The Swap and Expansion Moves Revisited and Fused

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Abstract. Many solutions to computer vision and image processing problems involve the minimization of multi-label energy functions with up to \( K \) variables in each term. In the minimization process, the swap and the expansion are two types of commonly used moves. This paper re-derives the optimal swap and expansion moves for \( K = 2 \) in a short manner by using the original solution to the pseudo-Boolean quadratic function minimization problem as a “black box”, and reveals that the found minima w.r.t. expansions are actually also minima w.r.t. swaps. This is repeated for \( K = 3 \).

The minima-related result is extended to all objective functions under the condition that they are reduced into submodular ones, which makes it applicable to all expansion move algorithms. These may explain the prevalent impression that expansion algorithms are more effective than swap ones.

To have a larger search space, the exwap – a generalization of the expansion and the swap move types – is introduced. Efficient algorithms for minimizing w.r.t. it for \( K = 2 \) (including a ‘truncation’ procedure), \( K = 3 \) and the \( P^n \) Potts model are provided. Its capabilities to reach lower energies than those reached by the expansion algorithm are demonstrated for image denoising and stereo matching benchmark problems.

1 Introduction

Many of the proposed solutions to computer vision and image processing problems involve the minimization of multi-label energy functions - functions \( E(x) \) where the variables \( x = (x_1, \ldots, x_n) \) may assume labels from the label set \( L = \{l_0, l_1, \ldots, l_{L-1}\} \). These functions are commonly formed as a sum of terms:

\[
E(x) = \sum_{i=1}^{n} E_i(x_i) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i, x_j) + \sum_{1 \leq i < j < k \leq n} E_{ijk}(x_i, x_j, x_k) + \ldots + \sum_{1 \leq i_1 < i_2 < \ldots < i_K \leq n} E_{i_1i_2\ldots i_K}(x_{i_1}, x_{i_2}, \ldots, x_{i_K}). \quad (1)
\]

The energy function is often viewed as the negative log likelihood of a Gibbs distribution (up to an addition and a multiplication by constants), which interprets the minimizer of (1) as the maximum a posteriori estimate. Under this view, each term in the function is viewed as a potential function of variables that constitute a clique in the neighborhood system of the corresponding Markov random field (or conditional random field [15]) [6]. Under the restricted case where \( K = 2 \) and where the labels have a linear ordering and the bivariate terms are convex functions of the difference of the labels’ ordinal numbers, the energy function (1) can minimized fast and exactly [9]. Generally, however, minimizing (1) is NP-hard already for \( K = L = 2 \). This is easily seen from the fact that MAX-2-SAT is NP-hard (see, for example, [2]) and has an immediate reduction to the former problem.

In the context of multi-label energy function minimization, the swap and the expansion are two types of commonly used moves from one labeling to the next, where the minimization is carried out by a sequence of optimal moves of these types. Given a labeling \( x \in \mathcal{L}^n \) and
a pair of labels $\alpha$ and $\beta$, an $\alpha$-$\beta$-swap is any move (that is, changes applied on the labeling) that consists of only alterations of variables between $\alpha$ and $\beta$. Given a labeling and a label $\alpha$, an $\alpha$-expansion is any move that consists of only alterations of variables to $\alpha$. By seeking a minimum $s$-$t$-cut of a specially constructed weighted graph, an $\alpha$-$\beta$-swap that yields the biggest decrease of the function (1) can be found efficiently, and similarly for the expansion move type. There are different graph constructions, all suitable for different conditions related to the energy terms and some suitable only for a maximal $K$. By repeatedly performing such optimal moves of one type, each time for a different label (for expansion moves) or label pair (for swap moves) in a fixed or random order and until no decrease of the function is possible for any move of the corresponding type, the function is minimized with respect to this type of move.

The swap and expansion move types were introduced in [3, 4], and they became the two most popular graph-cut based minimization algorithms [18] and yielded good results for many problems (e.g., [10, 12, 18]). The first schemes for finding the optimal $\alpha$-$\beta$-swap and $\alpha$-expansion moves were proposed in [3, 4] for $K = 2$ provided that the function’s bi-variates terms satisfy certain conditions. In [13]–[14], the conditions related to the bivariate terms were relaxed. These papers also provided a graph-cut based algorithm for solving under certain conditions the pseudo-Boolean energy function minimization problem for $K = 3$, which was used for finding the optimal $\alpha$-expansion move for $K = 3$. Proofs that there are polynomial-time algorithms (not necessarily based on graph-cuts) for finding the optimal $\alpha$-$\beta$-swap move and the optimal $\alpha$-expansion move for certain classes of energy functions of arbitrary $K$ were provided in [10]. This work also developed graph constructions for these tasks for a certain family of functions of arbitrary $K$ ($\mathbb{P}^n$ Potts model). Graph constructions for finding the optimal $\alpha$-$\beta$-swap and $\alpha$-expansion moves for another family of functions of arbitrary $K$ (Robust $\mathbb{P}^n$ model) were proposed in [11, 12].

This paper re-derives the optimal swap and expansion moves for $K = 2$ in a short manner by using the original solution to the pseudo-Boolean quadratic function minimization problem [16] as a “black box”, and reveals that the found minima with respect to expansion moves are actually also minima with respect to swap moves. This may explain the observation that better experimental results were obtained for expansion moves than swap moves in previous works (see, e.g., [18]). It also provides some justification for the fact that the expansion move algorithm is preferred by various image stitching applications [18]. The above is repeated for $K = 3$ by using the solution to the pseudo-Boolean energy function minimization problem for $K = 3$ [14].

The above minima-related result is then extended to all objective functions under the condition that they are reduced into submodular ones. Since all expansion algorithms work under this condition (as the swap algorithms are), this result applies to all of them, including those for arbitrary $K$. This suggests that minimization with respect to expansion moves should be more effective than that with respect to swap moves.

In order to have in the minimization procedure a search space that is larger than that of the swap and the expansion move types combined, the exwap move type is introduced. The exwap move type is a generalization of the expansion and the swap move types that allows all the variables to change in one move. Efficient algorithms for minimizing with respect to this move type for $K = 2$, $K = 3$ and the $\mathbb{P}^n$ Potts model, all applicable under the same conditions corresponding to the expansion move algorithms, are provided. In order to deal

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1 Although [14] already contains a derivation for the expansion move and the derivation for the swap move is very similar given [14]'s solution for $L = 2$, we discuss these for the sake of completeness. The minima-related result for $K = 3$ is novel.
with general functions of $K = 2$, a ‘truncation’ procedure for this move type is provided as well. The proposed move type is demonstrated to be capable of reaching lower energy values than those reached by the expansion move type in stereo matching and in image denoising and inpainting benchmark problems.

