$d$-Dimensional Variants of Heilbronn’s Triangle Problem

Jonathan Naor
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Jonathan Naor

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Abstract

Heilbronn’s triangle problem asks for the maximal possible area of the triangle of smallest area formed by $n$ points in the unit square. In this thesis we show lower and upper bounds for a generalization of Heilbronn’s triangle problem to $d$ dimensions. Namely, we show that there exists a set $S_1$ (resp., $S_2$) of $n$ points in the $d$-dimensional unit cube such that the minimum-area triangle (embedded in $d$ dimensions) defined by some three points of $S_1$ (resp., $S_2$) has an area of $\Omega(d^{1-1/(2(d-1))}/n^{2/(d-1)})$ (resp., $O(d/n^{2/d})$). We then generalize the applied methods and show that there exists a set $S_3$ (resp., $S_4$) of $n$ points in the $d$-dimensional unit cube such that the minimum-volume $k$-dimensional simplex (embedded in $d$ dimensions, for $2 \leq k \leq d$) defined by some $k+1$ points of $S_3$ (resp., $S_4$) has volume $\Omega(f(k,d)/n^{k/(d-k+1)})$, where $f(k,d)$ is independent of $n$ (resp., $O(k^{k/d}d^{k/2}/(k! n^{k/d})$)).
Abbreviations and Notations

Γ(\(x\)) — the Gamma function

\(\mathcal{H}(k, d)\) — Heilbronn’s function for \(k\)-simplices in \(d\) dimensions

\(\alpha(G)\) — the size of the largest independent set in the graph \(G\)
Chapter 1

Introduction

In the 1950’s the Jewish German mathematician Hans Heilbronn posed the following question: what is the maximal area, over all configurations of \( n \) points in the unit square, of the smallest of the \( \binom{n}{3} \) triangles formed by the points of the configuration? We denote this value \( H_{2,2}(n) \): the first index stands for two-dimensional simplices, i.e., triangles, whereas the second index indicates that these simplices are sought in the two-dimensional unit square. Exact values for \( H_{2,2}(n) \) are known only for \( n = 3, 4, 5 \) and \( 6 \) (respectively \( \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{3}}{9}, \frac{1}{8} \)). Even for \( n = 7 \) the exact answer is unknown [6]. Heilbronn conjectured that \( H_{2,2}(n) = O(1/n^2) \).

Erdős showed by an example [12] that if this bound had been true, then it would have been tight, namely, he demonstrated that \( H_{2,2}(n) = \Omega(1/n^2) \) by observing that no three distinct points on the parabola \( (x, x^2) \) are collinear for \( x \in \mathbb{Z}_n \), \( n \) prime. Heilbronn’s conjecture was disproved, however, thirty years later by Komlós, Pintz, and Szemerédi [9]. They showed by a probabilistic construction that \( H_{2,2}(n) = \Omega(\log n/n^2) \). By discretizing the problem — restricting the points to lie on a grid — and transforming it into an independent-set problem
on hypergraphs, Bertram-Kretzberg, Hofmeister, and Lefmann [4] provided a deterministic polynomial-time algorithm that computes a specific configuration of \(n\) points that achieves this lower bound, which is currently the best known. As for the upper bound, it is trivial to show that \(\mathcal{H}_{2,2}(n) = O(1/n)\): any triangulation of any point set (in general position) in the unit square admits \(\Theta(n)\) triangles.

Roth [13] showed that \(\mathcal{H}_{2,2}(n) = O(1/(n\sqrt{\log \log n})\), which was improved by Schmidt [16] to \(\mathcal{H}_{2,2}(n) = O(1/(n\sqrt{\log n})\). Roth further improved the upper bound to \(O(1/(n^{\mu - \varepsilon})\), originally [14] with \(\mu = 2 - 2/\sqrt{5} = 1.1055 \ldots\) and later [15] with \(\mu = (17 - \sqrt{65})/8 = 1.1172 \ldots\). The best currently known upper bound, due to Komlós, Pintz, and Szemerédi [8], is \(\mathcal{H}_{2,2}(n) = O(2^{\sqrt{\log n}}/n^{8/7})\), where \(c\) is some positive constant. Jiang, Li, and Vitany [7] showed that for \(n\) points distributed randomly and uniformly in the unit square, the expected area of the smallest triangle formed by any three of these points is \(\Theta(1/n^3)\).

