Symmetries of non-rigid shapes

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Symmetries of non-rigid shapes

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Abstract

Symmetry and self-similarity is the cornerstone of Nature, exhibiting itself through the shapes of natural creations and ubiquitous laws of physics. Since many natural objects are symmetric, the absence of symmetry can often be an indication of some anomaly or abnormal behavior. Therefore, detection of asymmetries is important in numerous practical applications, including crystallography, medical imaging, and face recognition, to mention a few. Conversely, the assumption of underlying shape symmetry can facilitate solutions to many problems in shape reconstruction and analysis. Traditionally, symmetries are described as extrinsic geometric properties of the shape. While being adequate for rigid shapes, such a description is inappropriate for non-rigid ones. Extrinsic symmetry can be broken as a result of shape deformations, while its intrinsic symmetry is preserved. Here we explore the problem of evaluating intrinsic symmetries in non-rigid objects. We define and visualize this new concept of symmetry, which can be considered as a generalization of the well studied Euclidean symmetries. We present an efficient mathematical framework for the evaluation of the level of symmetry.
## List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
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<tbody>
<tr>
<td>$\mathbb{R}^n$</td>
<td>Euclidean $n$ dimension space</td>
</tr>
<tr>
<td>$M_n(\mathbb{R})$</td>
<td>$n \times n$ matrix with elements in $\mathbb{R}$</td>
</tr>
<tr>
<td>$X$</td>
<td>Two dimensional Riemannian manifold</td>
</tr>
<tr>
<td>$d_X$</td>
<td>geodesic metric of $X$</td>
</tr>
<tr>
<td>$d_{\mathbb{R}^2}(X)$</td>
<td>Euclidean metric of $X$</td>
</tr>
<tr>
<td>$F(X)$</td>
<td>Symmetry space of $X$</td>
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<tr>
<td>$d_F(X)$</td>
<td>Symmetry space metric</td>
</tr>
<tr>
<td>$X_r$</td>
<td>$r$ sampling of $X$</td>
</tr>
<tr>
<td>$x_i$</td>
<td>A mesh point in $X_r$</td>
</tr>
<tr>
<td>$d_{ij}$</td>
<td>Geodesic distance between $x_i$ and $x_j$</td>
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Chapter 1

Introduction

“Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend the created order, beauty, and perfection” [30]. These words of Hermann Weyl, one of the greatest twentieth century mathematician, reflect the importance symmetry has in all aspects of our life. Often referred to as self-similarity or invariance, symmetries can be found in many fields, ranging from shapes of natural creations through mathematical equations, and physical laws. The interest in symmetries of shapes dates back to the dawn of the human civilization. Early evidences that our predecessors attributed importance to symmetries can be found in many cultural heritages, ranging from monumental architecture of the Egyptian pyramids to traditional ancient Greek decorations. Johannes Kepler was among the first who attempted to give a geometric formulation to symmetries in his treatise On the six-cornered snowflake [17] in as early as 1611. A few centuries later, the study of symmetric shapes became a cornerstone of crystallography. Finally, symmetries of more complicated higher-dimensional objects underlie modern physics theories about the nature of matter, space and time.
Why do we like symmetries? Researchers believe that the human mind interprets symmetry as beauty in general. It was shown that babies spend more time staring at symmetric pictures, and adults rate symmetric faces as the prettiest. Since symmetric facial features are often associated with beauty and aesthetics [21], facial symmetry is important in craniofacial plastic surgery [16]. As many natural objects are symmetric, symmetry breaking can often be an indication of some anomaly or abnormal behavior. Therefore, detection of asymmetries arises in numerous practical problems, among which medical applications are probably the first to come in mind. Furthermore, facial asymmetry can also be an indication of various syndromes and disorders [15]. Conversely, the assumption of symmetry can be used as a prior knowledge in many problems. It may facilitate, for example, the reconstruction of surfaces [29], face detection, recognition and feature extraction [26, 28]. In the computer vision and pattern recognition literature, there exists a significant number of papers dedicated to symmetries. A wide spectrum of methods were employed for detecting symmetries, some handled two-dimensional shapes in the plane, other in images, and in recent years several new methods were developed to handle symmetries of three-dimensional shapes in space. We can find several papers on points wise symmetries, polygonal and planner shapes by Wolter et al. [31], Atallah [2], Alt et al [1], while Marola [20] showed an efficient method for finding symmetry axes in planner images. The past decades gave rise to methods based on dual spaces [8], genetic algorithms [13], moments calculation [6], pair matching [19, 7], and local shape descriptors [32]. Traditionally, symmetries are considered extrinsic geometric properties of shapes, i.e., related to the way the shape is represented in the Euclidean space. Rotations and reflections are often used to describe possible symmetries. Indeed, more complicated transforms were explored, like skew symmetries [5], but those refer to non-bendable (rigid) shapes. Though adequate for rigid shapes, Euclidean transformations are inadequate for non-rigid or deformable ones.
Due to the possible deformations of such shapes, the extrinsic symmetries may be lost, while intrinsically the shape still remains symmetric. Considering as an example the human body in Figure 1 (left). Extrinsic bilateral symmetry of the body is broken when the body assumes different postures. Yet, from the point of view of intrinsic geometry, the new shape remains almost identical; as such a deformation does not significantly change its metric structure. In this sense, intrinsic symmetries are a superset of the extrinsic ones. Considering intrinsic rather than extrinsic symmetries allows us to characterize the object self-similarity that is invariant to deformations. If we resort to our previous medical illustration, we can detect tumors as irregular, non-symmetric objects, no matter how the tissues and the symmetric organs surrounding them are bent. Similarity of non-rigid shapes was considered in recent papers of Elad and Kimmel [9], Mémoli and Sapiro [22], and Bronstein et al. [4]. Using non-rigid embedding methods we can embed our surface into itself for exploring the self similarity. Here we address the problem of finding intrinsic symmetries of non-rigid shapes. We define criteria of global and local asymmetry and present a practical method for their computation. The rest of this thesis is organized as follows: In Chapter 2 we define intrinsic and extrinsic symmetries. Chapter 3 is devoted to numerical schemes for symmetry computation. Experimental results are presented in Chapter 4 and conclusions in Chapter 5.
Figure 1.1: Symmetric or not? Visualization of the difference between extrinsic and intrinsic symmetry: extrinsically symmetric shape is also intrinsically symmetric (left), however, an isometry of the shape is intrinsically symmetric but extrinsically asymmetric (right).
Chapter 2

Extrinsic and intrinsic symmetries

Let us begin our exploration using a toy example, which emphasizes the difference between the orthodox definition of extrinsic symmetries and the intrinsic ones. Consider a square cut of a piece of paper (Figure 2.1) as a regular surface in space. The symmetry group of the square consists of eight members which can be generated from a 90 degrees rotation and one reflection. It is called the Dihedral group $D_4$. If we bend one of the corners only one reflection beside the trivial identity member remains. The symmetry group becomes $G$ which is the only group of order two. Intrinsically nothing changed due to the bended corner. Assuming an insect walking on this piece of paper, it would ignore the bending. From the insect’s point of view distances measured on the piece of paper are invariant under bending. Intrinsic symmetries capture such invariants. In contrast to the extrinsic symmetry group, the intrinsic symmetry group remains $D_4$.

