ is not a son of $z$.$\square$

We transform $F_G$ into a tree rooted at $\text{root}(t_1)$, called oriented form of $F_G$, by the following process: For every $j=2, 3, \ldots, r$, let $r_{u,k}$ be the leg of an $n$-gon-interval-filament $a(u)$, $u \in t_1 \cup t_2 \cup \ldots \cup t_{j-1}$, which is at the left and closest to the left endpoint of $i(\text{root}(t_j))$, i.e., there are no endpoints of intervals $i(v), v \in F_G$, in the interval between $r_{u,k}$ and $i(\text{root}(t_j))$. We attach $\text{root}(t_j)$ to $u$ by a dashed edge as a dummy son of $u$ at $r_{u,k}$. In Fig. 2 $u'$ is a dummy son of $u$. Since the trees of $F_G$ are ordered by the left endpoints of their roots, it follows that $F_G$ has exactly $r-1$ dummy sons. Also, for the rightmost endpoint $r_{u,m(u)} \in EP, u \in t_1$, on $L$, $u$ has attached as successors by dummy edges all the trees of $F_G$ having intervals at the right of $r_{u,m(u)}$.

For the remaining of this Section we assume that $F_G$ is in oriented (tree) form rooted at $\text{root}(t_1)$; we denote by $F_G(u)$ the subtree containing $u$ and its successors in $F_G$.

Let $z$ be the father of a vertex $w$ and assume that $w$ is a non-dummy son of $z$ (Fig. 2). By Property 2, every $n$-gon-interval-filament $a(u)$ having $i(u)$ between the two legs of an arc of $a(z)$, does not intersect $a(z)$ and $u$ is a potential successor of $w$. Conversely, the interval of every successor of $w$ is contained between the two legs of an arc of $a(z)$, or is at the left of $i(z)$ or is at the right of $i(z)$.

If $i(w)$ is contained in a line-segment of $z$, then $w$ has no successors in $F_G$ and $F_G(w)=\{w\}$ (see $w',w''$ in Fig. 2). Assume that $i(w)$ is not contained in a line-segment of $z$.\[7]
As explained above, the intervals of the successors of \( w \) appear at the left of \( r_{z,1} \) or at the right of \( r_{z,m(z)} \), or are properly contained between the two (consecutive) legs of an arc of \( z \), since they do not intersect \( z \). For every two legs \( r_{z,j}, r_{z,j+1}, 1 \leq j \leq m(z) - 1 \), of an arc of \( z \) such that \( w \) has at least a leg or a (or part of a) line-segment between \( r_{z,j} \) and \( r_{z,j+1} \), let \([x_{w,j}, y_{w,j}]\) be the interval delimiting the subforest defined by the sons of \( w \) and by their successors whose intervals are contained between the legs \( r_{z,j}, r_{z,j+1} \) of \( z \) (Fig. 2a). Also, let \([x_{w,0}, y_{w,0}]\) be the interval delimiting the subforest, if it exists, defined by the sons of \( w \) and by their successors whose intervals are at the left of the leg \( r_{z,1} \) of \( z \) and let \([x_{w,m(z)}, y_{w,m(z)}]\) be the interval delimiting the subforest, if it exists, defined by the sons of \( w \) and by their successors whose intervals are at the right of the leg \( r_{z,m(z)} \) of \( z \). By Lemma 5 \( i(z) \not\subset i(w) \), hence only one of \([x_{w,0}, y_{w,0}], [x_{w,m(z)}, y_{w,m(z)}]\) may exist. Note that an interval \([x_{w,j}, y_{w,j}]\) may contain or intersect line-segments of \( w \). Let \( j(1), \ldots, j(s) \) be the sequence of indices for which the intervals \([x_{w,j(i)}, y_{w,j(i)}]\) exist. We denote by \( IS_{z,w} \) the sequence, from left to right, of intervals \([x_{w,j(1)}, y_{w,j(1)}], \ldots, [x_{w,j(s)}, y_{w,j(s)}]\); by their definition, the intervals of \( IS_{z,w} \) are disjoint. Clearly, \( n \)-gon-interval-filaments with intervals in different \([x_{w,j(i)}, y_{w,j(i)}], [x_{w,j(j)}, y_{w,j(j)}]\) do not
intersect. We say that the subforest $F_G(w)$ is delimited by the intervals of $IS_{z,w}$. If $i(w)$ is contained in a line-segment of $z$, then $F_G(w) = \{w\}$ and $IS_{z,w} = \emptyset$.

