Sampling and Noise in Compressive Sensing

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Abstract
We will examine the effect of noise in the acquisition process of compressive sampling on the reconstruction error, assuming correct identification of the active coefficient of the sparse signal. We will show that while accepted random sampling matrices may have small restricted isometry constants, they differ greatly in the reconstruction error due to noise in the acquisition process. We will focus on noises that are typical to photons sensing.

1 Introduction
In this article we will consider the effect of different types of noise in the acquisition process on the accuracy of signal reconstruction from compressed samples. We will assume practical setting and limitations, and examine the choice of different sampling matrices on the reconstruction error that is due to noise.

1.1 Compressed Sensing
In compressed sensing (compressive sampling, CS) [1] [5] we sample a signal $x$ by the sampling matrix (or operator) $\Phi$ to obtain the measurements vector $y$: $y = \Phi x + n$, were $n$ is the noise term. The sparse signal $x$ is then recovered by solving:

$$\min_x \|x\|_0 \quad s.t. \quad \|\Phi x - y\|_2^2 < \epsilon^2$$

Solving (1) is infeasible, so we would usually either use a greedy suboptimal algorithm [7], [8], or convex relaxation [2]. In many cases $x$ is dense, but has some sparse representation in some fixed base $\Psi$: $x = \Psi z$, $z$ is sparse. In this case we will solve:

$$\min_z \|z\|_0 \quad s.t. \quad \|\Phi \Psi z - y\|_2^2 < \epsilon^2$$

The sampling matrix, $\Phi$, should be such that the Restricted Isometry Property (RIP) constant of $\Phi \Psi$ were $\Psi$ is the base in which $x$ is sparse.
will be as small as possible (in [1] the term UUP is used, but the meaning is very close). Random matrices are known to have low RIP constant with a fixed dictionary with high probability [6]. Noise in the acquisition process decreases the success probability of CS reconstruction, which is the ability to recognize correctly the active coefficients. CS reconstruction in the presence of noise is treated in [4]. Even when the reconstruction is successful, there is still error due to noise in the acquisition process. This work will deal with that error assuming different noise types and practical sampling schemes.

2 Sampling matrices, noise and reconstruction error

While in theory any $M \times N$ matrix with small RIP constant can be used for sampling, in many real-life settings the entries of the sampling matrix are in $[0, 1]$, or even ${0, 1}$. A value of 1 means that all the photons in the corresponding element of $x$ are added to the corresponding sensor element (and thus to $y$). A sampling matrix with negative entries can be implemented for example by using additional "average" sample, and then subtract the result from $y$. This will be developed further in subsection 2.5, but note that the noise is added in the true acquisition stage only. We will test how does the choice of the sampling matrix affects the accuracy of CS reconstruction in the presence of noise for typical noise types, and common sampling matrices.

2.1 Method

Assumption: we will assume that the reconstruction algorithm recognized the active dictionary atoms correctly, and thus the reconstruction will now be done by minimizing a least-squares problem. Denote: $I_a$ is a list of $x$'s indices that are different from zero (active elements). Denote $\tilde{x} = x_{I_a}$, the active elements of $x$. Also, $\tilde{\Phi}$ is the $s$ columns from $\Phi$ listed in $I_a$, and we get $\tilde{\Phi}x = \tilde{\Phi}\tilde{x}$. Assuming $I_a$ is known, the best estimation of $\tilde{x}$ in a least-squares sense is $\hat{\tilde{x}} = \tilde{\Phi}^\dagger y$, were $\tilde{\Phi}^\dagger$ is the pseudo-inverse of $\tilde{\Phi}$. Note that $\tilde{\Phi} \in \mathbb{R}^{s \times N}$. Let us look at the reconstruction error: assuming (again) that $I_a$ is correct, the reconstruction error is due to noise in the acquisition process: $y = \Phi x + n$. Define the reconstruction error vector $e$: $e = \hat{x} - \tilde{x}$, then:

$e = \Phi^\dagger y - \tilde{x} \Rightarrow e = \tilde{\Phi}^\dagger (\tilde{\Phi}\tilde{x} + n) - \tilde{x}$

$\Rightarrow e = \tilde{x} + \tilde{\Phi}^\dagger n - \tilde{x} \Rightarrow e = \tilde{\Phi}^\dagger n$

We are interested in the variance (energy) of the error $e$, compared to the variance of the signal $x$. Denote the covariance matrix of the noise, $n$, by $\Sigma_n$: $\Sigma_n = E(nn^T)$. We already know that $e = \tilde{\Phi}^\dagger n$, and thus the covariance matrix of the error $e$, $\Sigma_e \equiv \text{cov}(e)$, is obtained by:

$$\Sigma_e = \text{cov}\left(\tilde{\Phi}^\dagger n\right) = \tilde{\Phi}^\dagger \Sigma_n \left(\tilde{\Phi}^\dagger\right)^T \quad (3)$$
The last result can be verified by using the linearity of the expectation operator:

\[ \Sigma_e = E\left(\tilde{\Phi}^\dagger n n^T \left(\tilde{\Phi}^\dagger\right)^T\right) = \tilde{\Phi}^\dagger E\left(n n^T\right) \left(\tilde{\Phi}^\dagger\right)^T = \tilde{\Phi}^\dagger \Sigma_n \left(\tilde{\Phi}^\dagger\right)^T \] (4)

The mean energy of \( e, E(e^T e) \) is the main interest of this work. Assuming \( E(n) = 0 \), we get that \( E(e) = 0 \) and \( E(e^T e) = Tr(\Sigma_e) \). The relation between \( \Sigma_n \), the sampling matrix \( \tilde{\Phi} \) and \( Tr(\Sigma_e) \) will be explored in the next subsection.

In subsection 2.2 we will find the relations between \( Tr(\Sigma_e), \Sigma_n \) and \( \tilde{\Phi} \), and in subsection 2.3 we will expand the results to signals that are sparse in an orthogonal basis. In subsection 2.4 we will analyze three noise types that are typical to photon sensing, and in subsection 2.5 we will define different sampling schemes and their effect on the noise in \( y, n \). Subsections 2.4 and 2.5 results will be combined in subsection 2.6 to calculate the noise distribution in \( y, n \). The dynamic range of \( y \) will be analyzed in 2.7. Subsection 2.8 will combine the results of 2.6, 2.7, and 2.2 to obtain the error energy due to three typical detector noises for various sampling schemes. We will use simulations to verify the results in section 3.

### 2.2 Error energy as a function of \( \Sigma_n \) and \( \tilde{\Phi} \)

The expected energy of the error, \( Tr(\Sigma_e) \), is a function of both \( \Sigma_n \) and \( \tilde{\Phi} \). In the simple case when \( \Sigma_n \) is a scalar matrix \( \Sigma_n = \sigma_n^2 I \), and \( \tilde{\Phi} \)’s columns are orthogonal we get:

\[ Tr(\Sigma_e) = \sigma_n^2 Tr\left(\tilde{\Phi}^T \tilde{\Phi}^{-1}\right) \] (5)

Denote \( m_i = \tilde{\Phi}_i^T \tilde{\Phi}_i \) the squared norm of \( \tilde{\Phi} \)’s \( i \)’th column, and remember that \( \tilde{\Phi} \)’s columns are orthogonal, meaning that \( \tilde{\Phi}^T \tilde{\Phi} \) is a diagonal matrix, then

\[ Tr(\Sigma_e) = \sigma_n^2 \sum_i m_i^{-1} \] (6)