The most common symmetries are those calculated in $\mathbb{R}^n$. The induced Euclidean metric enforces a strict definition of symmetries. We are familiar with all possible symmetries of an object in Euclidean space. The need to preserve distances, which are computed according to straight lines, allows only translations, rotations and reflections to be optional
CHAPTER 2. EXTRINSIC AND INTRINSIC SYMMETRIES

Figure 2.1: Two squares embedded in \( \mathbb{R}^3 \). The extrinsic symmetry group of the left square is \( D_4 \) and the extrinsic symmetry group of the right square is \( G \). The intrinsic symmetry groups of both squares are \( D_4 \). The geodesic distances remain invariant under bending, thus, it will have no effect on the intrinsic symmetry group.

symmetries. More formally, let \( V \subset \mathbb{R}^n \) be a shape in space, then the only possible symmetries of \( V \) are orthonormal operators from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). Let \( M_n(\mathbb{R}) \) be an \( n \times n \) matrix with elements in \( \mathbb{R} \), then, \( \{ S \in M_n(\mathbb{R}) \text{ such that } S^t S = I \} \) are the only optional symmetries. Note that \( \text{det}(S) \) can be either 1 or -1, representing rotations and reflections, respectively. Since the essence of symmetry is the object’s invariance with respect to \( S \), we are looking for a possible \( S \) such that \( S(V) = V \). Under these conventions only rotations and reflections are optional, and we refer to them as compact symmetries. In order to somehow disconnect our shape from its embedding in Euclidean space let us redefine symmetries in a more flexible way. We model a non-rigid shape \( X \) as a two-dimensional smooth compact connected and complete Riemannian surface (possibly with boundary) embedded in \( \mathbb{R}^3 \). The manifold can be considered as a metric space with the geodesic metric \( d_X : X \times X \to \mathbb{R}^+ \) measuring the lengths of the shortest paths on \( X \), where the length structure is induced by the Euclidean metric \( d_{\mathbb{R}^3} \). Using these notations we write the induced Euclidean metric as \( d_{\mathbb{R}^3} |_X \). Here we consider \( \mathbb{R}^3 \) to be the metric space, and the distances on \( X \) are induced from it. In simple words, the distance between two points is the length of the straight line in \( \mathbb{R}^3 \) which is stretched between them. We broadly refer
2.1. SYMMETRY SPACE

to properties described in terms of the metric \(d_X\) as to the **intrinsic geometry** of \(X\), and to properties associated with \(d_{\mathbb{R}^3}|_X\) as the **extrinsic geometry**. A transformation \(g : X \rightarrow \mathbb{R}^3\) preserving the extrinsic geometry of a shape \(X\) is called a **congruence**, and \(X\) and \(g(X)\) are called **congruent**. If in addition \(g(X) = X\), such a \(g\) is called a **self-congruence** or an **extrinsic symmetry** of \(X\). Extrinsic symmetries form a group related to function composition,

\[
\text{ESym}(X) = \{g : X \xrightarrow{1:1} X : d_{\mathbb{R}^3}|_X = d_{\mathbb{R}^3}|_X \circ (g \times g)\}.
\] (2.1)

which we call the **extrinsic symmetry group** of \(X\). \text{ESym}(X)\) contains a subset of rotations and reflections transformations in \(\mathbb{R}^3\), to which the shape \(X\) is invariant. Analogously, a transformation \(g : X \rightarrow \mathbb{R}^3\) preserving the intrinsic geometry of a shape \(X\) is called an **isometry**, and the shapes \(X\) and \(g(X)\) are referred to as **isometric**. An isometry mapping \(X\) onto itself is called a **self-isometry** or an **intrinsic symmetry** of \(X\). Like their extrinsic counterparts, intrinsic symmetries form the **intrinsic symmetry group**,

\[
\text{ISym}(X) = \{g : X \rightarrow X : d_X = d_X \circ (g \times g)\}.
\] (2.2)

The intrinsic symmetry group is invariant to isometries of the shape. Since every extrinsic symmetry is also an intrinsic symmetry, \text{ESym}(X)\) is a subset of \text{ISym}(X)\). This fact is visualized in Figure 1, showing that an extrinsically symmetric shape is also intrinsically symmetric, yet an intrinsically symmetric shape might be extrinsically asymmetric.

2.1 Symmetry space

Since rotations and reflections are the only candidates for extrinsic compact symmetries, their representation is a simple one. Unfortunately, intrinsic symmetries do not have this luxury, and must be defined explicitly. One can find an example in Figure 2.2.
In contrast to extrinsic symmetries which can be easily described, intrinsic symmetries need to be defined explicitly. For example, a rotational symmetry of a square with a bended corner is defined explicitly for each point.

We explore the intrinsic symmetry space and its features by enforcing a metric form on it. We denote by $\mathcal{F}(X)$ the space of all continuous mappings $g : X \to X$ (not necessarily bijective), and define a metric on $\mathcal{F}(X)$ as

$$
d_{\mathcal{F}(X)}(f, g) = \sup_{x \in X} d_X(f(x), g(x))
$$

for all $f, g \in \mathcal{F}(X)$. Since $X$ is a compact metric space, and $f, g$ are continuous bounded functions, $\mathcal{F}(X)$ is a complete metric space. One can find a visual example of this metric in Figure 2.3. We refer to the set

$$
B_{\mathcal{F}(X)}(g, r) = \{ f \in \mathcal{F}(X) : d_{\mathcal{F}(X)}(g, f) \leq r \}
$$

as to the closed metric ball of radius $r$ centered at $g$ (as a matter of notation, we will omit $r$ referring to a ball of some unspecified radius). A ball forms a closed neighborhood of $g$.

We also define the **distortion** of a symmetry

$$
\text{dist}(g) = \max_{x, x' \in X} |d_X(x, x') - d_X(g(x), g(x'))|
$$

(2.5)
2.1. SYMMETRY SPACE

Figure 2.3: A visual example of the symmetry metric. One point is embedded back to the surface in different locations by two different functions (left and middle). We quantify the difference between the two functions regarding a given point according to the geodesic distance between the two embeddings (dotted curve, right frame). The difference between the two functions is the supremum of all embeddings.

For every mapping \( g \in F(X) \). In this context, the distortion is a function \( \text{dist} : F \rightarrow \mathbb{R}^+ \) measuring how \( d_X \) differs from \( d_X \circ (g \times g) \) in terms of preserving geodesic distances. In these terms, an intrinsic symmetry can be defined as a mapping \( g \in F(X) \) having \( \text{dist}(g) = 0 \). Moreover, intrinsic symmetries are also local minima of the distortion, in the sense that for every \( g \in \text{ISym}(X) \), there exists a sufficiently small neighborhood \( B_{F(X)}(g) \subset F(X) \), such that any \( f \in B_{F(X)}(g) \) has \( \text{dist}(f) \geq \text{dist}(g) \).

