MINIMAL MODEL SEMANTICS FOR
FIRST-ORDER GROUND NON-MONOTONIC
MODAL LOGIC

BENJAMIN GRIMBERG
MINIMAL MODEL SEMANTICS FOR FIRST-ORDER GROUND NON-MONOTONIC MODAL LOGIC

RESEARCH THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN COMPUTER SCIENCE

BENJAMIN GRIMBERG

SUBMITTED TO THE SENATE OF THE TECHNION – ISRAEL INSTITUTE OF TECHNOLOGY

HESHVAN, 5767 HAIFA NOVEMBER 2006
THE RESEARCH THESIS WAS DONE UNDER THE SUPERVISION OF ASSOC. PROF. MICHAEL KAMINSKI IN THE FACULTY OF COMPUTER SCIENCE.

I would like to thank Michael Kaminski for his excellent guidance throughout the course of this research.

THE GENEROUS FINANCIAL HELP OF THE TECHNION IS GRATEFULLY ACKNOWLEDGED.
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Abstract

Non-monotonic logics are intended to simulate the process of human reasoning by providing a formalism for deriving consistent conclusions from an incomplete description of the world. Roughly speaking, the main idea lying behind non-monotonic modal logics is using an inference rule of the form

“If ¬φ is not derivable conclude that φ is possible.”

Ground non-monotonic modal logics are based on the idea of characterizing the knowledge of the agent by allowing him/her to make non-monotonic assumptions only with respect to his/her knowledge about the world.

As of today, the main effort has been invested into the investigation of propositional non-monotonic logics. Only recently, Zbar came with the first sound and well-based definition of first-order ground non-monotonic modal logic. Her definition characterizes the knowledge of the agent by a fixpoint equation.

In this work we present an alternative semantical characterization of first-order ground non-monotonic modal logic, by defining a preference relation on possible-world models. The preference relation is based on the minimization of knowledge expressed by non-modal formulas. We prove the equivalence of our definition to Zbar’s definition. We also show that other well-known semantical characterizations of propositional ground non-monotonic modal logic, which are also based on the minimization of knowledge expressed by non-modal formulas, are not suitable for the first-order case.

In addition, we present two semantical definitions of minimal sets for first-order default theories. The definitions are similar to the definitions of extensions for first-order default theories. We prove that our two definitions are equivalent and that the relationship between minimal sets and extensions in the first-order case is similar to that of the propositional case.

Finally, we prove that the relationship between minimal sets for first-order default theories and first-order ground non-monotonic modal logic based on S4 and S5 is similar to the corresponding one in the propositional case.
## List of symbols

- **Th**: A theory
- **Tr**: A set of terms
- **$L_0$**: The language of a modal-free logic
- **$L$**: The language of a modal logic
- **$GFm$**: The set of all propositional modal-free formulas
- **$Fm$**: The set of all propositional modal formulas
- **$GST$**: The set of all first-order modal-free sentences
- **$St$**: The set of all first-order modal sentences
Chapter 1

Introduction

Non-monotonic logics are intended to simulate the process of human reasoning by providing a formalism for deriving consistent conclusions from an incomplete description of the world. Roughly speaking, the main idea lying behind non-monotonic logics is an inference from what is possible to what should actually hold. In particular, non-monotonic modal logics, introduced by McDermott and Doyle ([20, 19]), use an inference rule of the form

\[ \text{if } \neg \varphi \text{ is not derivable conclude that } \varphi \text{ is possible} \]

(1.1)

that corresponds to negative introspection of a rational agent, whereas positive introspection is provided by classical modal necessitation.

Ground non-monotonic modal logics ([9, 12, 23, 26, 27]) are based on the idea of characterizing the knowledge of the agent by allowing him/her to make non-monotonic assumptions only with respect to his/her knowledge about the world. That is, negative introspection (1.1) is bound to ground (i.e., modal-free) formulas \( \varphi \) only.

Another important non-monotonic formalism is Reiter’s default logic ([22]). This logic deals with rules of inference called defaults, which are expressions of the form

\[ \alpha : \beta_1, \ldots, \beta_m \]

\[ \gamma \]

(1.2)

where \( \alpha, \beta_1, \ldots, \beta_m, m \geq 0 \), and \( \gamma \) are propositional formulas.\(^1\) Roughly speaking, the intuitive meaning of a default is as follows. If \( \alpha \) is believed,

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\(^1\)Actually, in [22] the default rules are written in the form \( \alpha : M\beta_1, \ldots, M\beta_m \). In this work we employ a bit different notation to avoid a possible confusion of \( M \) that appears in default rules with modal operator \( M \).
and $\beta$s are consistent with one’s beliefs, then one is permitted to deduce $\gamma$ and add it to the “belief set”.

According to Reiter’s and McDermott and Doyle’s approaches, the knowledge of the agent is characterized in terms of a fixpoint equation, whose solutions are sets of formulas non-monotonically derivable from the agent’s initial knowledge. The syntactical nature of both approaches, together with the circularity of a fixpoint construction, has some deficiencies which made these approaches insufficiently clear.

Schwarz ([24]) proposed an intuitively clear Kripke-style semantics of a class of propositional non-monotonic modal logics. He introduced some partial order between Kripke interpretations, called a preference relation. Given a set of formulas $A$, which represents the agent’s initial knowledge, a “non-monotonic model” of $A$ is a Kripke model of $A$ which is minimal with respect to Schwarz’s preference relation. Schwarz proved that for a class of modal logics, a set of formulas $E$ is a fixpoint of McDermott and Doyle’s equation (called an expansion) if and only if $E$ is the theory of a “non-monotonic model” of $A$.

Donini, Nardi and Rosati ([2]) presented a semantical characterization of a class of propositional ground non-monotonic modal logics. Similarly to Schwarz, they defined a preference relation on Kripke interpretations, which is based on the minimization of knowledge expressed by modal-free formulas. Their minimality criterion is stronger than Schwarz’s, since every model which is minimal in a class of models according to their criterion is minimal in that class according to Schwarz’s criterion as well. The converse, however, does not hold in general.

Another semantical characterization of a class of propositional ground non-monotonic modal logics was presented by Marek and Truszczyński ([17]). They showed that for a class of modal logics, ground expansions can be characterized as expansions satisfying some minimality criterion regarding the knowledge expressed by modal-free formulas.

As of today, the main effort has been invested into the investigation of propositional non-monotonic logics. Only recently, Kaminski and Rey ([10]) presented the first sound and well-based definition of first-order non-monotonic modal logic. Their definition is based on a preference relation similar to that of Schwarz. Zbar ([28]) presented an alternative definition of first-order non-monotonic modal logic, and showed that her definition is equivalent to Kaminski and Rey’s definition. She also presented the first sound and well-based definition of first-order ground non-monotonic modal
logic. Roughly speaking, her new definitions result from the corresponding syntactical definitions in the propositional case, which characterize the knowledge of the agent by a fixpoint equation, by replacing provability relation $\vdash$ with semantical entailment $|=\$.

In this work we present an alternative semantical characterization of a class of first-order ground non-monotonic modal logics, by defining a preference relation on possible-world models. The preference relation is based on the minimization of knowledge expressed by non-modal formulas. Our characterization applies to the same class of modal logics as Donini, Nardi and Rosati’s characterization in the propositional case. We prove the equivalence of our definition to Zbar’s definition ([28]). We also show that other well-known semantical characterizations of propositional ground non-monotonic modal logic, namely Donini, Nardi and Rosati’s characterization and Marek and Truszczyński’s characterization, which are also based on the minimization of knowledge expressed by non-modal formulas, do not translate to the first-order case.

In addition, we present two semantical definitions of minimal sets for first-order default theories, similarly to the definitions of extensions for first-order default theories ([9, 13]). We prove that our two definitions are equivalent and that the relationship between minimal sets and extensions in the first-order case is the same as the corresponding relationship in the propositional logic.

Finally, we prove that the relationship between minimal sets for first-order default theories and first-order ground non-monotonic modal logic based on modal logics S4 and S5 is similar to the corresponding one in the propositional case ([8]).

The rest of this thesis is organized as follows. In Chapter 2 we recall the definitions of the ordinary (monotonic) propositional modal logic and some propositional non-monotonic logics. In this chapter we also recall the semantical characterizations of propositional ground non-monotonic modal logic presented by Donini, Nardi and Rosati, and Marek and Truszczyński. Finally, this chapter also contains the embedding of propositional minimal sets into propositional ground non-monotonic modal logic based on S5 and S4 presented by Kaminski ([8]). Chapter 3 deals with definitions of first-order non-monotonic logics. In Chapter 4 we present our first contribution: a new minimal model semantics for a class of first-order ground non-monotonic modal logics, along with the proof that our definition is equivalent to the definition of [28]. We conclude this chapter by showing that other well-
known semantical characterizations of propositional ground non-monotonic modal logic, namely Donini, Nardi and Rosati’s characterization and Marek and Truszczynski’s characterization, are not suitable for the first-order case. In Chapter 5 we present two semantical definitions of minimal sets for first-order default theories, prove that they are equivalent and that they preserve important properties of the propositional logic. In Chapter 6 we prove that the relationship between minimal sets for first-order default theories and first-order ground non-monotonic modal logic based on modal logics S4 and S5 is similar to the corresponding one in the propositional case.
Chapter 2

Propositional non-monotonic logic

In this chapter we recall the definitions of propositional stable theories, propositional non-monotonic modal logic, propositional ground non-monotonic modal logic, and propositional default logic, and state some of their basic properties and inter-translatability results. We start with the definition of the ordinary propositional (monotonic) modal logic.

2.1 Propositional modal logic

The language $L$ of modal logic is obtained from the language of the propositional logic $L_0$, by extending it with a modal connective $L$ (necessarily). The dual connective, $M$ (possibly), is defined by $\neg L\neg$. Formulas not containing modal connectives are called ground. The set of all propositional modal formulas will be denoted by $Fm$ and the set of all propositional ground formulas will be denoted by $GFm$.

In this work we shall mainly deal with the weakest normal modal logic $K$ which results from the classical propositional logic by adding the rule of inference

$\text{NEC } \varphi \vdash L\varphi,$

called $necessitation$, and the axiom scheme

$k: L(\varphi \supset \psi) \supset (L\varphi \supset L\psi).$
The “classical” modal logics are obtained by adding to K all instances of some axiom schemes, e.g.,

\[ t \quad L\phi \supset \phi, \]
\[ d \quad L\phi \supset M\phi, \]
\[ 4 \quad L\phi \supset LL\phi, \]
\[ f \quad (\phi \land ML\psi) \supset L(M\phi \lor \psi), \]
\[ 5 \quad M\phi \supset LM\phi. \]

Adding \( t \) to K results in T, adding \( 4 \) to T results in S4, adding \( f \) to S4 results in S4F, and adding \( 5 \) to S4 results in S5, etc., see [17, p. 197]. Modal logics containing NEC and \( k \) are called normal.

For a modal logic S and a set of formulas \( A \), called (proper) axioms, we define the (monotonic) theory of \( A \), denoted by \( \text{Th}_S(A) \), as \( \text{Th}_S(A) = \{ \phi : A \vdash_S \phi \} \). As usual, we write \( A \vdash_S \phi \), if and only if there exists a sequence of formulas \( \psi_1, \psi_2, \ldots, \psi_n \) containing \( \phi \) such that each \( \psi_i \) is an axiom from S, or belongs to \( A \), or is obtained from some of the formulas \( \psi_1, \psi_2, \ldots, \psi_{i-1} \) by modus ponens or necessitation. Note that any normal modal logic S can be embedded in K by extending the set of proper axioms with S. That is, \( A \vdash_S \phi \) if and only if \( A, S \vdash_K \phi \). This is one of the reasons the logic K is of a special interest.

The Kripke semantics of propositional modal logic is defined as follows. A Kripke interpretation is a triple \( M = \langle U, R, I \rangle \), where \( U \) is a non-empty set of possible worlds, \( R \) is an accessibility relation on \( U \), and \( I \) is an assignment to each world in \( U \) of a set of propositional variables.

**Definition 1** ([17]) Let \( M = \langle U, R, I \rangle \) be a Kripke interpretation, \( u \in U \), and \( \varphi \in \text{Fm} \). We say that the pair \( (M, u) \) satisfies \( \varphi \), denoted by \( (M, u) \models \varphi \), if the following holds.

- If \( \varphi \) is a propositional variable \( p \), then \( (M, u) \models \varphi \) if and only if \( p \in I(u) \);
- \( (M, u) \models \neg \varphi \) if and only if \( (M, u) \not\models \varphi \);
- \( (M, u) \models \varphi \supset \psi \) if and only if \( (M, u) \not\models \varphi \) or \( (M, u) \models \psi \);
• $(\mathcal{M}, u) \models L\phi$ if and only if for each $v$ such that $uRv$, $(\mathcal{M}, v) \models \phi$.

We say that a Kripke interpretation $\mathcal{M}$ satisfies a formula $\phi$, denoted by $\mathcal{M} \models \phi$, if and only if for every $u \in U$, $(\mathcal{M}, u) \models \phi$, and we say that $\mathcal{M}$ satisfies a set of formulas $X$ or $\mathcal{M}$ is a model of $X$, denoted by $\mathcal{M} \models X$, if and only if $\mathcal{M} \models \phi$ for every $\phi \in X$. The set of all formulas satisfied by $\mathcal{M}$ is called the theory of $\mathcal{M}$ and is denoted by $\text{Th}(\mathcal{M})$. That is, $\text{Th}(\mathcal{M}) = \{ \phi : \mathcal{M} \models \phi \}$.

Finally, we say that a set of formulas $X$ semantically entails (respectively, semantically $S$-entails) a formula $\phi$, denoted by $X \models \phi$ (respectively, $X \models_S \phi$), if and only if every Kripke model of $X$ (respectively, every Kripke model of $X$ that satisfies $S$) satisfies $\phi$.

The above semantics is sound and complete for $K$. That is, $X \vdash_K \phi$ if and only if $\phi$ is satisfied by all Kripke interpretations which satisfy $X$. Kripke interpretations with a reflexive accessibility relation are sound and complete for $T$, Kripke interpretations with a reflexive and transitive accessibility relation are sound and complete for $S4$, and Kripke interpretations where the accessibility relation is an equivalence relation are sound and complete for $S5$, see [7, Section 2]. Kripke interpretations $\mathcal{M} = \langle U, R, I \rangle$ with the total accessibility relation $R = U \times U$ are called $S5$-models.

### 2.2 Propositional stable theories

We precede the definitions of propositional non-monotonic modal logic and propositional ground non-monotonic modal logic with the definition of propositional stable theories and the statement of one of their basic properties.

The notion of a stable theory naturally arises in context of non-monotonic reasoning and, in propositional logic, stable theories are tightly related to (ground) non-monotonic modal logic. Stable theories are supposed to represent belief sets of a rational agent possessing a full power of introspection.

**Definition 2** ([17]) A set of propositional modal formulas $X$ is called stable if it satisfies the following three conditions.

1. $X$ is closed under propositional consequence $\vdash$.

2. (Positive introspection) For every modal formula $\varphi$, if $\varphi \in X$, then $L\varphi \in X$. 

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3. (Negative introspection) For every modal formula \( \varphi \), if \( \varphi \notin X \), then \( \neg L\varphi \in X \).

Below we recall a basic property of propositional stable theories on which we shall rely in the sequel.

**Theorem 3** ([17, Theorem 8.10, p. 228 and Theorem 8.12, p. 229]) A consistent set of propositional modal formulas is stable if and only if it is a theory of an S5-model.

### 2.3 Propositional non-monotonic modal logic

In this section we recall the definition of propositional non-monotonic modal logic based on the McDermott and Doyle fixpoint equation. The definition is a relativization of McDermott’s original definition ([19]) based on the classical modal logics T, S4, or S5 to a normal modal logic S. A general form of McDermott’s definition is as follows.

**Definition 4** ([19, 17]) Let S be a normal modal logic and \( A \) be a set of propositional modal formulas (axioms). For a set of formulas \( X \) we denote the (modal) theory of \( A \cup \{ M\varphi : X \not\models S \neg \varphi \} \) by \( \text{NM}_S(X) \). That is,

\[
\text{NM}_S(X) = \text{Th}_S(A \cup \{ M\varphi : X \not\models S \neg \varphi \}).
\]

Fixpoints of the operator \( \text{NM}_S \) are called S-expansions for \( A \).

**Remark 5** By the soundness and completeness of the Kripke semantics, \( \text{NM}_S(X) \) can be defined as \( \text{Th}_S(A \cup \{ M\varphi : X \not\models S \neg \varphi \}) \).

Roughly speaking, fixpoints of the operator \( \text{NM}_S \) correspond to the “deductive closure” of \( A \) in S extended with “rule of inference” (1.1), which is also referred to as possibilitation.\(^1\) In presence of modal scheme \( k \), possibilitation corresponds to negative introspection, whereas positive introspection is provided by NEC. Thus, a fixpoint of \( \text{NM}_S \) can be considered as an acceptable set of beliefs which a rational agent may hold about an incompletely specified world.

\(^1\)Like all non-monotonic rules, possibilitation is self-referring and, therefore, is ill-defined.
2.4 Propositional ground non-monotonic modal logic

In this section we recall the definition of propositional ground non-monotonic modal logic ([9, 12, 23, 26, 27]) that bounds negative introspection (1.1) to ground (i.e., modal-free) formulas as described below.

**Definition 6** ([9, 12, 23, 26, 27]) Let $S$ be a normal modal logic and $A$ be a set of propositional modal formulas (axioms). For a set of formulas $X$ we denote the (modal) theory of $A \cup \{M\varphi : X \not\models_S \neg\varphi, \ \varphi \in GFm\}$ by $\text{GNM}_S^A(X)$.\(^2\) That is,

$$\text{GNM}_S^A(X) = \text{Th}_S(A \cup \{M\varphi : X \not\models_S \neg\varphi, \ \varphi \in GFm\}).$$

Fixpoints of the operator $\text{GNM}_S^A$ are called ground $S$-expansions for $A$.

**Remark 7** Similarly to Remark 5, $\text{GNM}_S^A(X)$ can be defined as

$$\text{Th}_S(A \cup \{M\varphi : X \not\models_S \neg\varphi, \ \varphi \in GFm\}).$$

**Theorem 8** ([17, Theorem 11.30, p. 342]) Let $S$ be a normal modal logic and $A$ be a set of propositional modal formulas (axioms). Then, any ground $S$-expansion for $A$ is also an $S$-expansion for $A$.

The converse of Theorem 8 is false, see [2, Example 2.5].

**Remark 9** Given a normal modal logic $S$ and a set of propositional modal formulas $A$, it is easy to see that any $S$-expansion for $A$ is stable. Therefore, by Theorem 8, any ground $S$-expansion for $A$ is also stable.

\(^2\)Recall that $GFm$ denotes the set of all propositional ground formulas.
### 2.4.1 Minimal model semantics

Below we recall a semantical characterization of a class of propositional ground non-monotonic modal logics, presented by Donini, Nardi and Rosati in [2]. Similarly to [24], the authors define a preference relation on Kripke interpretations, which is based on the minimization of knowledge expressed by modal-free formulas.

**Definition 10** ([2]) Given two Kripke interpretations $\mathcal{M}' = \langle U', R', I' \rangle$ and $\mathcal{M}'' = \langle U'', R'', I'' \rangle$, we write $\mathcal{M}' \supset_G \mathcal{M}''$ if $U' = U''$, $I' = I''$ and $R' \supset R''$.