\( \tilde{\Phi} \) is a \( M \times s \) matrix taken from the \( M \times N \) \( \Phi \) matrix, which means it is not orthogonal (but it is well conditioned \([1]\)). In order to analyze \( Tr(\Sigma_e) \) in the general case we will use the Singular Values Decomposition (SVD). Assume that \( \tilde{\Phi} \) is full-rank and let \( \tilde{\Phi} = USV^T \) be the reduced SVD decomposition of \( \tilde{\Phi} : U \in R^{M \times s} \) and \( V \in R^{s \times s} \) are orthogonal matrices, and \( S \in R^{s \times s} \) is the diagonal matrix of the singular values. The pseudo-inverse of \( \tilde{\Phi} \) is obtained by \( \tilde{\Phi}^\dagger = VS^{-1}U^T \). The expected energy of the error, \( Tr(\Sigma_e) \) becomes:

\[ Tr(\Sigma_e) = Tr\left(V S^{-1}U^T \Sigma_n US^{-1} V^T\right) \] (7)

Using the property \( Tr(AB) = Tr(BA) \) and the fact that \( V \) is orthogonal, we get:

\[ Tr(\Sigma_e) = Tr\left(S^{-1} U^T \Sigma_n U S^{-1}\right) \] (8)
In many cases $\Sigma_n$ is diagonal and even a scalar matrix. We can deal with diagonal $\Sigma_n$ in (8) by taking $\sigma_{\text{max}}^2 = \max(\text{diag}(\Sigma_n))$. Let $\Sigma_{\text{diff}} = \sigma_{\text{max}}^2 I - \Sigma_n$. Then $\Sigma_{\text{diff}}$ is a diagonal matrix with non-negative elements, and thus:

$$\text{Tr} \left( S^{-1} U^T \Sigma_{\text{diff}} U S^{-1} \right) \geq 0$$

$$\Rightarrow \sigma_{\text{max}}^2 \text{Tr} \left( S^{-1} U^T US^{-1} \right) - \text{Tr} \left( S^{-1} U^T \Sigma_n US^{-1} \right) \geq 0$$

$$\Rightarrow \text{Tr} \left( S^{-1} U^T \Sigma_n US^{-1} \right) \leq \sigma_{\text{max}}^2 \text{Tr} \left( S^{-1} U^T US^{-1} \right)$$

By using $U^T U = I$ we get:

$$\text{Tr} \left( \Sigma_e \right) \leq \sigma_{\text{max}}^2 \text{Tr} \left( S^{-2} \right) = \sigma_{\text{max}}^2 \sum_{i=1}^{s} \lambda_i^{-2} \left( \tilde{\Phi} \right)$$

were $\lambda_i \left( \tilde{\Phi} \right)$ is the $i$'th singular value of $\tilde{\Phi}$. From now on we assume $m_i = m$ (column norms are fixed). Define $\kappa \left( \tilde{\Phi} \right)$:

$$\kappa \left( \tilde{\Phi} \right) = s^{-1} m \sum_{i=1}^{s} \lambda_i^{-2} \left( \tilde{\Phi} \right) = s^{-1} m \text{Tr} \left( S^{-2} \right)$$

In [3] it is shown that the maximal and minimal singular values of $\tilde{\Phi}$ converge to $1 + \sqrt{s/M}, 1 - \sqrt{s/M}$ respectively for both Gaussian and $\pm 1$ column-normalized random matrices. If we take the column norms to be fixed and equal to $\sqrt{m}$, we get:

$$\lambda_{\text{max}} \left( \tilde{\Phi} \right) \rightarrow \sqrt{m} \left( 1 + \sqrt{s/M} \right)$$

$$\lambda_{\text{min}} \left( \tilde{\Phi} \right) \rightarrow \sqrt{m} \left( 1 - \sqrt{s/M} \right)$$

the results are asymptotical, and simulations were done in order to assess the value of $\kappa \left( \tilde{\Phi} \right)$ (see figure 1). According to the simulation it seems safe to assume that the values of $m^{-1/2} \lambda_i \left( \tilde{\Phi} \right), \ i = 1, \ldots, s$ are equally distributed in $\left[ 1 - \sqrt{s/M}, 1 + \sqrt{s/M} \right]$, and $\kappa \left( \tilde{\Phi} \right)$ can be easily evaluated as a function of $s$ and $M$ by $\kappa \left( \tilde{\Phi} \right) \approx \tilde{\kappa} \left( s, s/M \right)$, were:

$$\tilde{\kappa} \left( s, r \right) = s^{-1} \sum_{k=1}^{s} \left( 1 + 2 \frac{k - 1}{s - 1} \sqrt{r} - \sqrt{r} \right)^{-1}$$

This is an approximation, but according to figure 1 it is quite accurate (2% error).

When we use a binary $\{0, 1\}$ sampling matrix the results are somewhat different. The formulas in (12) no longer apply. A reasonable approximation is the following: let $A_1 = USV^T$ be the reduced SVD of $A_1$. Let $A_2 = A_1 + 1$, were $1$ is an "offset" matrix of ones. We are interested in the singular values of $A_2$. An approximation of the SVD decomposition of $A_2$ can be obtained in the following way: assume that the diagonal of
Figure 1: $\kappa \left( \tilde{\Phi} \right)$ (11) for different $s/M$ ratios. $M = 512$, a: Gaussian $\tilde{\Phi}$, b: Rademacher $\tilde{\Phi}$, c: Binary $\tilde{\Phi}$. Each point represents the average of 100 realizations of $\tilde{\Phi}$, were the upper and lower error bars represent the maximum and minimum value of $\kappa \left( \tilde{\Phi} \right)$ respectively. The x markers represent the approximation $\hat{\kappa} \left( s, s/M \right)$ (13). All the experiments are within 5% of the approximation in the Gaussian $\tilde{\Phi}$ case, 2% in the Rademacher $\tilde{\Phi}$ case, and 9% in the Binary $\tilde{\Phi}$ case.
$S_1$ is sorted such that $S_1(s, s)$ is the smallest singular value. The SVD of $A_1$ can be written as:

$$A_1 = \sum_{i=1}^{s} U_i S_1(i, i) V_i^T$$  \hspace{1cm} (14)$$

An approximation of $A_2$ based on (14) would be:

$$\hat{A}_2 = \sum_{i=1}^{s-1} U_i S_1(i, i) V_i^T + 1 1^T$$  \hspace{1cm} (15)$$

the singular values of $\hat{A}_2$ will be equal to those of $A_1$ except for one which will be much larger. Therefor we would expect that $Tr \left( S_1^{-2} \right) \approx Tr \left( \hat{S}_2^{-2} \right)$, and therefor $Tr \left( S_1^{-2} \right) \approx Tr \left( S_2^{-2} \right)$ were the difference is approximately $1/s$ of $Tr \left( S_1^{-2} \right)$. Returning to the binary $\{0, 1\}$ case, $\hat{\Phi}^B = \frac{1}{2} \hat{\Phi}^R + \frac{1}{2} 1$, which means that $Tr \left( S_2^{-2} \right) \approx 4Tr \left( S_2^{-2} \right)$ were $S_B$ and $S_R$ are the singular values diagonal matrices of $\Phi^B$ and $\Phi^R$ respectively. The four factor comes from the multiplication by $1/2$ of $\Phi^R$. Numerical simulations of $Tr \left( S_2^{-2} \right)$ are displayed in figure 1.c and for $s/M = 1/8$ in table 1. The approximation of $s^{-1}mTr \left( \hat{S}^{-2} \right)$, $\hat{\kappa}(s, s/M)$ (13) is accurate up to 9%.