\( F(X) \) is an abstract space which holds the symmetry’s characteristics. A surface may or may not have intrinsic symmetries. It may have a discrete set of symmetries, an infinite number of symmetries or a combinations of both. Visualizing \( F(X) \) allows us to peek into the world of intrinsic symmetries and to define several symmetry classes. Since this process involves a numerical procedure we first present an approximation scheme of the whole process. In chapter 4 we present a visualization approach for approximated symmetry spaces.
2.2 Approximate symmetries

In real applications, due to acquisition and representation inaccuracies, a perfect symmetry rarely exists, even for a truly symmetric shape. Non-symmetric shapes have a trivial intrinsic symmetry group, containing only the identity mapping $\text{Identity}(x) = x \ \forall x$. However, while not symmetric in the strict sense, a shape can still be approximately symmetric. Symmetry was once considered a binary state, meaning, a shape is either symmetric or not. [32] showed that extrinsic asymmetry can be measured as a continuous function. We adopt a similar approach regarding intrinsic symmetries. An intuitive way to understand the difference between the two definitions, is by thinking of a non-symmetric shape as obtained by applying a deformation to some other symmetric shape. Such a deformation may break the symmetries of the shape: if previously a symmetry was a self-isometry, we now have mappings which have non-zero distortion. A good example of an approximate extrinsic symmetry is our face. The face is not truly symmetric, but there is a tremendous resemblance between the left and the right sides. A trivial method which reveals extrinsic facial asymmetries is based on point matching between the face and its reflection after using a family of algorithms called iterative closest point (ICP).

Since we are looking for all possible approximate symmetries we must first define a threshold for our asymmetry. For a large enough threshold, it is clear that every function is a possible symmetry, and for a zero threshold only the identity mapping is a possible symmetry as numerical inaccuracy, and sampling issue usually eliminate alternative options. We call a map $g \in \mathbb{F}(X)$ with $\text{dist}(g) \leq \epsilon$ an $\epsilon$-self-isometry, and, by analogy with symmetries, define the intrinsic $\epsilon$-symmetries of $X$ as the set of $\epsilon$-self-isometries. The problem with this definition is its lack of consistency with true symmetries. Each true symmetry is represented by an infinite number of approximate symmetries. One can think of the best
candidate as a local minimum of the distortion in space of all mappings. More formally, approximate intrinsic symmetries are local minima of the distortion, in the sense that for a \( g \in \mathbb{F}(X) \) such that \( \text{dist}(g) \leq \epsilon \) exists a sufficiently small neighborhood \( B_{\epsilon}(X)(g) \subset \mathbb{F}(X) \), such that any \( f \in B_{\epsilon}(X)(g) \) has \( \text{dist}(f) \geq \text{dist}(g) \). We can define the \( \epsilon \) ISym set as

\[
\text{ISym}(X, \epsilon) = \left\{ g \in \mathbb{F}(X) : \epsilon \geq \text{dist}(g) = \min_{f \in B_{\epsilon}(X)(g)} \text{dist}(f) \right\}.
\]

Visual examples can be found in Chapter 4. Unlike symmetries, \( \epsilon \)-symmetries are not closed under function composition and therefore do not form a group.

In practice, \( \epsilon \) is unknown a priori, and choosing different values of \( \epsilon \) results in different structures of \( \text{ISym}(X, \epsilon) \). Experiments show that choosing \( \epsilon \) to be twice the sampling radius of discrete shapes produces meaningful results.

In order to quantify how a point on \( X \) contributes to the global shape asymmetry, we define the local shape asymmetry, with respect to mapping \( g \), as

\[
\text{asym}(X, x) = \max_{x' \in X} |d_X(x, x') - d_X(g(x), g(x'))|,
\]  

(2.6)

Measuring the distortion of \( g \) at a point \( x \). In addition, we can quantify the global shape asymmetry, for a given symmetry \( g \), as the normalized total asymmetry,

\[
\text{asym}(X) = \frac{\int_X \text{asym}(X, x)dS(x)}{\text{Perim} \times \int_X dS(x)}.
\]  

(2.7)

Where \( dS(x) \) is the infinitesimal area around \( x \), and \( \text{Perim} \sqrt{\int_X dS(x)} \) for uniform scaling. Note that other \( L_p \) norms could be used in (2.6). In practice, lower \( p \) values appear to be more robust to noise.
Chapter 3

Computation of intrinsic symmetries

There are several ways to represent a surface. Here we consider a simple one known as cloud of points. The surface $X$ is sampled at $N$ points, constituting an $r$-covering (i.e., $X = \bigcup_{n=1}^{N} B_X(x_n, r)$, where $B_X$ denotes a closed metric ball on $X$). We denote this sampling by $X_r = \{x_1, ..., x_N\} \subseteq X$. A good sampling of the surface can be achieved using the farthest point sampling algorithm [9, 11, 25, 27], which guarantees that $X_r$ is also $r$-separated, i.e. $d_X(x_i, x_j) \geq r$ for any $i \neq j$. The extrinsic geometry of $X$ is approximated by a triangular mesh $\hat{X}$ built upon the vertices $X_r$. In order to approximate the intrinsic geometry, we use the fast marching method [18], which produces a first-order approximation for the geodesic distances between $X_r$ on $\hat{X}$.

Our goal is to approximate (2.7) and (2.6), and find mappings of minimal distortions. In general, point matching [23] is an NP-hard combinatorial problem. Since a straightforward search is not feasible, a multi-resolution scheme is required. First we solve the problem on a coarse grid and then perform a multi-resolution refinement. For solving the problem on a coarse grid we present three optional methods. The refinement stage is an optimization scheme. In [4], Bronstein et al. proposed a relaxation for problems of this
CHAPTER 3. COMPUTATION OF INTRINSIC SYMMETRIES

type by means of continuous optimization on a triangular mesh \( \hat{X} \), in the spirit of the multidimensional scaling (MDS) problem. Their framework is referred to as \textit{generalized MDS} (GMDS). Here we adopt this method and use it to measure the error of embedding a given shape into itself.

3.1 Coarse initialization

Given \( X_r \), we sub-sample it with a larger radius \( R \), producing a sparser sampling \( X_R \subset X_r \) containing \( M \ll N \) points. We denote by \( \mathcal{F}(X_R) \) the set of all mappings \( \pi : X_R \rightarrow X_R \) (permutations on the discrete set \( X_R \)), which can be represented as \( M \)-tuples \( \pi = (\pi_1, ..., \pi_M) \in \{1, ..., M\}^M \). Without loss of generality, we set \( \pi_1 = (1, 2, ..., M) \) to be the identity map.

Solving the original problem on a coarse grid is still a hard combinatorial problem. We present three alternative methods for coarse initialization. The first method is based on Branch-and-Bound family of algorithms. The second method focuses on the shape’s boundary, if such exists. We find possible symmetries for geodesics between points along the boundary, and then interpolate each symmetry to the entire surface. The third method uses Euclidean embedding of intrinsic symmetries into that of extrinsic ones. For that we can use known symmetry search methods.

3.1.1 Low dimension search

Finding all \( M \)-tuples with similar metric behavior as \( \pi_1 \) potentially requires computing the distortion of \( \mathcal{O}(M^M) \) mappings. However, the search space can be reduced by ruling out mappings that are unlikely to have low distortion.