In the on-line version of the problem, the number of points is not “known” in advance, and points are chosen in the unit square one after the other until the process “suddenly” stops at \(n\). Clearly this is more restrictive than the off-line version, namely \(\mathcal{H}_{2,2}^{\text{on-line}}(n) = O(\mathcal{H}_{2,2}^{\text{off-line}}(n))\). The best known lower bound is \(\mathcal{H}_{2,2}^{\text{on-line}}(n) = \Omega(1/n^2)\), no better than the Erdős moment-curve example.

A higher-dimensional variant of Heilbronn’s problem was first investigated by Barequet [2]. Given a fixed dimension \(d\), we seek the maximal possible value, over all configurations of \(n\) points in the \(d\)-dimensional unit cube, of the smallest-volume \(d\)-dimensional simplex formed by any \(d+1\) of these points (in our notation, \(\mathcal{H}_{d,d}(n)\)). He proved the lower bound \(\mathcal{H}_{d,d}(n) = \Omega(1/n^d)\), both by generalizing the Erdős moment-curve example and by a probabilistic construction. This bound was later improved by Lefmann [10] to \(\mathcal{H}_{d,d}(n) = \Omega(\log n/n^d)\), again by transforming
the problem into a hypergraph independent-set problem. A triangulation argument shows the trivial upper bound $H_{d,d}(n) = O(1/n)$. Brass [5] gave the first nontrivial upper bound $H_{d,d}(n) = O(1/n^{1+1/(2d)})$ for odd $d \geq 3$.

In this thesis we remove the relation between the dimensions of the simplices and the space in which they reside. That is, we investigate the general case of $k$-dimensional simplices in the $d$-dimensional unit cube. We demonstrate the methods used on triangles in a general dimension, obtaining $H_{2,d}(n) = \Omega \left( \frac{d^{d-2(d+1)}}{n^{d+1}} \right)$ for a lower bound and $H_{2,d}(n) = O \left( \frac{d}{n^{d+1}} \right)$ for an upper bound, and show that in the general case $H_{k,d}(n) = \Omega \left( \frac{c_k d}{n^{d-k+1}} \right)$, and $H_{k,d}(n) = O \left( \frac{k^n d^k}{k! n^d} \right)$.

The main results of this thesis appeared in [3].
Chapter 2

The Lower Bound

In this chapter we prove that $\mathcal{H}_{2,d}(n) = \Omega\left(\frac{d^{d-1}n^{1-d}}{d-1}\right)$ and $\mathcal{H}_{k,d}(n) = \Omega\left(\frac{e_{k,d}n^{1-d}}{d-1}\right)$.

In the first section we prove a lemma which provides the foundation for the proofs of the lower bounds. In the two following sections we prove the bounds for the triangle and the general case.

2.1 A Probabilistic Lemma

We first prove a lemma, which is a generalization of a probabilistic argument of Alon and Spencer [1]. A slightly weaker version of this lemma was proven in [2].

**Lemma 1** Let $H(P_1, P_2, \ldots, P_m)$ be a mapping from $m$-tuples of points $P_1, P_2, \ldots, P_m$ in some domain $\mathcal{D}$ to $\mathbb{R}^+ \cup \{0\}$. If there exist constants $c_1 > 0$, $c_2$ such that for every $\varepsilon > 0$

$$\text{Prob}[H(P_1, P_2, \ldots, P_m) \leq \varepsilon] \leq c_1 \varepsilon^{c_2},$$

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where $P_1, P_2, \ldots, P_m$ are chosen randomly, uniformly, and independently in $D$, then for every integral $n$ there exists a set $S$ of $n$ points in $D$ such that

$$\min_{P_{i_1}, P_{i_2}, \ldots, P_{i_m} \in S} H(P_{i_1}, P_{i_2}, \ldots, P_{i_m}) > c_3 n^{-\frac{m-1}{c_2}},$$

where $c_3 = (\frac{m!}{2^m c_1})^{1/c_2}$.

**Proof** Let $P_1, P_2, \ldots, P_{2n}$ be a set of $2n$ points selected randomly, uniformly, and independently in $D$. Fix $c_3 = (\frac{m!}{2^m c_1})^{1/c_2}$. Let the random variable $X$ count the number of $m$-tuples $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$ for which $H(P_{i_1}, P_{i_2}, \ldots, P_{i_m}) \leq c_3 n^{-\frac{m-1}{c_2}}$. Then we have

$$E[X] \leq \binom{2n}{m} c_1 (c_3 n^{-\frac{m-1}{c_2}})^c < \left(\frac{2n}{m!}\right)^m \cdot \frac{m!}{2^m n^{m-1}} = n.$$

Therefore, there exists a specific set of $2n$ points with fewer than $n$ $m$-tuples $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$ for which $H(P_{i_1}, P_{i_2}, \ldots, P_{i_m}) \leq c_3 n^{-\frac{m-1}{c_2}}$. Remove one point of the set from each such $m$-tuple. (The same point may be deleted more than once but this only helps.) This leaves at least $n$ points and now all $m$-tuples $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$ satisfy $H(P_{i_1}, P_{i_2}, \ldots, P_{i_m}) > c_3 n^{-\frac{m-1}{c_2}}$. \hfill $\square$

Alon and Spencer [1] proved the special case of Lemma 1 in which $c_2 = 1$ and $m = 3$, and used it for showing that $\mathcal{H}_{2,2}(n) = \Omega(1/n^2)$.