### 2.3 Orthogonal sparsity bases

In most cases $x$ is not sparse in itself, but has a sparse representation in some fixed dictionary: $x = \Psi z$ were $z$ is sparse and $\Psi$’s columns form the basis. $\Psi$ may be given implicitly as a linear transformation, in which case $\Psi$ is the inverse transformation. Define $A = \Phi \Psi$, and let $\tilde{\Psi} = \Psi I_n$, $\tilde{A} = A I_n$ be $\Psi$’s and $A$’s columns (respectively) listed in $I_n$. Let $\tilde{z} = z I_n$. We get $\tilde{A} = \Phi \tilde{\Psi}$, and $y = A z = \tilde{A} \tilde{z}$.

By substituting: $x \leftarrow z$, $\tilde{x} \leftarrow \tilde{z}$, $\Phi \leftarrow A$, and $\hat{\Phi} \leftarrow \tilde{A}$ in subsection 2.2, we will get similar results to (8) and (10) regarding the error energy. There are key differences, though: first, the error is now $\tilde{z} - z$ and not $\tilde{x} - x$.

Since $\Psi$ is orthogonal, according to Parseval’s theorem the energy of $\tilde{z} - z$ and $\tilde{x} - x$ is equal. The second difference is that the distribution of the singular values of $\tilde{A}$ is different from that of $\hat{\Phi}$’s singular values. In [6] it is shown based on the concentration inequality that the isometry constants of $A = \Phi \Phi$ is equal to that of $\Phi$ when $\Psi$ is an orthogonal matrix. Equal restricted isometry means that the extremal eigenvalues of $A$ and $\Phi$ are equal. The value of $s^{-1}M Tr \left( S^{-2} \right)$ is estimated in simulations, see figure 2. The results for $s/M = 1/8$ are summarized in table 1. We can see from the graph and table that the estimation is quite accurate, including when we use an orthogonal sparsifying base.

We now have an estimate of $Tr \left( S^{-2} \right)$ for the relevant sampling matrices both with and without multiplication by an orthogonal basis. Those results will be used to estimate the expected reconstruction error in different sampling schemes in subsection 2.8.
Figure 2: Values of $\kappa(\tilde{A})$ (11) were $A = \Phi \Psi$, $\tilde{A} = A_{Ia}$ for different $s/M$ ratios and three different dictionaries: none ($I$), DCT base, and wavelets db2 base. $M = 512$, a: Gaussian $\Phi$, b: Rademacher $\Phi$, c: Binary $\Phi$. The x markers represent the approximation $\tilde{\kappa}(s, s/M)$ (13).
Table 1: Values of $\kappa(\tilde{A})$ (11) were $A = \Phi \Psi$, $\tilde{A} = A I_a$ for different $\Phi$'s (Gaussian, Rademacher, Binary), and $\Psi$’s (none-I, DCT, WT-db2). The mean, STD, minimum and maximum of 100 realizations are displayed, as well as the $\hat{\kappa}(s, s/M)$ estimation. $M = 512$, $s = 64$ ($s/M = 1/8$).

2.4 Detector noise

When we deal with photon sensing, for example by an imager, we get three types of noises: thermal noise, shot noise, and quantization noise. (We also have offset which will not be treated here). Thermal noise is a Gaussian i.i.d noise on the detector elements with zero mean: $\Sigma_{Therm} = \sigma^2 I$. Shot noise is independent between different detector elements but not identical - the variance in each detector element equals the mean: $\text{var}(y_i) = E(y_i)$. The physics behind shot-noise is that the number of photons reaching the detector element $i$ is a Poisson random variable with $\lambda$ parameter equals the average photons number on the detector, which means the the noise has zero mean and $y_i$ variance were $y_i$ is the “clean” value of the $i$th element. The covariance matrix of Shot noise when the mean signal is $y$ is $\Sigma_{Shot} = \text{diag}(y)$, and $y$’s units are # photons. For large $y$ it is possible to use Gaussian distribution as an approximation - we will not use this fact since we only deal with second order statistics here.

Quantization noise is the inaccuracy due to quantization. If we have $Q$ quantization levels to quantify the range $[0, FW]$, then the quantization step, $q$, is: $q = FW/(Q - 1)$. $FW$ stands for ”full-well”, which is the maximum amount of photons that can be received in a single detector element. The quantization noise $n_q$ obeys: $n_q \sim U(-q/2, q/2)$ and is independent. The covariance due to quantization noise is $\Sigma_{Quant} = \frac{1}{12} q I$. Equation (16) concentrates the three detector noise types covariance matrices:

$$
\begin{align*}
\Sigma_{Therm}^{\text{new}} &= \sigma^2 I \\
\Sigma_{Shot}^{\text{new}} &= \text{diag}(y) \quad \text{(were } y \text{ is the signal’s mean)} \\
\Sigma_q^{\text{new}} &= \frac{FW}{12(Q-1)} I
\end{align*}
$$

The assumption $\Sigma_{Shot} = \text{diag}(y)$ still needs to be justified: In compressive sensing $y$ is not a direct measure of some physical phenomena - it is the result of some mixed sampling technique that implements $y = \Phi x$. Shot noise is the fluctuations of the amounts of photons reaching each element of the detector, and we must understand the fluctuations in $y$ as...
a result of the sampling technique we use. A naive approach of viewing
\(n_{i}^{\text{shot}} = \Phi n_{s}^{\text{shot}}\) will lead us to think that \(\Sigma^{\text{Shot}} = \Phi \text{ cov} (n_{x}) \Phi^T\). That result
is false. The reason is that each detector element, \(y_{i} = \sum_{j=1}^{N} \Phi_{i,j} x_{j}\), is not really obtained by multiplying the elements of \(x\) by constants, to-
gether with their noise. \(y_{i}\) is obtained for example by setting the "on" time of the mirror array element \(i,j\) to \(\Phi_{i,j}\) time units. If the distribution
of \(x_{j}\) is known to be \(x_{j} \sim \text{Pois}(x_{0}^{j})\), then sampling \(x_{j}\) for \(\Phi_{i,j}\) time units
(instead of 1 time unit) will give us:

\[
\Phi_{i,j} x_{j} \sim \text{Pois} \left( \Phi_{i,j} \cdot x_{0}^{j} \right)
\]  

(17)

\(\Phi_{i,j} x_{j}\) here symbols sampling of \(x_{j}\) for \(\Phi_{i,j}\) time units, were \(\Phi_{i,j} \cdot x_{0}^{j}\)
is the multiplication of the two values. (17) is correct because when a Poisson
random variable with parameter \(\lambda\) describes the number of occurrences
of an event in a fixed period of time \(t\), the number of occurrences of
that event in \(\alpha t\) time will have a Poisson distribution with \(\alpha \lambda\) parameter.
Moreover, if \(t\) is divided into two non-overlapping intervals, the number
of occurrences is independent between the two intervals. The variance of
a Poisson random variable is its mean, and thus \(\text{var} (\Phi_{i,j} x_{j}) = \Phi_{i,j} \cdot x_{0}^{j}\),
and \(\Sigma^{\text{Shot}} = \text{diag}(y)\) as was already written in (16).