We observe that in order for \( \pi \) to be a good candidate for an approximate symmetry, the
3.1. COARSE INITIALIZATION

intrinsic properties of the surface, such as the behavior of the metric \( d_X \) around every \( x_i \) should be similar to that around \( x_{\pi_i} \). In order to quantify this behavior, for each \( x_i \in X_R \) we compute the histogram \( h_i = \text{hist}(\{\hat{d}_{ij} : \hat{d}_{ij} \leq \rho\}) \) of the approximate geodesic distances in a \( \rho \)-ball centered \( x_i \) (in our implementation, the parameter \( \rho \) was set to \( \infty \)). In the last definition we used the convention \( \hat{d}_{ij} = d_X(x_i, x_j) \). Using the vectors \( h_i \) as local descriptors of the points in \( X_R \), we compute the dissimilarity of two points \( x_i, x_j \in X_R \). Our practice shows that a straightforward use of the Euclidean distance \( \|h_i - h_j\|_2 \) between the descriptors may be inaccurate due to binning errors. To account for distances between the bins, we use the weighted histogram distance, \( d(h_i, h_j) = \sqrt{(h_i - h_j)^T A (h_i - h_j)} \), where \( A_{mn} \) are the distances between the bins \( m \) and \( n \).

For each point \( x_i \) in \( X_R \), we construct a set \( C_i \subset \{1, \ldots, M\} \) of indices of \( K \) points having the most similar descriptors. \( K \) is selected to be a small number, typically significantly smaller than \( M \). We define the reduced search space \( \mathcal{F}_{\text{init}} = C_1 \times C_2 \times \ldots \times C_M \). Mappings copying any \( x_i \) to \( x_{\pi_i} \notin C_i \) are excluded from the search space. Even though the coarse sample size \( M \) and the number of initial matches for every point are relatively small, \( \mathcal{F}_{\text{init}} \) has still \( \mathcal{O}(K^M) \) mappings, making an exhaustive search prohibitively expensive. However, following [12], we can use the following hierarchical greedy algorithm for selecting a reasonably good mapping from \( \mathcal{F}_{\text{init}} \). We must remember that the following procedure is not mandatory. Searching for all possible symmetries will take us directly to the branch-and-bound search. If we search for the best symmetry (e.g. a reflective symmetry in the face) then the suggested initialization is usually sufficient.

1. **Pairing:** For each pair \( (i, j) \in \{1, \ldots, M\}^2 \), choose the best pair \( (m, n) \in C_i \times C_j \) minimizing the distortion \( |\hat{d}_{ij} - \hat{d}_{mn}| \). This establishes a two-point correspondence \( (i, j) \mapsto (m, n) \). The outcome of this step is the set of \( \mathcal{O}(M^2) \) two-point correspondences \( E_2 \), which we sort in increasing order of distortion.
2. **Merging:** The pairs are merged into four-point correspondences. Taking the first two-point correspondence \( e \in E_2 \), we find another two-point correspondence having a disjoint domain and minimizing the distortion of the obtained four-point correspondence. We remove all correspondences sharing the same domain from \( E_2 \) and continue until \( E_2 \) becomes empty. The merging continuous hierarchically, producing \( E_{2k} \) from \( E_k \), stopping typical at \( E_8 \) or \( E_{16} \).

3. **Completion:** We select the minimum distortion correspondence \( (i_1, \ldots, i_k) \mapsto (\pi_{i_1}, \ldots, \pi_{i_k}) \) from the last produced \( E_k \), and complete it to a full \( M \)-point correspondence by adding the missing indices \( \{i_{k+1}, \ldots, i_M\} = \{1, \ldots, M\} \setminus \{i_1, \ldots, i_k\} \) and their images \( \pi_{i_{k+1}}, \ldots, \pi_{i_M} \). For each added point \( j \), we select

\[
\pi_j = \arg \min_{\pi_j \in \{1, \ldots, M\}} \max_{i \in \{i_1, \ldots, i_k\}} |\hat{d}_{ij} - \hat{d}_{\pi_i, \pi_j}|.
\]

Return the found mapping \( \pi \) and its distortion \( \epsilon_{\min} \).

Since the algorithm never backtracks, it may produce a sub-optimal mapping \( \pi \). However, practice shows that if some good pairs are found at Step 1, the algorithm tends to produce a very good estimate for the minimum distortion mapping on \( \mathbb{F}_{\text{init}} \).

A guaranteed global minimum on \( \mathbb{F}_{\text{init}} \) can be computed by using a branch and bound algorithm similar in spirit to that presented in [12] for improving convergence of ICP-based extrinsic surface alignment.

1. Given a correspondence of \( k - 1 \) feature points \( (1, \ldots, k - 1) \mapsto (\pi_1, \ldots, \pi_{k-1}) \), we would like to establish \( k \mapsto \pi_k \).

2. **Prune:** For each potential correspondence \( \pi_k \in C_k \), evaluate \( \max_{i=1, \ldots, k} |\hat{d}_{ik} - \hat{d}_{\pi_i, \pi_k}| \). If the obtained distortion is larger than \( \epsilon_{\min} \), discard the potential correspondence.
3. **Branch:** For each remaining $\pi_k$, recursively invoke Step 1 with $(1, \ldots, k) \mapsto (\pi_1, \ldots, \pi_k)$.

4. **Bound:** If $k = M$, compute the distortion $\text{dis}(\pi)$. If $\text{dis}(\pi) < \epsilon_{\text{min}}$, set $\epsilon_{\text{min}} = \text{dis}(\pi)$ and $\pi_{\text{min}} = \pi$.

Note that if we look for all possible coarse initializing then $\epsilon_{\text{min}}$ should not change (step 4). $\epsilon$ remains the unchanged parameter of acceptable asymmetry value.

Using P4 3GHz computer on approximately 3000 points mesh in search of about 5 points, coarse match is done in several seconds using non-optimized Matlab code.

### 3.1.2 Dimensionality reduction

It is important to reduce the search space to a feasible one. Since an intrinsic symmetry is actually a self-isometry of a metric space, we know that the boundary must be mapped onto itself. Restricting the intrinsic symmetries to the boundary alone can be done easily and efficiently. Interpolating the resulting symmetry to the entire surface could, in some cases, provide an interesting result.

Is the data along the boundary sufficient for intrinsic symmetries? The answer can not be known apriori, but we can think of a heuristic approach. Let us examine two problems which lead towards good heuristics. Imagine a square with an asymmetric bump in the middle. The geodesic curves between any two boundary points do not pass through the asymmetric part. Another example deals with a surface without a boundary. For example, a sphere. Punching a hole in the sphere changes its topology, producing a boundary. All the geodesics between the boundary points are extremely short and hold little information about the rest of the surface.

A good heuristic indicator that measures the suitability of this method is the partial area
created by minimal geodesic curves between all boundary samples. Let

\[ \text{Geodesic}(x_i, x_j) = \arg\min_{C(x_i, x_j)} \text{Length}(C(x_i, x_j)), \]  

where \( C(x_i, x_j) \) is a continuous curve on the surface \( X \) connecting \( x_i \) with \( x_j \) and \( \text{Length}(C) \) is its length. Note that a geodesic curve between \( x_i \) and \( x_j \) does not have to be a curve with a minimum length, but we do choose the minimum in definition (3.1).