Let us now assume that $w$ is a dummy son of $z$ at its leg $r_{z,j}$ implying that $i(w)$ is contained in the arc between the legs $r_{z,j}, r_{z,j+1}$ of $z$. Then, the subforest defined by $w$ and its successors is delimited by an interval $[r_{z,j}, y_{z,1}]$, $y_{z,1} \in (r_{z,j}, r_{z,j+1})$, $y_{z,1} \in EP$; we denote by $IS_{z,w}$ the sequence with the unique interval $[r_{z,j}, y_{z,1}]$.

In both above cases, we denote by $fil_{z,w}(F_G, IS_{z,w})$ the interval-filament formed by the union of the $n$-gon-interval-filament $a(w)$ and the intervals in $IS_{z,w}$. If $i(w)$ is contained in a line-segment of $z$ then $IS_{z,w} = \emptyset$ and $fil_{z,w}(F_G, IS_{z,w}) = a(w)$. In Fig. 2a, $fil_{z,w}(F_G, IS_{z,w}) = a(w')$ and $fil_{z,w}(F_G, IS_{z,w}) = a(w) \cup [x_{w,2}y_{w,2}] \cup [x_{w,3}y_{w,3}] \cup [x_{w,5}y_{w,5}] \cup [x_{w,6}y_{w,6}]$.

Let us consider now the sons $v$ of $w$: we define $IS_{w,v}$ and $fil_{w,v}(F_G, IS_{w,v})$ as above. By the above definitions, $i(v)$ and $i(fil_{w,v}(F_G, IS_{w,v}))$ are contained in an interval $[x,y]$ in $IS_{z,w}$; we also denote $fil_{w,v}(F_G, IS_{w,v}) = fil_{w,v}(F_G, IS_{w,v}, [x,y])$. In Fig. 2a, $fil_{w,v}(F_G, IS_{w,v}) = (a(v) \cup [x_v y_v]) \subseteq [x_{w,2}y_{w,2}]$ and $fil_{w,v}(F_G, IS_{w,v}) = (a(v) \cup [x_{u,3}y_{u,3}]) \subseteq [x_{w,2}y_{w,2}]$.

Thus, we can partition the sons of $w$ according to the interval of $IS_{z,w}$ in which their intervals are contained. For such an interval $[x,y]$, let $FI(F_G(w), [x,y])$ be the subforest of $F_G(w)$ defined by

$$FI(F_G(w), [x,y]) = \cup_v \{ F_G(v) | v \text{ son of } w \text{ and } i(v) \subseteq [x,y] \in IS_{z,w} \}. \quad (1)$$

Since the intervals in $IS_{z,w}$ are disjoint, the interval-filaments with intervals contained in different intervals of $IS_{z,w}$ do not intersect and their corresponding subforests $FI(F_G(w), [x,y])$ are disjoint. Therefore:

$$F_G(w) = \cup_{[x,y]} \{ FI(F_G(w), [x,y]) | [x,y] \in IS_{z,w} \} \cup \{w\}. \quad (2)$$

We denote

$$weight(fil_{z,w}(F_G, IS_{z,w})) = |F_G(w)| = 1 + \sum_{[x,y]} |FI(F_G(w), [x,y])| \mid [x,y] \in IS_{z,w}. \quad (3)$$

When $i(w)$ is contained in a line-segment of $z$, then it has no successors, hence $F_G(w) = \{w\}$, $IS_{z,w} = \emptyset$, $fil_{z,w}(F_G, IS_{z,w}) = a(w)$ and $weight(fil_{z,w}(F_G, IS_{z,w})) = 1$.

Note (Fig. 2a) that for two sons $u, v$ of $w$, $fil_{w,u}(F_G, IS_{w,u}, [x_{z,j}])$ and $fil_{w,v}(F_G, IS_{w,v}, [x_{z,j}])$ are non-intersecting and the intervals $i(fil_{w,u}(F_G, IS_{w,u}, [x_{z,j}]))$, $i(fil_{w,v}(F_G, IS_{w,v}, [x_{z,j}]))$ either (a) are contained one in the other, when the set of $w$ legs contained by one is a subset of the other's set (see $u, v$ in Fig. 2a), or (b) are disjoint when they contain disjoint sets of legs of $w$ or one of them is contained in a line-segment of $w$ (see $u, v'$ in Fig. 2a).

Consider a vertex $v$ such that an interval $i(u)$ is contained in $i(fil_{w,v}(F_G, IS_{w,v}, [x_{z,j}]))$, but $i(u)$ is contained between the legs of an arc of $w$ and is not a dummy son of $w$ (see $u, u'$
in Fig. 2a). Then \(u\) cannot be a son of \(w\), since by Property 2, \(a(u)\) cannot intersect \(a(w)\).

Hence, \(u\) is not a son of \(w\) and is a successor of \(v\) in \(F_G\).