There are other noises that can be viewed as noise in \(x\) rather than \(y\). Physical noises in \(x\) will be called "clatter", and their influence on \(y\) depends on the sampling paradigm. We examine two paradigms of
implementing \(y = \Phi x\): simultaneous sampling and sequential sampling:

**Simultaneous sampling:** \(y = \Phi x\) is acquired simultaneously, i.e.
in a single frame. In this case the noisy measurements vector will be
\(y = \Phi (x + n_{x})\) were \(n_{x}\) is the noise term. Assuming \(I_{x}\) are identified, we
will have \(\hat{x} = \Phi^\dagger y = \Phi^\dagger \Phi \hat{x} + \Phi^\dagger \Phi n_{x}\) and the error in
\(\hat{x}\) will be \(\Phi^\dagger \Phi n_{x}\). On the practical level, in simultaneous sampling the
photons from \(x_{j}\) are not multiplied, but they can be divided to different
elements of the detector \(y\), meaning that \(\sum_{i=1}^{M} \Phi_{i,j} \leq 1\), \(1 \leq j \leq N\).
Practical examples of simultaneous sampling in CS include ???.

**Sequential sampling:** \(y_{i}, \ i = 1, \ldots, M\) are acquired sequentially:
\(y_{i} = \Phi x\) were \(\Phi_{i}\) is the \(i\)'th row of \(\Phi\). An example for practical imple-
mentation of \(\Phi_{i,j} x_{j}\) is the use of the mirror array [9]. There is no single
noise vector \(n_{x}\) that is multiplied by \(\Phi_{i}\) for all \(i\): \(n_{i} = \Phi n_{x}^{i}\) were \(n_{x}^{i}\)
is the \(i\)'th realization of the noise vector. This means that \(n\) is independent
noise (assuming the \(n_{x}^{i}\) are independent for different \(i\)'s). The distribu-
tion of \(n\) must be determined according to \(n_{x}\)'s source and the acquisition
 technique.

We will not deal with noises in \(x\) further (other than shot-noise, which
can be viewed as noise in \(x\)) and thus equation (16) will describe all the
noises we will deal with.
2.5 Sampling matrices

We will examine several types of sampling matrices: binary \{0, 1\} matrices, Rademacher (±1) matrices, random Gaussian matrices, and Fourier ensemble. We will use practical implementations assuming photon sensing (or more generally non-negative sensing only). \(\Phi\) is not scale invariant in the sampling phase - we assume that "1" in \(\Phi_{i,j}\) means that all the photons from \(x_j\) are added to \(y_i\). We will also add a "gain" factor, \(c \in \mathbb{R}_+\) - \(c\) represents the relation between the sampling time per element of \(x\) and \(y\).

If we want to keep the overall sampling time constant, we set \(c = M/N\). An alternative description is that the overall amount of signal energy is fixed. \(c\) will be incorporated into the sampling by setting \(\Phi \leftarrow c\Phi\).

### 2.5.1 Binary sampling matrix

A binary sampling matrix, \(\Phi^B\) is obtained by realizing:

\[
\Phi^B_{i,j} = \begin{cases} 
c, & w.p. \ p \\
0, & w.p. \ 1 - p
\end{cases}
\]

were \(p \in [0, 1]\) is the density of \(\Phi^B\). Sampling by a binary matrix can be done directly by the mirror-array optical setting. The measurement vector \(y\) is obtained by \(y = \Phi^B x + n\).

### 2.5.2 Rademacher sampling matrix

A Rademacher sampling matrix, \(\Phi^R\) is obtained by realizing:

\[
\Phi^R_{i,j} = \begin{cases} 
c, & w.p. \ 1/2 \\
-c, & w.p. \ 1/2
\end{cases}
\]

Sampling by a Rademacher matrix cannot be done directly by the mirror-array optical setting. There are two ways to implement \(\Phi\) in practice:

**\(\tilde{x}\) method:** The first way is to add another measurement, \(\tilde{x} = c1^T x\), were \(1\) is a vector of ones. \(\tilde{x} = c1^T x\) is the sum of \(x\) multiplied by \(c\). Let \(\tilde{y} = 1\tilde{x}\) be a replication of \(c\bar{x}\) to match the length of \(y\). Let \(y_0 = \Phi^B x\). Then the measurement vector \(y\) is obtained by \(y = 2y_0 - \tilde{y}\).

**Noise:** The noise in this method is the summation of the noise in acquiring \(y_0, (n_0)\), and in acquiring \(\tilde{y}\). The noise of \(y, n\), is \(n = 2n_0 + \bar{n}\). The covariance matrix of the noise is thus \(\text{cov}(n) = 4\text{cov}(n_0) + \text{cov}(\bar{n})\). Assuming the noise source is the detector (thermal noise, shot noise, quantization), \(\text{cov}(n_0)\) is diagonal. \(\bar{n}\), however, is a replication of a single measurement \(\tilde{x}\) with single noise \(n_{\tilde{x}}\): \(\bar{n} = 1n_{\tilde{x}} \Rightarrow \text{cov}(\bar{n}) = 1\text{var}(n_{\tilde{x}}) 1^T\). We get: \(\text{cov}(n) = 4\text{cov}(n_0) + \text{var}(n_{\tilde{x}}) 1 1^T\).

**Two sensors method:** The second method of implementing a ±\(c\) sampling matrix by a mirror-array is to sample the input from the mirror-array both in the "on" direction of the mirrors and in the "off" direction. Again, if \(\Phi^B\) is a \{0, c\} binary matrix, we will acquire two measurements vectors simultaneously: \(y_+ = \Phi^B x\), and \(y_- = (c - \Phi^B)x\). \(y\) is obtained by \(y = y_+ - y_-\).
Noise: Denote \( n_+ \) as the noise in acquiring \( y_+ \), and \( n_- \) as noise on \( y_- \). Then \( \text{cov}(n) = \text{cov}(n_+) + \text{cov}(n_-) \). In this case both terms are diagonal matrices.

2.5.3 Random Gaussian sampling matrix

A random Gaussian sampling matrix \( \Phi^G \) is defined by:

\[
\Phi^G_{i,j} \sim N(0, \frac{c^2}{9})
\]

(20)

A value different from 1 or \( c \) in \( \Phi^G_{i,j} \) is implemented by controlling the time in which it is "on" and "off" in each cycle, or changing the duty cycle.

\( \bar{x} \) method: Set \( \bar{x} = c \mathbf{1}^T x \) were \( \mathbf{1} \) is a vector of ones, and let \( \bar{y} = 1 \bar{x} \). Let \( y_G = \Phi^G \mathbf{x} \). Then \( y = 2y_G - \bar{y} = \Phi^G \mathbf{x} \). This is true because \( \Phi^G = 2\Phi^G - c \mathbf{1} \mathbf{1}^T \).

Noise: Let \( n_G \) be the noise in acquiring \( y_G \), and let \( n \) be the noise in acquiring \( \bar{y} \). Again, we get that \( \text{cov}(n) = 4\text{cov}(n_G) + \text{var}(n_\bar{x}) \mathbf{1} \mathbf{1}^T \).

Two sensors method: Here we will use two sensors, the first were the first is in the "on" direction and the second is in the "off" direction. We will acquire two sampling vectors simultaneously: \( y_+ = \Phi^G \mathbf{x} \), and \( y_- = (c - \Phi^G) \mathbf{x} \). \( y \) is obtained by \( y = y_+ - y_- \), and thus \( y = \Phi^G \mathbf{x} \) because \( \Phi^G - (c - \Phi^G) = 2\Phi^G - c = \Phi^G \).

Noise: As in the Rademacher \( \Phi \) case, we get: \( \text{cov}(n) = \text{cov}(n_+) + \text{cov}(n_-) \) were \( n_+ \) and \( n_- \) are the noises in acquiring \( y_+ \) and \( y_- \) respectively.