**Definition 11** ([2]) Let $\mathcal{M}' = \langle U', R', I' \rangle$ and $\mathcal{M}'' = \langle U'', R'', I'' \rangle$ be Kripke interpretations such that $U' \cap U'' = \emptyset$. The **concatenation** of $\mathcal{M}'$ and $\mathcal{M}''$, denoted by $\mathcal{M}' \odot \mathcal{M}''$, is a Kripke interpretation $\mathcal{M} = \langle U, R, I \rangle$, where $U = U' \cup U''$, $R = R' \cup (U' \times U'') \cup R''$, and $I$ is defined by

$$I(u) = \begin{cases} I'(u) & \text{if } u \in U' \\ I''(u) & \text{if } u \in U'' \end{cases}$$

**Definition 12** ([2]) Let $\mathcal{M}' = \langle U', R', I' \rangle$ and $\mathcal{M}'' = \langle U'', R'', I'' \rangle$ be Kripke interpretations. We write $\mathcal{M}'' \sqsubseteq_G \mathcal{M}'$ if there exists a Kripke interpretation $\mathcal{M}$ such that:

1. $\mathcal{M}'' \supset_G \mathcal{M} \odot \mathcal{M}'$;
2. there exists a world $u \in U'' \setminus U'$ such that for each world $u' \in U'$, $I''(u') \neq I''(u)$.

**Definition 13** ([2]) Let $\mathcal{C}$ be a class of Kripke interpretations and $S$ be a modal logic. We say that $S$ is **characterized** by $\mathcal{C}$ if the following holds. For every set of propositional modal formulas $A$ and every propositional modal formula $\varphi$, $A \vdash_S \varphi$ if and only if for every Kripke interpretation $\mathcal{M} \in \mathcal{C}$, $\mathcal{M} \models A$ implies $\mathcal{M} \models \varphi$.

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3In the sequel, when dealing with a number of Kripke interpretation, by renaming their worlds, if necessary, we may always assume that sets of worlds of Kripke interpretations under consideration are pairwise disjoint.
Definition 14 ([2]) Let \( A \) be a set of modal formulas (axioms) and \( S \) be a normal modal logic characterized by the class of Kripke interpretations \( \mathcal{C} \). A Kripke interpretation \( \mathcal{M} \in \mathcal{C} \) is a ground \( \mathcal{C} \)-minimal model for \( A \) if \( \mathcal{M} \models A \) and for every Kripke interpretation \( \mathcal{M}' \in \mathcal{C} \) such that \( \mathcal{M}' \models A \), \( \mathcal{M}' \not\preceq_G \mathcal{M} \).

Definition 15 ([2]) A Kripke interpretation \( \mathcal{M} \) is called cluster decomposable if it is either an S5-model or is of the form \( \mathcal{M}' \odot \mathcal{M}'' \), where \( \mathcal{M}' \) is a Kripke interpretation and \( \mathcal{M}'' \) is an S5-model.

Definition 16 ([2]) A class \( \mathcal{C} \) of Kripke interpretations is called cluster decomposable if every Kripke interpretation in \( \mathcal{C} \) is cluster decomposable, and for every interpretation \( \mathcal{M} \odot \mathcal{M}' \in \mathcal{C} \) such that \( \mathcal{M}' \) is an S5-model, and every S5-model \( \mathcal{M}'' \), the interpretation \( \mathcal{M} \odot \mathcal{M}'' \) is in \( \mathcal{C} \).

It is easy to see that modal logics S5, KD45, S4F and SW5 are all characterized by a cluster-decomposable class of Kripke interpretations (see [2, Section 3]).

Definition 17 ([2]) The canonical model for a stable theory \( T \) is an S5-model \( \mathcal{M} = \langle U, R, I \rangle \) such that \( U \) consists of all sets of propositional variables in which all formulas of \( T \cap GFm \) are true, and \( I(u) = u \) for every \( u \in U \).

Theorem 18 ([2, Theorem 3.19]) Let \( S \) be a normal modal logic characterized by a cluster decomposable class of Kripke interpretations \( \mathcal{C} \), \( A \) be a set of modal formulas, \( T \) be a stable theory and \( \mathcal{M} \) be the canonical model for \( T \). Then \( T \) is a ground \( S \)-expansion for \( A \) if and only if \( \mathcal{M} \) is a ground \( \mathcal{C} \)-minimal model for \( A \).

2.4.2 Relationship between propositional ground non-monotonic modal logic and propositional non-monotonic modal logic

Below we recall a relationship between a class of propositional ground non-monotonic modal logics and a class of propositional non-monotonic modal logics, presented by Marek and Truszczynski in [17]. It is shown in [17] that for several modal logics \( S \), ground \( S \)-expansions can be characterized as \( S \)-expansions satisfying some minimality criterion.
Definition 19 ([17]) Let $T$ and $T'$ be two stable theories. We write $T \preceq T'$ if $T \cap GFm \subseteq T' \cap GFm$.

Definition 20 ([17]) Let $A$ be a set of modal formulas and $S$ be a normal modal logic. An $S$-expansion $E$ for $A$ is called minimal if for every stable theory $T$ such that $A \subseteq T$, $E \preceq T$.

Definition 21 ([17]) A Kripke interpretation $\mathfrak{M} = \langle U, R, I \rangle$ satisfies the terminal cluster property if for every world $u \in U$ there is a terminal cluster for $u$, i.e., a maximal subset $U'$ of $U$ such that $U' \times U' \subseteq R$ and:

1. for every $u' \in U'$, $(u, u') \in R$;
2. for every $u' \in U'$ and every $u'' \in U \setminus U'$, $(u', u'') \notin R$.

Definition 22 ([17]) A logic $S$ which is characterized by a class of Kripke interpretations $C$ satisfies the terminal cluster property if every $\mathfrak{M} \in C$ satisfies the terminal cluster property.

It is easy to see that modal logics $S5$, $KD45$, $S4F$ and $SW5$ satisfy the terminal cluster property (see [2, Section 3]).

Theorem 23 ([17, Theorem 11.38]) Let $S$ be a normal modal logic contained in $S5$ and satisfying the terminal cluster property and $A$ be a set of modal formulas. Then a theory $E$ is a ground $S$-expansion for $A$ if and only if $E$ is a minimal $S$-expansion for $A$.

Corollary 24 below follows immediately from Theorem 23.

Corollary 24 ([8, Theorem 1], see also [17, Theorem 11.36]) Let $A$ be a set of modal formulas. Then a theory $E$ is a ground $S5$-expansion for $A$ if and only if $E$ is a minimal $S5$-expansion for $A$.

Corollary 25 below follows immediately from Theorem 23 and Corollary 24.

Corollary 25 ([2, Proposition 3.9]) Let $S$ be a normal modal logic contained in $S5$ and satisfying the terminal cluster property and $A$ be a set of modal formulas. Then a theory $E$ is a ground $S$-expansion for $A$ if and only if $E$ is both an $S$-expansion for $A$ and a ground $S5$-expansion for $A$. 

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2.5 Propositional default logic

In this section we recall the definition of propositional default logic. This logic deals with rules of inference called defaults which are expressions of the form (1.2). The formula $\alpha$ is called the prerequisite of the default rule, the formulas $\beta_1, \ldots, \beta_m$ are called the justifications, and the formula $\gamma$ is called the conclusion. A default theory is a pair $(D, A)$, where $D$ is a set of defaults and $A$ is a set of propositional formulas (axioms).

Next we recall the definition of an extension for a default theory, introduced by Reiter in [22].

**Definition 26 ([22])** Let $(D, A)$ be a default theory. For a set of formulas $X \subseteq GFm$ let $\Gamma_{(D,A)}(X)$ be the smallest set of formulas $B \subseteq GFm$ (beliefs) that satisfies the following three properties.

- $A \subseteq B$.
- If $B \models \varphi$, then $\varphi \in B$, i.e., $B$ is deductively closed.
- If $\frac{\alpha: \beta_1, \ldots, \beta_m}{\gamma} \in D$, $\alpha \in B$, and $\neg \beta_i \notin X$, $i = 1, 2, \ldots, m$, then $\gamma \in B$.

A set of formulas $E$ is an extension for $(D, A)$ if $\Gamma_{(D,A)}(E) = E$, i.e., if $E$ is a fixpoint of the operator $\Gamma_{(D,A)}$.

Now we present an alternative semantical definition of extensions for default theories.

**Definition 27 ([6])** Let $(D, A)$ be a default theory. For any class of models $W$ let $\Sigma_{(D,A)}(W)$ be the largest class $V$ of models of $A$ that satisfies the following condition.

If $\frac{\alpha: \beta_1, \ldots, \beta_m}{\gamma} \in D$, $V \models \alpha$, and $W \models \neg \beta_i$, $i = 1, \ldots, m$, then $V \models \gamma$.\(^4\)

It is known from [6] that the definition of extensions as the theories of the fixpoints of $\Sigma$ is equivalent to Reiter’s original definition (Definition 26). That is, a set of formulas is an extension for a default theory $(D, A)$ if and only if $E = Th_{\Sigma_0}(W)$ for some fixpoint $W$ of $\Sigma_{(D,A)}$.

\(^4\)This largest class $\Sigma_{(D,A)}(W)$ always exists, see [13, Proposition 1].
2.5.1 Propositional minimal sets

Some default theories suffer from incoherence problem, i.e., they have no extensions. To deal with this issue, another structure, called minimal set, was introduced in [15].

**Definition 28 ([15])** Let \((D, A)\) be a default theory. A minimal set for \((D, A)\) is a consistent minimal set of formulas \(B \subseteq GFm\) that satisfies properties PT1-PT3 below.\(^5\)

\((PT1)\) \(A \subseteq B.\)

\((PT2)\) If \(B \vdash \varphi\), then \(\varphi \in B\), i.e., \(B\) is deductively closed.

\((PT3)\) If \(\alpha ; \beta_1, \ldots, \beta_m \in D\), \(\alpha \in B\), and \(\neg \beta_i \notin B\), \(i = 1, 2, \ldots, m\), then \(\gamma \in B\).

It was proved in [15] that for every default theory there exists a minimal set.

Theorem 29 below states that every extension for a default theory \((D, A)\) is a minimal set for \((D, A)\). It was also shown in [29] that the converse of Theorem 29 does not hold.

**Theorem 29 ([29, Theorem 1])** Let \((D, A)\) be a default theory. Then every extension for \((D, A)\) is a minimal set for \((D, A)\).

2.5.2 Relationship between propositional minimal sets and propositional ground non-monotonic modal logic

Below we recall the embedding of propositional minimal sets into propositional ground non-monotonic modal logic based on S5 and S4, presented by Kaminski in [8].

We start with the embedding of propositional minimal sets into propositional ground non-monotonic modal logic based on S5, which is based on the interpretation of defaults in modal logic proposed in [12]. The same interpretation has been used in [15] and [16] for different structures. The interpretation of defaults in modal logic is given by the following definition.

\(^5\)PT stands for Propositional Theory.
Definition 30 ([8]) For a propositional default \( d = \alpha : \beta_1, \ldots, \beta_m \) we denote by \( \Theta(d) \) the modal formula \((L\alpha \land \bigwedge_{i=1}^m M\beta_i) \supset \gamma\) and for a set of propositional defaults \( D \) we denote the set of modal formulas \( \{\Theta(d) : d \in D\} \) by \( \Theta(D) \).

Finally, for a propositional default theory \( \Delta = (D, A) \) we define \( \Theta(\Delta) \subseteq Fm \) by \( \Theta(\Delta) = A \cup \Theta(D) \).

Theorem 31 ([8, Theorem 3]) Let \( \Delta = (D, A) \) be a propositional default theory. Then \( E \) is a minimal set for \( \Delta \) if and only if there exists a ground S5-expansion \( T \) for \( \Theta(\Delta) \) such that \( E = T \cap GSt \).

We conclude this section with the embedding of propositional minimal sets into propositional ground non-monotonic modal logic based on S4, see [8].

Definition 32 ([8]) For a propositional default \( d = \alpha : \beta_1, \ldots, \beta_m \) we denote the modal formula \((ML\alpha \land \bigwedge_{i=1}^m M\beta_i) \supset L\gamma\) by \( \Theta'(d) \) and for a set of propositional defaults \( D \) we denote the set of modal formulas \( \{\Theta'(d) : d \in D\} \) by \( \Theta'(D) \).

Finally, for a propositional default theory \( \Delta = (D, A) \) we define \( \Theta'(\Delta) \subseteq Fm \) by \( \Theta'(\Delta) = A \cup \Theta'(D) \).

Theorem 33 ([8, Theorem 5]) Let \( \Delta = (D, A) \) be a propositional default theory. Then \( E \) is a minimal set for \( \Delta \) if and only if there exists a ground S4-expansion \( F \) for \( \Theta'(\Delta) \) such that \( E = F \cap GSt \).
Chapter 3

First-order non-monotonic logic

In this chapter we recall the definitions of first-order modal logic, first-order ground non-monotonic modal logic and first-order default logic. We start with the definition of the ordinary first-order (monotonic) logic.

3.1 First-order logic

In this section we recall the definition of the ordinary first-order (monotonic) logic and its semantics. See [21, Chapter 2] for more details.

We assume that the language $\mathcal{L}_0$ of first-order logic contains only two classical propositional connectives: a logical constant $\bot$ (false) and a binary connective $\supset$ (implication), together with the universal quantifier $\forall$. Connectives $\top$ (true), $\neg$ (negation), $\land$ (conjunction), $\lor$ (disjunction), $\equiv$ (equivalence), and the existential quantifier $\exists$ are defined in a usual manner, e.g., $\neg \varphi$ is $\varphi \supset \bot$. In addition, it contains a finite, possibly empty, or denumerable set of function symbols and a finite or denumerable, non-empty, set of predicate symbols (which come together with their arities). We treat the language constants as 0-place function symbols.

The semantics of first-order logic is as follows.

A first-order interpretation $M$ consists of a non-empty domain $D_M$, an assignment to each $n$-place function symbol $f$ of the underlying language of an $n$-place function $f^M : D^M_n \rightarrow D_M$, and an assignment to each $n$-place predicate symbol $P$ of the underlying language of an $n$-place relation $P^M$ on

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1We reserve $\mathcal{L}$ to denote the language of first-order modal logic.
Given such an interpretation, variables are thought of as ranging over the set $D_M$.

Let $D_M$ be the domain of some interpretation $M$ of $L_0$. We shall denote by $\Sigma_{D_M}$ the set of all denumerable sequences of the elements of $D_M$. For a first-order formula $\varphi \in L_0$ we shall define what it means for a sequence $s = (s_1, s_2, \ldots) \in \Sigma_{D_M}$ to satisfy $\varphi$ in $M$. As a preliminary step, for a given $s \in \Sigma_{D_M}$ we shall define a function $s^*$ that assigns to each term $t$ of $L_0$ an element $s^*(t)$ in $D_M$.

**Definition 34 ([21])** Let $M$ be an interpretation of $L_0$, $D_M$ be the domain of $M$, and let $s = (s_1, s_2, \ldots) \in \Sigma_{D_M}$. Then the function $s^*$ assigns to each term $t$ of $L_0$ an element $s^*(t) \in D_M$ as follows.

- If $t$ is a variable $x_j$ then $s^*(t)$ is $s_j$, and
- if $t$ is of the form $f(t_1, \ldots, t_n)$, where $f$ is an $n$-place function symbol and $t_1, \ldots, t_n$ are terms, then $s^*(t)$ is $f^M(s^*(t_1), \ldots, s^*(t_n))$.

Now we proceed to the definition of satisfaction.

**Definition 35 ([21])** Let $\varphi$ be a first-order formula, $M$ be an interpretation of $L_0$, $D_M$ be the domain of $M$, and let $s = (s_1, s_2, \ldots) \in \Sigma_{D_M}$. We say that the sequence $s$ satisfies $\varphi$ in $M$, or the “pair” $(M, s)$ satisfies $\varphi$, denoted $(M, s) \models \varphi$, if the following holds.

- If $\varphi$ is an atomic formula $P(t_1, \ldots, t_n)$ then $(M, s) \models \varphi$ if and only if $(s^*(t_1), \ldots, s^*(t_n)) \in P^M$;
- $(M, s) \not\models \bot$;
- $(M, s) \models \varphi \supset \psi$ if and only if $(M, s) \not\models \varphi$ or $(M, s) \models \psi$; and
- $(M, s) \models (\forall x_j) \varphi(x_j)$ if and only if for every $s' \in \Sigma_{D_M}$ which differs from $s$ in at most the $j$th component, $(M, s') \models \varphi(x_j)$.

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2 We do not assume that $=^M$ is identity, but just a congruence relation. That is, interpretations are not necessarily normal, see [21, p. 100] for details.

3 Recall that we treat the constants of $L_0$ as zero place function symbols.
Lemma 36 ([21, Property VIII, p. 61]) Let \( \varphi(x) \), \( x = x_1, \ldots, x_n \), be a first-order formula all whose free variables are among \( x_1, \ldots, x_n \), and let \( M \) be an interpretation of \( \mathcal{L}_0 \). Also, let \( s' = (s'_1, s'_2, \ldots) \) and \( s'' = (s''_1, s''_2, \ldots) \) be such that \( s'_i = s''_i \), \( i = 1, \ldots, n \). Then, \( (M, s') \models \varphi(x) \) if and only if \( (M, s'') \not\models \varphi(x) \).

Lemma 37 ([21, Lemma 2, p. 63]) Let \( \varphi(x_i) \) be a first-order formula, \( M \) be an interpretation of \( \mathcal{L}_0 \), \( s = (s_1, s_2, \ldots) \) be such that \( s \in \Sigma_{DM} \), and let \( t \) be free for \( x_i \) in \( \varphi(x_i) \). In addition, let \( s' \in \Sigma_{DM} \) be a sequence obtained from \( s \) by substituting \( s^*(t) \) for \( s_i \) in the \( i \)th place. Then, \( (M, s) \models \varphi(t) \) if and only if \( (M, s') \models \varphi(x_i) \).

In what follows we use an alternative equivalent semantics of first-order logic.

Definition 38 Let \( t(x) \) be a term of \( \mathcal{L}_0 \) whose free variables are among \( x = x_1, \ldots, x_n \). Let \( M \) be a first-order interpretation and let \( d = d_1, \ldots, d_n \subseteq D_M \). Below we define the value of \( t(x) \) at \( d \), denoted \( t^M(d) \).

- If \( t \) is a variable \( x_i \), \( i = 1, 2, \ldots, n \), then \( t^M(d) = d_i \); and
- If \( t \) is of the form \( f(t_1, \ldots, t_m) \), where \( f \) is an \( m \)-place function symbol and \( t_1, \ldots, t_m \) are terms, then \( t^M(d) = f^M(t^M_1(d), \ldots, t^M_m(d)) \). Recall that we treat constants as 0-place function symbols.

That is, \( t^M(d) \) is the value of \( t(x) \) under the assignment of \( d_i \) to \( x_i \), \( i = 1, \ldots, n \).

Definition 39 Let \( \varphi(x) \) be a first-order formula whose free variables are among \( x = x_1, \ldots, x_n \). Let \( M \) be a first-order interpretation and let \( d = d_1, \ldots, d_n \subseteq D_M \). We say that \( M \) satisfies \( \varphi(x) \) at \( d \), or \( M \) satisfies \( \varphi(x) \) under the assignment of \( d_i \) to \( x_i \), \( i = 1, \ldots, n \), or the “pair” \((M, x/d)\) satisfies \( \varphi(x) \), denoted \( M \models \varphi(d) \), if the following holds.

- If \( \varphi(x) \) is an atomic formula \( P(t_1(x), \ldots, t_m(x)) \), then \( M \models \varphi(d) \) if and only if \((t_1^M(d), \ldots, t_m^M(d)) \in P^M \);
- \( M \not\models \perp \); and
- \( M \models \varphi(d) \not\models \psi(d) \) if and only if \( M \not\models \varphi(d) \) or \( M \models \psi(d) \); and
• $M \models \forall x \varphi(x, d)$ if and only if for all $d' = d'_1, \ldots, d'_n \subseteq D_M$, $M \models \varphi(d', d)$.

For an interpretation $M$, we say that $M$ satisfies a first-order formula $\varphi(x)$ whose free variables are among $x = x_1, \ldots, x_n$, denoted by $M \models \varphi(x)$, if and only if for every assignment $d = d_1, \ldots, d_n \subseteq D_M$, $M \models \varphi(d)$. We say that $M$ satisfies a set of formulas $\Gamma$, denoted by $M \models \Gamma$, if and only if for every $\varphi \in \Gamma$, $M \models \varphi$. We say that a set of first-order formulas $\Gamma$ entails a first-order formula $\varphi$, denoted by $\Gamma \models \varphi$, if and only for every interpretation $M'$ such that $M' \models \Gamma$, $M' \models \varphi$.