The paper proceeds as follows: Sec. 2 provides the short derivations of the optimal swap and expansion moves for $K = 2$ and proves the minima-related relation in this case. These are repeated for $K = 3$ in Sec. 3. The generalization of the minima-related relation to any $K$ is provided in Sec. 4. Sec. 5 introduces the exwap move type and derives the algorithms for finding the optimal moves of this type for the different cases of energy functions. A truncation procedure for this move type is provided as well in this section. Experiments are provided in Sec. 6, and Sec. 7 summarizes the paper.

## 2 $K = 2$

### 2.1 The Original Solution to the Binary Case ($L = 2$)

Given $n$ binary variables, $b = (b_1, b_2, \ldots, b_n)$, $b_i \in \{0, 1\}$, Picard et al. have provided in [16] an efficient solution to the pseudo-Boolean quadratic function minimization problem

$$
\arg \min_b \sum_{i=1}^{n} \lambda_i b_i + \sum_{1 \leq i < j \leq n} \beta_{ij} b_i b_j, \quad \beta_{ij} \leq 0.
$$

The solution is obtained by solving for a minimum $s$-$t$-cut on a related graph. The condition $\beta_{ij} \leq 0$ is required for the weights of the edges in the graph to be nonnegative, which allows for an efficient minimum $s$-$t$-cut computation. A similar solution to (2) was provided in [5].

The solution to (2) may be easily utilized for minimizing the energy function (1) for $K = L = 2$ by assigning $l_0 = 0$ and $l_1 = 1$ and expressing each energy term as a function of its 0-1 variables:

$$
\arg \min_x \sum_{i=1}^{n} E_i(x_i) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i, x_j)
= \arg \min_x \sum_{i=1}^{n} a_i x_i + \sum_{1 \leq i < j \leq n} \{e_{ij} x_i + d_{ij} x_j + e_{ij} x_i x_j\},
$$

where

$$
a_i = E_i(l_1) - E_i(l_0),
$$

$$
e_{ij} = E_{ij}(l_1, l_1) - E_{ij}(l_0, l_0),
$$

$$
d_{ij} = E_{ij}(l_0, l_1) - E_{ij}(l_0, l_0),
$$

(4)

(Note that the additive constants in all terms were eliminated as they are irrelevant in the optimization.) Opening the parentheses and grouping common terms results in an equivalent optimization problem in the form of (2) with the coefficients

$$
\lambda_i = a_i + \sum_{j=1}^{i-1} e_{ji} + \sum_{j=i+1}^{n} d_{ij},
$$

$$
\beta_{ij} = e_{ij}.
$$

(5)

Since (2) is restricted by $\beta_{ij} \leq 0$, we obtain that the energy function can be minimized using Picard et al.’s solution to (2) as long as $e_{ij} \leq 0$, that is, as long as

$$
E_{ij}(l_1, l_1) + E_{ij}(l_0, l_0) \leq E_{ij}(l_1, l_0) + E_{ij}(l_0, l_1).
$$

(6)
This is the regularity condition obtained in [14]. Greig et al.’s algorithm in [8] used Picard et al.’s solution to (2) and was probably the first in computer vision that employed graph cuts.

2.2 The Optimal Swap Move for $K = 2$

Denote the current labeling (for which the optimal $\alpha$-$\beta$-swap is sought) by $\{x^0_i\}_{i=1}^n$ and the index-set of all variables currently labeled $\alpha$ or $\beta$ by $I = \{i : x^0_i \in \{\alpha, \beta\}\}$. The problem of finding the optimal $\alpha$-$\beta$-swap is

$$
\arg \min_{x_i \in \{\alpha, \beta\}} \sum_{i \in I} E_i(x_i) + \sum_{i \notin I} E_i(x^0_i) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i, x^0_j) + \sum_{1 \leq i < j \leq n} E_{ij}(x^0_i, x_j).
$$

Eliminating the second and last sums, which are constants, and grouping common terms results in the equivalent minimization problem

$$
\arg \min_{x_i \in \{\alpha, \beta\}} \sum_{i \in I} E'_i(x_i) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i, x_j),
$$

where

$$
E'_i(x_i) = E_i(x_i) + \sum_{1 \leq j < i \atop j \notin I} E_{ji}(x^0_j, x_i) + \sum_{i < j \leq n \atop j \notin I} E_{ij}(x_i, x^0_j).
$$

The objective function is of the same structure as the one of the energy function (1) for $L = 2$ (associate $\alpha$ with $l_0$ and $\beta$ with $l_1$, or vice versa). Therefore, it may be minimized using Picard et al.’s solution in the manner described in Sec. 2.1, provided that each $E_{ij}$ term fulfills (6) for labels $\alpha$ and $\beta$. Since the overall minimization procedure iterates over all label pairs, this condition should hold for all of them, that is,

$$
\forall \alpha, \beta \in \mathcal{L} \quad E_{ij}(\alpha, \alpha) + E_{ij}(\beta, \beta) \leq E_{ij}(\alpha, \beta) + E_{ij}(\beta, \alpha).
$$

2.3 The Optimal Expansion Move for $K = 2$

As before, denote the current (pre-move) labeling by $\{x^0_i\}_{i=1}^n$. The problem of finding the optimal $\alpha$-expansion is

$$
\arg \min_{x_i \in \{\alpha, x^0_i\}} E(\{x_i\}_{i=1}^n) = \arg \min_{x_i \in \{\alpha, x^0_i\}} \sum_{i=1}^n E_i(x_i) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i, x_j).
$$
As in the previous case, the objective function has the same structure as the one of the energy function (1) for $L = 2$ (for each variable $x_i$, associate $\alpha$ with $l_0$ and $x_i^0$ with $l_1$, or vice versa). Therefore, it may be minimized using Picard et al.’s solution (Sec. 2.1) as well, provided that each $E_{ij}$ term fulfills (6) for the labels feasible by an $\alpha$-expansion from the current labeling. For each $x_i$, these feasible labels are $\{\alpha, x_i^0\}$. Either associating $\alpha$ with $l_0$ for all variables or associating $\alpha$ with $l_1$ for all variables results in the following condition:

$$E_{ij}(\alpha, \alpha) + E_{ij}(x_i^0, x_j^0) \leq E_{ij}(\alpha, x_j^0) + E_{ij}(x_i^0, \alpha). \quad (12)$$

Generally $x_i^0$ and $x_j^0$ might equal any labels $\beta$ and $\gamma$, and the overall minimization procedure iterates over all labels $\alpha$. Therefore, the above condition should be fulfilled for all label triplets, that is,

$$\forall \alpha, \beta, \gamma \in \mathcal{L} \quad E_{ij}(\alpha, \alpha) + E_{ij}(\beta, \gamma) \leq E_{ij}(\alpha, \gamma) + E_{ij}(\beta, \alpha). \quad (13)$$

Substituting $\beta$ for $\gamma$ in (13) yields (10), which shows that the swap move-related condition is a relaxation of the expansion move-related condition.

### 2.4 Minima w.r.t. Expansions are Minima w.r.t. Swaps when $K = 2$

Following we prove that under the swap move-related condition (10) (which, as was explained, is more general than the expansion move-related one), a minimum of the energy function (1) with respect to expansion moves is also a minimum with respect to swap moves when $K = 2$.