2.5.4 Random Fourier sampling matrix

TBD

2.6 Noise in \( y \)

The detector noise was defined in subsection 2.4, and the relation between the detector noise and the noise in \( y \) was defined in section 2.5. We now combine the results to obtain the covariance of the noise in \( y \). n. We will use the notations from the previous 2.5 subsection:
Binary Φ noise covariance:

\[ \Sigma_{n+}^{\text{Therm}} = \sigma^2 I, \quad \Sigma_{n-}^{\text{Therm}} = \sigma^2 I \]
\[ \Rightarrow \Sigma_n^{\text{Therm}} = 2\sigma^2 I \]

\[ \Sigma_{n+}^{\text{Shot}} = \text{diag}(y_+), \quad \Sigma_{n-}^{\text{Shot}} = \text{diag}(y_-) \]
\[ \Rightarrow \Sigma_n^{\text{Shot}} = c\bar{x}I \]

\[ \Sigma_{n+}^q = FW/(Q-1)I, \quad \Sigma_{n-}^q = FW/(Q-1)I \]
\[ \Rightarrow \Sigma_n^q = \frac{2FW}{Q(Q-1)}I \]

Rademacher Φ and Gaussian Φ two-sensors method noise covariance: The noise \( n \) distribution as a function of \( \sigma^2 \), \( x \), and \( FW \), \( Q \) is the same for the Rademacher and Gaussian Φ’s. Note, however, that \( FW \) might be different.

\[ \Sigma_{n+}^{\text{Therm}} = \sigma^2 I, \quad \Sigma_{n-}^{\text{Therm}} = \sigma^2 I \]
\[ \Rightarrow \Sigma_n^{\text{Therm}} = 2\sigma^2 I \]

\[ \Sigma_{n+}^{\text{Shot}} = \text{diag}(y_+), \quad \Sigma_{n-}^{\text{Shot}} = \text{diag}(y_-) \]
\[ \Rightarrow \Sigma_n^{\text{Shot}} = c\bar{x}I \]

\[ \Sigma_{n+}^q = FW/(Q-1)I, \quad \Sigma_{n-}^q = FW/(Q-1)I \]
\[ \Rightarrow \Sigma_n^q = \frac{2FW}{Q(Q-1)}I \]

Rademacher Φ and Gaussian Φ \( \bar{x} \) method noise covariance:

\[ \Sigma_{n+}^{\text{Therm}} = \sigma^2 I, \quad \Sigma_{n-}^{\text{Therm}} = \sigma^2 \bar{x}\bar{x}^T \]
\[ \Rightarrow \Sigma_n^{\text{Therm}} = 2\sigma^2 I + \sigma^2 \bar{x}\bar{x}^T \]

\[ \Sigma_{n+}^{\text{Shot}} = \text{diag}(y_+), \quad \Sigma_{n-}^{\text{Shot}} = \bar{x}\bar{x}^T \]
\[ \Rightarrow \Sigma_n^{\text{Shot}} = 4\text{diag}(y_0) + c\bar{x}\bar{x}^T \]

\[ \Sigma_{n+}^q = FW/(Q-1)I, \quad \Sigma_{n-}^q = FW/(Q-1)\bar{x}\bar{x}^T \]
\[ \Rightarrow \Sigma_n^q = 4FW/(Q-1)I + FW/(Q-1)\bar{x}\bar{x}^T \]

Random Fourier Φ noise covariance:

\[ TBD \] (25)

An immediate conclusion from the analysis above is that when it comes to acquisition noise, the two-sensors method is much better than the \( \bar{x} \) method. The \( \sigma^2\bar{x}\bar{x}^T \) full covariance matrix will also result in large reconstruction error. One way of dealing with this problem is to measure \( \bar{x} \) more than once and take the average of the results, but it will cost in additional measurements.

### 2.7 Dynamic range and \( y \)'s distribution

In any practical system the dynamic range of the system’s sensors must be chosen. The dynamic range, along with the finite accuracy of the A2D converter determine the quantization error of the system. It is also
important that the signal will "use" the dynamic range properly. Let us look at \( y \)'s distribution: let \( \mu_z, \Sigma_z \) be the expectation and the covariance of the underlying sparse signal \( z \). From \( y = \Phi \Psi z \) we get:

\[
\mu_y = \Phi \Psi \mu_z \\
\Sigma_y = \Phi \Psi \Sigma_z \Psi^T \Phi^T.
\]

If we assume \( \Sigma_z = \sigma_z^2 I \), and the very common case \( \Psi^{-1} = \Psi^T \) (\( \Psi \) is a tight frame \([10]\)), we get:

\[
\Sigma_y = \sigma_z^2 \Phi \Phi^T.
\]

We are interested in the diagonal of \( \Sigma_y \), which is the variance of \( y \)'s elements. If we take \( \Phi_j \Phi_j^T = \nu \) for all \( j \), meaning that row norms are all equal to \( \sqrt{\nu} \), we get that \( \text{var}(y_i) = \nu \sigma_z^2 \), \( i = 1, \ldots, M \).

As to \( \mu_y \), we would like it to be as uniform as possible. Large deviation in \( \mu_y \) will force us to use large dynamic range compared to \( y \)'s variance, resulting in larger quantization error. \( \mu_y = \Phi \mu_x \) is a function of \( \Phi \), but we can look at \( E_{\Phi}(\mu_y) \) - the expectation of \( \mu_y \) w.r.t \( \Phi \):

\[
E_{\Phi}(\mu_y) = E_{\Phi}(\Phi \mu_x) = E(\Phi) \mu_x.
\]

For random \( \Phi \), \( E(\Phi_{i,j}) = \bar{c} \) were \( \bar{c} = E(\Phi_{i,j}) \). We remember that for practical \( \Phi \), \( \Phi_{i,j} \geq 0 \), and thus \( \bar{c} > 0 \). We get:

\[
E_{\Phi}(\mu_y(i)) = \bar{c} \bar{\mu}_x, \quad i = 1, \ldots, M
\]

Now we can compare the behavior of the \( x \) and \( y \) signals. Assuming a "well behaved" \( x \), \( \mu_x(i) = \mu \), and \( \text{var}(x(j)) = \sigma_x^2 \). We get \( \mu_y(i) = \bar{c} N \mu \), and \( \text{var}(y(i)) = \nu \sigma_z^2 \). The quantization error in \( y \) can be calculated: We take the full-well of the sensor that senses \( y \) to be:

\[
FW = \mu_y + 3\text{std}(y) = \bar{c} N \mu + 3\sqrt{\nu \sigma_x^2}.
\]

For the Binary and the Rademacher sampling matrices we have:

\[
\bar{c} = c/2, \quad \nu = c^2 N/2.
\]

For the random Gaussian \( \Phi^C \), \( \bar{c} = c/2 \). As for \( \nu \):

\[
\nu = N \cdot E \left( \left( \Phi_{i,j}^C \right)^2 \right) = N \left( \text{var}(\Phi_{i,j}^C) + E(\Phi_{i,j}^C)^2 \right) \\
\Rightarrow \nu = N \left[ \frac{c^2}{36} + \frac{c^2}{4} \right] = c^2 N \cdot 10/36
\]

Now we can plug the result for \( FW \) into the quantization noise formulas in subsection 2.6.
2.8 Reconstruction error due to noise

In this section we will combine the results from the previous sections, and find the mean energy of the error due to sampling noises, \( e \): 

\[
\text{Tr} (\Sigma_e) = \text{Tr} \left( S^{-1} U^T \Sigma_m U S^{-1} \right)
\]

We will abuse the notation a little bit and write \( \kappa(s/M) = \mathbb{E} \left( \bar{\Phi}^T \kappa(s/M) \bar{\Phi} \right) \) - the mean of \( \kappa(s/M) \) which should not change much as long as \( s/M \) is fixed. We will use

\[
\text{Tr} \left( S^{-2} \right) \approx s/m \cdot \kappa(s/M) \tag{34}
\]

were \( \kappa(s/M) \) can be replaced either by \( \hat{\kappa}(s, s/M) \) (13) or by simulation results (see figures 1, 2, and table 1).