Thus, for an interval \([x,y] \in IS_{z,w}\), the family

\[
\{fil_{w,v}(F_G, IS_{w,v,[x,y]}) \mid v \text{ son of } w \text{ in } F_G(w) \text{ and } i(v) \subseteq [x,y]\}
\]

is a family of mutually non-intersecting interval-filaments.

Now, we consider the vertex \(w\) independently of its father \(z\). We will consider every interval \([x,y], x,y \in EP\), which intersects \(i(w)\) (see for example intervals \([x_w,2], [x_w,3], [x_w,5], [x_w,6], [y_w,2], [y_w,3], [y_w,5], [y_w,6]\) in Fig. 2a); the number of such intervals is \(O(4n^2|V|^2)\), since \(EP\) has \(O(2n|V|)\) elements. For an interval \([x,y], x,y \in EP\), which intersects \(i(w)\), we define:

\[
VI_{w}[x,y] = \{u \mid i(u) \subseteq [x,y]\} \cup \{w\}.
\]

(5)

Note that this definition includes the situation where \(x\) is the left leg \(r_{w,j}\) of an arc of \(w\) and \(y_{w,l} \in (r_{w,j}, r_{w,j+1})\), needed for the dummy sons of \(w\). Also, the definition includes the situation where \([x,y]\) is contained in (or partly intersects) a line-segment of \(w\) (\(v'',s,s'\) in Fig. 2a).

When \([x,y]\) is contained in a line-segment of \(w\) (\([x_{w,3}, y_{w,3}]\) in Fig. 2a), the family of sons of \(w\) corresponds to a family of mutually non-adjacent vertices. Note that when \(i(w)\) is contained in a line-segment of \(z\) then \(IS_{z,w} = \emptyset\) and \(fil_{z,w}(F_G, IS_{z,w}) = a(w)\) and \(F_G(w) = \{w\}\).

**Lemma 6.** For every interval \([x,y]\) in \(IS_{z,w}\), the subgraph \(FI(F_G(w), [x,y])\) is a maximum induced forest in oriented form, rooted at \(w\), in \(G(VI_w[x,y])\).

**Proof:** If not, we can replace it by a bigger forest, to obtain a bigger forest \(F_G\). □

Consider now a potential son \(v\) of \(w\): it fulfills \(i(w) \subset i(v)\), and there exists an interval \([x,y], x,y \in EP\), which intersects \(a(w)\) fulfilling \(i(v) \subseteq [x,y]\). We now wish to delimitate by a sequence \(IS_{w,v,[x,y]}\) of intervals in \([x,y]\) the possible subforest defined by a possible son \(v\) of \(w\). Every such sequence \(IS_{w,v,[x,y]}\) is composed of disjoint intervals contained in \([x,y]\), every interval intersecting \(v\), and not intersecting \(w\), i.e., contained between the legs of an arc of \(w\), or appearing at the left of \(r_{w,1}\) or at the right of \(r_{w,m(w)}\) (see \([x_v, y_v]\) for \(v\) and \([r_{u,3}, y']\) for \(u\) in Fig. 2a). We denote the interval-filament formed by the union of the \(n\)-gon-interval-filament \(a(v)\) and the intervals in \(IS_{w,v,[x,y]}\), by \(fil_{w,v}(IS_{w,v,[x,y]})\) and assign as

\[\text{weight}(fil_{w,v}(IS_{w,v,[x,y]}))\]

the cardinality of a maximum induced forest rooted at \(v\) and delimited by the intervals \(IS_{w,v,[x,y]}\). When \(i(v)\) is contained in a line-segment of \(w\) then

\[IS_{w,v,[x,y]} = \emptyset, fil_{w,v}(IS_{w,v,[x,y]}) = a(v)\]

and

\[\text{weight}(fil_{w,v}(IS_{w,v,[x,y]})) = 1\]

Let \(HI_{w}[x,y]\) be the weighted intersection graph of the family of weighted interval-filaments.
\( \{ \text{fil} \{ v, (IS_{w,v,[x,y]} ) \} | \text{v adjacent to } w, \text{i}(w) \subsetneq \text{i}(v), \text{and } \text{i}(v) \subseteq [x,y] \} \cup \{ [r_{w,j}, y'] | r_{w,j} \text{ is } r_{w,m(w)} \text{ or is the left leg of an arc of } w, y' \in (r_{w,j}, r_{w,j+1}), y' \in EP, [r_{w,j}, y'] \subseteq [x,y] \} \)

(6)

the left subset is for the non-dummy sons of \( w \), and the right subset is for the dummy sons of \( w \). We denote the vertex representing \( \text{fil} \{ w, (IS_{w,v,[x,y]} ) \} \) in \( HI_w[x,y] \) by \( v \); we denote the vertex representing \( [r_{w,j}, y'] \) by any \( u \) having \( \text{i}(u) \subseteq (r_{w,j}, y'] \). Note that every independent set of \( HI_w[x,y] \) is also an independent set of \( G \), since \( a(v) \subseteq \text{fil} \{ v, (IS_{w,v,[x,y]} ) \} \).