Proposition 40 below follows immediately from Lemma 36.

Proposition 40 Let $\varphi(x), x = x_1, \ldots, x_n$, be a first-order formula all whose free variables are among $x_1, \ldots, x_n$, and let $M$ be an interpretation of $\mathcal{L}_0$. Also, let $d = d_1, \ldots, d_n \subseteq D_M$, and let $s' = (s_1, s_2, \ldots) \in \Sigma_{D_M}$ be such that $s_i = d_i, i = 1, \ldots, n$. Then, $(M, s) \models \varphi(x)$ if and only if $M \models \varphi(d)$.

Therefore, the latter semantics is sound and complete for first-order logic.

Theorem 41 ([21]) (Soundness) Let $\Gamma$ and $\varphi$ be a set of first-order formulas and a first-order formula, respectively. Then, $\Gamma \vdash \varphi$ implies $\Gamma \models \varphi$.

Theorem 42 ([21]) (Completeness) Let $\Gamma$ and $\varphi$ be a set of first-order formulas and a first-order formula, respectively. Then, $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$.

3.2 Herbrand semantics of first-order logic

In this section we recall the Herbrand semantics of first-order logic, which lies in the basis of our semantical approach to first-order non-monotonic logics.

Recall that the language of the underlying first-order logic is denoted by $\mathcal{L}_0$ and the set of all closed formulas (sentences) over $\mathcal{L}_0$ will be denoted by $G\text{St}$. Let $b$ be a set that contains no symbols of $\mathcal{L}_0$. We denote by $\mathcal{L}_{0b}$ the language obtained from $\mathcal{L}_0$ by augmenting its set of constant symbols with all elements of $b$. The set of all closed terms of $\mathcal{L}_{0b}$, denoted by $T\text{r}_{c_{0b}}$, is

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4We reserve $\text{St}$ to denote the set of all modal sentences over $\mathcal{L}$. Thus, when dealing with modal logic, $G\text{St}$ refers to the subset of $\text{St}$ consisting of all ground (i.e., modal-free) sentences.
called the *Herbrand universe* of $\mathcal{L}_{0b}$, and closed formulas over $\mathcal{L}_{0b}$, denoted by $\text{GST}_{b}$, will be referred to as *b-sentences*. Note that b-sentences are of the form $\varphi(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n \in \text{Tr}_{\mathcal{L}_{0b}}$, and $\varphi(x_1, \ldots, x_n)$ is a formula of $\mathcal{L}_0$ whose free variables are among $x_1, \ldots, x_n$. A *Herbrand b-interpretation* of $\mathcal{L}_0$ is a set of atomic b-sentences.

**Definition 43** ([14]) Let $w$ be a Herbrand b-interpretation and let $\varphi$ be a b-sentence. We say that $w$ *satisfies* $\varphi$, denoted by $w \models \varphi$, if the following holds.

- If $\varphi$ is an atomic formula, then $w \models \varphi$ if and only if $\varphi \in w$;
- $w \models \neg \varphi$ if and only if $w \not\models \varphi$;
- $w \models \varphi \supset \psi$ if and only if $w \not\models \varphi$ or $w \models \psi$; and
- $w \models \forall x \varphi(x)$ if and only if for every $t \in \text{Tr}_{\mathcal{L}_{0b}}$, $w \models \varphi(t)$.

We can view a Herbrand b-interpretation $w$ as an ordinary first-order interpretation $M$ of $\mathcal{L}_0$ as follows.

- The domain $D_M$ of the interpretation is $\text{Tr}_{\mathcal{L}_{0b}}$.
- If $f$ is an $n$-place function symbol of $\mathcal{L}_0$ and $t_1, \ldots, t_n$ are terms, then $f^M(t_1, \ldots, t_n)$ is $f(t_1, \ldots, t_n)$.
- If $P$ is an $n$-place predicate symbol of $\mathcal{L}_0$ and $t_1, \ldots, t_n$ are terms, then $(t_1, \ldots, t_n) \in P^M$ if and only if $P(t_1, \ldots, t_n) \in w$.

Then, in view of Lemma 37 and Proposition 40, for a formula $\varphi(x_1, \ldots, x_n)$ and $t_1, \ldots, t_n \in D_M = \text{Tr}_{\mathcal{L}_{0b}}$, $M \models \varphi(t_1, \ldots, t_n)$ if and only if $w \models \varphi(t_1, \ldots, t_n)$.

For a Herbrand b-interpretation $w$ we define the *$\mathcal{L}_0$-theory* (respectively, *$\mathcal{L}_{0b}$-theory*) of $w$, denoted by $\text{Th}_{\mathcal{L}_0}(w)$ (respectively, $\text{Th}_{\mathcal{L}_{0b}}(w)$), as the set of all sentences of $\mathcal{L}_0$ (respectively, all b-sentences of $\mathcal{L}_{0b}$) satisfied by $w$. Let $X$ be a set of b-sentences. We say that $w$ is a (Herbrand) *b-model* of $X$, denoted by $w \models X$, if $X \subseteq \text{Th}_{\mathcal{L}_{0b}}(w)$, and for a set of Herbrand interpretations $W$ we write $W \models X$ if for every $w \in W$, $w \models X$. Also, for a set of Herbrand b-interpretations $W$ we denote by $\text{Th}_{\mathcal{L}_0}(W)$ (respectively,
$Th_{\mathcal{L}_0}(W)$) the set of all sentences of $\mathcal{L}_0$ (respectively, $b$-sentences of $\mathcal{L}_{0b}$) satisfied by all elements of $W$. That is, $Th_{\mathcal{L}_0}(W) = \bigcap_{w \in W} Th_{\mathcal{L}_0}(w)$ (respectively, $Th_{\mathcal{L}_{0b}}(W) = \bigcap_{w \in W} Th_{\mathcal{L}_{0b}}(w)$). Next, for a set of $b$-sentences $X$ we denote by $Mod_b(X)$ the set of all Herbrand $b$-models of $X$. Finally, we say that $X$ \textit{b-entails} a $b$-sentence $\varphi$, denoted by $X \models_b \varphi$, if every Herbrand $b$-model of $X$ also satisfies $\varphi$. The set of all $b$-sentences $b$-entailed by $X$ will be denoted by $Th_{\mathcal{L}_{0b}}(X)$.

**Remark 44** It is well-known that for an infinite set of new constant symbols $b$, Herbrand $b$-interpretations are complete and sound for first-order logic. That is, for a set of $\mathcal{L}$ sentences $X$ and a $\mathcal{L}$ sentence $\varphi$, $X \models \varphi$ if and only if $X \models_b \varphi$. In particular, Herbrand $b$-interpretations with an infinite $b$ naturally arise in the Henkin proof of the completeness theorem, see [21, Lemma 2.16, p. 89].

### 3.3 First-order modal logic

This section deals with first-order modal logic. See [4, 3, 5, 18] for more details.

In the classical case, starting from propositional logic, there is essentially only one way of extending to the first-order setting. There are, of course, some relatively mild variations, but the number of possibilities is severely limited. In the modal case, however, the variety of ways of extending any given logic to a first-order version is remarkably high.

The language $\mathcal{L}$ of first-order modal logic is obtained similarly to the language of propositional modal logic, i.e., $\mathcal{L}$ is obtained from the classical language by adding to it the modal connective $L$ (necessarily). The dual connective, $M$ (possibly), is defined by $\neg L \neg$.

In principle, the semantics of first-order modal logic could simply be obtained by extension of the propositional semantics: a first-order modal interpretation can be built on the base of a set of first-order classical interpretations (the “possible worlds”), connected by a binary relation (the

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5Note that if the operator $Th_{\mathcal{L}_{0b}}$ is applied to a set of Herbrand $b$-interpretations $W$ it results in the set of all $b$-sentences satisfied by all elements of $W$, whereas if it is applied to a set of $b$-sentences $X$ it results in the set of all $b$-sentences $b$-entailed by $X$.\footnote{Note that if the operator $Th_{\mathcal{L}_{0b}}$ is applied to a set of Herbrand $b$-interpretations $W$ it results in the set of all $b$-sentences satisfied by all elements of $W$, whereas if it is applied to a set of $b$-sentences $X$ it results in the set of all $b$-sentences $b$-entailed by $X$.}
accessibility relation). However, a number of different possibilities can be considered, for example:

- **The object domains**: Is there any relation between the object domains of the different worlds? They can be required to be the same for all worlds (*constant domains*), they can bear no relation one to the other (*varying domains*), or the object domains can vary, but monotonically, i.e., if \( u' \) is accessible from \( u \), then the domain of \( u \) is included in the domain of \( u' \) (*cumulative domains*).

- **The designation of terms**: Is the designation of terms the same in all worlds, or can it vary? When the answer is positive, then designation is *rigid*, otherwise it is *non-rigid*.

- **The existence of objects**: Does the interpretation of a term at a world \( u \) lie always in that world’s domain or not? If the answer is positive then the terms are *local*, otherwise the terms are *non-local*.

Hence, several variants of first-order modal logic are possible, just by choosing different combinations of the cases considered above. As often happens in modal logics, different choices are appropriate for different applications. A discussion can be found in [4] and [5].

Now we recall the semantics of first-order modal logic, and define the possibilities mentioned above more formally.

**Definition 45** ([18]) A *first-order modal interpretation* \( \mathfrak{M} \) of \( \mathcal{L} \) is a tuple \( \langle U, R, D, \delta, \phi, \pi \rangle \) such that:

- \( U \) is a non-empty set (the set of “possible worlds”);
- \( R \) is a binary relation on \( U \) (the “accessibility relation”);
- \( D \) is a non-empty set (the object domain);
- \( \delta \) is a function assigning to each \( u \in U \) a non-empty subset of \( D \), \( \delta(u) \subseteq D \) (the domain of \( u \));
- \( \phi \) is the interpretation of function symbols in the language: for every \( u \in U \) and \( n \)-ary function symbol \( f \) in \( \mathcal{L} \), \( \phi(u, f) \in D^n \to D \); and
\[ \pi \text{ is the interpretation of predicate symbols: for every } u \in U \text{ and } n\text{-ary predicate symbol } P \text{ in } \mathcal{L}, \pi(u, P) \subseteq D^n \text{ is a set of } n\text{-tuples of elements in } D. \]

The interpretation function \( \phi \) is extended to terms in the usual way (like in the classical first-order case), and, by an abuse of notation, \( \phi(u, t) \) denotes the interpretation of a term \( t \) in a world \( u \).

**Definition 46** ([18]) Let \( \varphi(x) \) be a first-order formula whose free variables are among \( x = x_1, \ldots, x_n \). Let \( \mathcal{M} = \langle U, R, D, \delta, \phi, \pi \rangle \) be a first-order modal interpretation of \( \mathcal{L} \), \( u \in U \), and let \( d = d_1, \ldots, d_n \subseteq \delta(u) \). We say that \( \mathcal{M} \) satisfies \( \varphi(x) \) at \( d \) in a world \( u \), or the “triple” \((\mathcal{M}, u, x/d)\) satisfies \( \varphi(x) \), denoted \((\mathcal{M}, u) \models \varphi(d)\), if the following holds.

- If \( \varphi(x) \) is an atomic formula \( P(t_1(x), \ldots, t_m(x)) \), then \((\mathcal{M}, u) \models \varphi(d)\) if and only if \((\phi(u, t_1(d)), \ldots, \phi(u, t_m(d))) \in \pi(u, P)\);
- \((\mathcal{M}, u) \not\models \bot\);
- \((\mathcal{M}, u) \models \varphi(d) \supset \psi(d)\) if and only if \((\mathcal{M}, u) \not\models \varphi(d)\) or \((\mathcal{M}, u) \models \psi(d)\);
- \((\mathcal{M}, u) \models \forall x \varphi(x, d)\) if and only if for all \( d \in \delta(u) \), \((\mathcal{M}, u) \models \varphi(d, d)\); and
- \((\mathcal{M}, u) \models L\varphi(d)\) if and only if for each \( v \) such that \( uRv \), \((\mathcal{M}, v) \models \varphi(d)\).

**Definition 47** ([18]) Let \( \mathcal{M} = \langle U, R, D, \delta, \phi, \pi \rangle \) be a first-order modal interpretation of \( \mathcal{L} \).

- **Constant domains:** For all \( u, u' \in U \), \( \delta(u) = \delta(u') \).
- **Varying domains:** For all \( u, u' \in U \), \( u \neq u' \), there is no restriction on the relation between \( \delta(u) \) and \( \delta(u') \).
- **Cumulative domains:** For all \( u, u' \in U \), if \( uRu' \), then \( \delta(u) \subseteq \delta(u') \).
- **Rigid designation:** For all \( u, u' \in U \) and for every function symbol \( f \), \( \phi(u, f) = \phi(u', f) \).

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• **Non-rigid designation:** No restriction on the relation between \( \phi(u, f) \) and \( \phi(u', f) \), for any \( u, u' \in U, u \neq u' \), and any function symbol \( f \).

• **Local terms:** For every \( u \in U \) and for every \( n \)-ary function symbol \( f \), if \( d_1, \ldots, d_n \in \delta(u) \), then \( \phi(u, f)(d_1, \ldots, d_n) \in \delta(u) \).

• **Non-local terms:** For every \( u \in U \), there is no restriction on the closure of the domain \( \delta(u) \) with respect to the application of the designation of function symbols.

To conclude this section, we provide some motivation for our choice of the Kripke semantics of first-order modal logic, which is presented in the next section.

Consider the axiom scheme

\[
\text{BF} \quad \forall x L \varphi \supset L \forall x \varphi,
\]
called the Barcan formula.\(^6\)

**Lemma 48** ([3, Section 2.2]) *If all instances of the axiom scheme \( ML \varphi \supset \varphi \) are provable in a first-order modal logic \( S \), then the Barcan formula must be a theorem in \( S \).*

**Lemma 49** ([3, Section 2.2]) *The Barcan formula is valid in every first-order modal interpretation with constant domains.*

It was also shown in [3, Section 2.2] that the Barcan formula is not valid in all first-order modal interpretations with cumulative domains. Hence, it follows from Lemmas 48 and 49 that every first-order modal interpretation which satisfies modal logic S5 should have constant domains.

The following example demonstrates why we prefer first-order modal interpretations with rigid designation.

**Example 50** Let \( \mathcal{L} \) be a first-order language which contains one 0-place function symbol \( c \) and one 1-place predicate symbol \( P \).

Let \( \mathcal{M} = \langle U, R, D, \delta, \phi, \pi \rangle \) be a first-order modal interpretation of \( \mathcal{L} \), such that \( U = \{ u, v, w \} \), \( R = \{(u, v), (u, w)\} \), \( D = \{d_1, d_2\} \), \( \delta(u) = \delta(v) = \)

\(^6\)The name can be traced to Barcan (Marcus) [1], although the actual formulas considered there were different in form.
\[ \delta(w) = D, \phi(u, c) = d_1, \phi(v, c) = d_1, \phi(w, c) = d_2, \pi(u, P) = \{d_1, d_2\}, \]
\[ \pi(v, P) = \{d_1\}, \text{and} \pi(w, P) = \{d_2\}. \]

Since \( \phi(v, c) \neq \phi(w, c) \), \( M \) has non-rigid designation. In addition, it is easy to see that \( M \) has constant domains.

We shall see that the formula \( LP(c) \supset \exists x LP(x) \) is not valid in \( M \).

Since \( (M, v) \models P(c) \) and \( (M, w) \models P(c), (M, u) \models LP(c) \). However, since \( (M, v) \not\models P(d_2) \) and \( (M, w) \not\models P(d_1) \), \( (M, u) \not\models \exists x LP(x) \).

Consequently, as we shall see in the next chapter, our semantics of first-order modal logic will be based on first-order modal interpretations with constant domains, rigid designation and local terms. This is crucial for extending propositional non-monotonic modal logic to the first-order case.

### 3.4 Kripke semantics of first-order modal logic

In this section we recall the Kripke semantics of first-order modal logic.

All modal logics under consideration will contain the axiom scheme \( BF \) (Barcan formula).

Similarly to the propositional case, any modal logic \( S \) containing \( BF \) can be embedded in \( K + BF \) by extending the set of proper axioms with \( S \). That is, \( A \vdash S \varphi \) if and only if \( A, S \vdash K + BF \varphi \). This is one of the reasons the logic \( K + BF \) is of a special interest.

The set of all modal sentences over \( L_b \) will be referred to as (modal) \( b \)-sentences and will be denoted by \( St_b \). Recall that the the set of all ground (i.e., modal-free) \( b \)-sentences is denoted by \( GSt_b \).

The Kripke semantics of first-order normal modal logic containing the scheme \( BF \) is defined as follows. A Kripke \( b \)-interpretation is a triple \( M = \langle U, R, I \rangle \), where \( U \) is a non-empty set of possible worlds, \( R \) is an accessibility relation on \( U \), and \( I \) is an assignment to each world in \( U \) of a Herbrand \( b \)-interpretation (a set of atomic \( b \)-sentences).

**Definition 51** ([28, 10, 11]) Let \( M = \langle U, R, I \rangle \) be a Kripke \( b \)-interpretation, \( u \in U \), and let \( \varphi \) be a \( b \)-sentence. We say that the pair \( (M, u) \) satisfies \( \varphi \), denoted by \( (M, u) \models \varphi \), if the following holds.

- If \( \varphi \) is an atomic formula \( P(t_1, \ldots, t_n) \), then \( (M, u) \models \varphi \) if and only if \( P(t_1, \ldots, t_n) \in I(u) \).

\[ ^7 \]We shall identify a first-order modal logic \( S \) with the set of the universal closures of its axioms, also denoted by \( S \).

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• \((M, u) \models \neg \varphi\) if and only if \((M, u) \not\models \varphi\);

• \((M, u) \models \varphi \supset \psi\) if and only if \((M, u) \not\models \varphi\) or \((M, u) \models \psi\);

• \((M, u) \models \forall x \varphi(x)\) if and only if for every \(t \in Tr_Lb\), \((M, u) \models \varphi(t)\); and

• \((M, u) \models L \varphi\) if and only if for every \(v\) such that \(uRv\), \((M, v) \models \varphi\).

We say that a Kripke \(b\)-interpretation \(M\) satisfies a \(b\)-sentence \(\varphi\), denoted by \(M \models \varphi\), if and only if for all \(u \in U\), \((M, u) \models \varphi\). We say that \(M\) satisfies a set of \(b\)-sentences \(X\) or \(M\) is a \(b\)-model of \(X\), denoted by \(M \models X\), if and only if \(M \models \varphi\) for every \(\varphi \in X\). We say that \(M\) is an S-Kripke \(b\)-interpretation if and only if it satisfies a modal logic \(S\). We say that \(M\) is an S-Kripke \(b\)-model of \(X\) if and only if it is both an S-Kripke \(b\)-interpretation and a \(b\)-model of \(X\). The set of all \(b\)-sentences satisfied by \(M\) will be denoted by \(Th_{Lb}(M)\). That is, \(Th_{Lb}(M) = \{ \varphi \in St_b : M \models \varphi \}\). We say that a set of modal \(b\)-sentences \(X\) \(b\)-entails a modal \(b\)-sentence \(\varphi\), denoted by \(X \models_b \varphi\), if and only if every Kripke \(b\)-model of \(X\) also satisfies \(\varphi\). We write \(X \models^S_b \varphi\) if and only if every S-Kripke \(b\)-model of \(X\) also satisfies \(\varphi\). Finally, the set of all modal \(b\)-sentences \(b\)-entailed by \(X\) will be denoted by \(Th_{Lb}(X)\).