**Proof.** Assume the labeling $\{x_i^0\}_{i=1}^n$ is a minimum with respect to all expansion moves. We will show that under condition (10) any $\alpha$-$\beta$-swap move from this labeling will not decrease the energy function (1).

Denote the labeling after the $\alpha$-$\beta$-swap move by $\{x_i^+\}_{i=1}^n$, the alteration of variable $x_i$ from label $l^0$ to label $l^+$ as part of an $\alpha$-$\beta$-swap move by $x_i^l = l^0 \rightarrow l^+$, and denote the set of all other possible alterations of this variable by $x_i^l \in [l^0, l^+]$ (note that alterations of the type $x_i^l = l \rightarrow l$, that is, no alteration de facto, are legitimate). To shorten the formulas, we denote $E_{ij}(x_i, x_j) \equiv E_{ij}(x_j, x_i)$ $(i \neq j)$ and perform all the double summations over unordered pairs of variables. The difference between the energy function after an $\alpha$-$\beta$-swap move and before the move is

$$E \left( \{x_i^+\}_{i=1}^n \right) - E \left( \{x_i^0\}_{i=1}^n \right) = \sum_{i \beta \rightarrow \alpha} E_i(\beta) - E_i(\alpha) + \sum_{i \beta \rightarrow \alpha} E_i(\alpha) - E_i(\beta)$$

$$+ \sum_{\{i, j\}, i \neq j \atop x_i^l = \alpha \rightarrow \beta} \left[ E_{ij}(\beta, x_j^+) - E_{ij}(\alpha, x_j^0) \right] + \sum_{\{i, j\}, i \neq j \atop x_j^l = \beta \rightarrow \alpha} \left[ E_{ij}(\alpha, x_i^+) - E_{ij}(\beta, x_i^0) \right]$$

$$+ \sum_{\{i, j\}, i \neq j \atop x_i^l = \alpha \rightarrow \beta} \left[ E_{ij}(\beta, \alpha) - E_{ij}(\alpha, \beta) \right]. \quad (14)$$

By inequality (10) we have the following lower bound for the term inside the last sum:

$$E_{ij}(\beta, \alpha) - E_{ij}(\alpha, \beta) \geq E_{ij}(\alpha, \alpha) + E_{ij}(\beta, \beta) - 2E_{ij}(\alpha, \beta). \quad (15)$$
Substituting this lower bound for the term inside the last sum and splitting the resulting sum into two sums results in the following lower bound for the difference in the energy function:

\[
E \left( \left\{ x_i^+ \right\}_{i=1}^n \right) - E \left( \left\{ x_i^0 \right\}_{i=1}^n \right) \geq \sum_{i,x_i = \alpha \to \beta} [E_i(\beta) - E_i(\alpha)] + \sum_{i,x_i = \beta \to \alpha} [E_i(\alpha) - E_i(\beta)] \\
+ \sum_{\{i,j\}, i \neq j} \left[ E_{ij}(\beta, x_j^+) - E_{ij}(\alpha, x_j^0) \right] + \sum_{\{i,j\}, i \neq j} \left[ E_{ij}(\alpha, x_j^+) - E_{ij}(\beta, x_j^0) \right] \\
+ \sum_{\{i,j\}, i \neq j} \left[ E_{ij}(\beta, \beta) - E_{ij}(\alpha, \beta) \right] + \sum_{\{i,j\}, i \neq j} \left[ E_{ij}(\alpha, \alpha) - E_{ij}(\alpha, \beta) \right].
\]

(16)

The first, third, and fifth sums comprise exactly the difference in the energy function resulting from making only the \( \alpha \)-to-\( \beta \) alterations in the considered \( \alpha \)-\( \beta \)-swap. These alterations are equivalent to a \( \beta \)-expansion from the labeling \( \left\{ x_i^0 \right\}_{i=1}^n \). The second, fourth, and sixth sums comprise exactly the difference in the energy function resulting from making only the \( \beta \)-to-\( \alpha \) alterations in the considered \( \alpha \)-\( \beta \)-swap. These alterations are equivalent to an \( \alpha \)-expansion from the labeling \( \left\{ x_i^0 \right\}_{i=1}^n \). By assumption all expansion moves from the labeling \( \left\{ x_i^0 \right\}_{i=1}^n \) do not decrease the energy function, and therefore the right-hand side of (16) is nonnegative. \( \square \)

3 \( K = 3 \)

All the above results were derived for energy functions whose terms are functions of up to two variables. In the following is discussed the generalization of these results to energy functions that contain terms that are functions of three variables as well. The generalization is accomplished by replacing the algorithm component that minimizes the energy function (1) for \( K = L = 2 \) (Sec. 2.1) with the one in [14] for minimizing the energy function for \( K = 3 \) and \( L = 2 \). The latter algorithm finds the global minimum under the condition that all the \( E_{ijk} \) terms satisfy constraint (6) as before, and that all six projections of two variables of each \( E_{ijk} \) term satisfy this constraint as well. As defined in [14], a projection of a function of binary variables is a function of a subset of these variables, where this function is obtained by fixing all the variables outside this subset. (Actually, the condition in [14] is that all projections of the whole function \( E(x) \) of two variables satisfy (6), but [14] shows that any such function can be rewritten in a form where each term is regular.)

3.1 The Optimal Swap Move for \( K = 3 \)

Using the notation that was used in the corresponding previous derivation (Sec. 2.2) and performing a similar derivation to the one performed there, but this time for \( K = 3 \), shows that the problem of finding the optimal \( \alpha \)-\( \beta \)-swap move is equivalent here to the minimization
arg \min_{x_i \in \{\alpha, \beta\}} \sum_{i \in I} E'_i(x_i) + \sum_{1 \leq i < j \leq n} E'_{ij}(x_i, x_j) + \sum_{1 \leq i < j < k \leq n} E_{ijk}(x_i, x_j, x_k). (17)

The terms \( E'_i(x_i) \) and \( E'_{ij}(x_i, x_j) \) are sums of projections of terms in (1) of the corresponding variables. (We use here the generalized notion of a projection of a function [10], which includes functions of variables that may assume more than two labels.)