Define,

\[ \tilde{X} = \{ x \in X \text{ if } \exists x_i, x_j \in \partial X \text{ } x \in \text{Geodesic}(x_i, x_j) \} \]  

\[ \text{Indicator}(X) = \frac{\int_X dS(x)}{\int_X dS(x)} = \frac{\text{Area}(\tilde{X})}{\text{Area}(X)}, \]  

where \( dS(x) \) is the infinitesimally area around \( x \). \( \text{Indicator}(X) = 1 \) implies that the set of all points, beside a set of measure zero, are influenced by the boundary (e.g. planner shapes), while \( \text{Indicator}(X) = 0 \) implies that the geodesic curves between boundary samples do not capture any information about the shape. We define the Boundary-ISym in a similar way to ISym. Distances are measured on the surface, yet the metric equality needs to hold only along the boundary.

\[ \text{Boundary-ISym}(X) = \{ g : \partial X \rightarrow \partial X : d_X = d_X \circ (g \times g) \}. \]  

Given an optional symmetry \( g \), we can define the boundary asymmetry by

\[ \text{asym}(X, x) = \max_{x' \in \partial X} |d_X(x, x') - d_X(g(x), g(x'))| \]

\[ \text{Basym}(X) = \max_{\partial X} \text{asym}(X, x) \]  

The numerical scheme needed to find a possible symmetry \( g \) which minimize (3.5) is a simple one. The combinatorial problem becomes a simple rotation and reflection search.

We sample the boundary using \( M \) points, \( x_i \in \partial X \) \( i = 1, \cdots, M \) such that

\[ d_X(x_i, x_{i+1}) = d_X(x_{i-1}, x_i). \]
where $x_i = x_{i+M}$. We can represent each possible symmetry as a shift between boundary samples, with and without a reflection. The discretization of (3.5) becomes

\[
\text{Rot-asym}(i) = \sum_{1 \leq j \leq M} \max_{1 \leq k \leq M} |d_X(x_j, x_k) - d_X(x_{j+i}, x_{k+i})| 
\]

(3.7)

\[
\text{Ref-asym}(i) = \sum_{1 \leq j \leq M} \max_{1 \leq k \leq M} |d_X(x_j, x_k) - d_X(x_{2-j+i}, x_{2-k+i})|.
\]

(3.8)

For a possible index $i$, Rot-asym represent the asymmetry value after a rotational shift, and Ref-asym represent the asymmetry value after a rotational shift followed by a reflection around sample $x_1$. Similar to intrinsic symmetries we can use other $L_p$ norms as well, which are more robust to noisy data.

In Figure 3.1 Rot-asym and Ref-asym of the square toy example are presented as a polar graph. The rotations are normalized to a possible angle, and the asymmetry is represented by the radius. All eight symmetries are represented as local minima. In Figure 3.2 we perform a similar task on a human face. The face has only one reflective symmetry beside the trivial. The two minima are easily detected in the graphs. On the human face we can see the robustness of this procedure since the pruning of the face is not accurate and yet the reflective symmetry can easily be detected.

Calculating the geodesic distances between samples is performed only once. Since the distances between boundary samples are similar, the distances after a cyclic shift is simply a cyclic shift of the distance matrix. If we define the matrix $(D)_{ij}$ to hold the geodesic distances between $x_i \in \partial X$ and $x_j \in \partial X$ then a cyclic shift of the boundary samples is the same as performing a cyclic shift of the columns followed by a cyclic shift of the rows (or vice versa).

We can summarize the procedure by
**Algorithm 1** Boundary Initializer

1. Resample the surface with equal distances between boundary samples.
2. Find the surface’s boundary, and check the method’s suitability.
3. Calculate geodesic distances between boundary samples.
4. Calculate Rot-asym and Ref-asym.
5. Locate local minima as best candidates.
6. Interpolate the boundary symmetry to the entire surface.

---

**Figure 3.1**: A polar graph representing the rotational (red) and reflective (blue) boundary asymmetry of the square toy example. The radius represents the asymmetry value and the angle is the normalized rotational shift between boundary samples. All eight minima, representing the eight intrinsic symmetries are found.
3.1. COARSE INITIALIZATION

Figure 3.2: Similar to Figure 3.1 we represent boundary asymmetry. The radios represents the asymmetry value and the angle is the normalized rotational shift between boundary samples. The two intrinsic symmetries (identity and reflective) are found. The face was pruned roughly with low accuracy, but it has little effect on the boundary’s asymmetry.

3.1.3 Canonical Forms

Even for shapes with simple intrinsic geometry, the complexity of $\mathbb{F}(X)$ is likely to be enormous. The lack of a simple parametrization, similar to the one available for describing extrinsic symmetries, makes the analysis of symmetries of non-rigid shapes significantly more difficult. A way to circumvent this difficulty is to try casting the intrinsic symmetries of $X$ into extrinsic symmetries of an alternative representation of $X$.

We define the extrinsic distortion in a similar way to the intrinsic one by

$$\text{edist}(g) = \max_{x,x' \in X} |d_{\mathbb{R}^3}|_{X}(x, x') - d_{\mathbb{R}^3}|_{X}(g(x), g(x'))|.$$  \hspace{1cm} (3.9)

We call a map $g \in \mathbb{F}(X)$ with $\text{edist}(g) \leq \epsilon$ an $\epsilon$-self-congruence, and denote the collection of such maps by

$$\text{Con}(X, \epsilon) = \{ g \in \mathbb{F}(X) : \text{edist}(g) \leq \epsilon \}.$$  \hspace{1cm} (3.10)
and the collection of all $\epsilon$-self-isometry by

$$\text{Iso}(X, \epsilon) = \{ g \in F(X) : \text{dist}(g) \leq \epsilon \}.$$  \hfill (3.11)

Let $(Z, d_Z)$ be a generic homogeneous metric space endowed with a simple metric (ideally, there should exist a closed form expression for $d_Z$); we require homogeneity to obtain a simple isometry group $\text{Iso}(X)$. For a moment, let us also assume that there exists an isometric embedding of $X$ into $Z$, that is, a map $\varphi : X \rightarrow Z$ with $d_X = d_Z \circ (\varphi \times \varphi)$. We refer to the image of $X$ under $\varphi$ as a canonical form of $X$ in $Z$ [10]. Clearly, canonical forms are defined up to an isometry in $Z$, since $d_Z \circ (\varphi \times \varphi) = d_Z \circ ((\varphi \circ i) \times (\varphi \circ i))$ for any $i \in \text{Iso}(Z)$. The canonical form $Z = \varphi(X)$ represents the intrinsic geometry of $X$ in the sense that the two metric spaces $(X, d_X)$ and $(Z, d_Z|_Z)$ are isometric and, consequently, have isomorphic intrinsic symmetry groups. Moreover, since the intrinsic geometry of $Z$ coincides with its extrinsic counterpart, the analysis of the intrinsic symmetry group of the shape reduces to the analysis of the extrinsic symmetry group of its canonical form. Therefore, if the embedding space $Z$ has a reasonably simple isometry group (preferably with a convenient parametrization), the search for intrinsic symmetries is greatly simplified. For example, if $Z$ is selected to be a low-dimensional Euclidean space, conventional extrinsic symmetry detection algorithms can be employed [24].