Like in the proofs in [7, Section 9], one can show that the Kripke semantics is sound and complete for \(K+BF\). That is, \(X \vdash_{K+BF} \varphi\) if and only if \(\varphi\) is satisfied by all Kripke \(b\)-interpretations which satisfy \(X\) (for any infinite set of new constant symbols \(b\)). Similarly, Kripke \(b\)-interpretations with a reflexive accessibility relation are sound and complete for \(T+BF\), Kripke \(b\)-interpretations with a reflexive and transitive accessibility relation are sound and complete for \(S4+BF\), and Kripke \(b\)-interpretations whose accessibility relation is an equivalence relation are sound and complete for \(S5\).

Remark 52 We can view a Kripke \(b\)-interpretation \(M = \langle U, R, I \rangle\) as an ordinary first-order modal interpretation \(M' = \langle U', R', D', \delta', \phi', \pi' \rangle\) as follows.

• \(U'\) is \(U\);

• \(R'\) is \(R\);

\(8^8\)Note that the same notation \(\models\) for \(b\)-entailment was also used in the ordinary first-order case. The reason for extending it to modal logic is that a ground \(b\)-sentence \(\varphi\) is \(b\)-entailed by a set of ground \(b\)-sentences \(X\) according to the Herbrand semantics if and only if it is \(b\)-entailed by \(X\) according to the Kripke semantics.

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• $D'$ is $Tr_{\mathcal{L}_b}$;
• For every $u' \in U'$, $\delta'(u') = D'$, i.e., constant domains;
• For every $u' \in U'$, every $n$-place function symbol $f$ and every $t_1, \ldots, t_n \in D'$ ($= Tr_{\mathcal{L}_b}$), $\phi'(u', f)(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$, i.e., rigid designation; and
• For every $u' \in U'$, if $P$ is an $n$-place predicate symbol and $t_1, \ldots, t_n$ are terms, then $(t_1, \ldots, t_n) \in \pi'(u', P)$ if and only if $P(t_1, \ldots, t_n) \in I(u')$.

In addition, it follows immediately from the above that $\mathfrak{M}'$ has local terms.

### 3.5 First-order ground non-monotonic modal logic

In this section we recall the definition of first-order ground non-monotonic modal logic from [28].

**Definition 53** ([28, Definition 35]) Let $S$ be a normal modal logic, $A$ be a set of first-order modal sentences (axioms), and $b$ be an infinite set of new constant symbols. For a set of sentences $X$ we denote by $\text{GNM}^b_A(S)$ the set of $b$-sentences $b$-entailed by $A \cup S \cup \{M\varphi : \neg \varphi \notin X, \varphi \in \text{GST}_b\}$. That is,

$$\text{GNM}^b_A(S) = \text{Th}_{\mathcal{L}_b}(A \cup S \cup \{M\varphi : \neg \varphi \notin X, \varphi \in \text{GST}_b\}).$$

A set of first-order modal sentences $E$ is called a ground $S$-expansion for $A$, if there exists an infinite set of new constant symbols $b$ and a fixpoint $E_b$ of $\text{GNM}^b_A(S)$ such that $E$ is the restriction of $E_b$ to $\text{St}$.

**Remark 54** By the soundness and completeness of the Kripke semantics, $\text{GNM}^b_A(S)$ can be defined as $\text{Th}_{\mathcal{L}_b}(A \cup S \cup \{M\varphi : X \not\models^b_b \neg \varphi, \varphi \in \text{GST}_b\}).$

**Theorem 55** Let $S$ be a normal modal logic, $A$ be a set of first-order modal sentences (axioms), $b$ be an infinite set of new constant symbols, and $X$ be a consistent set of $b$-sentences. Then $X = \text{GNM}^b_A(S)$ if and only if $X$ satisfies the following two conditions.

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For some \( G \subseteq GSt_b \), \( X = \text{Th}_{b}(A \cup S \cup \{M\varphi : \varphi \in G\}) \); and

(2) for any \( \varphi \in GSt_b \), either \( X \models^S \neg \varphi \) or \( X \models^b M\varphi \).

**Proof of Theorem 55** Assume that \( X = \text{GNM}^{b,A}_S(X) \), that is

\[
X = \text{Th}_b(A \cup S \cup \{M\varphi : X \not\models^S \neg \varphi, \varphi \in GSt_b\}).
\]

(3.1)

Let \( G = \{\varphi : X \not\models^S \neg \varphi, \varphi \in GSt_b\} \). Then, condition 1 immediately follows from (3.1). To show that condition 2 is also satisfied, let \( \varphi \in GSt_b \) be such that \( X \not\models^S \neg \varphi \). Then \( \varphi \in G \), and \( X \models^S M\varphi \) follows from condition 1.

Conversely, assume that \( X \) satisfies conditions 1 and 2. Since \( X \) is consistent, by condition 2, for each \( \varphi \in GSt_b \),

\[
X \not\models^S \neg \varphi \text{ if and only if } M\varphi \in X.
\]

(3.2)

Therefore, \( \{M\varphi : X \not\models^S \neg \varphi, \varphi \in GSt_b\} = \{M\varphi : X \models^S \neg \varphi, \varphi \in GSt_b\} \).

Since \( G \subseteq GSt_b \), by condition 1, \( \{M\varphi : \varphi \in G\} \subseteq \{M\varphi : \varphi \in GSt_b\} \).

Therefore, condition 1 implies

\[
X \subseteq \text{Th}_b(A \cup S \cup \{M\varphi : X \not\models^S \neg \varphi, \varphi \in GSt_b\}).
\]

On the other hand, by (3.2), \( \{M\varphi : X \not\models^S \neg \varphi, \varphi \in GSt_b\} \subseteq X \). Consequently, \( X = \text{Th}_b(A \cup S \cup \{M\varphi : X \not\models^S \neg \varphi, \varphi \in GSt_b\}) (= \text{GNM}^{b,A}_S(X)) \), which completes the proof.

### 3.6 First-order default logic: a semantical approach

First-order defaults are expressions of the form

\[
\frac{\alpha(x) : \beta_1(x), \ldots, \beta_m(x)}{\gamma(x)},
\]

where \( \alpha(x), \beta_1(x), \ldots, \beta_m(x) \), \( m \geq 0 \), and \( \gamma(x) \) are formulas of first-order logic whose free variables are among \( x = x_1, \ldots, x_n \). A default is *closed* if none of \( \alpha, \beta_1, \ldots, \beta_m \), and \( \gamma \) contains a free variable. Otherwise a default is called *open*. Similarly to the propositional case, the intuitive meaning of a first-order default is as follows. For every \( n \)-tuple of objects \( t = t_1, \ldots, t_n \),

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if $\alpha(t)$ is believed, and the $\beta_i(t)$s are consistent with one’s beliefs, then one is permitted to deduce $\gamma(t)$ and add it to the “belief set.” Thus, an open default can be thought of as a kind of a “default scheme,” where the free variables $x$ can be replaced by any of the theory’s objects. Various examples of deduction by defaults can be found in [22]. We remind the reader that a default theory is a pair $(D, A)$, where $D$ is a set of defaults and $A$ is a set of first-order sentences (axioms). A default theory is called closed, if all its defaults are closed. In the general case, it is called open.

**Definition 56 ([22])** Let $(D, A)$ be a closed default theory. For any set of sentences $X$ let $\Gamma_{(D,A)}(X)$ be the smallest set of sentences $B$ (beliefs) that satisfies the following three properties.

- $A \subseteq B$.
- $B = \{ \varphi : B \vdash \varphi \}$, i.e., $B$ is deductively closed.
- If $\frac{\alpha: \beta_1, \ldots, \beta_m}{\gamma} \in D$, $\alpha \in B$, and $\neg \beta_1, \ldots, \neg \beta_m \notin X$, then $\gamma \in B$.

A set of sentences $E$ is an extension for $(D, A)$ if $\Gamma_{(D,A)}(E) = E$, i.e., if $E$ is a fixpoint of the operator $\Gamma_{(D,A)}$.

Next, we present an equivalent semantical definition of extensions for closed default theories.

**Definition 57 ([6])** Let $(D, A)$ be a closed default theory. For any class of interpretations $W$ let $\Sigma_{(D,A)}(W)$ be the largest class $V$ of models of $A$ that satisfies the following condition.

If $\frac{\alpha: \beta_1, \ldots, \beta_m}{\gamma} \in D$, $V \models \alpha$, and $W \not\models \neg \beta_i$, $i = 1, \ldots, m$, then $V \models \gamma$.

It is known from [6] that the definition of extensions as the theories of the fixpoints of $\Sigma$ is equivalent to Reiter’s original definition (Definition 56). That is, a set of sentences $E$ is an extension for a closed default theory $(D, A)$ if and only if $E = \text{Th}_{L_0}(W)$ for some fixpoint $W$ of $\Sigma_{(D,A)}$.

Now, following [13] and [9], we define extensions for open default theories. We start with the intuition lying behind the definition.

There are two types of the domain objects of a default theory. One type consists of the fixed built in objects which are closed terms of $L_0$ and must

---

9This largest class $\Sigma_{(D,A)}(W)$ always exists, see [13, Proposition 1].
be present in any interpretation, and the other type consist of implicitly defined unknown objects which may vary from one interpretation to other. These objects generate other unknown objects by means of the function symbols of $L_0$. Thus, it seems natural to assume that the theory domain is a Herbrand universe of the original language extended with a set of new (unknown) objects, cf. [14, Chapter 1, Section 3]. Note that, in general, it is impossible to describe a Herbrand universe by means of a proof theory. The only exception is the case when the theory domain is provably finite.

The following definition of extensions for open default theories is a relativization of Definition 57 to Herbrand $b$-interpretations.

**Definition 58** ([9]) Let $b$ be an infinite set of new constant symbols and $(D, A)$ be an open default theory. For any set of (possible) Herbrand $b$-interpretations $W$ let $\Delta^b_{(D,A)}(W)$ be the largest class $V$ of (belief) Herbrand $b$-interpretations which are models of $A$ and satisfy the following condition.

For any $\alpha(x) : \beta_1(x), \ldots, \beta_m(x) \in D$, and any tuple $t$ of elements of $\text{Tr}_{L_0}$, if $V \models \alpha(t)$ and $W \not\models \neg\beta_i(t)$, $i = 1, \ldots, m$, then $V \models \gamma(t)$.$^{10}$

A set of sentences $E$ is called a $b$-extension for $(D, A)$ if $E = \text{Th}_{L_0}(W) \cap GSt$ for some fixpoint $W$ of $\Delta^b_{(D,A)}$.

It is known from [9, Theorem 42] that for closed default theories, Definition 58 is equivalent to Reiter’s original definition (Definition 26, p. 15).

**Remark 59** In Definition 58, the requirement for $b$ to be infinite is essential, because for a finite $b$, in some cases, extensions might contain an upper bound on the number of elements in the domain that does not follow from the axioms and defaults of the default theory, see [9, p. 295].

In view of Remarks 44 and 59, for the rest of this paper we assume that the set of new constant symbols $b$ augmenting $L$ or $L_0$ is infinite.

**Remark 60** In the above definitions we implicitly assume that there is no equality relation between the domain elements. Any such equality in an extension must follow from axioms and defaults. To some extent, this can be considered as a weak form of the unique name assumption, cf. [9, p. 304]. Note that the assignment to equality in Herbrand interpretations is a binary relation.

$^{10}$Similarly to [13, Proposition 3], it can be shown that this largest set $\Delta^b_{(D,A)}(W)$ always exists.
relation that does not have to be identity in the domain of the interpretation, but satisfies the equality first-order axioms. That is, equality is treated as an ordinary dyadic predicate.\footnote{Interpretations where the assignment to the equality relation is identity in the domain of the interpretation are called \textit{normal}, see [21, p. 78] for details.}
Chapter 4

Minimal model semantics for first-order ground non-monotonic modal logic

In this chapter we present our first contribution: A new minimal model semantics for a class of first-order ground non-monotonic modal logics and prove that for this class of logics, our definition is equivalent to Definition 53 (p. 29). We also show that the semantical characterizations of propositional ground non-monotonic modal logic, from Section 2.4.1 (p. 12) and Section 2.4.2 (p. 13) are not suitable for first-order ground non-monotonic modal logic.

4.1 A new minimal model semantics

In this section we present a new minimal model semantics for a class of first-order ground non-monotonic modal logics and prove that for this class of logics, our definition is equivalent to Definition 53 (p. 29). Our description involves the following notation and definitions.

- For an S5-Kripke $b$-interpretation $\mathcal{M} = \langle U, R, I \rangle$, we just write $\mathcal{M} = \langle U, I \rangle$. This is because $R$ is the total accessibility relation.

- Let $S$ be a normal modal logic characterized by a cluster-decomposable class $\mathcal{C}$ of Kripke $b$-interpretations. For an S-Kripke $b$-interpretation $\mathcal{M} \in \mathcal{C}$, $\hat{\mathcal{M}}$ denotes the following S5-Kripke $b$-interpretation.
If $\mathcal{M}$ is an S5-Kripke $b$-interpretation, then $\widehat{\mathcal{M}}$ is $\mathcal{M}$.

Otherwise, $\widehat{\mathcal{M}}$ is the final cluster of $\mathcal{M}$. That is, $\mathcal{M} = \mathcal{M}' \circ \widehat{\mathcal{M}}$, for some Kripke $b$-interpretation $\mathcal{M}'$.

- We shall denote by $MGSt_b$ the set of $b$-sentences $\{M\varphi : \varphi \in GSt_b\}$.

Definition 61 Let $S$ be a normal modal logic, $b$ be an infinite set of new constant symbols, and let $\mathcal{M}'$ and $\mathcal{M}''$ be S-Kripke $b$-interpretations. We write $\mathcal{M}' \leq \mathcal{M}''$ if $Th_{L_b}(\mathcal{M}') \cap GSt_b \subseteq Th_{L_b}(\mathcal{M}'') \cap GSt_b$, and write $\mathcal{M}' < \mathcal{M}''$ if $Th_{L_b}(\mathcal{M}') \cap GSt_b \subset Th_{L_b}(\mathcal{M}'') \cap GSt_b$.

Definition 62 Let $b$ be an infinite set of new constant symbols, $S$ be a normal modal logic characterized by a cluster-decomposable class of Kripke $b$-interpretations $\mathcal{C}$ and $A$ be a set of modal sentences. An S-Kripke $b$-interpretation $\mathcal{M} \in \mathcal{C}$ is called $A$-minimal if it satisfies properties A1-A4 below.

(A1) $\mathcal{M} \models A$

(A2) $Th_{L_b}(\mathcal{M}) \cap GSt_b = Th_{L_b}(\mathcal{M}) \cap GSt_b$.

(A3) There is no S-Kripke $b$-interpretation $\mathcal{M}' \in \mathcal{C}$ such that $\mathcal{M}' \models A$ and $\mathcal{M}' \not< \mathcal{M}$.

(A4) There is no S-Kripke $b$-interpretation $\mathcal{M}' \in \mathcal{C}$ such that $\mathcal{M}' \models A$ and $Th_{L_b}(\mathcal{M}') \cap GSt_b = Th_{L_b}(\mathcal{M}) \cap GSt_b$, but $\mathcal{M}' < \mathcal{M}$.

Remark 63 Since every S5-Kripke $b$-interpretation $\mathcal{M}$ trivially satisfies properties (A2) and (A4) of Definition 62, Definition 62 can be simplified for modal logic S5 as follows. An S5-Kripke $b$-interpretation $\mathcal{M}$ is called $A$-minimal if it satisfies $A$ and is minimal with respect to the partial ordering $\leq$ among all S5-Kripke $b$-models of $A$.

Theorem 64 Let $S$ be a normal modal logic characterized by a cluster-decomposable class of Kripke $b$-interpretations $\mathcal{C}$, and $A$ be a set of modal sentences (axioms). A set of sentences $E$ is a consistent first order ground $S$-expansion for $A$ if and only if there exist an infinite set of new constant symbols $b$, an $A$-minimal S-Kripke $b$-interpretation $\mathcal{M} \in \mathcal{C}$ and a set of $b$-sentences $E_b$ such that

$$E_b = Th_{L_b}(A \cup S \cup \{M\varphi : \mathcal{M} \not\models \varphi, \; \varphi \in GSt_b\}).$$
and \( E = E_b \cap St. \)

Theorem 64 immediately follows from Proposition 65 below and Definition 53 (p. 29).

**Proposition 65** Let \( b \) be an infinite set of new constant symbols, \( S \) be a normal modal logic characterized by a cluster-decomposable class of Kripke \( b \)-interpretations \( C \), and \( A \) be a set of modal sentences (axioms). A set of \( b \)-sentences \( E_b \) is a consistent fixpoint of \( GNM^{b,A}_S \) if and only if there exists an \( A \)-minimal \( S \)-Kripke \( b \)-interpretation \( \mathcal{M} \in C \) and

\[
E_b = \text{Th}_{\mathcal{L}_b}(A \cup S \cup \{ M \varphi : \mathcal{M} \nvDash \neg \varphi, \ \varphi \in GST_b \}).
\]

We shall now prove a number of lemmas which are needed for the proof of Proposition 65. In these lemmas, \( b \) is an infinite set of new constant symbols, \( S \) is a normal modal logic characterized by a cluster-decomposable class of Kripke \( b \)-interpretations \( C \), and \( A \) is a set of modal sentences (axioms).

**Lemma 66** Let \( E_b \) be a fixpoint of \( GNM^{b,A}_S \), and \( \mathcal{M} \) be an \( S \)-Kripke \( b \)-model of \( E_b \). Then, \( \text{Th}_{\mathcal{L}_b}(\mathcal{M}) \cap GST_b = E_b \cap GST_b. \)

**Proof** Since \( \mathcal{M} \models E_b, E_b \cap GST_b \subseteq \text{Th}_{\mathcal{L}_b}(\mathcal{M}) \cap GST_b. \) Assume to the contrary that \( E_b \cap GST_b \subset \text{Th}_{\mathcal{L}_b}(\mathcal{M}) \cap GST_b. \) Then, there exists a \( b \)-sentence \( \varphi \in GST_b \) such that \( \varphi \notin E_b \) and \( \mathcal{M} \models \varphi. \)

Since \( E_b = \text{Th}_{\mathcal{L}_b}(A \cup S \cup \{ M \varphi : \neg \varphi \notin E_b, \ \varphi \in GST_b \}) \) and \( \varphi \notin E_b, M \neg \varphi \notin E_b. \) This, together with \( \mathcal{M} \models E_b \), implies \( \mathcal{M} \models M \neg \varphi. \) Consequently, \( \mathcal{M} \models \varphi, \) in contradiction with \( \mathcal{M} \models \varphi. \)

**Lemma 67** Let \( \mathcal{M}' \) and \( \mathcal{M}'' \) be \( S \)-Kripke \( b \)-interpretations such that \( \mathcal{M}' \trianglelefteq \mathcal{M}'' \).

Then, \( \text{Th}_{\mathcal{L}_b}(\mathcal{M}'') \cap MSGSt_b \subseteq \text{Th}_{\mathcal{L}_b}(\mathcal{M}') \cap MSGSt_b. \)

**Proof** Let \( \psi \in GST_b \) be such that \( \mathcal{M}'' \models M \psi. \) Then, \( \mathcal{M}'' \models \neg \psi, \) because \( \mathcal{M}'' \) is cluster decomposable. This, together with \( \mathcal{M}' \trianglelefteq \mathcal{M}'' \), implies \( \mathcal{M}' \models \neg \psi. \) Consequently, \( \mathcal{M}' \models M \psi. \)

\[1\] Note that this is no longer a fixpoint equation, cf. Definition 53 (p. 29).

\[2\] In Lemma 66, \( S \) can be any normal modal logic.
Lemma 68  Let $\mathcal{M}'$ and $\mathcal{M}''$ be S-Kripke $b$-interpretations such that

$$Th_{L_b}(\mathcal{M}') \cap MGSt_b \subseteq Th_{L_b}(\mathcal{M}'') \cap MGSt_b.$$  

Then, $\widehat{\mathcal{M}''} \subseteq \widehat{\mathcal{M}'}$.