### 2.2 Triangles in $d$ Dimensions

We introduce our technique by first using it for planar simplices, that is, triangles.

**Theorem 1** $\mathcal{H}_{2,d}(n) = \Omega(d^{1-1/(2(d-1))}/n^{2/(d-1)})$. 

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Proof We first bound the probability of the area of a random triangle being smaller than $\varepsilon$. Let $P_{i_0}, P_{i_1}, P_{i_2}$ be three points chosen randomly and independently in the unit cube, and let $A(P_{i_0}, P_{i_1}, P_{i_2})$ be the area of the triangle defined by $P_{i_0}$, $P_{i_1}$, and $P_{i_2}$. To upper bound $\text{Prob}[A(P_{i_0}, P_{i_1}, P_{i_2}) \leq \varepsilon]$, let $x$ be the distance from $P_{i_0}$ to $P_{i_1}$. Then,

$$\text{Prob}[b \leq x \leq b + db] \leq d(\pi^{d/2}b^d/\Gamma(d/2 + 1)) = (\pi^{d/2}db^{d-1}/\Gamma(d/2 + 1)) db,$$

the difference\(^1\) between the volumes of the corresponding balls in $\mathbb{R}^d$ (see Figure 2.1).\(^2\) Given $P_{i_0}$ and $P_{i_1}$ at distance $b$, the altitude $h$ from $P_{i_2}$ to the line defined by $P_{i_0}$ and $P_{i_1}$ satisfies $bh/2 \leq \varepsilon$, i.e., $h \leq 2\varepsilon/b$. Thus, $P_{i_2}$ must lie within a $d$-dimensional cylinder whose height is at most $\sqrt{d}$ and whose cross-section is a

\(^1\)We use the non-italicized symbol ‘d’ to denote the differentiation operator, in order to avoid confusion with the italicized symbol ‘$d$’ that denotes the dimension.

\(^2\)Recall that the volume of a $d$-dimensional ball, with radius $r$, is $\pi^{d/2}r^d/\Gamma(d/2 + 1)$, where $\Gamma(\cdot)$ is the continuous generalization of the factorial function, for which $\Gamma(x) = (x - 1)\Gamma(x - 1)$, $\Gamma(1/2) = \sqrt{\pi}$, and $\Gamma(1) = 1$. It is easy to see that $\text{vol}(B^{2k}) = \pi^k/k!$ and $\text{vol}(B^{2k+1}) = \pi^{k+1/2}/\Gamma(k + 3/2)$ (where $B^d$ denotes the unit $d$-dimensional ball), for an integral $k \geq 1$. 

Figure 2.1: The difference between the volumes of concentric balls
(d − 1)-dimensional ball whose volume is \( \pi^{(d-1)/2} (2\varepsilon/b)^{d-1} / \Gamma((d+1)/2) \). (See Figure 2.2). This occurs with probability at most \( \pi^{(d-1)/2} \sqrt{d} (2\varepsilon/b)^{d-1} / \Gamma((d+1)/2) \).

Figure 2.2: The height of the cylinder is at most the length of the diagonal

Since \( 0 \leq b \leq \sqrt{d} \),

\[
\Pr[A(P_{i_0}, P_{i_1}, P_{i_2}) \leq \varepsilon] \leq \int_0^{\sqrt{d}} \left( \frac{\pi^{d/2} db^{d-1}}{\Gamma(d/2 + 1)} \right) \left( \frac{\pi^{(d-1)/2} \sqrt{d} (2\varepsilon/b)^{d-1}}{\Gamma((d+1)/2)} \right) db
\]

\[
= \frac{\pi^{d-1/2} 2^{d-1} d^{d-1} \varepsilon^{d-1}}{\Gamma(d/2 + 1) \Gamma((d+1)/2)}.
\]