The above objective function is of the same structure as the one of the energy function (1) for \( K = 3 \) and \( L = 2 \) (as before, associate \( \alpha \) with \( l_0 \) and \( \beta \) with \( l_1 \), or vice versa). Therefore, if all the \( E_{ij} \) terms and all projections of all the \( E_{ijk} \) terms of two variables fulfill (6) for any pair of labels, the minimization algorithm in [14] can be used to solve the minimization problem (17). These conditions may be summarized as follows: for all terms that are functions of two or three variables and for all \( \alpha, \beta, \gamma \in L \),

\[
\begin{align*}
E_{ij}(\alpha, \alpha) + E_{ij}(\beta, \beta) & \leq E_{ij}(\alpha, \beta) + E_{ij}(\beta, \alpha), \\
E_{ijk}(\alpha, \alpha, \gamma) + E_{ijk}(\beta, \beta, \gamma) & \leq E_{ijk}(\alpha, \beta, \gamma) + E_{ijk}(\beta, \alpha, \gamma), \\
E_{ijk}(\alpha, \gamma, \alpha) + E_{ijk}(\beta, \gamma, \beta) & \leq E_{ijk}(\alpha, \gamma, \beta) + E_{ijk}(\beta, \alpha, \gamma), \\
E_{ijk}(\gamma, \alpha, \alpha) + E_{ijk}(\gamma, \beta, \beta) & \leq E_{ijk}(\gamma, \alpha, \beta) + E_{ijk}(\gamma, \beta, \alpha). 
\end{align*}
\]

(18)

3.2 The Optimal Expansion Move for \( K = 3 \)

As in the corresponding previous derivation (Sec. 2.3), denote the current labeling by \( \{x_i^0\}_{i=1}^n \). The problem of finding the optimal \( \alpha \)-expansion this time is

\[
\begin{align*}
\arg \min_{x_i \in \{\alpha, x_i^0\}} \sum_{i=1}^n E_i(x_i) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i, x_j) + \sum_{1 \leq i < j < k \leq n} E_{ijk}(x_i, x_j, x_k). (19)
\end{align*}
\]

As in the previous case, the objective function has the same structure as the one of the energy function (1) for \( K = 3 \) and \( L = 2 \) (for each variable \( x_i \), associate \( \alpha \) with \( l_0 \) and \( x_i^0 \) with \( l_1 \), or vice versa). Therefore, it may be minimized using the minimization algorithm in [14]. Provided that all the \( E_{ij} \) terms and all the projections of the \( E_{ijk} \) terms of two variables fulfill (6), where the variables in (6) and the fixed variables in the projections may assume the labels feasible by an \( \alpha \)-expansion from the current labeling. For each \( x_i \) these feasible labels are \( \{\alpha, x_i^0\} \). As in the corresponding previous derivation (Sec. 2.3), we either associate \( \alpha \) with \( l_0 \) for all variables or associate \( \alpha \) with \( l_1 \) for all variables. Generally, \( x_i^0 \), \( x_j^0 \), and \( x_k^0 \) might equal any labels \( \beta \), \( \gamma \), and \( \delta \), respectively, and the overall minimization procedure iterates over all labels \( \alpha \). Therefore, the conditions under which the proposed minimization procedure works may be summarized as follows: for all terms that are functions of two or three variables and for all \( \alpha, \beta, \gamma, \delta \in L \),

\[
\begin{align*}
E_{ij}(\alpha, \alpha) + E_{ij}(\beta, \gamma) & \leq E_{ij}(\alpha, \gamma) + E_{ij}(\beta, \alpha), \\
E_{ijk}(\alpha, \alpha, \delta) + E_{ijk}(\beta, \gamma, \delta) & \leq E_{ijk}(\alpha, \gamma, \delta) + E_{ijk}(\beta, \alpha, \delta), \\
E_{ijk}(\alpha, \beta, \alpha) + E_{ijk}(\beta, \delta, \gamma) & \leq E_{ijk}(\alpha, \delta, \gamma) + E_{ijk}(\beta, \alpha, \gamma). 
\end{align*}
\]

(20)
Substituting $\beta$ for $\gamma$ and substituting $\gamma$ for $\delta$ in (20) yields (18), which shows that the obtained swap move-related conditions are a relaxation of the obtained expansion move-related conditions also for energy functions containing terms of three variables.

### 3.3 Minima w.r.t. Expansions are Minima w.r.t. Swaps when $K = 3$

Given an energy function consisting of terms that are functions of up to two variables, we proved in Sec. 2.4 that under the swap move-related condition (10), a minimum with respect to expansion moves is also a minimum with respect to swap moves. In the following we extend the proof for energy functions that consist of terms that are functions of up to three variables. The proof is extended under the swap move-related extended conditions (18).

**Proof Extension.** Denote the right-hand side of (14) by $A$. The difference between the energy function after an $\alpha$-$\beta$-swap move and the move becomes

$$E \left( \{ x_i^+ \}_{i=1}^n \right) - E \left( \{ x_i^0 \}_{i=1}^n \right) = A + \sum_{\{i, j, k\}, \text{ all distinct}} \left[ E_{ijk} \left( \beta, x_j^+, x_k^+ \right) - E_{ijk} \left( \alpha, x_j^0, x_k^0 \right) \right]$$

$$+ \sum_{\{i, j, k\}, \text{ all distinct}} \left[ E_{ijk} \left( \alpha, x_j^+, x_k^+ \right) - E_{ijk} \left( \beta, x_j^0, x_k^0 \right) \right]$$

$$+ \sum_{\{i, j, k\}, \text{ all distinct}} \left[ E_{ijk} \left( \alpha, x_j^0, x_k^0 \right) - E_{ijk} \left( \beta, x_j^0, x_k^0 \right) \right].$$

(21)

By inequality (18) we obtain the following lower bound for the term inside the second sum from the end:

$$E_{ijk}(\beta, \beta, \alpha) - E_{ijk}(\alpha, \alpha, \beta) \geq E_{ijk}(\beta, \alpha, \alpha) + E_{ijk}(\beta, \beta, \beta) - E_{ijk}(\beta, \alpha, \beta) - E_{ijk}(\alpha, \alpha, \beta)$$

$$\geq E_{ijk}(\beta, \alpha, \beta) + E_{ijk}(\alpha, \alpha, \alpha) - E_{ijk}(\alpha, \alpha, \beta) + E_{ijk}(\beta, \beta, \beta) - E_{ijk}(\beta, \beta, \beta) - E_{ijk}(\alpha, \alpha, \beta)$$

$$= E_{ijk}(\alpha, \alpha, \alpha) + E_{ijk}(\beta, \beta, \beta) - 2E_{ijk}(\alpha, \alpha, \beta).$$

(22)

Similarly, we obtain the following lower bound for the term inside the last sum:

$$E_{ijk}(\beta, \alpha, \alpha) - E_{ijk}(\alpha, \beta, \beta) \geq E_{ijk}(\alpha, \alpha, \alpha) + E_{ijk}(\beta, \beta, \beta) - 2E_{ijk}(\alpha, \beta, \beta).$$

(23)