This conclusion was based on the assumption that $\varphi$ is an isometric embedding, and the reader may argue whether such an embedding exists at all. Unfortunately, the answer is negative in most cases. However, we can assume that $\varphi$ is a minimum distortion embedding of $X$ into $Z$,

$$\varphi = \arg \min_{\varphi : X^{\text{1-1}} \rightarrow Z} \sup_{x, x' \in X} |d_X(x, x') - d_Z(\varphi(x), \varphi(x'))|,$$  \hfill (3.12)

and repeat our reasoning replacing the assumption $d_X = d_Z \circ (\varphi \times \varphi)$ with

$$\sup_{x, x' \in X} |d_X(x, x') - d_Z(\varphi(x), \varphi(x'))| \leq \delta.$$  \hfill (3.13)
3.1. COARSE INITIALIZATION

Let $X$ be a shape, and let $Z$ be its canonical form created by the embedding $\varphi : X \xrightarrow{1:1} Z$ with distortion $\delta$. Then, for every $f \in \text{Iso}(X, \epsilon)$, $\varphi \circ f \circ \varphi^{-1} \in \text{Con}(Z, \epsilon + 2\delta)$; and for every $g \in \text{Con}(Z, \epsilon)$, $\varphi^{-1} \circ g \circ \varphi \in \text{Iso}(X, \epsilon + 2\delta)$.

An alternative way to write the latter result in terms of relations between the symmetry spaces is

$$
\varphi \circ \text{Iso}(X, \epsilon) \circ \varphi^{-1} \subseteq \text{Con}(Z, \epsilon + 2\delta),
\varphi^{-1} \circ \text{Con}(Z, \epsilon) \circ \varphi \subseteq \text{Iso}(X, \epsilon + 2\delta).
$$

Observe that in the particular case where $\delta = 0$, the two spaces are equivalent; furthermore, if $\epsilon = 0$, $\varphi$ is a group isomorphism. We conclude that the applicability of intrinsic symmetry analysis based on canonical forms relies inherently on the ability to produce a low-distortion embedding $\varphi$. For example, if $Z = \mathbb{R}^2$, the approach is suitable for nearly “flat” shapes with small Gaussian curvature. The method is usually unsuitable for complicated intrinsic geometries, which cannot be faithfully represented as subsets of generic embedding spaces.

Let us show some experimental results. Figure 3.3 presents the embedding of the human body in $\mathbb{R}^2$ and in $\mathbb{R}^3$. In $\mathbb{R}^2$ the human’s intrinsic symmetries are revealed, while the embedding in $\mathbb{R}^3$ does not poses extrinsic symmetries. As expected, the legs are spread in one direction while the hands are spread orthogonally to the legs. In Figure 3.4 we view all stages of embedding a giraffe with a long leg in $\mathbb{R}^3$, and the coarse pair matching due to a possible reflective symmetry. Apriori we can not predict that due to the high embedding error the symmetry analysis would still be valid. Note that this experiment as well as others shows high rate of success regardless of the embedding error.

We can summarize the procedure in the following algorithm.
Algorithm 2 Canonical Form Initializer

1. Calculate geodesic distances between mesh samples.
2. Embed the surface in an Euclidean space.
3. Calculate extrinsic symmetries.
4. Generate sample pairs according to each extrinsic symmetry.
5. Map the pairs back to the surface.

Figure 3.3: The human body holds intrinsic reflective symmetry. The embedding in the Euclidean space $\mathbb{R}^2$ (middle figure) reveals this symmetry, while the embedding in $\mathbb{R}^3$ (right figure) does not.

3.2 Local refinement

The second stage of our approach is a local refinement. We use the first stage as a coarse-grid initialization for a multi-resolution optimization scheme used to solve the GMDS problem.

We optimize over the images $x'_i = g(x_i)$,

$$
\min_{\{x'_1,\ldots,x'_N\} \subset \hat{X}} \max_{j>1} \left| \hat{d}_{ij} - \hat{d}_X(x'_i, x'_j) \right|,
$$

(3.15)

where $\hat{d}_{ij}$ is the geodesic distance between points $x_i$ and $x_j$ of $\hat{X}$, and $\hat{d}_X$ is the geodesic distance on $\hat{X}$ between two samples, not necessary mesh points. An alternative approach to solve (3.15) is to use an iterative reweighted scheme which approximates the GMDS
3.2. LOCAL REFINEMENT

Figure 3.4: Summarizing the Canonical Form intializer. In the first row we present the embedding in $\mathbb{R}^3$ and the reflective plane found. In the second row we couple up several pairs according to the extrinsic symmetry, and point map them back to the original surface. The pair $(i, g(i))$ represent a matched pair.

The problem written as a min-max problem. More details can be found in [3]. Here we will elaborate on the $L_2$ norm solution of the GMDS problem,

$$
\min_{\{x'_1, \ldots, x'_N\} \in \mathcal{X}} \max_{j \geq i} \left( \hat{d}_{ij} - \hat{d}_X(x'_i, x'_j) \right)^2.
$$

(3.16)

In the following experimental results we used a similar cost function based on the summed asymmetry,

$$
\min_{\{x'_1, \ldots, x'_N\} \in \mathcal{X}} \sum_{j > i} \left( \hat{d}_{ij} - \hat{d}_X(x'_i, x'_j) \right)^2.
$$

(3.17)

Since there is no analytic expression for the geodesic distances of each possible symmetry, we need to numerically approximate them. Each sample falls inside a face of a triangulated mesh, and we interpolate the unknown distances using known mesh distances.
which were computed once. Following [3] we use *barycentric coordinates* to represent each point which allows us to detach ourself from the embedding space. We can rewrite a point $x'_i$ on our surface as a convex combination of the three triangle vertices surrounding it. Let $t_i$ be the triangle in which $x'_i$ is present, and $x_{t_i,j} \ 1 \leq j \leq 3$ its three vertices. We can rewrite,

$$x'_i = u_i x_{t_i,1} + v_i x_{t_i,2} + (1-u_i-v_i)x_{t_i,3}. \quad (3.18)$$

In order to approximate $\hat{d}_X(x'_i, x'_j)$ we first calculate $\hat{d}_X(x'_i, x_{t_j,k})$ for $1 \leq k \leq 3$ using the barycentric coefficients. We use a linear interpolation

$$\hat{d}_X(x'_i, x_{t_j,k}) = u_i \hat{d}_X(x_{t_i,1}, x_{t_j,k}) + v_i \hat{d}_X(x_{t_i,2}, x_{t_j,k}) + (1-u_i-v_i)\hat{d}_X(x_{t_i,3}, x_{t_j,k}). \quad (3.19)$$

Note that all the geodesic distances in the above expression are between fixed vertices, and can be computed once. We can now interpolate

$$\hat{d}_X(x'_i, x'_j) = u_j \hat{d}_X(x'_i, x_{t_j,1}) + v_j \hat{d}_X(x'_i, x_{t_j,2}) + (1-u_j-v_j)\hat{d}_X(x'_i, x_{t_j,3}). \quad (3.20)$$

Inserting (3.20) into (3.17) yields the cost function

$$\sigma(u_1, ..., u_N, v_1, ..., v_N, t_1, ..., t_N) = \sum_{j>i} (\hat{d}_X(x'_i, x'_j) - \hat{d}_X(x_i, x_j))^2. \quad (3.21)$$

For each $v_i$ and $u_i$ we fix all other variables, thus we have a quadratic function which can be solved analytically using a $2 \times 2$ Newton system. Note that we may receive an illegal barycentric coordinates. To avoid this problem we solve a *constrained* quadratic problem, in which the minimum can be achieved on the boundary as well.