Let \( FIN_v(IS_{w,v,[x,y]} ) \) be a maximum induced forest rooted at \( v \) and delimited by the intervals of \( IS_{w,v,[x,y]} \). When \( i(v) \) is contained in a line-segment of \( w \) then \( IS_{w,v,[x,y]} = \phi \) and \( FIN_v(IS_{w,v,[x,y]} ) = \{ v \} \).

**Lemma 7.** For every interval \( [x,y] \in IS_{z,w} \), the vertex sets corresponding to
\n\( \{ \text{fil} \{ v, (FG, IS_{w,v,[x,y]} ) \} | v \text{ son of } w, \text{i}(v) \subseteq [x,y] \} \) is a maximum weight independent set of \( HI_w[x,y] \). Conversely, every maximum weight independent set of \( HI_w[x,y] \), together with \( w \), defines an induced forests which can replace \( FI(FG(w),[x,y]) \) in \( FG \) to obtain a maximum induced forest.

**Proof:** Consider an interval \( [x,y] \in IS_{z,w} \). Clearly, \( \{ \text{fil} \{ v, (FG, IS_{w,v,[x,y]} ) \} | v \text{ son of } w, \text{i}(v) \subseteq [x,y] \} \) is contained in the set of filaments in formula (6) which defines \( HI_w[x,y] \). The filaments in \( \{ \text{fil} \{ v, (FG, IS_{w,v,[x,y]} ) \} | v \text{ son of } w, \text{i}(v) \subseteq [x,y] \} \) are mutually non-intersecting and the corresponding vertex set \( X \) in \( HI_w[x,y] \) is an independent set fulfilling \( |X| = |\{ v | v \text{ son of } w, \text{i}(v) \subseteq [x,y] \}| \).

Conversely, let \( X \) be a maximum weight independent set of \( HI_w[x,y] \). For every interval \( [r_{w,j}, y'] \), \( r_{w,j} \) is the left leg of an arc of \( w \), \( y' \in (r_{w,j}, r_{w,j+1}), y' \in EP, [r_{w,j}, y'] \subseteq [x,y] \), corresponding to a vertex in \( X \), let \( FI_w(r_{w,j}, y') \) be a maximum induced forest in oriented form in \( \{ u | \text{i}(u) \subsetneq (r_{w,j}, y'] \} \) and let \( d_i \) be its vertex with leftmost left endpoint; thus \( |FI(r_{w,j}, y')| = \text{weight}([r_{w,j}, y']) \). Let \( X' \) be the set of vertices in \( G \) corresponding to the vertices in \( X \), with the vertex, corresponding to \( [r_{w,j}, y'] \) replaced by \( d_i \) (to be a dummy son of \( w \)); clearly \( |X'| = |X| \) and \( X' \) is a maximum independent set in \( HI_w[x,y] \).

Since every \( \text{a(v)} \subseteq \text{fil} \{ v, (IS_{w,v,[x,y]} ) \} \), the interval-filaments in \( \{ \text{a(v)} | v \in X' \} \) are mutually non-intersecting and they all intersect \( a(w) \). Hence \( X' \) is an independent set in \( G \) with all its vertices adjacent to \( w \).

In \( FG(w) \), we replace the forest \( FI(FG(w),[x,y]) \) with the forest obtained by attaching as son of \( w \) each vertex \( v \in X' \) and a maximum induced forest rooted at \( v \) and delimited by the intervals \( IS_{w,v,[x,y]} \), whose weight is \( \text{weight} \{ \text{fil} \{ v, (IS_{w,v,[x,y]} ) \} \} \). We obtain a
new forest \( F_G \), since \( FI(F_G(w),[x,y]) - \{w\} \) has interval-filaments delimited by \([x,y]\) and disjoint from any other interval-filament of \( F_G - FI(F_G(w),[x,y]) \).

Therefore, every maximum weight independent set of \( HI_w[x,y] \), together with \( w \), defines an induced forest which can replace \( FI(F_G(w),[x,y]) \) in \( F_G \) to obtain a maximum induced forest. By the proof above and Lemma 6, the vertex set corresponding to \( \{\text{fil}_{w,v}(F_G, IS_{w,v,[x,y]}) | v \text{ son of } w, \ i(v) \subseteq [x,y] \} \) also is a maximum weight independent set of \( HI_w[x,y] \).