In case of a binary \( \Phi \), \( m \) is not the squared column norm of \( \Phi \) but of \( \Phi - 1 \), which is \( m = 1/4 \). Also, we should compensate for \( c \neq 1 \). Let us now calculate \( \Sigma_{\text{Therm}} \), \( \Sigma_{\text{Shot}} \), and \( \Sigma_{\text{q}} \) for the different sampling schemes:

**Binary \( \Phi \):**

\[
\Sigma_{\text{Therm}} = \bar{\Phi}^T \sigma^2 I \left( \bar{\Phi} \right)^T
\]

\[
\Sigma_{\text{Shot}} = \bar{\Phi}^T \text{diag}(y_0) \left( \bar{\Phi} \right)^T
\]

\[
\Sigma_{\text{q}} = \bar{\Phi}^T \text{FW}_{12}(Q - 1) \left( \bar{\Phi} \right)^T
\]

and thus the error energy is:

\[
\text{Tr} \left( \Sigma_{\text{Therm}} \right) = sM \kappa(s/M) \sigma^2
\]

\[
\text{Tr} \left( \Sigma_{\text{Shot}} \right) \approx 2sN \kappa(s/M) \bar{x}
\]

\[
\text{Tr} \left( \Sigma_{\text{q}} \right) = 2s \cdot \kappa(s/M) \mu_x + \sqrt{2} \sigma_x \tag{37}
\]

If we take \( c = N/M \) (constant overall sampling time), we get:

\[
\text{Tr} \left( \Sigma_{\text{Therm}} \right) = 4sN \kappa(s/M) \sigma^2
\]

\[
\text{Tr} \left( \Sigma_{\text{Shot}} \right) \approx 4sN \kappa(s/M) \bar{x}
\]

\[
\text{Tr} \left( \Sigma_{\text{q}} \right) = 2s \cdot \kappa(s/M) \mu_x + s \sqrt{2} \sigma_x \tag{38}
\]

We can see that for fixed \( c = N/M \), the error due to thermal noise linearly decreases with \( M \), and the error due to shot-noise and quantization noise are invariant to the number of measurements. \( \kappa(s/M) \) is in the range \([1, 1.5]\), with higher \( \kappa(s/M) \) \( \Phi \) is "less orthogonal".
We can start from increasing the variance of $\Phi^{\text{Rademacher}}$ and even binary matrices. The error can be decreased if we draw back of using random Gaussian matrices for sampling compared to different constants. The errors when using $\Phi^{\text{Rademacher}}$ are similar to the Binary and Rademacher cases, but with different constants. The errors’ energy is half compared to the binary $\Phi$ case (38) - it is reasonable because we have twice the number of photons and twice the number of measurements.

Rademacher $\Phi$ (two-sensors method):

$$
\Sigma_\Phi^{\text{therm}} = \tilde{\Phi}^\dagger 2\sigma^2 I (\tilde{\Phi}^\dagger)^T \\
\Sigma_\Phi^{\text{shot}} = \tilde{\Phi}^\dagger c \bar{x} I (\tilde{\Phi}^\dagger)^T \\
\Sigma_\Phi = \tilde{\Phi}^\dagger \frac{\text{FW}}{12(Q-1)} I (\tilde{\Phi}^\dagger)^T
$$

the error energy:

$$
\begin{align*}
\text{Tr} (\Sigma_\Phi^{\text{therm}}) &= 2 \frac{c}{M} \kappa (s/M) \sigma^2 \\
\text{Tr} (\Sigma_\Phi^{\text{shot}}) &= \frac{c}{M} \kappa (s/M) \bar{x} \\
\text{Tr} (\Sigma_\Phi^{\text{shot}}) &= \frac{c}{M} \kappa (s/M) \frac{\text{FW}}{12(Q-1)}
\end{align*}
$$

For a Rademacher $\Phi$ we have: $m = Mc^2$, $\nu = c^2 N/2$, $\text{FW} = c/2 \cdot N \mu_x + 3c\sqrt{N/2}\sigma_z$.

$$
\begin{align*}
\text{Tr} (\Sigma_\Phi^{\text{therm}}) &= 2 \frac{c}{M} \kappa (s/M) \sigma^2 \\
\text{Tr} (\Sigma_\Phi^{\text{shot}}) &= \frac{c}{M} \kappa (s/M) \bar{x} \\
\text{Tr} (\Sigma_\Phi^{\text{shot}}) &= \frac{c}{M} \kappa (s/M) \frac{\text{FW}}{12(Q-1)}
\end{align*}
$$

We take $c = N/M$ and get:

$$
\begin{align*}
\text{Tr} (\Sigma_\Phi^{\text{therm}}) &= 2 \frac{c}{M} \kappa (s/M) \sigma^2 \\
\text{Tr} (\Sigma_\Phi^{\text{shot}}) &= \frac{c}{M} \kappa (s/M) \bar{x} \\
\text{Tr} (\Sigma_\Phi^{\text{shot}}) &= \frac{c}{M} \kappa (s/M) \frac{\text{FW}}{12(Q-1)}
\end{align*}
$$

Again, only the error due to thermal noise decreases with $M$ for fixed $c = N/M$. The errors’ energy is half compared to the binary $\Phi$ case (38) - it is reasonable because we have twice the number of photons and twice the number of measurements.

Random Gaussian $\Phi$ (two-sensors method): We can start from equation (40), and substitute the values of $m$ and $\text{FW}$ of Random Gaussian $\Phi$: $m = Mc^2/9$, $\nu = c^2 N \cdot 10/36$, $\text{FW} = c/2 \cdot N \mu_x + 3c\sqrt{10N/36}\sigma_z$. We get:

$$
\begin{align*}
\text{Tr} (\Sigma_\Phi^{\text{therm}}) &= 18 \frac{c}{M} \kappa (s/M) \sigma^2 \\
\text{Tr} (\Sigma_\Phi^{\text{shot}}) &= \frac{c}{M} \kappa (s/M) \bar{x} \\
\text{Tr} (\Sigma_\Phi^{\text{shot}}) &= \frac{c}{M} \kappa (s/M) \frac{\text{FW}}{12(Q-1)}
\end{align*}
$$

By setting $c = N/M$ we get:

$$
\begin{align*}
\text{Tr} (\Sigma_\Phi^{\text{therm}}) &= 18 \frac{c}{M} \kappa (s/M) \sigma^2 \\
\text{Tr} (\Sigma_\Phi^{\text{shot}}) &= \frac{c}{M} \kappa (s/M) \bar{x} \\
\text{Tr} (\Sigma_\Phi^{\text{shot}}) &= \frac{c}{M} \kappa (s/M) \frac{\text{FW}}{12(Q-1)}
\end{align*}
$$

The results are similar to the Binary and Rademacher cases, but with different constants. The errors when using $\Phi^G$ are 9 times larger than when using $\Phi^B$ and 4.5 times larger compared to using $\Phi^B$. This is a serious drawback of using random Gaussian matrices for sampling compared to Rademacher and even binary matrices. The error can be decreased if we increase the variance of $\Phi^G_{i,j}$, and as we increase the variance and limit $\Phi^G_{i,j}$ to be in $[0,c]$ we get $\Phi^G \rightarrow \Phi^B$. 