Proof  Let $\psi \in GSt_b$ be such that $\widehat{\mathcal{M}'} \not\models \psi$. Then, $\mathcal{M}' \models M\neg \psi$, because $\mathcal{M}'$ is cluster decomposable. This, together with $Th_{L_b}(\mathcal{M}') \cap MGSt_b \subseteq Th_{L_b}(\mathcal{M}'') \cap MGSt_b$, implies $\mathcal{M}'' \models M\neg \psi$. Consequently, $\widehat{\mathcal{M}''} \not\models \psi$. 

Lemma 69  Let $E_b$ be a fixpoint of $\text{GNM}_S^{b,A}$, and let $\mathcal{M} \in \mathcal{C}$ be an S-Kripke $b$-interpretation of $E_b$. Then, $Th_{L_b}(\mathcal{M}) \cap GSt_b = Th_{L_b}(\widehat{\mathcal{M}}) \cap GSt_b$.

Proof  By the definition of $\widehat{\mathcal{M}}$, $Th_{L_b}(\mathcal{M}) \cap GSt_b \subseteq Th_{L_b}(\widehat{\mathcal{M}}) \cap GSt_b$. Assume to the contrary that $Th_{L_b}(\mathcal{M}) \cap GSt_b \subset Th_{L_b}(\widehat{\mathcal{M}}) \cap GSt_b$. Then, there exists a $b$-sentence $\psi \in GSt_b$ such that $\mathcal{M} \not\models \psi$, but $\widehat{\mathcal{M}} \models \psi$. Since $\mathcal{M} \models E_b$ and $\widehat{\mathcal{M}} \not\models \psi$, by Lemma 66, $\psi \not\in E_b$. Hence, by the definition of $E_b$, $M\neg \psi \in E_b$. However, since $\mathcal{M}$ is cluster decomposable, $\widehat{\mathcal{M}} \models \psi$ implies $\mathcal{M} \not\models M\neg \psi$, in contradiction with $\mathcal{M} \models E_b$.

Lemma 70  Let $E_b$ be a fixpoint of $\text{GNM}_S^{b,A}$, and $\mathcal{M} \in \mathcal{C}$ be an S-Kripke $b$-interpretation that satisfies $E_b$. Then, there is no S-Kripke $b$-interpretation $\mathcal{M}' \in \mathcal{C}$ such that $\mathcal{M}' \models A$ and $\mathcal{M}' \not\prec \mathcal{M}$.

Proof  Let $\mathcal{M}' \in \mathcal{C}$ be an S-Kripke $b$-interpretation which satisfies $A$ and assume to the contrary that $\mathcal{M}' \prec \mathcal{M}$. Then, for some $\psi \in GSt_b$, $\mathcal{M}' \not\models \psi$, but $\widehat{\mathcal{M}} \models \psi$.

Since $\mathcal{M} \models E_b$ and $E_b$ is a fixpoint of $\text{GNM}_S^{b,A}$,

$$\mathcal{M} \models A \cup S \cup \{M\varphi : \neg \varphi \not\in E_b, \ \varphi \in GSt_b\}.$$  

In addition, by Lemma 67, $Th_{L_b}(\mathcal{M}) \cap MGSt_b \subseteq Th_{L_b}(\mathcal{M}') \cap MGSt_b$, implying $\mathcal{M}' \models A \cup S \cup \{M\varphi : \neg \varphi \not\in E_b, \ \varphi \in GSt_b\}$. Therefore, since $\mathcal{M}' \not\models \psi$ implies $\mathcal{M}' \not\models \psi$, and $E_b$ is a fixpoint of $\text{GNM}_S^{b,A}$, $\psi \not\in E_b$.

Since $\mathcal{M}$ is cluster decomposable, $\widehat{\mathcal{M}} \models \psi$ implies $\mathcal{M} \not\models M\neg \psi$. This, together with $\mathcal{M} \models E_b$, implies $\psi \in E_b$, in contradiction with $\psi \not\in E_b$. 

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Lemma 71 Let \( E_b \) be a fixpoint of \( \text{GNM}^b_S \), and \( \mathcal{M} \in C \) be an S-Kripke \( b \)-interpretation that satisfies \( E_b \). Then, there is no S-Kripke \( b \)-interpretation \( \mathcal{M}' \in C \) such that \( \mathcal{M}' \models A \) and \( \text{Th}_{L_b}(\mathcal{M}') \cap GSt_b = \text{Th}_{L_b}(\mathcal{M}) \cap GSt_b \), but \( \mathcal{M}' \prec \mathcal{M} \).

Proof Let \( \mathcal{M}' \in C \) be an S-Kripke \( b \)-interpretation such that \( \mathcal{M}' \models A \) and \( \text{Th}_{L_b}(\mathcal{M}') \cap GSt_b = \text{Th}_{L_b}(\mathcal{M}) \cap GSt_b \). Assume to the contrary that \( \mathcal{M}' \prec \mathcal{M} \).

Since \( \text{Th}_{L_b}(\mathcal{M}) \cap GSt_b = \text{Th}_{L_b}(\mathcal{M}') \cap GSt_b \), by Lemmas 67 and 68,
\[
\text{Th}_{L_b}(\mathcal{M}) \cap \text{MGSt}_b = \text{Th}_{L_b}(\mathcal{M}') \cap \text{MGSt}_b.
\]
Since \( E_b \) is a fixpoint of \( \text{GNM}^b_S \) and \( \mathcal{M} \models E_b \),
\[
\mathcal{M} \models A \cup S \cup \{ M\varphi : \neg \varphi \not\in E_b, \varphi \in GSt_b \}.
\]
Therefore,
\[
\mathcal{M}' \models \{ M\varphi : \neg \varphi \not\in E_b, \varphi \in GSt_b \}.
\]
Hence, since \( \mathcal{M}' \) is an S-Kripke \( b \)-model of \( A \),
\[
\mathcal{M}' \models A \cup S \cup \{ M\varphi : \neg \varphi \not\in E_b, \varphi \in GSt_b \}.
\]
Consequently, since \( \mathcal{M}' \not\models \psi \),
\[
\psi \not\in \text{Th}_{L_b}(A \cup S \cup \{ M\varphi : \neg \varphi \not\in E_b, \varphi \in GSt_b \}).
\]
Therefore, since \( E_b \) is a fixpoint of \( \text{GNM}^b_S \), \( \psi \not\in E_b \).

However, by Lemma 66, \( \mathcal{M} \models \psi \) implies \( \psi \in E_b \), in contradiction with \( \psi \not\in E_b \).

Proposition 72 Let \( E_b \) be a fixpoint of \( \text{GNM}^b_S \), and \( \mathcal{M} \in C \) be an S-Kripke \( b \)-model of \( E_b \). Then, \( \mathcal{M} \) is A-minimal.

Proof The proof immediately follows from Lemmas 69, 70 and 71.

Lemma 73 Let \( \mathcal{M} \in C \) be an A-minimal S-Kripke \( b \)-interpretation and let
\[
X = \text{Th}_{L_b}(A \cup S \cup \{ M\varphi : \mathcal{M} \not\models \neg \varphi, \varphi \in GSt_b \}).
\]
Then,
\[
\{ \varphi : \mathcal{M} \not\models \varphi, \varphi \in GSt_b \} = \{ \varphi : \varphi \not\in X, \varphi \in GSt_b \}.
\]
Proof We start with the proof of the inclusion

\[\{ \varphi : M \not\models \varphi, \ \varphi \in GSt_b \} \subseteq \{ \varphi : \varphi \not\in X, \ \varphi \in GSt_b \}.\]

Let \(\psi \in GSt_b\) be such that \(M \not\models \psi\). Since \(M\) is \(A\)-minimal, \(M \models A\) and \(Th_{L_b}(M) \cap GSt_b = Th_{L_b}(M) \cap GSt_b\). Therefore, since \(M\) is an \(S\)-Kripke \(b\)-interpretation,

\[M \models A \cup S \cup \{ M\varphi : M \not\models \neg \varphi, \ \varphi \in GSt_b \},\]

implying \(M \models X\). Consequently, \(\psi \not\in X\).

For the converse inclusion

\[\{ \varphi : \varphi \not\in X, \ \varphi \in GSt_b \} \subseteq \{ \varphi : M \not\models \varphi, \ \varphi \in GSt_b \},\]

let \(\psi \in GSt_b\) be such that \(\psi \not\in X\). Assume to the contrary that \(M \models \psi\). Then, by the definition of \(X\), there exists an \(S\)-Kripke \(b\)-interpretation \(M' \in C\) such that

\[M' \models A \cup S \cup \{ M\varphi : M \not\models \neg \varphi, \ \varphi \in GSt_b \},\]

but \(M' \not\models \psi\).

We observe that \(Th_{L_b}(M) \cap MGSt_b \subseteq Th(M') \cap MGSt_b\). Indeed, let \(\psi \in GSt_b\) be such that \(M \models \psi\). Then, by the definition of \(\models\), \(M \not\models \neg \psi\). Therefore, since \(M' \models \{ M\varphi : M \not\models \neg \varphi, \ \varphi \in GSt_b \},\) \(M' \models \psi\).

By our observation and Lemma 68, \(M' \preceq M\). Therefore, since \(M\) is \(A\)-minimal, \(Th_{L_b}(M) \cap GSt_b = Th_{L_b}(M) \cap GSt_b\). In addition, since \(M'\) is cluster decomposable, \(M' \preceq M',\) implying \(M' \preceq M\).

Since \(M\) is \(A\)-minimal, \(Th_{L_b}(M) \cap GSt_b = Th_{L_b}(M) \cap GSt_b\). Hence, \(M' \preceq M\). However, this, together with \(M \models \psi\) and \(M' \not\models \psi\), implies \(M' \not\preceq M\), in contradiction with the \(A\)-minimality of \(M\).

Proof of Proposition 65 Let \(E_b\) be a consistent fixpoint of \(GNM^A_{S}\). That is,

\[E_b = Th_{L_b}(A \cup S \cup \{ M\varphi : \neg \varphi \not\in E_b, \ \varphi \in GSt_b \}).\]

Since \(E_b\) is consistent and \(S\) is characterized by \(C\), there exists in \(C\) an \(S\)-Kripke \(b\)-model of \(E_b\), which will be denoted by \(M\). Hence, By Proposition 72, \(M\) is \(A\)-minimal. By Lemma 73,

\[\{ M\varphi : \neg \varphi \not\in E_b, \ \varphi \in GSt_b \} = \{ M\varphi : M \not\models \neg \varphi, \ \varphi \in GSt_b \}.\]
Consequently,
\[ E_b = Th_{\mathcal{L}_b}(A \cup S \cup \{ M\varphi : M \not\models \neg\varphi, \ \varphi \in GSt_b \}). \]

Conversely, let \( M \in \mathcal{C} \) be an \( A \)-minimal S-Kripke \( b \)-interpretation and let
\[ E_b = Th_{\mathcal{L}_b}(A \cup S \cup \{ M\varphi : M \not\models \neg\varphi, \ \varphi \in GSt_b \}). \]

Then, by Lemma 73,
\[ E_b = Th_{\mathcal{L}_b}(A \cup S \cup \{ M\varphi : M \not\models \neg\varphi, \ \varphi \in GSt_b \}), \]

i.e., \( E_b \) is a fixpoint of \( \text{GNM}_{S,b}^{A} \).

Finally, to prove that \( E_b \) is consistent, it suffices to show that
\[ M \models A \cup S \cup \{ M\varphi : M \not\models \neg\varphi, \ \varphi \in GSt_b \}. \]

Since \( M \) is \( A \)-minimal, by property (A1) of Definition 62, \( M \models A \). Since \( M \) is an S-Kripke \( b \)-interpretation, \( M \models S \). By property (A2) of Definition 62, for any \( \varphi \in GSt_b \), \( M \not\models \neg\varphi \) implies \( \widehat{M} \not\models \neg\varphi \). Hence, there exists a world \( u \) in \( \widehat{M} \) such that \( (\widehat{M}, u) \models \varphi \). Therefore, since every world in \( M \) is connected to every world in \( \widehat{M} \), \( M \models M\varphi \). Consequently,
\[ M \models \{ M\varphi : M \not\models \neg\varphi, \ \varphi \in GSt_b \}, \]

which completes the proof.

4.2 On the suitability of semantical characterizations of propositional ground non-monotonic modal logic for the first-order case

In this section we show that the semantical characterizations of propositional ground non-monotonic modal logic, presented in Section 2.4.1 (p. 12) and Section 2.4.2 (p. 13), are not suitable for the first-order case.

We start with the minimal model semantics of propositional ground non-monotonic modal logic, presented in Section 2.4.1.
Definitions 10–16 (pages 12–13) of Section 2.4.1 naturally extend to the first-order case, by replacing Kripke semantics of propositional modal logic (Definition 1, p. 8) with Kripke semantics of first-order modal logic (Definition 51, p. 27). We omit the details.

Example 74 below shows that for normal modal logic KD45,\(^3\) which can be characterized by a cluster-decomposable class of Kripke \(b\)-interpretations \(C_{KD45}\), ground \(C_{KD45}\)-minimal \(b\)-models are not suitable to characterize first-order ground KD45-expansions.

**Example 74** Let \(b\) be an infinite set of new constant symbols. Then, there exists a set of modal sentences (axioms) \(A\), for which there are no consistent first-order ground KD45-expansions, but there is a ground \(C_{KD45}\)-minimal \(b\)-model of \(A\).

The proof of Example 74 is based on [28, Example 3 (p. 47)]. It involves the following notations.

- The underlying language \(L\) consists of one unary predicate symbol \(P\) and infinitely many constant symbols \(c_1, c_2, \ldots\);
- \(b\) is an infinite set of new constant symbols;
- \(u_A\) is a Herbrand \(b\)-interpretation defined by \(u_A = \{P(c_i)\}_{i=1,2,...};\)
- \(U_A\) is a set of all Herbrand \(b\)-interpretations \(u\) which include \(u_A\), i.e., \(U_A = \{u : u_A \subseteq u\};\)
- \(U\) is a set of Herbrand \(b\)-interpretations that is the result of removing \(u_A\) from \(U_A\), i.e., \(U = U_A \setminus \{u_A\} = \{u : u_A \subset u\};\)
- \(\xi\) is the modal sentence \(\exists x(\neg P(x) \land LP(x)) \supset L\exists x(P(x) \land \neg LP(x));\)
- \(A\) is a set of modal sentences defined by \(A = \{\xi\} \cup \{LP(c_i)\}_{i=1,2,...};\)
- \(M_A = \langle U_A, I_A \rangle\), where for every \(u \in U_A\), \(I_A(u) = u.\)
- \(M = \langle U, I \rangle\), where for every \(u \in U\), \(I(u) = u;\)

\(^3\)Modal logic KD45 is characterized by the class \(C_{KD45}\) of all Kripke interpretations of the form \(\langle\{u\} \cup U, R, I\rangle\), where \(U\) is non-empty, \(\{u\}\) and \(U\) are not necessarily disjoint and \(R = (\{u\} \cup U) \times U\) (see [24, p. 38]).
We shall now prove a number of lemmas which are needed for the proof of Example 74.

Lemma 75 ([28, Example 3]) $\text{Th}_{L_{E}}(M) \cap GSt_{b} = \text{Th}_{L_{E}}(M_{A}) \cap GSt_{b}$.

Lemma 76 $M_{A} \models A$.

Proof Since $M_{A}$ is an S5-Kripke $b$-interpretation, for every $u \in U_{A}$ and for every $t \in Tr_{L_{r}}$, $(M_{A}, u) \models LP(t)$ implies $(M_{A}, u) \models P(t)$. Therefore, $M_{A} \not\models \exists x(\neg P(x) \land LP(x))$. Hence, $M_{A} \models \xi$.

By the definition of $U_{A}$, $U_{A} \models \{P(c_{i})\}_{i=1,2,...}$. That is, for every $u \in U_{A}$, $(M_{A}, u) \models \{P(c_{i})\}_{i=1,2,...}$. Consequently, $M_{A} \models \{LP(c_{i})\}_{i=1,2,...}$.

Therefore, since $A = \{\xi\} \cup \{LP(c_{i})\}_{i=1,2,...}$, $M_{A} \models A$.

Lemma 77 $M_{A}$ is a ground $C_{KD45}$-minimal $b$-model for $A$.

Proof By Lemma 76, $M_{A}$ is a $b$-model of $A$. Assume to the contrary that there exists a Kripke $b$-interpretation $M' = \langle U', R', I' \rangle \in C_{KD45}$ such that $M' \models A$ and $M' \supseteq G \ M_{A}$. Then, there exists a world $u' \in U'$ such that for every $u \in U_{A}$, $I'(u') \not\models I_{A}(u)$. Consequently, since $U_{A}$ is the set of all Herbrand $b$-interpretations which include $\{P(c_{i})\}_{i=1,2,...}$, there exists some $c \in \{c_{i}\}_{i=1,2,...}$ for which $P(c) \not\models I'(u')$. Hence, $(M', u') \not\models P(c)$. Also, $M' \models A$ implies $M' \models LP(c)$. Consequently,

$(M', u') \models \exists x(\neg P(x) \land LP(x))$.

This, together with $M' \models \xi$, implies

$(M', u') \models L\exists x(P(x) \land \neg LP(x))$.

Since $M' \supseteq G \ M_{A}$, there exists a Kripke $b$-interpretation $M''$ such that $M'' \supseteq G \ M'' \cap M_{A}$. Therefore, for each world $u \in U'$, $(u, u_{A}) \in R'$. This, together with $(M', u') \models L\exists x(P(x) \land \neg LP(x))$, implies

$(M', u_{A}) \models \exists x(P(x) \land \neg LP(x))$.

However, for every $t \in Tr_{L_{E}}$, $(M', u_{A}) \models P(t)$ implies $t \in \{c_{i}\}_{i=1,2,...}$. Consequently, for some $c \in \{c_{i}\}_{i=1,2,...}$, $(M', u_{A}) \models \neg LP(c)$, in contradiction with $M' \models \{LP(c_{i})\}_{i=1,2,...}$.

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Proof of Example 74 By Lemma 77, $\mathcal{M}_A$ is a ground $C_{KD45}$-minimal b-model for $A$. Therefore, by Theorem 64, to prove that there are no consistent first-order ground KD45-expansions for $A$, it suffices to prove that $C_{KD45}$ contains no $A$-minimal Kripke $b$-interpretations. Assume to the contrary that there exists an $A$-minimal Kripke $b$-interpretation $\mathcal{M}' = \langle U', R', I' \rangle \in C_{KD45}$. Let $u_1$ be a Herbrand $b$-interpretation such that $P(c_1) \notin u_1$, i.e., $u_1 \neq P(c_1)$. Consider a Kripke $b$-interpretation $\mathcal{M}'' = \langle U'', R'', I'' \rangle \in C_{KD45}$, where $U'' = \{u_1\} \cup U$, $R'' = \{\{u_1\}\} \times U \cup (U \times U)$, and for every $u \in U''$, $I''(u) = u$. We shall prove that $\mathcal{M}'' \models A$ and

$$Th_{\mathcal{L}_b}(\widehat{\mathcal{M}'}) \cap GSt_b = Th_{\mathcal{L}_b}(\widehat{\mathcal{M}'}) \cap GSt_b,$$

but $\mathcal{M}'' \lhd \mathcal{M}'$, implying that $\mathcal{M}'$ does not satisfy condition (A4) of Definition 62.

First, we shall prove that $\mathcal{M}'' \models A$. By the definition of $U$, for every $u \in U'' \setminus \{u_1\}$, $(\mathcal{M}'', u) \models \{P(c_i)\}_{i=1,2,\ldots}$. Also, for every $u \in U''$, $(u, u_1) \notin R''$. Consequently, $\mathcal{M}'' \models \{LP(c_i)\}_{i=1,2,\ldots}$.

For every $u \in U$, there exist some $c \in b$ and $u' \in U$ such that $P(c) \in u$ and $P(c) \notin u'$. Therefore, for every $u \in U$, $(\mathcal{M}'', u) \models \exists x (P(x) \land \neg LP(x))$. Hence, $\mathcal{M}'' \models L \exists x (P(x) \land \neg LP(x))$, implying $\mathcal{M}'' \models \xi$. Consequently, $\mathcal{M}'' \models A$.