In order to accelerate this procedure and to avoid shallow minima produced in the optimization method we adopt a multi-resolution scheme. We work with hierarchy of problems,
starting from the coarse initialization points, which were chosen in the previous section. The coarse solution is interpolated to the next resolution level, and is used as an initialization for that level’s optimization. Small local minima tend to disappear while substantial minima, like the actual intrinsic symmetries, are preserved. Such a data hierarchy can be constructed using the *farthest point sampling* strategy [11].
CHAPTER 3. COMPUTATION OF INTRINSIC SYMMETRIES
Chapter 4

Results

4.1 Symmetry detection

In order to assess the accuracy of our method, we performed an experiment of intrinsic symmetry detection on a data set of four different non-rigid shapes (dog, giraffe, man and crocodile). Each shape appeared in three instances: extrinsically symmetric (serving as a reference), intrinsically symmetric but not extrinsically symmetric (obtained by means of a near-isometric deformation of the reference shapes) and asymmetric (obtained by local non-isometric deformation). The local asymmetric features included enlarged ear of the dog, elongated leg of the giraffe, cropped hand of the man and enlarged foot of the crocodile.

Figure 4.3 visualizes our intrinsic symmetry detection method. The color represents the local measure of asymmetry ($\text{asym}(\hat{X}, x_i)$). The numbers represent the global measure of asymmetry ($\text{asym}(\hat{X})$). We were able to identify symmetric shapes and correctly detect the symmetry breaking features (marked with arrows in Figure 4.3).

We can visualize the accuracy of the procedure on the “Stanford Bunny” mesh. This
bunny is a well known scan of a clay bunny. The bunny is extrinsically asymmetric due to the rotated head and inaccuracies of the model. A recent paper [14] introduces extrinsic shape symmetrization of its different parts which are almost symmetric. Here we claim the bunny is almost intrinsically symmetric. In Figure 4.1 we can see that only after enlarging one of the ears we actually break the intrinsic reflective symmetry. In figure 4.2 we can see that the neck is indeed stretched due to the side pose, one front leg is a bit larger than the other and the ears have a different shape due to sampling.

### 4.2 Visualization of the space of symmetries

Extrinsic symmetries of a shape form a sub-group of the isometry group of \( \mathbb{R}^3 \), and thus its members can be easily parametrized (for example, in case of a rotational symmetry, such parameters can be a unit direction vector and an angle). This allows a simple description of the extrinsic symmetry group. Unfortunately, such a straightforward parametrization is
4.2. VISUALIZATION OF THE SPACE OF SYMMETRIES

Figure 4.2: Visualizing the intrinsic symmetry while stretching the colormap allows us to emphasize intrinsically asymmetric features. We can view asymmetry in the front legs, stretched neck and the different shape of the ears.

generally unavailable for intrinsic symmetries, and even less for their approximate counterparts. Information about the intrinsic symmetries of a shape can be inferred from the structure of the space $\mathcal{F}(X)$ and the associated distortion $\text{dist} : \mathcal{F}(X) \to \mathbb{R}$. Despite the very high dimensionality of $\mathcal{F}(X)$, its metric structure can be approximately visualized as a configuration of points in the Euclidean space, where each point represents a map $g \in \mathcal{F}(X)$, while the Euclidean distance between each pair of points approximates $d_{\mathcal{F}(X)}$ (Eq. 2.3). Such a representation can be straightforwardly created using MDS techniques.

An approximation of the distortion function is obtained by projecting the values of $\text{dist}(g)$ onto corresponding points in the representation space. The level sets of the approximate distortion function reveal the structure of the intrinsic symmetry group, exhibiting a pattern of local minima. Clearly pronounced local minima correspond each to a different symmetry.

Figure 4.4 visualizes the structure of the approximate symmetries space $\mathcal{F}(X)$ of a shape obtained by isometrically folding a planar patch. Such a shape has eight intrinsic
symmetries which are described by a dihedral group $D_4$, as opposed to only one extrinsic symmetry obtained by reflection with respect to a diagonal. The figure represent a level set sampling of the distortion for different epsilon values.

Finding the best $\epsilon$ is not needed since we are looking for stress minima. Choosing $\epsilon$ in the order of the geodesic distance between neighbors sampling of the surface produces good visible results. In the following figures we present the symmetry space using $\epsilon$ as described here.

Figure 4.5 presents another symmetry space sampling of the toy square example. Eight minima represent the eight intrinsic symmetries. The spacial arrangement of the four rotations versus the reflection can be easily detected in $\mathbb{R}^3$.

In Figure 4.6 we present the influence of $\epsilon$ on the symmetry space. For $\epsilon = 0$ only the identity symmetry is an option. For a small $\epsilon$ the identity cluster is formed, and as $\epsilon$ grows we receive the reflective symmetry cluster.

A continuous set of symmetries can be visualized by a distorted ring in $\mathbb{R}^2$ or a trimmed distorted cone in $\mathbb{R}^3$. The distortion depends on other discrete or non-discrete symmetries. Figure 4.7 presents the symmetry space of a torus knot. It has an almost intrinsic continuous symmetry, which is visualized as a 2D ring in the symmetry space. A double structure is created due to a reflective symmetry. A more detailed analysis of the data shows that the rotational symmetries should be represented by a torus and not by a ring. Due to embedding error in $\mathbb{R}^3$ we can not visualize the torus. A second representation problem is related to the ratio between the width and length of the torus knot. A second reflection symmetry is present along side the torus knot, which should be represented by another double form. Since the torus is narrow, the embedding errors overcome such reflective symmetry and we can not separate the structures.
4.2. VISUALIZATION OF THE SPACE OF SYMMETRIES

Figure 4.3: Detection of symmetries of non-rigid shapes. Color encodes the local measure of asymmetry (red corresponds to large local asymmetry). Numbers represent the global asymmetry measure (larger means less symmetric).
Figure 4.4: Symmetry space levelset sampling representation of a bended square. The local minima are visualized as separate clusters. The colors represent the symmetries.

Figure 4.5: Symmetry space representation of a bended square. Choosing $\epsilon$ in the order of the mesh sampling neighbors reveals the eight intrinsic symmetries. On the left we can see the square surface, in the middle we colored the identity mapping, and on the right we visualize all eight clusters.
4.2. VISUALIZATION OF THE SPACE OF SYMMETRIES

Figure 4.6: $\epsilon$ size has a crucial effect on the symmetry space sampling. On the left you can view the crocodile surface, in the middle we colored the identity mapping, and on the right we present three different symmetry spaces for different $\epsilon$. 
Figure 4.7: Continuous intrinsic symmetry within the torus knot is visualized as a ring. On the left is the torus knot surface, in the middle we colored the identity mapping, and the symmetry space is on the right. A reflective symmetry along side the torus creates a double form. Not all symmetries can easily be visualized in this method. We do not see another double form due to a reflective symmetry across the torus knot, and the topology of each ring should actually resembles a torus.

