Minima w.r.t. Expansions are Minima w.r.t. Swaps - A Proof

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Abstract. Many proposed solutions to computer vision and image processing problems involve the minimization of multi-label functions - functions whose variables may assume labels from a finite set of labels. In the context multi-label function minimization, the swap and the expansion are two types of commonly used moves from one labeling to the next. Efficient algorithms for finding a local minimum with respect to each of these two moves have been developed. These algorithms are restricted by different limits to the maximal number of variables allowed in one term of the objective function and by different conditions related to the function terms. These algorithms work under the condition that the objective functions are reduced into submodular ones, which have been shown to be minimized in polynomial time. This report provides a proof that under this condition minima with respect to expansion moves are also minima with respect to swap moves. This suggests that minimization with respect to the former move type should be more effective than with respect to the latter. This may also give an explanation for obtaining better experimental results for expansion moves than swap moves in previous works.

1 Introduction

The problem discussed in this report is the minimization of functions $E(x)$ where the variables $x = (x_1, \ldots, x_n)$ may assume labels from the label set $L = \{l_0, l_1, \ldots, l_{L-1}\}$. Many proposed solutions to computer vision and image processing problems involve the minimization of such functions, which are commonly formed as a sum of terms:

$$E(x) = \sum_{k=1}^{n} E_k(x_k) + \sum_{1 \leq k_1 < k_2 \leq n} E_{k_1,k_2}(x_{k_1}, x_{k_2}) + \sum_{1 \leq k_1 < k_2 < k_3 \leq n} E_{k_1,k_2,k_3}(x_{k_1}, x_{k_2}, x_{k_3}) + \ldots + \sum_{1 \leq k_1 < k_2 < \ldots < k_K \leq n} E_{k_1,k_2,\ldots,k_K}(x_{k_1}, x_{k_2}, \ldots, x_{k_K}). \quad (1)$$

Given a labeling $x \in L^n$ and a pair of labels $\alpha$ and $\beta$, an $\alpha$-$\beta$-swap is any move (that is, changes applied on the labeling) that consists of only alterations of variables between $\alpha$ and $\beta$. Given a labeling and a label $\alpha$, an $\alpha$-expansion is any move that consists of only alterations of variables to $\alpha$. These two types of moves were introduced in [1, 2] and are the two most popular graph-cut based minimization algorithms [9]. These papers proposed an efficient scheme for finding the $\alpha$-$\beta$-swap that yields the biggest decrease of the function (1) for $K = 2$ provided that the function’s bivariate terms satisfy certain conditions, and similarly for the expansion move type. By repeatedly performing such optimal moves of one type,
each time for a different label (for expansion moves) or label pair (for swap moves) in a fixed or random order and until no decrease of the function is possible for any move of the corresponding type, the function is minimized with respect to this type of move.

In [7]–[8], the conditions related to the bivariate terms were relaxed. These papers also proposed an efficient method for minimizing (1) with respect to expansion moves for $K = 3$ provided that the function terms satisfy certain conditions. Classes of functions (1) for arbitrary $K$ that can be minimized with respect to swap moves and with respect to expansion moves in polynomial time were proposed in [4], where also efficient algorithms for these tasks were proposed for a certain family of functions ($P^n$ Potts model). Efficient algorithms for minimizing another family of functions (1) of arbitrary $K$ (Robust $P^n$ model) with respect to swap and expansion moves were proposed in [5, 6].

All the above methods rely on the basic condition that when the objective function is restricted to all possible moves of a particular type and for a specific label (for expansion moves) or label pair (for swap moves), it is reduced into a submodular function (defined in the following), which has been shown to be minimized in polynomial time (see, e.g., [3]). In the following we show that under this condition a minimum with respect to expansion moves is in fact also a minimum with respect to swap moves. This suggests that minimization with respect to the former move type should be preferred over the latter. This may also explain the observation that better experimental results were obtained for expansion moves than swap moves in previous works (see, e.g., [9]) and the occasional preference of using the expansion moves for the minimization task [9].

## 2 Preliminaries

As aforementioned, the methods for minimizing multi-label functions with respect to swap (expansion) moves operate by repeatedly finding the $\alpha$-$\beta$-swap ($\alpha$-expansion) move that produces the biggest decrease of the function out of all possible $\alpha$-$\beta$-swap ($\alpha$-expansion) moves. As will be explained, finding this optimal $\alpha$-$\beta$-swap ($\alpha$-expansion) move relies on a simple reduction of this problem into the minimization of a submodular function.

**Definition:** A set function $f : 2^V \to \mathbb{R}$ is called submodular if

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$$

is satisfied for all subsets $X$ and $Y$ of a base set $V$ of $n$ elements.

A set function $f : 2^V \to \mathbb{R}$ of subsets $X$ of $V = \{1, 2, \ldots, n\}$ can be viewed as a function $B : \{0, 1\}^n \to \mathbb{R}$ of variables $(b_1, b_2, \ldots, b_n)$, where the variable $b_i$ is 1 if $i$ is included in $X$ and 0 otherwise (or vice versa). It is easy to verify that the submodularity of the set function is equivalent to the following condition on the corresponding 0-1 function:

$$\forall i_1, i_2 \in \{1, 2, \ldots, n\} : B\left(\big\{b_i^{0}\big\}_{i=1}^{n}\right) + B\left(\big\{b_i^{2}\big\}_{i=1}^{n}\right) \geq B\left(\big\{\max\{b_i^{1}, b_i^{2}\}\big\}_{i=1}^{n}\right) + B\left(\big\{b_i^{1} \cdot b_i^{2}\big\}_{i=1}^{n}\right).$$

(3)

It can be shown that this condition is equivalent to the definition of function regularity in [8].