Next, we shall prove that

$$Th_{\mathcal{L}_b}(\widehat{\mathcal{M}'}) \cap GSt_b = Th_{\mathcal{L}_b}(\widehat{\mathcal{M}'}) \cap GSt_b = Th_{\mathcal{L}_0b}(\{P(c_i)\}_{i=1,2,\ldots}).$$

By the definition of $\mathcal{M}_A$, $Th_{\mathcal{L}_b}(\widehat{\mathcal{M}'}) \cap GSt_b = Th_{\mathcal{L}_0b}(U_A)$. Since $U_A$ is the set of all Herbrand $b$-interpretations which include $\{P(c_i)\}_{i=1,2,\ldots}$, $Th_{\mathcal{L}_0b}(U_A) = Th_{\mathcal{L}_0b}(\{P(c_i)\}_{i=1,2,\ldots})$. Hence, $Th_{\mathcal{L}_b}(\widehat{\mathcal{M}'}) \cap GSt_b = Th_{\mathcal{L}_0b}(\{P(c_i)\}_{i=1,2,\ldots})$. Consequently, since $\widehat{\mathcal{M}''}$ is $\mathcal{M}$, by Lemma 75,

$$Th_{\mathcal{L}_b}(\widehat{\mathcal{M}'}) \cap GSt_b = Th_{\mathcal{L}_0b}(\{P(c_i)\}_{i=1,2,\ldots}).$$

Since $\widehat{\mathcal{M}''} \models \{LP(c_i)\}_{i=1,2,\ldots}$ and since $\widehat{\mathcal{M}'}$ is an $S5$-Kripke $b$-interpretation, $Th_{\mathcal{L}_0b}(\{P(c_i)\}_{i=1,2,\ldots}) \subseteq Th_{\mathcal{L}_b}(\widehat{\mathcal{M}'}) \cap GSt_b$. In addition, since $\mathcal{M}'$ is an $A$-minimal element of $C_{KD45}$, by condition (A2) of Definition 62,

$$Th_{\mathcal{L}_b}(\mathcal{M}') \cap GSt_b = Th_{\mathcal{L}_b}(\widehat{\mathcal{M}'}) \cap GSt_b.$$

\footnote{Recall that $U = U_A \setminus \{u_A\} = \{u : u_A \subset u\}$.}
Finally, as we have seen, \( \mathcal{M}_A \models A \) and
\[
Th_{\mathcal{L}_b}(\mathcal{M}_A) \cap GSt_b = Th_{\mathcal{L}_b}(\{P(c_i)\}_{i=1,2,...}).
\]
Therefore, were
\[
Th_{\mathcal{L}_b}(\{P(c_i)\}_{i=1,2,...}) \subset Th_{\mathcal{L}_b}(\hat{\mathcal{M}}') \cap GSt_b,
\]
we would have
\[
Th_{\mathcal{L}_b}(\hat{\mathcal{M}}') \cap GSt_b \subset Th_{\mathcal{L}_b}(\hat{\mathcal{M}}'') \cap GSt_b,
\]
in contradiction with condition (A3) of Definition 62. Consequently,
\[
Th_{\mathcal{L}_b}(\hat{\mathcal{M}}'') \cap GSt_b = Th_{\mathcal{L}_b}(\{P(c_i)\}_{i=1,2,...}).
\]
Finally, we shall prove that \( \mathcal{M}'' \models \mathcal{M}' \). Since \( \mathcal{M}' \) is \( \mathcal{M}'' \mathcal{M}' \). We have seen that
\[
Th_{\mathcal{L}_b}(\mathcal{M}') \cap GSt_b = Th_{\mathcal{L}_b}(\{P(c_i)\}_{i=1,2,...}),
\]
and
\[
Th_{\mathcal{L}_b}(\hat{\mathcal{M}}') \cap GSt_b = Th_{\mathcal{L}_b}(\{P(c_i)\}_{i=1,2,...}),
\]
Therefore,
\[
Th_{\mathcal{L}_b}(\mathcal{M}) \cap GSt_b = Th_{\mathcal{L}_b}(\mathcal{M}') \cap GSt_b.
\]
This, together with \( \mathcal{M}' \models \mathcal{M}'' \), implies \( \mathcal{M}' \models \mathcal{M}' \).

Since \( u_1 \not\models P(c_1) \), \( (\mathcal{M}'', u_1) \not\models P(c_1) \). Therefore,
\[
P(c_1) \not\in Th_{\mathcal{L}_b}(\mathcal{M}'') \cap GSt_b.
\]
However, since \( Th_{\mathcal{L}_b}(\mathcal{M}') \cap GSt_b = Th_{\mathcal{L}_b}(\{P(c_i)\}_{i=1,2,...}) \),
\[
P(c_1) \in Th_{\mathcal{L}_b}(\mathcal{M}') \cap GSt_b.
\]
Consequently, \( \mathcal{M}' \models \mathcal{M}' \).

In conclusion, we show that the characterization of propositional ground S-expansions as S-expansions satisfying some minimality criterion, presented in Section 2.4.2 (p. 13) is not suitable for the first-order case. The reason is that, unlike in the propositional case, \(^5\) first-order ground S-expansions are not necessarily S-expansions.

First we shall recall the definition of first-order S-expansions, from [10].

\(^5\)See Theorem 8 (p. 11).
Definition 78 ([10]) Let $\mathcal{M} = \langle U, I \rangle$ and $\mathcal{M}' = \langle U', I' \rangle$ be an S5-Kripke $b$-interpretation and a Kripke $b$-interpretation, respectively. We say that $\mathcal{M}' \circ \mathcal{M}$ is weakly preferred over $\mathcal{M}$, denoted by $\mathcal{M}' \circ \mathcal{M} \not<_{w} \mathcal{M}$, if there is a (not necessary ground) $b$-sentence $\Theta$ such that $\mathcal{M}' \circ \mathcal{M} \not|= \Theta$, but $\mathcal{M} |= \Theta$.

Definition 79 ([10]) Let $b$ be an infinite set of new constant symbols, $C$ be a class of Kripke $b$-interpretations and $A$ be a set of $b$-sentences. An S5-Kripke $b$-interpretation $\mathcal{M} \in C$ is called strongly $C$-minimal for $A$ if $\mathcal{M} |= A$ and for every Kripke $b$-interpretation $\mathcal{M}'$ such that $\mathcal{M}' \circ \mathcal{M} \in C$ and $\mathcal{M}' \circ \mathcal{M} |= A$, $\mathcal{M}' \circ \mathcal{M} \not<_{w} \mathcal{M}$.

Definition 80 ([10]) Let $S$ be a normal modal logic and $A$ be a set of first-order modal sentences (axioms) over $L$. A set of first-order modal sentences $E$ is called an $S$-expansion for $A$ if there exists an infinite set of new constant symbols $b$ and a Kripke $b$-interpretation $\mathcal{M}$ strongly minimal for $A \cup S$ such that $E = Th_{L_b}(\mathcal{M}) \cap St$.

Consider a normal modal logic $S$. Since strongly minimal Kripke $b$-interpretations are necessarily S5-Kripke $b$-interpretations, all first-order $S$-expansions contain S5. However, the following example shows that first-order ground $S$-expansions do not necessarily contain S5, even if $S$ is characterized by a class of cluster decomposable Kripke interpretations. Therefore, it is impossible to characterize first-order ground $S$-expansions as $S$-expansions satisfying some minimality criterion.

Example 81 Let $L$, $b$, $\mathcal{M}_A$ and $\mathcal{M}$ be defined as in page 41. Also, let $A = \{P(c_i)\}_{i=1,2,...}$ and let
$$E = Th_b(A \cup S4F \cup \{M \varphi : A \not|=^{S4F} \varphi, \varphi \in GSt_b\}).$$

By Theorem 55 (p. 29), to prove that $E$ is a ground S4F-expansion for $A$, it suffices to show that the set of modal $b$-sentences
$$A \cup S4F \cup \{M \varphi : A \not|=^{S4F} \varphi, \varphi \in GSt_b\}$$
is $b$-consistent. Obviously, $\mathcal{M}_A |= A$, and since $S4F \subseteq S5$, $\mathcal{M}_A |= S4F$. Let $\varphi \in GSt_b$ be such that $A \not|=^{S4F} \varphi$. Then, since $\mathcal{M}_A$ consists of all

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6Modal logic S4F is characterized by the class of all Kripke interpretations of the form $\langle U_1 \cup U_2, R, I \rangle$, where $U_1$ and $U_2$ are disjoint, $U_2$ is non-empty and $uRv$ if and only if $v \in U_2$ or $u, v \in U_1$ (see [25] and [24, p. 38]).
Herbrand $b$-interpretations which include $A$, for some $u$ in $\mathcal{M}_A$, $(\mathcal{M}_A, u) \models \neg \varphi$. Therefore, since $\mathcal{M}_A$ is an S5-Kripke $b$-interpretation, $\mathcal{M}_A \models M\varphi$. Hence, $\mathcal{M}_A \models A \cup S4F \cup \{M\varphi : A \not\models^b_{S4F} \neg \varphi, \varphi \in GSt_b\}$, which implies that $E$ is consistent.

Similarly, $\mathcal{M} \odot \mathcal{M}_A \models A \cup S4F \cup \{M\varphi : A \not\models^b_{S4F} \neg \varphi, \varphi \in GSt_b\}$. However, $\mathcal{M} \odot \mathcal{M}_A \not\models S5$. Consequently, $E$, which is a ground S4F-expansion for $A$, does not contain S5, even though S4F is characterized by a class of cluster decomposable Kripke interpretations.
Chapter 5

First-order minimal sets

In this chapter we present two semantical definitions of minimal sets for first-order default theories, which are similar to the definitions of extensions for first-order default theories. We prove that our two definitions are equivalent and that the relationship between minimal sets and extensions in the first-order case is the same as the corresponding relationship in the propositional case.

We start with an alternative semantical definition of minimal sets for propositional default theories.

5.1 Propositional minimal sets: a semantical approach

Below we present an alternative semantical definition of propositional minimal sets for default theories, and show that our definition is equivalent to Definition 28 (p. 16).

Definition 82 Let \((D, A)\) be a propositional default theory. A maximal class of models of \((D, A)\) is a maximal class of interpretations \(V\) that satisfies properties PM1-PM2 below.\(^1\)

- (PM1) \(V \models A\), i.e., \(V\) is a class of models of \(A\).
- (PM2) For any \(\frac{\alpha; \beta_1, \ldots, \beta_m}{\gamma} \in D\), if \(V \models \alpha\) and \(V \not\models \neg \beta_i\), \(i = 1, \ldots, m\), then \(V \models \gamma\).

---

\(^1\)PM stands for Propositional Models.
**Theorem 83** Let \((D, A)\) be a propositional default theory. A set of sentences \(E\) is a minimal set for \((D, A)\) if and only if there exists a maximal class of models of \((D, A)\), \(V\), such that \(E = \text{Th}_{\mathcal{L}_0}(V)\).

Theorem 83 follows from a more general result - Theorem 86 in the next section.

### 5.2 First-order minimal sets: a semantical approach

In this section we present two semantical definitions of minimal sets for first-order open default theories, show the equivalence of the two definitions and that they preserve the relationship between minimal sets for propositional default theories and extensions for propositional default theories.

**Definition 84** Let \((D, A)\) be a first-order open default theory and \(b\) be an infinite set of new constant symbols. A \(\Gamma^b\)-minimal set for \((D, A)\) is a consistent minimal set of \(b\)-sentences \(B \subseteq GSt_b\) that satisfies properties FT1-FT3 below.\(^2\)

\((\text{FT1})\) \(A \subseteq B\).

\((\text{FT2})\) If \(B \models_b \varphi\), then \(\varphi \in B\), i.e., \(B\) is closed under semantical \(b\)-entailment.

\((\text{FT3})\) For any \(\alpha(x) : \beta_1(x), \ldots, \beta_m(x) \in D\), and any tuple \(t\) of elements of \(\text{Tr}_{\mathcal{L}_0b}\), if \(\alpha(t) \in B\) and \(\neg \beta_i(t) \notin B\), \(i = 1, \ldots, m\), then \(\gamma(t) \in B\).

A set of sentences \(E\) is called a \(\Gamma\)-minimal set for \((D, A)\) if there exists a \(\Gamma^b\)-minimal set for \((D, A)\), \(U\), such that \(E = U \cap GSt\).

**Definition 85** Let \((D, A)\) be a first-order open default theory and \(b\) be an infinite set of new constant symbols. A \(\Delta^b\)-maximal class of \(b\)-models of \((D, A)\) is a maximal class of Herbrand \(b\)-interpretations \(V\) that satisfies properties FM1-FM2 below.\(^3\)

\((\text{FM1})\) \(V \models A\), i.e., \(V\) is a class of \(b\)-models of \(A\).

\(^2\)FT stands for First-order Theory.
\(^3\)FM stands for First-order Models.
(FM2) For any \( \alpha(x): \beta_1(x), \ldots, \beta_m(x) \in D \), and any tuple \( t \) of elements of \( Tr_{\mathcal{L}b} \), if \( V \models \alpha(t) \) and \( V \notmodels \neg \beta_i(t), \ i = 1, \ldots, m \), then \( V \models \gamma(t) \).

A set of sentences \( E \) is called a \( \Delta \)-minimal set for \((D, A)\) if there exists a \( \Delta^b \)-maximal class of \( b \)-models of \((D, A)\), \( V \), such that \( E = Th_{\mathcal{L}b}(V) \cap GSt \).

Similarly to [9, Theorem 42], it can be shown that for closed default theories Definitions 84 and 85 are equivalent. The following theorem states that they are equivalent for first-order open default theories as well.

**Theorem 86** Let \((D, A)\) be a first-order open default theory, \( b \) be an infinite set of new constant symbols and \( E \) be a set of sentences over \( \mathcal{L} \). Then \( E \) is a \( \Delta \)-minimal set for \((D, A)\) if and only if \( E \) is a \( \Gamma \)-minimal set for \((D, A)\).

The proof of Theorem 86 is based on Lemmas 87– 90 below. In these lemmas, \( b \) is an infinite set of new constant symbols, and \((D, A)\) is a first-order open default theory.

**Lemma 87** Let \( B \) be the a set of \( b \)-sentences that satisfies conditions (FT1-FT3) of Definition 84. Then, \( Mod_b(B) \) satisfies conditions (FM1-FM2) of Definition 85.

**Proof** Since \( A \subseteq B \), every \( b \)-model of \( B \) is a \( b \)-model of \( A \). Consequently, \( Mod_b(B) \) is a class of \( b \)-models of \( A \). Let \( \frac{\alpha(x): \beta_1(x), \ldots, \beta_m(x)}{\gamma(x)} \in D \) and let \( t \) be a tuple of elements of \( Tr_{\mathcal{L}b} \) such that \( Mod_b(B) \models \alpha(t) \) and \( Mod_b(B) \notmodels \neg \beta_i(t), \ i = 1, \ldots, m \). Then, \( \alpha(t) \in Th_{\mathcal{L}b}(Mod_b(B)) \) and \( \neg \beta_i(t) \notin Th_{\mathcal{L}b}(Mod_b(B)), \ i = 1, \ldots, m \). Since \( B \) is closed under semantical \( b \)-entailment \( \models_b \), \( \alpha(t) \in B \) and \( \neg \beta_i(t) \notin B, \ i = 1, \ldots, m \). Therefore, since \( B \) satisfies condition (FT3) of Definition 84, \( \gamma(t) \in B \). Consequently, \( Mod_b(B) \models \gamma(t) \).

**Lemma 88** Let \( V \) be a class of \( b \)-interpretations that satisfies conditions (FM1-FM2) of Definition 85. Then, \( Th_{\mathcal{L}b}(V) \) satisfies conditions (FT1-FT3) of Definition 84.

**Proof** Since \( V \) is a class of \( b \)-models of \( A \), \( Th_{\mathcal{L}b}(V) \) includes \( A \). By the definition of the operator \( Th_{\mathcal{L}b} \), \( Th_{\mathcal{L}b}(V) \) is closed under semantical \( b \)-entailment \( \models_b \). Let \( \frac{\alpha(x): \beta_1(x), \ldots, \beta_m(x)}{\gamma(x)} \in D \) and let \( t \) be a tuple of
Let $\Gamma$ be a $\Delta^b$-minimal set for $(D, A)$. Then, $Th_{\xi_0}(V)$ is a $\Gamma^b$-minimal set for $(D, A)$.

**Proof** Let $V$ be a $\Delta^b$-maximal class of $b$-models of $(D, A)$. Then, by Lemma 88, $Th_{\xi_0}(V)$ satisfies conditions (FT1-FT3) of Definition 84. Therefore, there exists a $\Gamma^b$-minimal set for $(D, A)$, $B$, such that $B \subseteq Th_{\xi_0}(V)$. Assume to the contrary that $B \not\subseteq Th_{\xi_0}(V)$. Then, $Mod_b(Th_{\xi_0}(V)) \subset Mod_b(B)$. This, together with $Mod_b(Th_{\xi_0}(V)) = V$, implies $V \subset Mod_b(B)$. By Lemma 87, $Mod_b(B)$ is a class of $b$-interpretations that satisfies conditions (FM1-FM2) of Definition 85, in contradiction with the maximality of $V$. Consequently, $Th_{\xi_0}(V)$ is a $\Gamma^b$-minimal set for $(D, A)$.

**Lemma 90** Let $U$ be a $\Gamma^b$-minimal set for $(D, A)$. Then, $Mod_b(U)$ is a $\Delta^b$-maximal class of $b$-models of $(D, A)$.

**Proof** Let $U$ be a $\Gamma^b$-minimal set for $(D, A)$. Then, by Lemma 87, $Mod_b(U)$ is class of $b$-models of $A$ and satisfies condition (FM2) of Definition 85. Therefore, there exists a maximal class of $b$-models of $(D, A)$, $V$, such that $Mod_b(U) \subset V$. Assume to the contrary that $Mod_b(U) \subset V$. Then, $Th_{\xi_0}(V) \subset Th_{\xi_0}(Mod_b(U))$. Since $U$ is closed under semantical $b$-entailment $|_b, Th_{\xi_0}(V) \subset U$. By Lemma 88, $Th_{\xi_0}(V)$ satisfies conditions (FT1-FT3) of Definition 84, in contradiction with the minimality of $U$. Consequently, $Mod_b(U)$ is a $\Delta^b$-maximal class of $b$-models of $(D, A)$.

**Proof of Theorem 86** Let $E$ be a $\Delta$-minimal set for $(D, A)$. Then, there exists a maximal class of $b$-models of $(D, A)$, $V$, such that $E = Th_{\xi_0}(V) \cap GST$. By Lemma 89, $Th_{\xi_0}(V)$ is a $\Gamma^b$-minimal set for $(D, A)$. Consequently, $E$ is a $\Gamma$-minimal set for $(D, A)$.

Conversely, let $E$ be a $\Gamma$-minimal set for $(D, A)$. Then, there exists a $\Gamma^b$-minimal set for $(D, A)$, $U$, such that $E = U \cap GST$. By Lemma 90, $Mod_b(U)$ is a $\Delta^b$-maximal class of $b$-models of $(D, A)$. Since $U$ is closed.
under semantical $b$-entailment $|=b$, $\text{Th}_{L_{0b}}(\textbf{Mod}_{b}(U)) = U$. Therefore, $E$ is a $\Delta$-minimal set for $(D, A)$.

Finally, we prove that our definitions of first-order minimal sets preserve the relationship between minimal sets for propositional default theories and extensions for propositional default theories, stated by Theorem 29 (p. 16).

Theorem 91 Let $(D, A)$ be a first-order open default theory, $b$ be an infinite set of new constant symbols. Then every $b$-extension for $(D, A)$ is a $\Delta$-minimal set for $(D, A)$.