Now apply Lemma 1 with \( c_1 = \pi^{d-1/2} 2^{d-1} d^{d-1} \varepsilon^{d-1} / (\Gamma(d/2 + 1) \Gamma((d+1)/2)) \), \( c_2 = d - 1 \), and \( m = 3 \), and conclude that there exists a set \( S \subset [0, 1]^d \) of \( n \) points for which

\[
\min_{P_{i_0}, P_{i_1}, P_{i_2} \in S} A(P_{i_1}, P_{i_2}, P_{i_3}) > c_3 / n^{2/(d-1)},
\]

where

\[
c_3 = \left( \frac{3! \Gamma(d/2 + 1) \Gamma((d+1)/2)}{2^{d+2} \pi^{d-1/2} d^2} \right) \pi^{1/2} = \Theta(d^{1-1/(2(d-1))})
\]

(by applying Stirling’s asymptotic approximation \( \Gamma(x + 1) \sim \sqrt{2\pi x} (x/e)^x \)). That is, \( \mathcal{H}_{2,d}(n) = \Omega(d^{1-1/(2(d-1))} / n^{2/(d-1)}) \).
2.3 \( k \)-Dimensional Simplices

We now use the technique of Section 2.2 to show the following:

**Theorem 2** \( \mathcal{H}_{k,d}(n) = \Omega(f(k,d)/n^{k/(d-k+1)}) \), where \( f(k,d) \) is a function of only \( k \) and \( d \) that is independent of \( n \).

**Proof** Let \( P_0, P_1, \ldots, P_k \) be \( k+1 \) points chosen randomly and independently in the \( d \)-dimensional unit cube, and let \( V(P_0, P_1, \ldots, P_k) \) denote the volume of the \( k \)-dimensional simplex defined by these points. Also denote by \( x_i \) (for \( 1 \leq i \leq k \)) the distance from \( P_i \) to \( E_{i-1} \), the \((i-1)\)-dimensional flat defined by \( P_0, P_1, \ldots, P_{i-1} \).

Let us begin with upper bounding \( \text{Prob}[V(P_0, P_1, \ldots, P_k) < \varepsilon] \). First,

\[
\text{Prob}[b_1 \leq x_1 \leq b_1 + db_1] \leq d \left( \frac{\pi^{\frac{d}{2}}b_1^d}{\Gamma \left( \frac{d}{2} + 1 \right)} \right) = \frac{\pi^{\frac{d}{2}}db_1^{d-1}}{\Gamma \left( \frac{d}{2} + 1 \right)} db_1,
\]

the difference between the volumes of the corresponding balls. Second,

\[
\text{Prob}[b_2 \leq x_2 \leq b_2 + db_2] \leq d \left( \sqrt{d} \frac{\pi^{\frac{d-1}{2}}b_2^{d-1}}{\Gamma \left( \frac{d-1}{2} + 1 \right)} \right)
= \sqrt{d} \frac{\pi^{\frac{d-1}{2}}(d-1)b_2^{d-2}}{\Gamma \left( \frac{d-1}{2} + 1 \right)} db_2,
\]

the difference between the volumes of the corresponding cylindrical shapes obtained by sweeping \((d-1)\)-dimensional balls along a straight path whose length is at most that of the main diagonal of the unit cube. The general probability term is thus

\[
\text{Prob}[b_i \leq x_i \leq b_i + db_i] \leq d \left( d^{\frac{d-i+1}{2}} \frac{\pi^{\frac{d-i+1}{2}}b_i^{d-i+1}}{\Gamma \left( \frac{d-i+1}{2} + 1 \right)} \right)
= d^{\frac{d-i+1}{2}} \frac{\pi^{\frac{d-i+1}{2}}(d-i+1)b_i^{d-i}}{\Gamma \left( \frac{d-i+1}{2} + 1 \right)} db_i.
\]
For the second-to-last point we have
\[
\operatorname{Prob}\left[ b_{k-1} \leq x_{k-1} \leq b_{k-1} + db_{k-1} \right] \leq d\left( d^{\frac{k-2}{2}} \frac{\pi^{\frac{d-k+2}{2}} b_{k-1}^{d-k+2}}{\Gamma\left( \frac{d-k+2}{2} + 1 \right)} \right)
\]
\[
= d^{\frac{k-1}{2}} \frac{\pi^{\frac{d-k+2}{2}} (d - k + 2) b_{k-1}^{d-k+1}}{\Gamma\left( \frac{d-k+2}{2} + 1 \right)} \cdot db_{k-1}.
\]

For the last point we have the condition
\[
\prod_{i=1}^{k} \frac{b_i}{k!} \leq \varepsilon,
\]
that is,
\[
b_k \leq \frac{k! \varepsilon}{\prod_{i=1}^{k-1} b_i}.
\]

Therefore, \( P_k \) must lie in a shape whose volume is at most
\[
d^{\frac{k+1}{2}} \frac{\pi^{\frac{d-k+2}{2}}}{\Gamma\left( \frac{d-k+2}{2} + 1 \right)} \left( \frac{k! \varepsilon}{\prod_{i=1}^{k-1} b_i} \right)^{d-k+1}
\]
(the product of a \((k - 1)\)-dimensional slab, each of whose dimensions is at most \(\sqrt{d}\), and a \((d - k + 1)\)-dimensional ball).