Assume we are given a function $E : \mathcal{L}^n \to \mathbb{R}$, a current labeling $\{x_{i}^{0}\}_{i=1}^{n}$, and a pair of labels $\alpha, \beta \in \mathcal{L}$, and we restrict the allowed labelings $\{x_{i}^{0}\}_{i=1}^{n}$ to those obtained by $\alpha$-$\beta$-swap moves from $\{x_{i}^{0}\}_{i=1}^{n}$. Then, the function over all allowed labelings under this restriction can be viewed as a function of 0-1 variables, one variable $b_i$ per $x_{i}^{0} \in \{\alpha, \beta\}$, by
associating all $b_i = 0$ with $x_i^+ = \alpha$ and all $b_i = 1$ with $x_i^+ = \beta$ (or vice versa). Similarly, if we restrict the allowed labelings to those obtained by $\alpha$-expansion moves from $\{x_i^0\}_{i=1}^n$, then the function over all allowed labelings under this restriction can be viewed as a function of 0-1 variables, one variable $b_i$ per $x_i^0 \neq \alpha$, by associating all $b_i = 0$ with $x_i^+ = \alpha$ and all $b_i = 1$ with $x_i^+ = x_i^0$ (or vice versa).

### 3 Minima w.r.t. Expansion Moves are Minima w.r.t. Swap Moves

The methods for minimizing $E$ with respect to expansion moves and with respect to swap moves rely on the fulfillment of (3) by the functions (or, equivalently, on the submodularity of the corresponding set functions) obtained from the above simple reductions. As we will now prove under this condition, if a labeling $\{x_i^0\}_{i=1}^n$ is a minimum of a function $E : \mathbb{L}^n \rightarrow \mathbb{R}$ with respect to expansion moves, then this labeling is a minimum with respect to swap moves as well.

**Proof:** Assume the labeling $\{x_i^0\}_{i=1}^n$ is a minimum of $E$ with respect to all expansion moves. We will show that any $\alpha$-$\beta$ swap move from this labeling will result in a labeling $\{x_i^+\}_{i=1}^n$ for which $E$ is increased.

Denote the set of indices of the fixed (prohibited from changing) variables by $p = \{i : x_i^0 \notin \{\alpha, \beta\}\}$ and their corresponding labeling by $P = \{x_i^0\}_{i \notin p}$. For all $i \notin p$ associate $x_i^0$ with a 0-1 variable $b_i^0$ and associate $x_i^+$ with a 0-1 variable $b_i^+$. As explained, perform these associations by associating the label $\alpha$ with the binary value ‘0’ and the label $\beta$ with the binary value ‘1’, or vice versa. Denote the 0-1 function obtained by the reduction of $E$ by $E_P \left( \{b_i\}_{i \notin p}\right)$, which corresponds to the function $E \left( P, \{x_i\}_{i \notin p}\right) : \{\alpha, \beta\}^{n-|p|} \rightarrow \mathbb{R}$. By the assumption that $E_P$ satisfies (3), we have the inequality

$$E_P \left( \{b_i^+\}_{i \notin p}\right) + E_P \left( \{b_i^0\}_{i \notin p}\right) \geq E_P \left( \{\max \{b_i^+ , b_i^0\}\}_{i \notin p}\right) + E_P \left( \{b_i^+ b_i^0\}_{i \notin p}\right).$$  

(4)

Deducting $2E_P \left( \{b_i^0\}_{i \notin p}\right)$ from both sides of the equation yields

$$E_P \left( \{b_i^+\}_{i \notin p}\right) - E_P \left( \{b_i^0\}_{i \notin p}\right) \geq E_P \left( \{\max \{b_i^+ , b_i^0\}\}_{i \notin p}\right) - E_P \left( \{b_i^+ b_i^0\}_{i \notin p}\right) + E_P \left( \{b_i^0\}_{i \notin p}\right) - E_P \left( \{b_i^+\}_{i \notin p}\right).$$  

(5)

The left hand-side of this inequality equals the difference in the value of $E$ due to an $\alpha$-$\beta$ swap move from $\{x_i^0\}_{i=1}^n$. The first two terms in the right hand-side constitute the difference in the value of $E_P$ due to $0 \rightarrow 1$ alterations only, which equals to the difference of $E$ due to $\alpha \rightarrow \beta$ alterations only (or $\beta \rightarrow \alpha$ alterations only). The last two terms in the right hand-side constitute the difference in the value of $E_P$ due to $1 \rightarrow 0$ alterations only, which equals to the difference of $E$ due to $\beta \rightarrow \alpha$ alterations only (or $\alpha \rightarrow \beta$ alterations only). By the assumption that the labeling $\{x_i^0\}_{i=1}^n$ is a minimum with respect to all expansion moves, the latter two differences are nonnegative, and therefore the difference in the value of $E$ due to any $\alpha$-$\beta$-swap move is nonnegative as well. $\alpha$ and $\beta$ are arbitrary, which implies that $\{x_i^+\}_{i=1}^n$ is a minimum with respect to all swap moves.
References