3. Algorithm for maximum induced forests in circle \( n \)-gon graphs

Our purpose is to describe a dynamic programming algorithm to find a maximum induced forest \( F \) in oriented form of a graph \( G(V,E) \). The algorithm works on circle \( n \)-gon graphs represented by a family of \( n \)-gon-interval-filaments on a line \( L \), as described in Section 2. The main idea of the algorithm is the following: We consider with a vertex \( w \) every interval \([x,y]\) intersecting \( a(w) \) (as possible interval between the legs of an arc of \( w \)'s father \( z \)). Also, for every vertex \( v \) adjacent to \( w \) such that \( i(v) \subseteq [x,y] \) we consider every sequence \( IS_{w,v,[x,y]} \) of intervals contained in \([x,y]\), each interval in \( IS_{w,v,[x,y]} \) intersecting \( v \) and appearing between the legs of an arc of \( w \). For each \( w, [x,y], v, IS_{w,v,[x,y]} \) we find a maximum induced forest rooted at \( v \) and delimited by \( IS_{w,v,[x,y]} \), we assign the cardinality of such a forest as \( \text{weight}(\text{fil}_{w,v}(IS_{w,v,[x,y]})) \) and we find a maximum weight independent set in the intersection graph \( HI_w[x,y] \) of the family of interval-filaments defined by formula (6).

For every \( w, v \) and \([x,y]\), the time complexity of the algorithm is bounded by the number of sequences \( IS_{w,v,[x,y]} \) which is exponential in the number \( n(w) \) of arcs of \( w \), but polynomial when \( n \) is a constant.

3.2 We describe now the algorithm in more detail. We partition the vertex set \( V \) into a family of subsets as follows: denote by \( A_0 \) the set containing the vertices with minimal intervals in \( I \), delete \( A_0 \) from \( V \), denote by \( A_1 \) the set containing the vertices with minimal intervals in the remaining set of intervals, and so on \( A_0, A_1, ..., A_l \); for every \( l \), denote \( V_l = A_0 \cup A_1 \cup ... \cup A_l \), \( EP_l = \{x \mid x \text{ leg of } v, \ v \in V_l \} \). For every \( w \in V_l \) let \( VI_{w,l}[x,y] = VI_w[x,y] \cap \lor V_l \) and let \( HI_{w,l} \) be defined on \( VI_{w,l}[x,y], VI_{w,l-1}[x,y] \), \( EP_l \).

The algorithm works by dynamic programming on the levels \( l \). For every \( l, 0 \leq l \leq k \), for every \( w \in V_l \), and for every interval \([x,y]\), \( x, y \in EP_l \) which intersects \( w \), the algorithm constructs maximum induced subforests denoted \( FIN_{w,l}[x,y] \), \( FIN_{w,l-1}[x,y] \) in oriented form rooted at \( w \) in \( G(VI_{w,l}[x,y]), G(VI_{w,l-1}[x,y]) \), respectively. The intervals \([x,y]\) are considered in three disjoint sets:
SET1: the intervals \([x,y]\) at the right of the leftmost leg \(r_{w,1}\) of \(w\), i.e., \(r_{w,1} \leq x < y\), to be considered by going on \(L\) from right to left, in order of rightmost right leg of \(n\)-gon-interval-filaments;

SET2: the intervals \([x,y]\) containing the leftmost leg \(r_{w,1}\) of \(w\) but not its rightmost leg \(r_{w,m(w)}\), i.e., \(x < r_{w,1} < y < r_{w,m(w)}\), to be considered by going on \(L\) from left to right, in order of leftmost left leg of \(n\)-gon-interval-filaments;

SET3: the intervals \([x,y]\) containing both the leftmost leg \(r_{w,1}\) and the rightmost leg \(r_{w,m(w)}\) of \(w\), i.e., \(i(w) \subseteq [x,y]\), to be obtained by combining the results for the intervals in SET1 and SET2.

A maximum induced forest of \(G\) is one of maximum cardinality among the induced forests \(FIN_{w,k}[LL,LR]\), \(LL\) and \(LR\) being the leftmost and the rightmost points in \(EP_k\). The basic sub-problems of the algorithm is to construct the maximum induced subforests \(FIN_{w,[x,y]} - \{w\}\), \(FIN_{w,[x,y]} - \{w\}\) whose sets of roots define (Lemma 7) a maximum weight independent set in \(HI_{w,[x,y]}\), \(HI_{w,[x,y]}\), respectively.