Denote the sum of the right-hand side of (16) and the first two sums following $A$ in (21) by $B$. Substituting these two lower bounds for the corresponding terms inside the last two sums
and splitting each of the resulting sums into two sums results in the following lower bound for the difference in the energy function:

\[
E \left( \{ x^+_i \}_{i=1}^n \right) - E \left( \{ x^0_i \}_{i=1}^n \right) \geq B + \sum_{\{i, j, k\}, \text{all distinct}} \left[ E_{ijk}(\beta, \beta, \beta) - E_{ijk}(\alpha, \alpha, \beta) \right] + \sum_{\{i, j, k\}, \text{all distinct}} \left[ E_{ijk}(\alpha, \alpha, \alpha) - E_{ijk}(\alpha, \alpha, \beta) \right] + \sum_{\{i, j, k\}, \text{all distinct}} \left[ E_{ijk}(\alpha, \alpha, \beta) - E_{ijk}(\alpha, \beta, \beta) \right].
\]

(24)

Similarly to the argument regarding (16), the sums in (24) (including those in \(B\)) can be partitioned into those comprising the difference in the energy function resulting from a \(\beta\)-expansion (first, third, fifth,...,eleventh sums) and those comprising the difference for an \(\alpha\)-expansion (second, fourth, sixth,...,twelfth sums). By assumption all expansion moves from the labeling \(\{ x^0_i \}_{i=1}^n \) do not reduce the energy function, and therefore the right-hand side of (24) is nonnegative. □

4 Minima w.r.t. Expansion Moves are Minima w.r.t. Swap Moves – A Proof for Any \(K\)

All the swap and expansion algorithms rely on the basic condition that when the objective function is restricted to all possible moves of a particular type and for a specific label (for expansion moves) or label pair (for swap moves), it is reduced into a submodular function (defined in the following), which has been shown to be minimized in polynomial time (see, e.g., [7]). Functions like (1) for \(K \in \{2, 3\}\) and \(L = 2\) (under condition (6) or its generalization for \(K = 3\) (Sec. 3)) or that in (2) are special cases of submodular functions. In the following we show that under the above expansion-related condition a minimum with respect to expansion moves is in fact also a minimum with respect to swap moves.

4.1 Preliminaries

As aforementioned, the methods for minimizing multi-label functions with respect to swap (expansion) moves operate by repeatedly finding the \(\alpha\)-\(\beta\)-swap (\(\alpha\)-expansion) move that produces the biggest decrease of the function out of all possible \(\alpha\)-\(\beta\)-swap (\(\alpha\)-expansion) moves.
As will be explained, finding this optimal $\alpha$-$\beta$-swap ($\alpha$-expansion) move relies on a simple reduction of this problem into the minimization of a submodular function.

**Definition:** A set function $f : 2^V \rightarrow \mathbb{R}$ is called submodular if

\[
f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)
\]

is satisfied for all subsets $X$ and $Y$ of a base set $V$ of $n$ elements.

A set function $f : 2^V \rightarrow \mathbb{R}$ of subsets $X$ of $V = \{1, 2, \ldots, n\}$ can be viewed as a function $B : \{0, 1\}^n \rightarrow \mathbb{R}$ of variables $(b_1, b_2, \ldots, b_n)$, where the variable $b_i$ is 1 if $i$ is included in $X$ and 0 otherwise (or vice versa). It is easy to verify that the submodularity of the set function is equivalent to the following condition on the corresponding 0-1 function:

\[
\forall b_1^i, b_2^i \in \{0, 1\} 
\begin{align*}
B \left( \left\{ b_1^i \right\}_{i=1}^n \right) + B \left( \left\{ b_2^i \right\}_{i=1}^n \right) \\
\geq B \left( \left\{ \max \left\{ b_1^i, b_2^i \right\} \right\}_{i=1}^n \right) + B \left( \left\{ b_1^i \cdot b_2^i \right\}_{i=1}^n \right) . \tag{26}
\end{align*}
\]

It can be shown that this condition is equivalent to the definition of function regularity in [14].

Assume we are given a function $E : \mathcal{L}^n \rightarrow \mathbb{R}$, a current (pre-move) labeling $\{x_i^0\}_{i=1}^n$, and a pair of labels $\alpha, \beta \in \mathcal{L}$, and we restrict the allowed labelings $\{x_i^+\}_{i=1}^n$ to those obtained by $\alpha$-$\beta$-swap moves from $\{x_i^0\}_{i=1}^n$. Then, the function over all allowed labelings under this restriction can be viewed as a function of 0-1 variables, one variable $b_i$ per $x_i^0 \in \{\alpha, \beta\}$, by associating all $b_i = 0$ with $x_i^+ = \alpha$ and all $b_i = 1$ with $x_i^+ = \beta$ (or vice versa). Similarly, if we restrict the allowed labelings to those obtained by $\alpha$-expansion moves from $\{x_i^0\}_{i=1}^n$, then the function over all allowed labelings under this restriction can be viewed as a function of 0-1 variables, one variable $b_i$ per $x_i^\alpha \notin \alpha$, by associating all $b_i = 0$ with $x_i^+ = \alpha$ and all $b_i = 1$ with $x_i^+ = x_i^\alpha$ (or vice versa).

It is now easy to observe the following:

**Observation:** If (26) is fulfilled by all reductions of the energy function into expansion-related 0-1 functions $B$, then (26) is fulfilled by all reductions of the energy function into swap-related 0-1 functions as well.

This is correct because a function $B$ corresponding to an $\alpha$-$\beta$-swap reduction from the labeling $\{x_i^0\}_{i=1}^n$ is identical to a projection of the function $B$ corresponding to an $\alpha$-expansion reduction from that labeling with all the $\alpha$ labels replaced with $\beta$. (The projection is of the variables labeled $\alpha$ or $\beta$ in $\{x_i^0\}_{i=1}^n$, where the other variables are fixed to the 0-1 label corresponding to $x_i^0$.) This shows that the aforementioned implications (expansion conditions) $\Rightarrow$ (swap conditions) for $K \in \{2, 3\}$ were no coincidence.

### 4.2 The Proof

The methods for minimizing $E$ with respect to expansion moves (like those with respect to swap moves) rely on the fulfillment of (26) by the functions (or, equivalently, on the submodularity of the corresponding set functions) obtained from the above simple reduction. As we will now prove under this condition, if a labeling $\{x_i^0\}_{i=1}^n$ is a minimum of a function $E : \mathcal{L}^n \rightarrow \mathbb{R}$ with respect to expansion moves, then this labeling is a minimum with respect to swap moves as well.
We will show that any \( \alpha \)-\( \beta \)-swap move from this labeling will result in a labeling \( \{x_i^+\}_{i=1}^n \) for which \( E \) is not decreased.

Denote the set of indices of the fixed (prohibited from changing) variables by \( P = \{i : x_i^0 \notin \{\alpha, \beta\}\} \) and their corresponding labeling by \( \mathcal{P} = \{x_i^0\}_{i \in \mathcal{P}} \). For all \( i \notin P \) associate \( x_i^0 \) with a 0-1 variable \( b_i^0 \) and associate \( x_i^+ \) with a 0-1 variable \( b_i^+ \). As explained, perform these associations by associating the label \( \alpha \) with the binary value ‘0’ and the label \( \beta \) with the binary value ‘1’, or vice versa. Denote the 0-1 function obtained by the reduction of \( E \) due to any \( \alpha \)-\( \beta \)-swap move from this labeling will result in a labeling \( \{x_i^0\}_{i=1}^n \) is a minimum of \( E \) with respect to all swap moves.