Figure 4.8: A combination of continuous and discrete symmetries. Continuous symmetries are visualized as a trimmed cone form, and the possible reflection is visualized by a double form. On the left one can view the double cone surface, in the middle we colored the identity mapping, and the symmetry space is on the right.
Figure 4.9: The rigid shape of a sphere is represented as a sphere in the symmetry space, both for extrinsic and intrinsic symmetries. On the left one can view the sphere surface, and on the right its rotational symmetry space. Note that a second, similar sphere, which represents the rotational symmetries after a reflection is not presented.
Chapter 5

Conclusions and Future work

We formulated the problem of detecting symmetries of non-rigid shapes. Our measure of symmetry relies on the intrinsic geometry of the shape, which allows to find symmetries that are insensitive to bending and deformation. We presented a practical approach for the numerical computation of intrinsic symmetries and demonstrate its accuracy on several examples. Since intrinsic symmetries can not be represented explicitly, we learn about their structure by embedding them in Euclid space.

Intrinsic symmetry and its approximation is a new concept introduced in this paper. During the research we had encountered several paths suitable for future work. We still need to explore examples in which extrinsic symmetries fail but intrinsic symmetries could tell an interesting story on the shape we analyze. Human abnormality, especially in newborns, and tumeral detection are good examples. We visualized the symmetry space but did not provide full classification theorem for all intrinsic symmetries. We calculated symmetries only for two dimensional Riemannian surfaces. Symmetrization process for extrinsic symmetries is a known procedure which was developed in recent papers, and we can adopt ideas for intrinsic symmetries as well. We feel that almost all results that deal with extrinsic
symmetries can be translated to bendable surfaces by using intrinsic symmetries instead of the traditional extrinsic ones. This claim is yet to be proven.
References


סימטריות של מערכות גמישים

ד"ר רביב
tighter constraints on misfolds

hypothesis about

length of cyclic loops regarding to the number of twists
then measured directly
by means of simulation

Dr. Rabbi

honoring lecturer - Technion - Technological University of Israel
December 2007
Haifa
CS-21-2007
המחוקר עשה בניית פורפורה וקילימל בקוליית הlemnיאט הנשתלמות.

הכרת תודה

אני מודה לך אלכסנדר ברונשטיין וליידר מייכל ברונשטיין על השתרפות המстроен במחקך.

יתר מוכלי安东尼 תודעה ענומקה לפורפourt קייל. כולם לאותר יאני תודעה.

כדי ההנהיה לא נשכח.

אני מודה ל-tankin על התמיכה הכספית הנדרשת בקוליית הlemnיאט.
תקציר

סימטריה וdęמיה עצמית היהי אובייקט בולט-equiv. מונחים ביכולת כל חלקי הפרוספקטיבות שונים ההפיסקיה. אובייקט החדר בלתי סימטרי ללכ הדרכים同時に על הלוט וסימטריה מותקנת על הא Tattoos ואלה מתוחכית.

הרהנה. זוהי חוסר סימטריה חשבונות יומיים כמה קירותרגליים, הדמויות תאיות, המיין ועיו.

באופף מתומי סימטריה מתוחכתי על פי מאפייני היצירתיות והתוכנות של האしょう. נוצר זה התוכנות

למשתנים קשיחים בין תוך צור משטחים ומושמים. סימטריה היצירת ידיעה לכל卜יקת שיני מרהיב

של האופקים ובSaga סימטריה הפרמץ שמר. שביעיה זו ניצי את משות הסימטריה הפרמיצת של

משטחים ומושמים. ניצי פתרון לכל תนำไปים מדויק סימטריה הפרמיצת תכלד התוכנות היא דיוק

המקומטיים את דיוד חוסר הסימטריה.

ואז מדגימים סוף דש של סימטריה, כל חפץ בחפץ את ייחודה. תבניות בתמונות הבחנה של גו הגזאם.

בצק שמאלה האופקים נמצאות במקוב מחוות סימטריות העצומות. כלומר, ה基辅 מתום המשק hội באופף מסתכלות את חצי חיים

על גב חצי השמאל. נאמר כי קלח זה סימטריה שחיה. בצק ימי הזהי האופקים את דיוד תכלד.

נשבר הסימטריה השכיפה החברוכית. אנוategori את הסימטריה המאפרים פניים של חוף האופקים ואינו בנ suscept

סימטריה ולכלו חלולו החברוכית, או בימיו אוחזרו בשコピー של חוף האופקים. גם שיפורים

כי חוף האופקים וינון אופי硬化 הסימטריה החברוכית מתאימים את בטפים תכלד. ינוי להרי כ bstית

התנהלות של חוף האופקים, הסימטריה הפרמיצת גוות כללות את החוף והשכיפה.

שمورה.Window במטסטסшим ומשתנה היה המרחבים וה оригинал ביניים. כלומר בקופס לימוד את

המרחבים בין שני נקודת עד פניהם של חוף ישיבת העבר ביינן, אך מתמודד את המרחבים על פי אורכי של

העתקים המוערכים ביותר עד פניהם. לדגון, ישון תנוחת הידידים והרגילים של חוף האופקים (כמסוי).
על מרחוקיםulled. הפועלים ושנים ידיקים מסומנים בהם דוגמיה וסימטריות ההצבה נותקות במחליק
ב_frontendנים ידועים שמרוחק יישום.

$$E_{(X)} = \{ g : X \rightarrow X \mid d'_{r} | x = d'_{r} | x \circ (g \times g) \}.$$

ככותרו כל הפונקציות הממשות על גלים אפורים או פונקציות המרחוק הממשות מגר損害 יסודי. המחבר
ואון مضındים מבמור מרחוקים גאוצידיים أي סימטריות הפרミטים חינך לפיתוש.

$$I_{(X)} = \{ g : X \rightarrow X \mid d'_{x} = d'_{x} \circ (g \times g) \}.$$
הפונקציה \( X \) שמתמשת ביריבות מדمهرجان ומקבל \( F(X) \) ב_white_גしていない פונקציות יצרות עם \( X \).\\n\\nמשתנה \( r \) המבוסס \( d_{F(X)} = \sup_{x \in X} d_x(f(x), g(x)) \) \( f, g : X \to X \)\\n\\nהמשתת癿 \( d_{F(X)}(g, r) = \{ f \in F(X) | d_{F(X)}(g, f) \leq r \} \)\\n\\nכ xmax всем \( g \) מתמשת ביריבות \( F(X) \) קיימים \( r \) לอ่าน العامة במפגש פונקציה.

6'עותור טכני\
בenny שלימי\

 april, 2007

שלטומטריות תר noreferrer יש ייצוג פושט הניתן על ידי סטנדרט שיקוף, או במכרה הפוןמי או טאליקם חצץ

שלטומטריות ביאף מפורש. בקובדה וה נקז ביאפ יוחאי מסף מרחבי סטנטוריה ברידיס, אינספסיים וילוב

קובדה והtestdata היגן דמל חשק, וחקור שלטומטריות. הובודה ליציאה ברארית

ה errorCode מספר כיוונים להבשך. כלכל, נקז ליום הכ קלבובousedown על מושתה לדישודים קשייה וטימטרית

ה랐ראון מיתוג להרחבה למישתים למישות וטימטריות פנייה. זו והכללה שדוי נזר לבוחר, כמי את

המנעולים במנודות שפעגון לתחום בעית בוחנוב שוני.