At level \(l\) we assume that for every \(v \in V_l\) and every interval \([x',y']\), \(x',y' \in EP_{l-1}\), which intersects \(v\), we have a maximum induced subforest \(FIN_{v,l-1}[x',y']\) in oriented form with \(v\) as root, including the case that \(x'\) is a leg of \(v\).

We go on \(L\) from right to left in order of rightmost right legs \(r_{w,m(w)}\) of \(w \in V_l\).

For every left leg \(r_{w,j}\) of an arc of \(w\) and every \(y', r_{w,j} < y' < r_{w,j+1}\) or for \(r_{w,m(w)} < y'\), \(y' \in EP_{l}\), we evaluate \(FIN_{w,[r_{w,j},y']}\) as follows: For every vertex \(v \in V_{l}\), \(i(v) \subset (r_{w,j},y']\), we have, by the recursion on \(l\) or by the right to left recursion, the maximum induced forest \(FIN_{v,[r_{v,j},y']}\), \(FIN_{v,[r_{v,j},y']}\), with leftmost left leg identical to the leftmost leg \(r_{v,j}\) of \(v\) and root \(v\). We take \(FIN_{w,[r_{w,j},y']}\) to be the largest among these maximum induced forests and set \(weight([r_{w,j},y'])\) to its size. When \(r_{w,j} < y' < r_{w,j+1}\), every \(v\) having \(i(v) \subset (r_{w,j},y']\) is in \(V_{l-1}\) and \(FIN_{w,[r_{v,j},y']} = FIN_{w,[r_{v,j},y']}\).

Now, for every interval \([x,y]\), at the right of \(r_{w,1}\) (in SET1 above) which intersects \(w\), we construct, as follows, the maximum induced forest \(FIN_{w,[x,y]}\), rooted at \(w\), in \(G(VI_{w,[x,y]})\). Every interval \([x',y'] \in IS_{w,[x,y]}\) is contained between the legs of one of the \(n(w)\) arcs of \(w\) and the number of intervals \([x',y']\) within each arc is \(O((n(w)|V|)^2)\), thus the number of sequences \(IS_{w,[x,y]}\) is the number of combinations of intervals \([x',y']\) within the different arcs of \(w\), which is \(O((|V|)^{2n(w)})\).

The vertices \(v \in V_l\) such that \(r_{w,m(w)} \in i(v) \subseteq [x,y]\) have right endpoints at the right of \(r_{w,m(w)}\). Consider any sequence \(IS_{w,[x,y]}\). If an interval \([x',y'] \in IS_{w,[x,y]}\) is at the left of \(r_{w,m(w)}\),
then \([x',y'] \subset i(w)\) and we already have \(FIN_{v,0}[x',y'] = FIN_{v,-1}[x',y']\), and if \([x',y'] \in IS_{w,0,v,y}\) is at the right of \(r_{w,m(w)}\), then we have \(FIN_{v,0}[x',y']\) by the right to left recursion.

The vertices \(v\) having \(i(v) \subset i(w)\) and \(i(v)\) not contained in a line-segment of \(w\), are in \(V_{l,1}\) and we already have \(FIN_{v,l}[x',y'] = FIN_{v,l-1}[x',y']\).

We assign
\[
FIN_{w,v}(IS_{w,v,[x,y]}) = \bigcup \{ FIN_{v,l}[x',y'] \mid [x',y'] \in IS_{w,v,[x,y]} \}, \tag{7}
\]
\[
weight(fil_{w,v}(IS_{w,v,[x,y]})) = |FIN_{w,v}(IS_{w,v,[x,y]})| = 1 + \sum\{ |FIN_{v,l}[x',y']| - \{v\} \mid [x',y'] \in IS_{w,v,[x,y]} \}. \tag{8}
\]

For the vertices \(v\) such that \(i(v)\) is contained in a line-segment of \(w\), we take \(IS_{w,v,[x,y]} = \emptyset\), \(fil_{w,v}(IS_{w,v,[x,y]}) = a(v)\), \(FIN_{w,v}(IS_{w,v,[x,y]}) = \{v\}\) and \(weight(fil_{w,v}(IS_{w,v,[x,y]})) = 1\). Note that when \([x,y]\) is contained in a line-segment of \(w\), then every \(FIN_{v,v}(IS_{w,v,[x,y]}) = \{v\}\) and \(HI_{w,[x,y]}\) is the intersection graph of the family of interval-filaments \(\{a(v) \mid i(v) \subseteq [x,y]\}, v \in V_{l}\) (see \([x_{w,5}, y_{w,5}\] in Fig. 2a).