The method for finding the optimal energy function with respect to this move type can be accomplished by repeatedly performing optimal moves of this type, each time for a different label pair and until no decrease of the energy function is possible for any move of this type.

5 The Exwap Move

A new type of large move that may alter the label of all the variables regardless of the current (pre-move) labeling is the \textit{exwap} move type. Given an ordered pair of labels \( (\alpha, \beta) \), we define an \( \alpha \)-\( \beta \)-exwap move to be the unification of an \( \alpha \)-expansion move and an \( \alpha \)-\( \beta \) swap move (hence, the name \textit{exwap}). That is, the label alterations allowed by this move are alterations from any label to \( \alpha \) and alterations from \( \alpha \) to \( \beta \). Note that this move type is a generalization of both the swap and the expansion move types. Like in the previous two move types, the minimization of an energy function with respect to this move type can be accomplished by repeatedly performing optimal moves of this type, each time for a different label pair and until no decrease of the energy function is possible for any move of this type.

5.1 The Optimal Exwap Move for \( K = 2 \)

The method for finding the optimal \( \alpha \)-\( \beta \)-exwap move for \( K = 2 \) will now be derived. As before, denote the current labeling by \( \{x_i^0\}_{i=1}^n \), and denote the index-set of all variables
currently labeled $\alpha$ by $\mathcal{I}_\alpha = \{ i : x_i^0 = \alpha \}$. The problem of finding the optimal $\alpha$-$\beta$-exwap is

$$\arg \min_{x_i \in \{\alpha, x_i^0\}, \ i \notin \mathcal{I}_\alpha} E(\{x_i\}_{i=1}^n)$$

$$= \arg \min_{x_i \in \{\alpha, x_i^0\}, \ i \notin \mathcal{I}_\alpha} \sum_{i=1}^n E_i(x_i) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i, x_j). \quad (29)$$

As in the case of the expansion move for $K = 2$ (Sec. 2.3), the objective function has the same structure as the one of the energy function (1) for $K = L = 2$ (for each variable $x_i$, associate $\alpha$ with $l_0$ and one of $x_i^0$ or $\beta$ with $l_1$, or vice versa). Therefore, it may be minimized using Picard et al.'s solution (Sec. 2.1), provided that all the $E_{ij}$ terms fulfill (6), where the variables in (6) may assume the labels feasible by an $\alpha$-$\beta$-exwap from the current labeling. For each variable $x_i$ these feasible labels are $\{\alpha, x_i^0\}$ (if $i \notin \mathcal{I}_\alpha$) or $\{\alpha, \beta\}$ (if $i \in \mathcal{I}_\alpha$). We either associate $\alpha$ with $l_0$ for all variables or associate $\alpha$ with $l_1$ for all variables. As in the case of the expansion move, $x_i^0$ and $x_i^0$ might equal any label pair, and the overall minimization procedure iterates over all label pairs ($\alpha$, $\beta$). By intersecting the four conditions obtained for the four possibilities $\{i \in \mathcal{I}_\alpha, i \notin \mathcal{I}_\alpha\} \times \{j \in \mathcal{I}_\alpha, j \notin \mathcal{I}_\alpha\}$ we obtain that the conditions under which the proposed minimization procedure works are the same as those for the expansion move type for $K = 2$ (13).

### 5.1.1 Modifying the Exwap Move Algorithm for General Energies of $K = 2$

There are applications such as Digital Tapestry [17] or instances of image denoising [18] that involve the minimization of energy functions of $K = 2$ that contain terms that do not fulfill (13). Therefore, these energy functions cannot be minimized by the expansion move and the above exwap move algorithms. In order to make the expansion move algorithm applicable for such energy functions, [17] suggested a procedure that consists of a certain “truncation” of the terms in the reduced (binary) functions (11) to make these terms fulfill the regularity condition (6). After this truncation procedure is applied, the resulting energy function can be minimized (as described in Sec. 2.1) and it is guaranteed that the value of the original function (11) will not increase after the move.

A similar truncation procedure can be applied for the above exwap move algorithm. However, the original procedure for the expansion move algorithm [17] exploits the property that the labels of all variables in (11) prior to the move correspond to $l_0$ (which is due to the association of $x_i^0$ with $l_0$ for all variables). Since the exwap move algorithm associates $x_i^0$ with $l_0$ for $i \in \mathcal{I}_\alpha$ and associates $x_i^0$ with $l_1$ for $i \notin \mathcal{I}_\alpha$, it does not have a similar property. Thus, in order to use the truncation method in the exwap move algorithm, the truncation has to be generalized to account for all four possibilities of current binary label pairs $(x_i^0, x_j^0) \in \{l_0, l_1\}^2$. Making this generalization results in the procedure where each $E_{ij}(x_i, x_j)$ term in (29) that violates the regularity condition (6) is modified (“truncated”) to a term $\tilde{E}_{ij}(x_i, x_j)$ in any way conforming to Table 1 such that $\tilde{E}_{ij}(x_i, x_j)$ will fulfill (6) with equality. Now the specific ways allowed to modify each energy term depend on its current labels $x_i^0, x_j^0$. The resulting energy function $\tilde{E}(\{x_i\}_{i=1}^n)$ can now be minimized (as described in Sec. 2.1) and it is guaranteed that the value of the original function $E(\{x_i\}_{i=1}^n)$ (29) will not increase after the exwap move (see proof in Appendix A).

An alternative to the above generalized truncation procedure is the association of $x_i^0$ with $l_0$ for all variables, which would allow using the original truncation procedure. However, such
an association is inconsistent with the association used for “exwap-regular” energy functions, which is required for the exwap-related condition (13) to be fulfilled. This suggests that using this alternative association would make the violation of the regularity condition (6) worse, which would necessitate larger truncations of the original energy functions.