Proof Let $E$ be a $b$-extension for $(D, A)$. Then, $E = \text{Th}_{L_{0b}}(W) \cap GSt$ for some fixpoint $W$ of $\Delta^b_{(D, A)}$. In other words, $W$ is the largest class of Herbrand $b$-interpretations which are $b$-models of $A$ and satisfy the following property. For any $\alpha(x): \beta_1(x), \ldots, \beta_m(x) \in D$, and any tuple $t$ of elements of $\text{Tr}_{L_b}$, if $W \models \alpha(t)$ and $W \not\models \neg \beta_i(t)$, $i = 1, \ldots, m$, then $W \models \gamma(t)$.

Hence, by Definition 85, $W$ is a $\Delta^b$-maximal class of $b$-models of $(D, A)$. Consequently, $E$ is a $\Delta$-minimal set for $(D, A)$.

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Chapter 6

Relationship between first-order default logic and first-order ground non-monotonic modal logic

In this chapter we establish a relationship between minimal sets for first-order default theories and first-order ground non-monotonic modal logic based on S5 and S4. The relationship is similar to the corresponding relationship in the propositional case, see Section 2.5.2 (p. 16).

6.1 Relationship between first-order minimal sets and first-order ground S5-expansions

In this section we establish a relationship between minimal sets for first-order default theories and first-order ground S5-expansions. The relationship is similar to the corresponding relationship in the propositional case, stated by Theorem 31 (p. 17).

In order to present the relationship, we first extend Definition 30 (p. 17) to the first-order case.

Definition 92 For a first-order default \( d(x) = \frac{\alpha(x) : \beta_1(x), \ldots, \beta_m(x)}{\gamma(x)} \) we
define $\Theta(d(x)) \in St$ by

$$\Theta(d(x)) = (L_\alpha(x) \land \bigwedge_{i=1}^{m} M_\beta_i(x)) \supset \gamma(x).$$

For a set of first-order defaults $D$ we denote the set of modal sentences \{\forall x \Theta(d(x)) : d \in D\} by $\Theta(D)$.

Finally, for a first-order default theory $\Delta = (D, A)$ we define $\Theta(\Delta) \subseteq St$ by $\Theta(\Delta) = A \cup \Theta(D)$.

We shall also need the following notation. For a non-empty set $U$ of Herbrand $b$-interpretations, we denote by $[U]$ the S5-Kripke $b$-interpretation $\langle U, I \rangle$, where, for every $u \in U$, $I(u) = u$.

**Theorem 93** Let $\Delta = (D, A)$ be a first-order default theory. Then $G$ is a consistent $\Gamma$-minimal set for $\Delta$ if and only if there exists a consistent ground S5-expansion $E$ for $\Theta(\Delta)$ such that $G = E \cap GSt$.

We shall now prove a number of lemmas, which are needed for the proof of Theorem 93.

**Lemma 94** Let $(D, A)$ be a first-order default theory, $b$ be an infinite set of new constant symbols, $M = \langle U, I \rangle$ be an S5-Kripke $b$-model of $\Theta(\Delta)$, and let $V = \{ I(u) : u \in U \}$. Then, $V$ is a class of Herbrand $b$-interpretations that satisfies conditions (FM1-FM2) of Definition 85.

**Proof** Since $M$ is a $b$-model of $A$ and $A \subseteq GSt_b$, for every $u \in U$, $(M, u) \models A$. Therefore, by the definition of $\models$, $A \subseteq I(u)$ for every $u \in U$. Hence, by the definition of $V$, $V \models A$.

Let $\frac{\alpha(x) : \beta_1(x), \ldots, \beta_m(x)}{\gamma(x)} \in D$ and $t$ be a tuple of elements of $Tr_{\alpha_b}$ such that $V \models \alpha(t)$ and $V \not\models \neg \beta_i(t)$, $i = 1, \ldots, m$. The condition $V \models \alpha(t)$ implies that for every $u \in U$, $\alpha(t) \in I(u)$. Therefore, for every $u \in U$, $(M, u) \models \alpha(t)$. Consequently, $M \models L_\alpha(t)$. For every $1 \leq i \leq m$, $V \not\models \neg \beta_i(t)$ implies that for some $u_i \in U$, $\beta_i(t) \in I(u_i)$. Therefore, $(M, u_i) \models \beta_i(t)$. Hence, since $M$ is an S5-Kripke $b$-interpretation, $M \models M_\beta_i(t)$. This, together with $(L_\alpha(x) \land \bigwedge_{i=1}^{m} M_\beta_i(x)) \supset \gamma(x) \in \Theta(D)$ and $M \models \Theta(D)$, implies $M \models \gamma(t)$. Since $\gamma(t) \in GSt$, this implies $\gamma(t) \in I(u)$ for every $u \in U$. Therefore, by the definition of $V$, $V \models \gamma(t)$. Q.E.D.
**Lemma 95** Let \((D, A)\) be a first-order default theory, \(b\) be an infinite set of new constant symbols, \(E\) be a consistent set of \(b\)-sentences that satisfies conditions (FT1-FT3) of Definition 84, and let \(V = \text{Mod}_b(E)\). Then, \([V] \models \Theta(\Delta)\).

**Proof** Since \(E\) is consistent, \(V\) is not empty. By the definition of \(E\), \(A \subseteq E\). Hence, by the definition of \(V\), for every \(v \in V\), \(v \models A\). Therefore, since \(A \subseteq \text{GST}\), for every \(v \in V\), \(A \subseteq v\). Consequently, for every \(v \in V\), \([V], v \models A\), implying \([V] \models A\).

Let \((L\alpha(x) \land \bigwedge_{i=1}^{m} M\beta_i(x)) \supset \gamma(x) \in \Theta(D)\) and let \(t\) be a tuple of elements of \(\text{Tr}_{L\alpha b}\) such that \([V] \models L\alpha(t)\) and \([V] \models M\beta_i(t), i = 1, \ldots, m\). The condition \([V] \models L\alpha(t)\) and \(\alpha(t) \in \text{GST}\) imply that for every \(v \in V\), \(\alpha(t) \in v\). For every \(1 \leq i \leq m\), \([V] \models M\beta_i(t)\) and \(\beta_i(t) \in \text{GST}\) imply that for some \(v_i \in V\), \(\beta_i(t) \in v_i\). Consequently, \(\alpha(t) \in \text{Th}_{L\alpha b}(\text{Mod}_b(E))\) and \(\neg \beta_i(t) \not\in \text{Th}_{L\alpha b}(\text{Mod}_b(E)), i = 1, \ldots, m\). Since \(E\) is closed under semantical \(b\)-entailment \(\models_b\), \(\alpha(t) \in E\) and \(\neg \beta_i(t) \not\in E, i = 1, \ldots, m\). Therefore, since \(E\) satisfies condition (FT3) of Definition 84 and \(\frac{\alpha(x) : \beta_1(x), \ldots, \beta_m(x)}{\gamma(x)} \in D\), \(\gamma(t) \in E\). Hence, by the definition of \(V\), \(V \models \gamma(t)\). This, together with \(\gamma(t) \in \text{GST}\), implies that for every \(v \in V\), \(\gamma(t) \in v\). Consequently, \([V] \models \gamma(t)\). 

**Lemma 96** Let \(\Delta = (D, A)\) be a first-order default theory, \(b\) be an infinite set of new constant symbols, \(E\) be a consistent set of \(b\)-sentences closed under semantical \(b\)-entailment \(\models_b\), and let \(V = \text{Mod}_b(E)\). Then, \(E\) is a \(\Gamma^b\)-minimal set for \((D, A)\) if and only if \([V] \models \Theta(\Delta)\)-minimal (see Definition 62).

**Proof** Since \(E\) is consistent, \(V\) is not empty. Hence, \([V]\) is a well-defined Kripke \(b\)-interpretation.

Let \(E\) be a \(\Gamma^b\)-minimal set for \((D, A)\). Then, by Remark 63 (p. 35), it suffices to prove that \([V]\) satisfies properties (A1) and (A3) of Definition 62 (p. 35).

By the definition of \(V\) and by Lemma 95, \([V] \models \Theta(\Delta)\).

Assume to the contrary that there exists some S5-Kripke \(b\)-interpretation \(\mathcal{M} = \langle U, I \rangle\) such that \(\mathcal{M} \models \Theta(\Delta)\) and \(\text{Th}_{L\alpha b}(\mathcal{M}) \cap \text{GST}_b \subset \text{Th}_{L\alpha b}(\mathcal{M}[V]) \cap \text{GST}_b\). Let \(W\) be \(\{I(u) : u \in U\}\). By Lemma 94, \(V\) and \(W\) are classes of Herbrand \(b\)-interpretations satisfying conditions (FM1-FM2) of Definition 85. Hence,
by Lemma 88, $\text{Th}_{\mathcal{L}_b}(V)$ and $\text{Th}_{\mathcal{L}_b}(W)$ satisfy conditions (FT1-FT3) of Definition 84. By the definitions of $[V]$ and $\models, \text{Th}_{\mathcal{L}_b}([V]) \cap \text{GST}_b = \text{Th}_{\mathcal{L}_b}(V)$. Since $E$ is closed under semantical $b$-entailment $\models_b$, by the definition of $V$,

$$\text{Th}_{\mathcal{L}_b}(V) = \text{Th}_{\mathcal{L}_b}(\text{Mod}_b(E)) = E.$$ 

Therefore, $\text{Th}_{\mathcal{L}_b}(\mathfrak{M}) \cap \text{GST}_b \subseteq E$. By the definitions of $\mathfrak{M}$ and $\models, \text{Th}_{\mathcal{L}_b}(\mathfrak{M}) \cap \text{GST}_b = \text{Th}_{\mathcal{L}_b}(V)$. Thus, $\text{Th}_{\mathcal{L}_b}(\mathfrak{M}) \cap \text{GST}_b$ satisfies conditions (FT1-FT3) of Definition 85. Since $\text{Th}_{\mathcal{L}_b}(\mathfrak{M})$ is $\text{GST}_b$-minimal, in contradiction with the minimality of $E$.

Conversely, let $[V]$ be $\Theta(\Delta)$-minimal. Then, by Lemma 94, $V$ satisfies conditions (FM1-FM2) of Definition 85.

If $V$ is a $\Delta^b$-maximal class of $b$-models of $(D, A)$, then by Lemma 89, $\text{Th}_{\mathcal{L}_b}(V)$ is a $\Gamma^b$-minimal set for $(D, A)$. Thus, by the definition of $V$ and since $E$ is closed under semantical $b$-entailment $\models_b, \text{Th}_{\mathcal{L}_b}(V) = E$, implying that $E$ is a $\Gamma^b$-minimal set for $(D, A)$.

Otherwise, assume to the contrary that $V$ is not a $\Delta^b$-maximal class of $b$-models of $(D, A)$. That is, there exists a set $W$ of Herbrand $b$-interpretations, which satisfies conditions (FM1-FM2) of Definition 85, such that $V \subset W$. Then, by Lemma 88, $\text{Th}_{\mathcal{L}_b}(W)$ satisfies conditions (FT1-FT3) of Definition 84. Since $W = \text{Mod}_b(\text{Th}_{\mathcal{L}_b}(W))$, by Lemma 95, $[W] \models \Theta(\Delta)$. By the definition of $\models, \text{Th}_{\mathcal{L}_b}([V]) \cap \text{GST}_b = V$ and $\text{Th}_{\mathcal{L}_b}([W]) \cap \text{GST}_b = W$. Consequently, $\text{Th}_{\mathcal{L}_b}([W]) \cap \text{GST}_b \subset \text{Th}_{\mathcal{L}_b}([V]) \cap \text{GST}_b$, in contradiction with the $\Theta(\Delta)$-minimality of $[V]$. 

**Proof of Theorem 93** Let $G$ be a consistent $\Gamma$-minimal set for $(D, A)$. Then, there exists an infinite set of new constant symbols $b$ and a $\Gamma^b$-minimal set for $(D, A), G_b$, such that $G = G_b \cap \text{GST}$. Since $G_b$ satisfies condition (FT2) of Definition 85, $G_b$ is closed under semantical $b$-entailment $\models_b$. Let $V = \text{Mod}_b(G_b)$. Since $G$ is consistent, so is $G_b$. Therefore, by Lemma 96, $[V]$ is $\Theta(\Delta)$-minimal.

Let $X = \text{Th}_{\mathcal{L}_b}(\Theta(\Delta)) \cup \text{S5} \cup \{M \varphi : [V] \not\models \varphi, \varphi \in \text{GST}_b\}$. Since $[V]$ is an S5-Kripke $b$-model of $\Theta(\Delta)$, $[V] \models X$. By Proposition 65, $X$ is a consistent fixpoint of $\text{GNM}_{\text{S5}}^{b, \Theta(\Delta)}$. Therefore, by Definition 53, $E = X \cap \text{St}$ is a consistent ground S5-expansion for $\Theta(\Delta)$.

By Lemma 66, $\text{Th}_{\mathcal{L}_b}([V]) \cap \text{GST}_b = X \cap \text{GST}_b$. Also, by the definition of $V$, $\text{Th}_{\mathcal{L}_b}([V]) \cap \text{GST}_b = b$. Consequently, $G_b \cap \text{GST} = X \cap \text{GST}$. Therefore, since $G_b \cap \text{GST} = G$ and $X \cap \text{GST} = (X \cap \text{St}) \cap \text{GST}$, $G = E \cap \text{GST}$.

Conversely, let $E$ be a consistent ground S5-expansion for $\Theta(\Delta)$ such that $G = E \cap \text{GST}$. Then, there exists an infinite set of new constant symbols

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b and a fixpoint $E_b$ of $\text{GNM}_{S5}^{b, \Theta(\Delta)}$, such that $E = E_b \cap \text{St}$. Since $E$ is consistent, so is $E_b$. By Theorem 64, there exists a $\Theta(\Delta)$-minimal Kripke $b$-interpretation $\mathfrak{M} = \langle U, I \rangle$ and a set of $b$-sentences $X$ such that

$$X = Th_{L_b}(\Theta(\Delta) \cup S5 \cup \{ M\varphi : \mathfrak{M} \models \neg \varphi, \ \varphi \in GSt_b \})$$

and $E = X \cap \text{St}$. Since $\mathfrak{M}$ is an S5-Kripke $b$-model of $\Theta(\Delta)$, $\mathfrak{M} \models X$. By Theorem 66, $Th_{L_b}(\mathfrak{M}) \cap GSt_b = X \cap GSt_b$, implying $Th_{L_b}(\mathfrak{M}) \cap GSt = X \cap GSt$. Since $X \cap GSt = (X \cap \text{St}) \cap GSt$, $Th_{L_b}(\mathfrak{M}) \cap GSt = E \cap GSt$. Hence, $Th_{L_b}(\mathfrak{M}) \cap GSt = G$.

Let $G_b^M = Th_{L_b}(\mathfrak{M}) \cap GSt_b$. Then, $G_b^M$ is closed under semantical $b$-entailment $\models_b$ and consistent. Let $I_b^M$ be $\{ I(u) : u \in U \}$. Then by the definition of $G_b^M$, $I_b^M = \text{Mod}_b(G_b^M)$. Therefore, by Lemma 96, $G_b^M$ is a consistent $\Gamma^b$-minimal set for $(D, A)$. Since

$$Th_{L_b}(\mathfrak{M}) \cap GSt = (Th_{L_b}(\mathfrak{M}) \cap GSt_b) \cap GSt,$$

$G_b^M \cap GSt = G$. Consequently, by Definition 84, $G$ is a consistent $\Gamma$-minimal set for $(D, A)$.

### 6.2 Relationship between first-order minimal sets and first-order ground S4-expansions

In this section we present a relationship between minimal sets for first-order default theories and first-order ground S4-expansions. The relationship is similar to the corresponding relationship in the propositional case, stated by Theorem 33 (p. 17).

In order to present the relationship, we first extend Definition 32 (p. 17) to the first-order case.

**Definition 97** For a first-order default $d(x) = \frac{\alpha(x)}{\gamma(x)} : \beta_1(x), \ldots, \beta_m(x)$ we denote by $\Theta'(d(x))$ the modal sentence $(ML\alpha(x) \land \bigwedge_{i=1}^{m} M\beta_i(x)) \supset L\gamma(x)$ and for a set of first-order defaults $D$ we denote the set of modal sentences $\{ \forall x \Theta'(d(x)) : d(x) \in D \}$ by $\Theta'(D)$.

Finally, for a first-order default theory $\Delta = (D, A)$ we define $\Theta'(\Delta) \subseteq St$ by $\Theta'(\Delta) = A \cup \Theta'(D)$.

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Theorem 98 Let $\Delta = (D, A)$ be a first-order default theory. Then, $G$ is a consistent $\Gamma$-minimal set for $\Delta$ if and only if there exists a consistent ground $S4$-expansion $E$ for $\Theta'(\Delta)$ such that $G = E \cap GSt$.

We shall now prove a number of lemmas and propositions which are needed for the proof of Theorem 98. In the sequel of this section we shall rely on the equivalence between $\Theta(d(x))$ and the modal formula
\[
M-\alpha(x) \lor \bigvee_{i=1}^{m} L-\beta_i(x) \lor \gamma(x)
\]
and the equivalence between $\Theta'(d(x))$ and the modal formula
\[
LM-\alpha(x) \lor \bigvee_{i=1}^{m} L-\beta_i(x) \lor L\gamma(x).
\]

Proposition 99 Let $\Delta = (D, A)$ be a first-order default theory, $S$ be a normal modal logic, and let $b$ be an infinite set of new constant symbols. Then the operators $\text{GNM}^b_{S5}(\Theta(\Delta))$ and $\text{GNM}^b_{S5}(\Theta'(\Delta))$ have the same fixpoints.

Proof It suffices to show that for any $\psi_0, \psi_1, \ldots, \psi_m, \psi_{m+1} \in GSt$,
\[
\text{Th}_{L^b}(S5 \cup \{\forall x (LM\psi_0(x) \lor \bigvee_{i=1}^{m} L\psi_i(x) \lor L\psi_{m+1}(x))\}) = \text{Th}_{L^b}(S5 \cup \{\forall x (M\psi_0(x) \lor \bigvee_{i=1}^{m} L\psi_i(x) \lor \psi_{m+1}(x))\}).
\]

Let $\mathfrak{M}$ be an $S5$-Kripke $b$-interpretation such that
\[
\mathfrak{M} \models \forall x LM\psi_0(x) \lor \bigvee_{i=1}^{m} L\psi_i(x) \lor L\psi_{m+1}(x).
\]
Then, by the definition of $\models$, for every $t \in Tr_{L^b}$ at least one of the following holds.

1. $\mathfrak{M} \models LM\psi_0(t)$. Then since in $\mathfrak{M}$ each world is connected to itself, $\mathfrak{M} \models M\psi_0(t)$. Consequently, $\mathfrak{M} \models M\psi_0(t) \lor \bigvee_{i=1}^{m} L\psi_i(t) \lor \psi_{m+1}(t)$. 


2. There exists $1 \leq i \leq m$ such that $\mathcal{M} \models L\psi_i(t)$. Then trivially, $\mathcal{M} \models M\psi_0(t) \lor \bigvee_{i=1}^{m} L\psi_i(t) \lor \psi_{m+1}(t)$.

3. $\mathcal{M} \models L\psi_{m+1}(t)$. Then since in $\mathcal{M}$ each world is connected to itself, $\mathcal{M} \models \psi_{m+1}(t)$. Consequently, $\mathcal{M} \models M\psi_0(t) \lor \bigvee_{i=1}^{m} L\psi_i(t) \lor \psi_{m+1}(t)$.

Therefore, $\mathcal{M} \models \forall x. M\psi_0(x) \lor \bigvee_{i=1}^{m} L\psi_i(x) \lor \psi_{m+1}(x)$.

For the converse inclusion, let $\mathcal{M}$ be an $S5$-Kripke $b$-interpretation such that $\mathcal{M} \models \forall x. M\psi_0(x) \lor \bigvee_{i=1}^{m} L\psi_i(x) \lor \psi_{m+1}(x)$.

Then, by the definition of $\models$, for every $t \in Tr_{\zeta_b}$ at least one of the following holds.

1. $\mathcal{M} \models M\psi_0(t)$. Then, by the definition of $\models$, $\mathcal{M} \models LM\psi_0(t)$. Consequently, $\mathcal{M} \models LM\psi_0(t) \lor \bigvee_{i=1}^{m} L\psi_i(t) \lor \psi_{m+1}(t)$.