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Random Gaussian $\Phi$ and Rademacher $\Phi - \bar{x}$ method:

\[
\Sigma_{\text{therm}} = \Phi^\dagger \left( 4\sigma^2 I + \sigma^2 11^T \right) \left( \Phi^\dagger \right)^T \\
\Sigma_{\text{shot}} = \Phi^\dagger \left( 4 \text{diag}(y_0) + c\bar{x} 11^T \right) \left( \Phi^\dagger \right)^T \\
\Sigma_{\delta} = \Phi^\dagger \left( 4 \frac{\epsilon}{\sqrt{Q-1}} I + \frac{\epsilon}{\sqrt{Q-1}} 11^T \right) \left( \Phi^\dagger \right)^T 
\]

(approximation is true if none of the left singular vectors of $\tilde{\Phi}$ is close to zero).

We do not know what is $\mu_y = c\bar{x}/2$, and separate the two terms, the error energy becomes:

\[
\begin{align*}
\text{Tr}(\Sigma_{\text{therm}}) &= 4 \frac{\sigma^2 \kappa(s/M)}{m} + \sigma^2 \text{Tr} \left( \Phi^\dagger 11^T \left( \Phi^\dagger \right)^T \right) \\
\text{Tr}(\Sigma_{\text{shot}}) &= 2 \frac{\sigma^2 \kappa(s/M)}{m} \bar{x} + c\bar{x} \text{Tr} \left( \Phi^\dagger 11^T \left( \Phi^\dagger \right)^T \right) \\
\text{Tr}(\Sigma_{\delta}) &= 4 \frac{\sigma^2 \kappa(s/M)}{m} \frac{\epsilon}{\sqrt{Q-1}} + \frac{\epsilon}{\sqrt{Q-1}} \text{Tr} \left( \Phi^\dagger 11^T \left( \Phi^\dagger \right)^T \right)
\end{align*}
\]

The error energy has the same structure as before plus an additional element that is due to the replication of $\bar{x}$. Define $\rho(\cdot)$:

\[
\rho \left( \Phi^\dagger \right) \equiv m \cdot s^{-1} \text{Tr} \left( \Phi^\dagger 11^T \left( \Phi^\dagger \right)^T \right) 
\]

then:

\[
s \cdot \rho \left( \Phi^\dagger \right) = m \text{Tr} \left( 1^T \left( \Phi^\dagger \right)^T \Phi^\dagger 1 \right) = m \text{Tr} \left( \Phi^\dagger 1 \right)^T \Phi^\dagger 1 = m \| \Phi^\dagger \|_2^2 
\]

Since $\Phi^\dagger$ is an $s \times M$ matrix with $s < M$, $\Phi^\dagger 1$ can be zero. $\| \Phi^\dagger \|_2^2$ can be bounded by:

\[
0 \leq \| \Phi^\dagger 1 \|_2^2 \leq \lambda_{\text{max}}^2 \left( \Phi^\dagger \right) \| 1 \|_2^2 
\]

Estimation of $\| \Phi^\dagger 1 \|_2^2$ can be done by splitting the vector $1$ into one part that is in the null-space of $\Phi^\dagger$ and the second orthogonal to it: Let $USV^T = \Phi$ be the reduced SVD decomposition of $\Phi$. Then $\Phi^\dagger = VS^{-1}U^T$. Divide $1$ into two vectors: $P^U(1)$ and $1 - P^U(1)$, were $P^U(\cdot)$ is an orthogonal projection operator onto span$(U)$. We get that $\Phi^\dagger 1 = \Phi^\dagger P^U(1)$, and:

\[
\lambda_{\text{max}}^{-2} \left( \Phi^\dagger \right) \| P^U(1) \|_2^2 \leq \| \Phi^\dagger 1 \|_2^2 \leq \lambda_{\text{min}}^{-2} \left( \Phi^\dagger \right) \| P^U(1) \|_2^2 
\]

We do not know what is $\| P^U(1) \|_2^2$ in advance, but we can estimate that it is approximately proportional to the dimension of span$(U)$. That assumption is true if none of the left singular vectors of $\Phi$ is close to $1$ (up to multiplication by a scalar), which is the case for balanced sampling matrices such as the Gaussian $\Phi$ and Rademacher $\Phi$. We get that $\| P^U(1) \|_2^2 \approx s/M \|1\|_2^2 = s$, and thus:

\[
s \lambda_{\text{max}}^{-2} \left( \Phi^\dagger \right) \leq \| \Phi^\dagger 1 \|_2^2 \leq s \lambda_{\text{min}}^{-2} \left( \Phi^\dagger \right)
\]
Substitute \( \| \hat{\Phi}^\dagger \|_2^2 \) by \( s \cdot m^{-1} \rho (\hat{\Phi}) \):

\[
m\lambda_{\text{max}} (\hat{\Phi}) \leq \rho (\hat{\Phi}) \leq m\lambda_{\text{min}} (\hat{\Phi})
\]

for large \( M \), and thus \( \rho (\hat{\Phi}) \in \left[ \left( 1 - \sqrt{s/M - \epsilon} \right)^2, \left( 1 + \sqrt{s/M + \epsilon} \right)^2 \right] \)

for some small \( \epsilon > 0 \). Simulation results for the value of \( \rho (\hat{\Phi}) \) for Gaussian and Rademacher \( \Phi \)'s are displayed in figure 3 a, b, as well as \( \rho (A) \) for \( A = \Phi \Psi \) were \( \Psi \) is three different dictionaries (3 c, d). The numerical results for \( M = 512, s = 64 \) are displayed in table 2. Returning to (46), the second term for each of the errors covariance matrices might be in the order of the first term, and it is due to a single measurement of \( \bar{x} \). It might be a good idea to pay more attention for that particular sample in order to decrease the noise. This can be done, for example, by repeating the measurement several times. We will assume that the noise for measuring \( \bar{x} \) is reduced by \( \alpha \in [0, 1] \), and by incorporating \( \alpha \) and \( \rho (\hat{\Phi}) \) into (46) we get:

\[
\begin{align*}
\text{Tr} (\Sigma_{\text{therm}}) &= 4 \frac{s}{m} \kappa (s/M) \sigma^2 + \alpha \frac{s}{m} \rho (s/M) \sigma^2 \\
\text{Tr} (\Sigma_{\text{shot}}) &= 2 \frac{s}{m} \kappa (s/M) \bar{x} + \alpha \frac{s}{m} \rho (s/M) \bar{x} \\
\text{Tr} (\Sigma_{\text{q}}) &= 4 \frac{s}{m} \kappa (s/M) \frac{F_2}{12 (Q - 1)} + \alpha \frac{s}{m} \rho (s/M) \frac{F_2}{12 (Q - 1)}
\end{align*}
\]

As both \( \kappa (s/M) \) and \( \rho (s/M) \) are close to 1, the second term is 2-4 times smaller than the first on average. The acquisition of \( \bar{x} \) should be done such that \( \alpha \) is as small as possible. Note that when \( \alpha \to 0 \) the error energy is double than when using the two-sensors method.