5.2 The Optimal Exwap Move for $K = 3$

The derivation of the optimal $\alpha$-$\beta$-exwap move for $K = 3$ is similar to the one for $K = 2$. The problem of finding the optimal $\alpha$-$\beta$-exwap is

$$\arg\min_{x_i \in \{\alpha, x^0_i\}, \ i \not\in I_0, \ x_i \in I_0, \ i \in I_0} \sum_{i=1}^{n} E_i(x_i) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i, x_j) + \sum_{1 \leq i < j < k \leq n} E_{ijk}(x_i, x_j, x_k).$$

(30)

As in the case of the expansion move for $K = 3$ (Sec. 3.2), the objective function has the same structure as the one of the energy function (1) for $K = 3$ and $L = 2$ (for each variable $x_i$, associate $\alpha$ with $l_0$ and one of $x^0_i$ or $\beta$ with $l_1$, or vice versa). Therefore, it may be minimized using the minimization algorithm in [14], provided that all the $E_{ij}$ terms and all the projections of the $E_{ijk}$ terms of two variables fulfill (6), where the variables in (6) and the fixed variables in the projections may assume the labels feasible by an $\alpha$-$\beta$-exwap from the current labeling. For each variable $x_i$ these feasible labels are $\{\alpha, x^0_i\}$ (if $i \not\in I_0$) or $\{\alpha, \beta\}$ (if $i \in I_0$). We either associate $\alpha$ with $l_0$ for all variables or associate $\alpha$ with $l_1$ for all variables. As in the case of the expansion move, $x^0_i, x^0_j$, and $x^0_k$ might equal any label triplets, and the overall minimization procedure iterates over all label pairs ($\alpha, \beta$). By intersecting the eight conditions obtained for the eight possibilities $\{i \in I_0, i \not\in I_0\} \times \{j \in I_0, j \not\in I_0\} \times \{k \in I_0, k \not\in I_0\}$ we obtain that the conditions under which the proposed minimization procedure works are the same as those for the expansion move type for $K = 3$ (20). An example of function for $K = 3$ that is potentially useful for multicamera scene reconstruction was provided in [14].

5.3 The Optimal Exwap Move for the $\mathcal{P}^n$ Potts Model

A family of energy terms that may contain an arbitrary number of variables is the $\mathcal{P}^n$ Potts Model, where $n$ denotes the number of variables in an energy term. These energy terms were introduced in [10], where graph constructions whose minimum $s$-$t$-cuts correspond to optimal swap and expansion moves were developed. A $\mathcal{P}^K$ Potts Model energy term in an energy function (1) is defined as

$$E_{i_1, i_2, \ldots, i_K}(x_{i_1}, x_{i_2}, \ldots, x_{i_K}) \triangleq \begin{cases} \gamma_t, & x_{i_1} = x_{i_2} = \ldots = x_{i_K} = l, \\ \gamma_{\text{max}}, & \text{otherwise}, \end{cases}$$

(31)
where \( \gamma_{\text{max}} \geq \gamma_l, \forall l \in \mathcal{L} \).

In the following we construct a graph whose minimum \( s-t \)-cuts correspond to optimal \( \alpha-\beta \)-exwap moves from a current labeling \( \{x_{1k}^0\}_{k=1}^K \) for one \( \mathcal{P}^K \) Potts Model energy term. By the additivity theorem of [14], the graph corresponding to the entire energy function is obtained by merging all the graphs corresponding to the different energy terms (which may belong to \( \mathcal{P}^K \) Potts Models as well or not).

The graph corresponding to a \( \mathcal{P}^K \) Potts Model energy term (31) is described in Fig. 1. The graph consists of \( K \) vertices \( v_{i1}, \ldots, v_{iK} \) that correspond to the \( K \) variables \( x_{i1}, \ldots, x_{iK} \), respectively, two auxiliary vertices \( M_s \) and \( M_t \), and the source and the sink. The graph has the same structure as the one used for computing optimal swap and expansion moves in [10], but the edge weights are different and, of course, the partitions correspond to the new move type. The edge weights can be set to various values. One possible way to set the weights will now be described. As before, denote the index-set of all variables currently labeled \( \alpha \) by \( \mathcal{I}_{\alpha} = \{i: x_i^0 = \alpha\} \). Given a cut \( (S, T) \) where the source is in \( S \) and the sink is in \( T \), let \( v_i \in S \) be associated with \( x_i = \alpha \), and let \( v_i \in T \) be associated with \( x_i = x_i^0 \) (if \( x_i^0 \notin \mathcal{I}_{\alpha} \)) or with \( x_i = \beta \) (if \( x_i^0 \in \mathcal{I}_{\alpha} \)). Let \( c \) be any value such that \( c < \min\{\gamma_l: l \in \mathcal{L}\} \), and set the edge weights as following:

\[
\begin{align*}
    w_t &= \gamma_{\alpha} - c, \\
    w_u &= \frac{\gamma_{\text{max}} - c}{K}, \\
    w_s &= \begin{cases} 
    \gamma_{\beta} - c, & x_i^0 \in \{\alpha, \beta\}, \forall k = 1, \ldots, K, \\
    E_{i1\ldots iK}(x_{i1}^0, x_{i2}^0, \ldots, x_{iK}^0) - c, & \text{otherwise.}
    \end{cases}
\end{align*}
\]

Fig. 1. The graph corresponding to a \( \mathcal{P}^K \) Potts Model energy term for computing an optimal exwap move.

Let us verify now that the graph construction is correct. First, it is immediate to see that all the edge weights are positive, which allows for the efficient computation of the minimum \( s-t \)-cut. The possible \( s-t \)-cuts in the graph (as part of a minimum \( s-t \)-cut in the entire graph corresponding to the whole energy function) are as follows: 1. the cut corresponding to the
edge connecting $M_s$ with the sink; 2. any cut corresponding to a mix consisting of both edges connecting $M_s$ with $v_i$s and edges connecting $M_t$ with the other $v_i$s, and no other edges; 3. the cut corresponding to the edge connecting $M_s$ with the source. (Note that any other cut cannot be obtained as part of a minimum $s$-$t$-cut in the entire graph. The reason is that for all other cuts there are cuts of weight at least as small in the above three cases that correspond to the same partitioning of the vertices $v_i$.) Obviously, for any possible $\alpha$-$\beta$-exwap move there is a corresponding cut in one of the cases 1-3. It is also easy to verify that in all cases 1-3 the weight of the cut equals the value of the energy term after the $\alpha$-$\beta$-exwap move minus $c$.

6 Experiments

Energy function minimization with respect to exwap moves was tested and compared to that with respect to expansion moves on benchmark problems from [18]. The problems we used consisted of the Image Denoising and Inpainting ones and the Middlebury stereo benchmarks used in [18]. (All these benchmarks are available at http://vision.middlebury.edu/MRF/.) These types of problems are pixel-labeling problems, where the goal is to assign to every image pixel $p$ a label $l_p$. In our experiments, we used the same energy functions (along with the same parameters $\lambda$, $k$ and $V_{\text{max}}$) as those in [18]. These energy functions have the form

$$E(l) = \sum_p d_p(l_p) + \lambda \sum_{\{p,q\} \in \mathcal{N}} \min \left( |l_p - l_q|^k, V_{\text{max}} \right),$$

(35)

where $l$ denotes the collection of all pixel label assignments, $\mathcal{N}$ denotes the set of all neighboring pixel pairs in the standard 4-connected neighborhood system, $d_p(l_p)$ is the data cost for pixel $p$, and $k \in \{1, 2\}$. The input images used in the experiments, the corresponding ground truths and results obtained by the exwap move algorithm are shown in Fig. 2.