To find \(FIN_{w,0}[x,y]\) it remains by Lemma 7 to find a maximum weight independent set in the weighted interval-filament graph \(HI_{w,[x,y]}\) which is the intersection graph of the family of interval-filaments
\[
\{ fil_{w,v}(IS_{w,v,[x,y]}), v \text{ adjacent to } w, i(w) \subset i(v), \text{ and } i(v) \subseteq [x,y]\} \cup \{ [r_{w,j}, y'] \mid r_{w,j} \text{ is } r_{w,m(w)} \}
\text{ or is the left leg of an arc of } w, y' \in (r_{w,j}, r_{w,j+1}), y' \in EP_{h}, [r_{w,j}, y'] \subseteq [x,y]\}. \tag{9}
\]

We can do this by the polynomial time algorithm in [7]. To improve its efficiency, we insert the dynamic programming process of [7] into the dynamic programming process of the present algorithm, as follows: A brother \(u\) of \(v\) (in a maximum induced forest rooted at \(w\)) having \(i(u) \subset i(v)\) has the interval \([x',y']\) delimiting its subforest, contained between two consecutive intervals in \(IS_{w,v,[x,y]}\) (see \(v\) in Fig. 2a). For such an interval \([x',y']\) equal to an interval between two consecutive intervals in \(IS_{w,v,[x,y]}\) and a maximum induced forest \(FIN_{w,l-1}[x',y']\), we keep only the vertices \(u\) with maximal intervals \(i(u)\), and pointers to their brothers with maximal intervals contained in \(i(u)\). For \(v\), we insert pointers to these sons \(u\) of \(w, v\) for all intervals between two consecutive intervals of \(IS_{w,v,[x,y]}\). We also assign to \(v\) a new weight \(weight1(v)\) equal to the sum of the maximum induced subforests under these brothers \(u\). To find a maximum weight independent set in \(HI_{w,[x,y]}\) we find a maximum weight, by \(weight1\), in the interval graph defined by the set of intervals
\[
\{ i(fil_{w,v}(IS_{w,v,[x,y]})), v \text{ adjacent to } w, i(w) \subset i(v), \text{ and } i(v) \subseteq [x,y]\} \cup \{ [r_{w,j}, y'] \mid r_{w,j} \text{ is } r_{w,m(w)} \}
\text{ or is the left leg of an arc of } w, y' \in (r_{w,j}, r_{w,j+1}), y' \in EP_{h}, [r_{w,j}, y'] \subseteq [x,y]\}. \tag{10}
\]
This can be done in linear time in the number of intervals.

When \(i = 0, FIN_{v,0}[x',y']\) is a collection of induced paths starting with \(v, FIN_{w,0}[x,y]\) is obtained by taking \(FIN_{v,0}[x',y']\) of maximum size and attaching \(v\) to \(w\).
When \( w \in V_{l-1} \) we evaluate \( \text{FIN}_{w,l-1}[x,y] \) by the above formulas (7)-(9) with the additional restriction that \( v \in V_{l-1} \), using the already evaluated \( \text{FIN}_{v,l-1}[x',y'] \) for every \( v \in V_{l-1} \), with \( i(v) \subseteq [x,y] \) and \( [x',y'] \subseteq [x,y] \). In addition, when \( [x,y] \) is contained in a line-segment of \( w \), every \( \text{FIN}_{w,v}(IS_{w,v}[x,y])=\{v\} \), \( \text{fil}_{w,v}(IS_{w,v}[x,y])=a(v) \) and \( \text{HI}_{w,l-1}[x,y] \) is the intersection graph of the family of interval-filaments \( \{a(v)| i(v) \subseteq [x,y], v \in V_{l-1}\} \) (see \([xw,5],yw,5\) in Fig. 2a).

Now, we go on \( L \) from left to right in order of leftmost left legs \( r_{w,1} \), \( w \in V_{l} \).

For every left leg \( r_{w,j} \) of an arc of \( w \) and every \( y', r_{w,j} < y' < r_{w,j+1}, y' \in EP \), we already have \( \text{FIN}_{w,l}[r_{w,j},y'] \) from the above right to left recursion.

Now, for every interval \( [x,y] \), containing \( r_{w,1} \) (in SET2 above) and not containing \( r_{w,m(w)} \), we evaluate the maximum induced forest \( \text{FIN}_{w,l}[x,y] \), rooted at \( w \), in \( G(VI_{w,l}[x,y]) \) as follows.