### 2.8.1 Simple sampling of \( x \)

**Sparsity in the signal domain:** We can compare the errors above with errors due to noise in directly sampling a sparse \( x \). \( x \) is reconstructed

<table>
<thead>
<tr>
<th>( \Phi ) basis</th>
<th>( \Phi^G )</th>
<th>( \Phi^G )</th>
<th>( \Phi^G )</th>
<th>( \Phi^R )</th>
<th>( \Phi^R )</th>
<th>( \Phi^R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>1.19</td>
<td>1.13</td>
<td>1.11</td>
<td>1.15</td>
<td>1.16</td>
<td>1.14</td>
</tr>
<tr>
<td>std</td>
<td>0.2</td>
<td>0.22</td>
<td>0.21</td>
<td>0.19</td>
<td>0.2</td>
<td>0.22</td>
</tr>
<tr>
<td>min</td>
<td>0.707</td>
<td>0.755</td>
<td>0.694</td>
<td>0.694</td>
<td>0.639</td>
<td>0.671</td>
</tr>
<tr>
<td>( \hat{\kappa} (s, s/M) )</td>
<td>1.15</td>
<td>1.15</td>
<td>1.15</td>
<td>1.15</td>
<td>1.15</td>
<td>1.15</td>
</tr>
</tbody>
</table>

Table 2: Values of \( \rho (\hat{\Phi}) \) (47) were \( A = \Phi \Psi, \hat{A} = A_L \) for different \( \Phi \)'s (Gaussian, Rademacher, Binary), and \( \Psi \)'s (none-I, DCT, WT-db2). The mean, STD, minimum and maximum of 100 realizations are displayed, as well as the \( \hat{\kappa} (s, s/M) \) estimation. \( M = 512, s = 64 \) (\( s/M = 1/8 \)).
Figure 3: Values of $\rho(s/M)$ with and without a dictionary for different $s/M$ ratios, $M = 512$. 

- **a, b:** values of $\rho(\tilde{\Phi})$ (47) for Gaussian (a) and Rademacher (b) $\Phi$’s. The line indicates the mean of 100 simulations, the small squares are one STD from the mean and the bars indicate the maximum and minimum values. 

- **c, d:** mean values of $\rho(A)$ were $A = \Phi \Psi$, $A = A_{I_a}$ for three different dictionaries: none ($I$), DCT base, and wavelets db2 base. The x markers represent the approximation $\hat{\kappa}(s, s/M)$ (13).
by choosing the $s$ largest elements. The reconstruction error energy will be:

\[
\begin{align*}
\text{Tr} (\Sigma^\text{Therm}_e) &= s\sigma^2 \\
\text{Tr} (\Sigma^\text{Shot}_e) &= \bar{x} \\
\text{Tr} (\Sigma^\text{Q}_e) &= \frac{s\mu_x + 3\sigma_x}{12(Q-1)}
\end{align*}
\]

This error should be compared with the $c = N/M$ case. The large error due to thermal noise and shot-noise is due to the fact that when $x$ is $s$-sparse we have only $s$ measurements - the rest of the $N-s$ measurements measure irrelevant entries of $x$. Under the CS scheme we have $M$ relevant measurements.

**Sparsity in a dictionary:** When $x$ is known to be $s$-sparse in some basis other than $I$, especially a basis with non localized atoms such as DCT or Fourier, all of the $N$ measurements ideally contribute to estimating $\tilde{z}$ were $x = \Psi \tilde{z}$. Let us check this: We identify $\tilde{\Psi}$, as $\Psi$’s columns corresponding to the active elements of $\tilde{z}$, and obtain the optimal $\tilde{\tilde{z}}$ by:

\[
\tilde{\tilde{z}} = \tilde{\Psi}^\dagger \tilde{x}
\]

We can repeat the calculations from subsection 2.5 with $\Phi = I$ and $M = N$. Since $\tilde{\Psi}$ is deterministic, orthogonal and with column norms equal to 1, we have: $\kappa(\tilde{\Psi}) = 1$ and $m = 1$. The sampling noise covariance is:

\[
\begin{align*}
\Sigma^\text{Therm}_n &= \sigma^2 I \\
\Sigma^\text{Shot}_n &= \text{diag}(x_0) \\
\Sigma^\text{Q}_n &= FW/((Q-1)I)
\end{align*}
\]

Since $\tilde{\Psi}^T = \Psi^T$, the reconstruction error covariance matrix will be:

\[
\Sigma_e = \tilde{\Psi}^T \Sigma_n \tilde{\Psi}
\]

When $\Sigma_n$ is a scalar matrix $\sigma I$, $\Sigma_e = \sigma^2 \tilde{\Psi}^T I \tilde{\Psi} = \sigma^2 \tilde{\Psi}^T \tilde{\Psi} = \sigma^2 I$. The error due to shot noise is dependent on $x_0$, so we can only estimate its average and use simulation results. In case of a uniform (fixed) $x$, $x_i = \frac{1}{N} \bar{x}$, we will have:

\[
\Sigma^\text{Shot}_n = \tilde{\Psi}^T \frac{1}{N} \bar{x} I \tilde{\Psi} = \frac{\bar{x}}{N} I
\]

And the error energy:

\[
\begin{align*}
\text{Tr} (\Sigma^\text{Therm}_e) &= s\sigma^2 \\
\text{Tr} (\Sigma^\text{Shot}_e) &= \frac{\bar{x}}{N} \\
\text{Tr} (\Sigma^\text{Q}_e) &= \frac{s\mu_x + 3\sigma_x}{12(Q-1)}
\end{align*}
\]

We can see that while the thermal noise error can be reduced by using mixed or compressed sampling, the shot-noise error remains the same or slightly higher due to the imperfect orthogonality of $\Phi$. If $x$ is sparse in the signal domain the shot noise error will decrease when using a sampling matrix.
3 Simulations

The results from section 2 were verified by numerical simulations. We fixed $N$, $M$ and $s$, and then realized 100 instances of sparse $z$ such that $x = \Psi z$ is non-negative. The sampling schemes described in subsection 2.5 including noise was simulated, and the reconstruction error of the sparse signal was averaged over the 100 realizations for the different types of errors. That process was repeated 10 times with different random sampling matrices for each sampling scheme, and the overall errors were averaged. The entire process was repeated for different values of $s$. The results are displayed in figure 4. The values of $N$ and $M$ are: $N = 1024$, $M = 256$. The theoretical values of the errors are displayed by the 'x' markers, and match the simulation results very well. We used other values of $M$ and still got very good match. The $\bar{x}$ method sampling schemes were not simulated. We used only DCT as the sparsity base. Note that the error due to shot-noise varies greatly when we directly sample $x$ as a function of the sparsity base. When $x$ is sparse in the signal domain the error will be $\bar{x}$ (54), compared to $\frac{s}{N}\bar{x}$ (59) when $x$ is sparse in a dictionary which is incoherent with $I$. DCT base is incoherent with $I$, and each sample of $x$ will contribute information, and thus we get the $s/N$ factor.
Figure 4: Error energy of CS reconstruction with three different sampling matrices and no CS sampling. The signal is sparse in the DCT basis. The lines are the means of the simulations results, and the x markers are the theoretical values from (37), (41), (43) and (59). a: error due to thermal noise. In the direct sampling (or "No CS") case the error is very high as expected from (54), and it is thus divided by 100 to fit the graph. b: error due to shot-noise.
4 Discussion

We have seen that different sampling schemes will affect the reconstruction error due to acquisition noise in different ways. The optimal choice among the schemes we analyzed is the Rademacher sampling scheme implemented using the two-sensors method. We saw that while errors due to thermal noise and quantization noise can be greatly reduced by a mixed sampling process, the error due to shot-noise can only increase compared to direct sampling of a sparse (in some dictionary) signal. One conclusion is that mixed and compressed sampling are useful when thermal (or background) noise is dominant, and not useful for noise reduction when shot-noise is the main noise source. Compressed sensing may be used when there is some difficulty in building a sensing system with high enough resolution - for example when the detector elements are expensive. In such case we saw that the increase in error due to shot noise is moderate when we use the Rademacher sampling scheme.

References