We note that although the exwap move type is a strict generalization of the expansion move type, a derivation similar to the one in Sec. 4 shows that under the corresponding exwap move-related condition, which is similar to the corresponding expansion move-related one, minima with respect to expansions are also a minima with respect to exwaps. However, when the expansion/exwap move-related condition is not fulfilled (that is, when a truncation procedure is required), this expansion-exwap relation does not hold. When this condition is fulfilled, using exwap moves still has the potential to decrease the value of the energy function in a higher rate than that obtained by using expansion moves, since the exwap move is a strict generalization of the expansion move.

**Image denoising and inpainting.** For the image denoising and inpainting experiments, we used the same noised and obscured versions of the “House” and “Penguin” images used in [18]. As in [18], the labels are the 256 intensities, the smoothness cost is quadratic ($k = 2$), and the data cost for each pixel is the squared difference between the observed intensity and the label, except in the obscured portions, where $d_p(l_p) \equiv 0$. Note that since $k = 2$, the expansion/exwap move-related condition (13) is not fulfilled. Therefore, as noted in [18], the expansion and exwap algorithms require truncation. Similarly to [18], for “House” we used $\lambda = 5$ and $V_{\text{max}} = \infty$, and for “Penguin” we used $\lambda = 25$ and $V_{\text{max}} = 200$. Being a strict generalization of the expansion, the exwap move reached lower energies for both images. However, as implied from the previous paragraph, the higher the likelihood for condition (13) (which reduces here to the triangle inequality) to be violated for label triplets, the greater exwap’s advantage. Therefore, the lower the truncation parameter $V_{\text{max}}$, the smaller exwap’s advantage. While in “House” ($V_{\text{max}} = \infty$) exwap reached an energy lower by 4.7% than expansion, in “Penguin” ($V_{\text{max}} \approx 14^2$) exwap reached an energy lower by 0.2% only.
Fig. 2. The input images, ground truths, and results obtained by the exwap move algorithm. (a) “House” (from left to right): input image, original image, and exwap’s result. (b) “Penguin” (from left to right): input image, original image, and exwap’s result. (c) “Venus” (from left to right): left reference image, ground truth disparities, ground truth disparities rounded to integer pixels, and disparities estimated by exwap.
When we used “Penguin” with $V_{\text{max}} = \infty$ as well, exwap’s advantage raised to 3.3%.

**Stereo matching.** For the stereo matching experiment, we used the “Venus” image pair used in [18], which was taken from the Middlebury stereo data set. As in [18], the labels are the disparities (1, . . . , 20 pixels), and the data costs are the squared color differences between corresponding pixels for each disparity\(^2\). As was done in [18], we used [1]’s variant of the color difference for increased robustness to image sampling. As in [18], we set $k = 2$ (and therefore, as before, both algorithms require truncation), $\lambda = 50$ and $V_{\text{max}} = 7$. As in the previous experiments, the exwap move yielded a lower energy than the expansion move. Due to the small truncation parameter ($V_{\text{max}} < 3^2$), exwap’s advantage was only 0.3%. When we raised $V_{\text{max}}$ to $10^2$, exwap’s advantage raised to 1.8%. The other two stereo matching experiments in [18] (“Tsukuba” and “Teddy”) used a linear smoothness cost ($k = 1$). This cost satisfies (13) and therefore exwap and expansion perform similarly for it.

7 Conclusion

The optimal swap and expansion moves for energy functions (1) with $K = 2$ and $K = 3$ were re-derived in a short manner by using as a “black box” the original solutions to the pseudo-Boolean quadratic function minimization problem [16] and the pseudo-Boolean energy function minimization problem for $K = 3$ [14], respectively. It was revealed that minima with respect to expansion moves are also minima with respect to swap moves for $K \in \{2, 3\}$ under the corresponding swap-related conditions, which were shown to be relaxations of the expansion-related ones. This provided an explanation for obtaining better experimental results by using the expansion move algorithm than by the swap move algorithm in previous works.

The above minima-related result was extended to all objective functions under the condition that they are reduced into submodular ones. All expansion (as well as swap) algorithms work under this condition, and therefore this result applies to all of them, including those for arbitrary $K$. This suggests that minimization with respect to expansion moves should generally be more effective than with respect to swap moves and reinforces the impression that “there never seems to be any reason to use Swap instead of Expansion” [18].

In order to enlarge the search space in the minimization procedure, the exwap move type was introduced. This move type is a generalization of the expansion and the swap move types. Efficient algorithms for finding optimal moves of this type for $K = 2$, $K = 3$ and the $P^n$ Potts model were provided under the same conditions corresponding to the expansion move algorithms. In addition, a ‘truncation’ procedure for finding the optimal exwap move for arbitrary energy functions of $K = 2$ was provided. The exwap move algorithm was compared to the expansion move algorithm using several benchmark problems, where the exwap move algorithm yielded results of lower energies.

A Proof for Generalized Truncation Procedure

Given an energy function $E(x)$ (1) of $K = L = 2$, its current (pre-move) labeling $x^0 = (x^0_1, \ldots, x^0_n)$ and a corresponding truncated energy function $\widetilde{E}(x)$ according to Table 1, we

---

\(^2\) Although not indicated in [18], in [18]’s results website for the “Venus” image pair (http://vision.middlebury.edu/MRF/results/venus/) the squared color differences are indicated rather than the absolute ones. Also, the former produce better results for this image pair.
prove in the following that if $x^* = (x_1^*, \ldots, x_n^*)$ minimizes $E(x)$ then $E(x^*) \leq E(x^0)$.

**Proof:**

\[
E(x^*) = \sum_{i=1}^{n} E_i(x_i^*) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i^*, x_j^*) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i^*, x_j^*) \\
\leq \sum_{i=1}^{n} \tilde{E}_i(x_i^*) + \sum_{1 \leq i < j \leq n} \tilde{E}_{ij}(x_i^*, x_j^*) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i^*, x_j^*) \\
+ \sum_{1 \leq i < j \leq n} E_{ij}(x_i^*, x_j^*) - \sum_{1 \leq i < j \leq n} \tilde{E}_{ij}(x_i^*, x_j^*) \\
\leq \tilde{E}(x^0) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i^*, x_j^*) - \sum_{1 \leq i < j \leq n} \tilde{E}_{ij}(x_i^*, x_j^*) \\
\leq \tilde{E}(x^0) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i^0, x_j^0) - \sum_{1 \leq i < j \leq n} \tilde{E}_{ij}(x_i^0, x_j^0) \\
\leq \tilde{E}(x^0) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i^0, x_j^0) - \sum_{1 \leq i < j \leq n} \tilde{E}_{ij}(x_i^0, x_j^0) \\
= \sum_{i=1}^{n} \tilde{E}_i(x_i^0) + \sum_{1 \leq i < j \leq n} \tilde{E}_{ij}(x_i^0, x_j^0) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i^0, x_j^0) \\
= \sum_{i=1}^{n} E_i(x_i^0) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i^0, x_j^0) + \sum_{1 \leq i < j \leq n} E_{ij}(x_i^0, x_j^0) = E(x^0) \square
\]

**References**