2. There exists $1 \leq i \leq m$ such that $\mathcal{M} \models L\psi_i(t)$. Then trivially, $\mathcal{M} \models LM\psi_0(t) \lor \bigvee_{i=1}^{m} L\psi_i(t) \lor \psi_{m+1}(t)$.

3. $\mathcal{M} \models \psi_{m+1}(t)$. Then, by the definition of $\models$, $\mathcal{M} \models L\psi_{m+1}(t)$. Consequently, $\mathcal{M} \models LM\psi_0(t) \lor \bigvee_{i=1}^{m} L\psi_i(t) \lor \psi_{m+1}(t)$.

Therefore, $\mathcal{M} \models \forall x. LM\psi_0(x) \lor \bigvee_{i=1}^{m} L\psi_i(x) \lor \psi_{m+1}(x)$, which completes the proof.

**Lemma 100** Let $\psi_1, \psi_2, \ldots, \psi_n \in St$ and $\mathcal{M}$ be an $S5$-Kripke $b$-interpretation such that $\mathcal{M} \models \forall x (L\psi_1(x) \lor \ldots \lor L\psi_n(x))$. Then, for every $t \in Tr_{\zeta_b}$ there exists $1 \leq i \leq n$ such that $\mathcal{M} \models L\psi_i(t)$.

**Proof** Since $\mathcal{M} \models \forall x (L\psi_1(x) \lor \ldots \lor L\psi_n(x))$, for every world $u$ in $\mathcal{M}$, $(\mathcal{M}, u) \models \forall x (L\psi_1(x) \lor \ldots \lor L\psi_n(x))$. Therefore, for every $t \in Tr_{\zeta_b}$ there exists $1 \leq i \leq n$, such that $(\mathcal{M}, u) \models L\psi_i(t)$. Since in $\mathcal{M}$ every two world are connected each to the other, $\mathcal{M} \models \psi_i(t)$. Consequently, $\mathcal{M} \models L\psi_i(t)$. ■

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Lemma 101  Let $\Delta = (D, A)$ be a first-order default theory, $b$ be an infinite set of new constant symbols, $M = \langle U, R, I \rangle$ be an $S4$-Kripke $b$-interpretation such that $M \models \Theta'(\Delta)$, and let $M' = \langle U, I \rangle$. Then, $M' \models \Theta'(\Delta)$.

Proof  Let $\xi \in \Theta'(\Delta)$ be such that $\xi \in GSt$. Since $M \models \xi$ and since $\xi$ is a sentence without modalities, for every $u \in U$, $\xi \in u$. Hence, by the definition of $M'$, $\xi$ is valid in every world in $M'$. Therefore, $M' \models \xi$.

By the definition of $\Theta'(\Delta)$, if $\xi \in \Theta'(\Delta) \setminus GSt$ then for some $\psi_0, \psi_1, \ldots, \psi_m, \psi_{m+1} \in GSt$,

$$
\xi = \forall x (LM\psi_0(x) \lor \bigvee_{i=1}^{m} L\psi_i(x) \lor L\psi_{m+1}(x)).
$$

Since $M \models \xi$, by Lemma 100, for every $t \in Tr_{\xi}$ at least one of the following holds.

1. $M \models LM\psi_0(t)$. Then, since every world in $M$ is connected to at least one world and since $\psi_0(t)$ is a sentence without modalities, there exists a world $u$ in $M$ in which $\psi_0(t)$ is valid. Hence, by the definition of $M'$, $\psi_0(t)$ is also valid in $u$ in $M'$. Therefore, since in $M'$ every two worlds are connected each to the other, $M' \models LM\psi_0(t)$. Consequently, $M' \models LM\psi_0(t) \lor \bigvee_{i=1}^{m} L\psi_i(t) \lor L\psi_{m+1}(t)$.

2. $M \models L\psi_i(t)$ for some $1 \leq i \leq m$. Then, since in $M$ each world is connected to itself and since $\psi_i(t)$ is a sentence without modalities, $\psi_i(t)$ is valid in every world $u$ in $M$. Hence, by the definition of $M'$, $\psi_i(t)$ is valid in every world $u$ in $M'$. Therefore, $M' \models L\psi_i(t)$. Consequently, $M' \models LM\psi_0(t) \lor \bigvee_{i=1}^{m} L\psi_i(t) \lor L\psi_{m+1}(t)$.

3. $M \models L\psi_{m+1}(t)$. Then, by exactly the same proof as in clause 2 above, $M' \models LM\psi_0(t) \lor \bigvee_{i=1}^{m} L\psi_i(t) \lor L\psi_{m+1}(t)$.

Therefore, $M' \models \xi$, which completes the proof. \[\square\]

---

1 That is, $R$ is reflexive and transitive.
Lemma 102 Let $b$ be an infinite set of new constant symbols and $\Psi$ be a set of first-order modal sentences of the form $\psi, M\psi$, or

$$\forall x (LM\psi_0(x) \lor \bigvee_{i=1}^{m} L\psi_i(x) \lor L\psi_{m+1}(x)),$$

where $\psi, \psi_0, \psi_1, \ldots, \psi_m, \psi_{m+1} \in GSt$. Then, for every $\varphi \in GSt_b$, $\Psi \models_{b} S5 \varphi$ if and only if $\Psi \models_{b} S4 \varphi$.

Proof Since every S5-Kripke $b$-interpretation is an S4-Kripke $b$-interpretation, $\Psi \models_{b} S4 \varphi$ implies $\Psi \models_{b} S5 \varphi$.

In order to prove the "only if" part of the lemma we assume that $\Psi \not\models_{b} S4 \varphi$ and show that $\Psi \not\models_{b} S5 \varphi$, either. The relation $\Psi \not\models_{b} S4 \varphi$ implies that there exists an S4-Kripke $b$-interpretation $M = \langle U, R, I \rangle$ such that $M \models \Psi$, but $M \not\models \varphi$. Consider the S5-Kripke $b$-interpretation $M' = \langle U, I \rangle$. The proof of the lemma will be complete if we show that $M' \models \Psi$, but $M' \not\models \varphi$.

Since $M \not\models \varphi$, there exists a world $u$ in $M$ such that $(M, u) \not\models \varphi$. Since $\varphi$ is a $b$-sentence without modalities and by the definition of $M'$, $(M', u) \not\models \varphi$. Hence, $M' \not\models \varphi$.

Let $\xi \in \Psi$. Then, by the definition of $\Psi$, one of the following holds.

1. $\xi \in GSt$. Then, since $M \models \xi$ and $\xi$ is a sentence without modalities, $\xi$ is valid in every world of $U$ in $M$. Hence, by the definition of $M'$, $\xi$ is also valid in every world in $M'$. In other words, $M' \models \xi$.

2. $\xi = M\psi$, $\psi \in GSt$. Then, there exists a world $u \in U$ such that $(M, u) \models \psi$. Therefore, since $\psi$ is a sentence without modalities, $(M', u) \models \psi$. Consequently, since in $M'$ every two worlds are connected each to the other, $M' \models \xi$.

3. $\xi = \forall x (LM\psi_0(x) \lor \bigvee_{i=1}^{m} L\psi_i(x) \lor L\psi_{m+1}(x))$, $\psi_0, \ldots, \psi_{m+1} \in GSt$.

Then, by Lemma 101, $M \models \xi$ implies $M' \models \xi$.

Therefore, $M' \models \Psi$, which completes the proof.

Proposition 103 Let $b$ be an infinite set of new constant symbols, $\Delta = (D, A)$ be a first-order default theory, and $E_b$ be a consistent fixpoint of $GNM_{S5}^{b, \Theta(\Delta)}$. Then, there exists a consistent fixpoint $F_b$ of $GNM_{S4}^{b, \Theta(\Delta)}$ such that $E_b \cap GSt_b = F_b \cap GSt_b$.  

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Proof Let \( E_b \) be a consistent fixpoint of \( GNM_{S4}^{b, \Theta'(\Delta)} \). Then, by Remark 54, \( E_b = Th_{L_b}(\Theta'(\Delta)) \cup S5 \cup \{ M \varphi : E_b \not\models^{S5}_b \neg \varphi, \ \varphi \in GSt_b \} \) and, by Theorem 55, \( E_b = Th_{L_b}(\Theta'(\Delta)) \cup S5 \cup (E_b \cap MGSt_b) \).

Let \( F_b = Th_{L_b}(\Theta'(\Delta)) \cup S5 \cup (E_b \cap MGSt_b) \). Since \( E_b \) is consistent, there exists an S5-Kripke \( b \)-interpretation \( \mathcal{M} \) such that \( \mathcal{M} \models E_b \). By the definition of \( E_b \) and \( F_b \) and since every S5-Kripke \( b \)-interpretation is an S4-Kripke \( b \)-interpretation, \( \mathcal{M} \models F_b \). Hence, \( F_b \) is consistent.

By Lemma 102, for any \( \varphi \in GSt_b \), \( \Theta'(\Delta) \cup (E_b \cap MGSt_b) \models^{S5}_b \varphi \) if and only if \( \Theta'(\Delta) \cup (E_b \cap MGSt_b) \models^{S4}_b \varphi \). Therefore, \( E_b \cap GSt_b = F_b \cap GSt_b \).

The proof will be complete if we show that \( F_b \) is a fixpoint of \( GNM_{S4}^{b, \Theta'(\Delta)} \). Since \( F_b \) is consistent, by Theorem 55 it suffices to show that \( F_b \) satisfies conditions (1) and (2) of Theorem 55.

We start by showing that \( F_b \) satisfies condition (1) of Theorem 55. By the definition of \( F_b \), \( E_b \cap MGSt_b \subseteq F_b \cap MGSt_b \). For the converse inclusion, let \( M \varphi \in F_b \cap MGSt_b \). Since \( F_b \) is consistent and closed under semantical \( b \)-entailment, \( F_b \models^{S4}_b \neg \varphi \). This, together with \( E_b \cap GSt_b = F_b \cap GSt_b \), implies \( E_b \not\models^{S5}_b \neg \varphi \). Therefore, by the definition of \( E_b \), \( E_b \models^{S5}_b M \varphi \), implying \( M \varphi \in E_b \cap MGSt_b \). Consequently, \( E_b \cap MGSt_b = F_b \cap MGSt_b \), implying \( F_b = Th_{L_b}(\Theta'(\Delta)) \cup S4 \cup (E_b \cap MGSt_b) \).

Finally, to show that \( F_b \) satisfies condition (2) of Theorem 55, let \( \varphi \in GSt_b \) be such that \( F_b \not\models^{S4}_b \neg \varphi \). Since \( E_b \cap GSt_b = F_b \cap GSt_b \) and since \( E_b \) is closed under semantical \( b \)-entailment, \( E_b \not\models^{S5}_b \neg \varphi \). Hence, by the definition of \( E_b \), \( M \varphi \in E_b \). This, together with \( F_b = Th_{L_b}(\Theta'(\Delta)) \cup S4 \cup (E_b \cap MGSt_b) \), implies \( F_b \models^{S4}_b M \varphi \).

\[ \square \]

Proposition 104 Let \( b \) be an infinite set of new constant symbols, \( \Delta = (D, A) \) be a first-order default theory, and \( E_b \) be a consistent fixpoint of \( GNM_{S4}^{b, \Theta'(\Delta)} \). Then, there exists a consistent fixpoint \( F_b \) of \( GNM_{S5}^{b, \Theta'(\Delta)} \) such that \( E_b \cap GSt_b = F_b \cap GSt_b \).

Proof Let \( E_b \) be a consistent fixpoint of \( GNM_{S4}^{b, \Theta'(\Delta)} \). Then, by Remark 54, \( E_b = Th_{L_b}(\Theta'(\Delta)) \cup S4 \cup \{ M \varphi : E_b \not\models^{S4}_b \neg \varphi, \ \varphi \in GSt_b \} \) and, by Theorem 55, \( E_b = Th_{L_b}(\Theta'(\Delta)) \cup S5 \cup (E_b \cap MGSt_b) \).

Let \( F_b = Th_{L_b}(\Theta'(\Delta)) \cup S5 \cup (E_b \cap MGSt_b) \). Since \( E_b \) is consistent, there exists an S4-Kripke \( b \)-interpretation \( \mathcal{M} = \langle U, R, I \rangle \) such that \( \mathcal{M} \models E_b \). Let \( \mathcal{M}' = \langle U, I \rangle \). By Lemma 101, \( M' \models \Theta'(\Delta) \).
We contend that $M' \models E_b \cap MGSt_b$. Let $M \varphi \in E_b \cap MGSt_b$. Then, there exists a world $u \in U$ such that $(\mathfrak{M}, u) \models \varphi$. Hence, by the definition of $\mathfrak{M}'$, $(\mathfrak{M}', u) \models \varphi$. Since in $\mathfrak{M}'$ every two worlds are connected each to the other, $\mathfrak{M}' \models M \varphi$. Hence, $M' \models E_b \cap MGSt_b$. Consequently, $M' \models F_b$, implying that $F_b$ is consistent.

By Lemma 102, for any $\varphi \in GSt_b$, $\Theta'(\Delta) \cup (E_b \cap MGSt_b) \models^{S5} \varphi$ if and only if $\Theta'(\Delta) \cup (E_b \cap MGSt_b) \models^{S4} \varphi$. Therefore, $E_b \cap GSt_b = F_b \cap GSt_b$.

The proof will be complete if we show that $F_b$ is a fixpoint of $\text{GNM}_{S5}^{b,\Theta'(\Delta)}$. Since $F_b$ is consistent, by Theorem 55 it suffices to show that $F_b$ satisfies conditions (1) and (2) of Theorem 55.

We start by showing that $F_b$ satisfies condition (1) of Theorem 55. By the definition of $F_b$, $E_b \cap MGSt_b \subseteq F_b \cap MGSt_b$. For the converse inclusion, let $M \varphi \in F_b \cap MGSt_b$. Since $F_b$ is consistent and closed under semantical $b$-entailment, $F_b \not\models^{S5}_b \varphi$. This, together with $E_b \cap GSt_b = F_b \cap GSt_b$, implies $E_b \not\models^{S4}_b \varphi$. Therefore, by the definition of $E_b$, $E_b \models^{S4}_b M \varphi$, implying $M \varphi \in E_b \cap MGSt_b$. Consequently, $E_b \cap MGSt_b = F_b \cap MGSt_b$, implying $F_b = Th_{\mathcal{L}_b}(\Theta'(\Delta) \cup S5 \cup (E_b \cap MGSt_b))$.

Finally, to show that $F_b$ satisfies condition (2) of Theorem 55, let $\varphi \in GSt_b$ be such that $F_b \not\models^{S5}_b \varphi$. Since $E_b \cap GSt_b = F_b \cap GSt_b$ and since $E_b$ is closed under semantical $b$-entailment, $E_b \not\models^{S4}_b \varphi$. Hence, by the definition of $E_b$, $M \varphi \in E_b$. This, together with $F_b = Th_{\mathcal{L}_b}(\Theta'(\Delta) \cup S5 \cup (E_b \cap MGSt_b))$, implies $F_b \models^{S5}_b M \varphi$.

**Proof of Theorem 98** Let $G$ be a consistent minimal set for $\Delta$. Then, by Theorem 93, there exist an infinite set of new constant symbols $b$ and a consistent fixpoint of $\text{GNM}_{S5}^{b,\Theta'(\Delta)}$, $E_b'$, such that $G = (E_b' \cap St) \cap GSt = E_b' \cap GSt$. By Proposition 99, $E_b'$ is a fixpoint of $\text{GNM}_{S5}^{b,\Theta'(\Delta)}$, and by Proposition 103, there exists a consistent fixpoint $E_b$ of $\text{GNM}_{S4}^{b,\Theta'(\Delta)}$ such that $E_b' \cap GSt_b = E_b' \cap GSt_b$. Therefore, $E = E_b \cap St$ is a consistent ground $S4$-expansion for $\Theta'(\Delta)$, such that

$$E \cap GSt = (E_b \cap St) \cap GSt = E_b \cap GSt = E_b' \cap GSt = G.$$  

Conversely, let $E$ be a consistent ground $S4$-expansion for $\Theta'(\Delta)$ and let $G = E \cap GSt$. Then, there exist an infinite set of new constant symbols $b$ and a fixpoint of $\text{GNM}_{S5}^{b,\Theta'(\Delta)}$, $E_b$, such that $E = E_b \cap St$. Since $E$ is consistent, so is $E_b$. Hence, by Proposition 104, there exists a consistent fixpoint $E_b'$ of $\text{GNM}_{S5}^{b,\Theta'(\Delta)}$ such that $E_b \cap GSt_b = E_b' \cap GSt_b$. By Proposition 99, $E_b'$ is
a fixpoint of $\text{GNM}_{ss}^{b,\Theta(\Delta)}$. Therefore, by Theorem 93, $(E_b' \cap St) \cap GST$ is a consistent minimal set for $\Delta$. Hence, since

$$(E_b' \cap St) \cap GST = E_b' \cap GST = E_b \cap GST = E \cap GST = G,$$

$G$ is a consistent minimal set for $\Delta$. $\blacksquare$
Bibliography


לברך, ונא מזכירוangel תכשיך בנו קבוצת מדליית עוזר ורות ברירית-🤘🏻 מוסד-ראשה
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בפרק הזפקת הפסק.

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S4, S5- ו S4, בפרק 4 Mוכil התוכי, התוכי וה仝יאונאות Kיריח
קצייר העבירה

(1.1)

\[ \beta_i \leq \text{어떤 } \phi \text{ 는 } -x \leq \phi \leq x \text{가 } \\
\gamma_i \text{뿐 }

(1.2)

\[ \alpha : \beta_1, \ldots, \beta_m \text{ }

<IMPORTANT>
תוכן עניניים

תקציר

רשימת סימונים

ג' נא

לינקא אל- американскית פסוקית
7 ........................... 2.1
8 ........................... 2.2
9 ........................... 2.3
10 ........................... 2.4
11 ........................... 2.4.1
12 ........................... 2.4.2
13 ........................... 2.5
14 ........................... 2.5.1
15 ........................... 2.5.2
16 ........................... 2.5.3

לינקא אל- מפר인데-ראשו
18 ........................... 3.1
19 ........................... 3.2
20 ........................... 3.3
21 ........................... 3.4
22 ........................... 3.5
23 ........................... 3.6
24 ........................... 3.7
25 ........................... 3.8
26 ........................... 3.9
27 ........................... 3.10
28 ........................... 3.11
29 ........................... 3.12
30 ........................... 3.13

מקבץ פסוקית של מדליות מרגניאליזם עבורי וולגניק פאלאטיאלאית אל-אמריקנית
34 ........................... 4.1
35 ........................... 4.2
36 ........................... 4.3
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38 ........................... 4.5
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קובץ מקבולי פאלאס-ראשו
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מקבץ פסוקית של מדליות מרגניאליזם עבורי וולגניק פאלאטיאלאית אל-אמריקנית
52 ........................... 6.1
53 ........................... 6.2
54 ........................... 6.3
55 ........................... 6.4
56 ........................... 6.5

רשימת ממקס

המודרך נציג במודיה פורו':// מיכאל קמנסקי
בפמליה למורים המורים.

אני מודה לימייאו קמנסקי על התנדבותו המצוינת לכל אורכ מורדים.

אני מודה לטכניון על התמיכת במיסים הנדרשת בפרסה.
סמנטייקה של מודלים מורפומטרליםولدיה
ל.osgiיה מודלית לא-מורפומטרית מלב�ת מסדר-ראשים

הרבר טלקה

לשמ מיולוי חלקי של הדרישות לكسبת החזון
מניטר למיתרים במדעי המחשב

ביימטר דריימברג

הרצות לספנן טכניקות - מרכז סכנวลגיט לישראלי
נובמבר 2006
היפה
השו"ע תשמ"ז
סדנתיך של מרדילם מונימאת┩י עבדר
לרבך מרדילם לא-מונומנטית מקובהת מסדר-ראשון

בירמי פרימברג