The vertices \( v \in V_{l} \) such that \( r_{w,j} \in i(v) \subseteq [x,y] \) have leftmost left leg at the left of \( r_{w,1} \). Consider any sequence \( IS_{w,v}[x,y] \). If an interval \( [x',y'] \in IS_{w,v}[x,y] \) is at the right of \( r_{w,1} \), then we already have \( \text{FIN}_{w,l}[x',y'] = \text{FIN}_{v,l-1}[x',y'] \) and if \( [x',y'] \in IS_{w,v}[x,y] \) is at the left of \( r_{w,1} \), then we have \( \text{FIN}_{v,l}[x',y'] \) by recursion from left to right. The vertices \( v \) having \( i(v) \subseteq i(w) \) are in \( V_{l-1} \) and we already have \( \text{FIN}_{v,l}[x',y'] = \text{FIN}_{v,l-1}[x',y'] \). We evaluate \( \text{FIN}_{w,v}(IS_{w,v}[x,y]) \) and \( \text{weight}(\text{fil}_{w,v}(IS_{w,v}[x,y])) \) by the above formulas (7)-(9). To find \( \text{FIN}_{w,l}[x,y] \) we find a maximum weight independent set in the weighted interval-filament graph \( \text{HI}_{w,l}[x,y] \) which is the intersection graph of the family of interval-filaments defined as in formula (9).

Finally, it remains to find \( \text{FIN}_{w,l}[x,y] \) when the interval \([x,y]\) contains both the leftmost leg \( r_{w,1} \) and the rightmost leg \( r_{w,m(w)} \) of \( w \) (in SET3 above), i.e., \( i(w) \subseteq [x,y] \). For this, we look again at the sons of \( w \) in the forest \( F_G \). Let \( v' \) be the filament with maximal interval containing \( r_{w,1} \); by Lemma 5, \( i(v') \) cannot contain the rightmost leg of \( w \). The sons of a vertex \( w \) in \( F_G \) can be partitioned into two sets, as follows:

\[
\begin{align*}
SL_{w}(F_G) &= \{ v \mid v \text{ son of } w \text{ fulfilling } i(v) \subseteq i(v') \}, \\
SR_{w}(F_G) &= \{ v \mid v \text{ son of } w \text{ fulfilling } i(v) \cap i(v') = \emptyset \}.
\end{align*}
\]

Clearly, the interval delimiting the subforest defined by the sons in \( SL_{w}(F_G) \) and the interval delimiting the subforest defined by the sons in \( SR_{w}(F_G) \) are disjoint. Let \( Y \in i(w) \) be the right endpoint of the interval delimiting the subforest defined by \( v' \). The point \( Y \) is in \( EP \) and separates between the interval delimiting the subforest defined by the sons in \( SL_{w}(F_G) \), and the forest defined by the sons in \( SR_{w}(F_G) \). Therefore, to find \( \text{FIN}_{w,l}[x,y] \), we consider every
point $Y \in [x, y] \cap EP_l$. $FIN_{w,l}[x, y]$ is the maximum forest among the induced forests $FIN_{w,l}[x, y] \cup FIN_{w,l}[Y, y]$.

A maximum induced forest of $G$ is one of maximum cardinality among the induced forests $FIN_{w,k}[LL, LR]$, $LL$ and $LR$ being the leftmost and the rightmost points in $EP_k$. In the algorithm we must consider every level $l$, with every vertex $w$ and every interval $[x, y]$, $x, y \in EP$, which intersects $w$ (their number is $O(n^2 |V|^2)$), which counts $O(n^4 |V|)$. In addition, for every set of these parameters, with every $w$ we must consider every $v$ adjacent to $w$, and every sequence of intervals $IS_{w,v,[x,y]}$. For a given $v$, every interval $[x', y'] \in IS_{w,v,[x,y]}$ is contained between the legs of one of the $n(w)$ arcs of $w$. The number of intervals $[x', y']$ between the two legs of an arc of $w$ is $O((n(w)|V|)^2)$, hence the number of sequences $IS_{w,v,[x,y]}$ is the number of combinations of intervals $[x', y']$ within the different arcs of $w$, which is $O(n |V|^{2n(w)})$. Thus, for all $v$'s we must consider $O(|V|(n |V|^{2n(w)}))$ sequences. In addition, for every such set of parameters we must find a maximum weight independent set in the interval graph defined by (10); this can be done in linear time in the number of intervals, which is also the number of sequences $O(|V|(n |V|^{2n(w)}))$, that is in time $O(|V|(n |V|^{2n(w)}))$. Therefore, the algorithm works in time $O(n^5 |V|^{5}(n |V|)^2n(w))$. For circle graphs the algorithm works in time $O(|V|^9)$, as in [6], since $n=1$. For circle trapezoid graphs the algorithm works in time $O(|V|^9)$, since $n=2$.

Assume that the vertices of $G$ are weighted. The lemmas and algorithms in Sections 3, 4 remain true if in the construction of an induced forest we maximize the sum of vertex weights instead of maximizing the number of vertices in a forest. If in the algorithm we do not include dummy sons, then we obtain a maximum weight induced tree of $G$.

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