The probability of obtaining a \(k\)-dimensional simplex of volume at most \(\varepsilon\) is thus upper bounded by

\[
\int_{0}^{\sqrt{d}} \int_{0}^{\sqrt{d}} \int_{0}^{\sqrt{d}} \ldots \left( \sum_{i=d-k+1}^{d} d! \cdot \prod_{i=d-k+1}^{d} \Gamma\left( \frac{i}{2} + 1 \right) \cdot (d - k + 1)! \right) \cdot (kd)^{d-k+1} \cdot \varepsilon^{d-k+1} db_1 db_2 \ldots db_k.
\]

By Stirling’s approximation,

\[
\prod_{i=d-k+1}^{d} \Gamma\left( \frac{i}{2} + 1 \right) \sim \prod_{i=d-k+1}^{d} \left( \sqrt{2\pi} \frac{i}{2} \cdot e^\frac{i}{e} \right) = \frac{\pi^{\frac{d}{2}} \sqrt{\frac{d!}{(2\pi)^{d-k+1}}} \prod_{i=d-k+1}^{d} \frac{i^i}{(2e)^k}}{(2e)^{(2d-k-1)/4}}.
\]
After substituting Eq. (2.3) in Eq. (2.1) and integrating \( k - 1 \) times, we conclude that the probability of obtaining a \( k \)-dimensional simplex of volume at most \( \varepsilon \) is at most

\[
\pi^{k(2d-k-1)/4} \cdot d^{k(k-1)/4} \cdot \sqrt{d!} \cdot (2e)^{k(2d-k+1)/4} \cdot (k!)^{d-k+1} \cdot \varepsilon^{d-k+1}.
\]

Finally, set \( c_1 = \frac{\pi^{k(2d-k-1)/4} \cdot d^{k(k-1)/4} \cdot \sqrt{d!} \cdot (2e)^{k(2d-k+1)/4} \cdot (k!)^{d-k+1}}{\sqrt{\prod_{i=d-k+1}^{d} i^i \cdot (d-k+1) \cdot \sqrt{(d-k)!} \cdot (k-1)!}} \), \( c_2 = d - k + 1 \), and \( m = k + 1 \), and apply Lemma 1. The lemma tells us that there exists a set of \( n \) points of which every subset of \( k + 1 \) points defines a \( k \)-dimensional simplex whose volume is at least \( c_3/n^{k/(d-k+1)} \), where \( c_3 = \left( \frac{(k+1)!}{2^{k+1} c_1} \right)^{1/(d-k+1)} \). Let us finally give a lower bound on \( c_3 \). By substituting \( c_1 \) in the above term, we see that

\[
c_3^{d-k+1} = \frac{(k+1)!(k-1)!}{(k!)^{d-k+1}}.
\]  

(2.4)

We write

\[
\frac{(k+1)!(k-1)!}{(k!)^{d-k+1}} = \frac{k+1}{k} \cdot \frac{1}{(k!)^{d-k-1}} \geq \frac{1}{(k!)^{d-k-1}}
\]

and

\[
(k!)^{d-k-1} \sim 2^{(d-k-1)/2} \pi^{(d-k-1)/2} e^{-k(d-k-1)} k^{(2k+1)(d-k-1)/2},
\]

and so,

\[
\frac{(k+1)!(k-1)!}{(k!)^{d-k+1}} \geq \frac{1}{2^{(d-k-1)/2} \pi^{(d-k-1)/2} e^{-k(d-k-1)} k^{(2k+1)(d-k-1)/2}}.
\]

In addition,

\[
\prod_{i=d-k+1}^{d} i^i \geq (d - k + 1)^{k(2d-k+1)/2}
\]
and
\[ \sqrt{\frac{d}{\prod_{i=d-k+1}^{d} i}} \leq d^{k/2}. \]

Substituting all these terms in Eq. (2.5), we obtain that
\[
 c_3^{d-k+1} \geq \frac{e^{k(2d-3k-5)/4}(d - k + 1)^{k(2d-k+1)/4+1}}{2(2kd-k^2+2d+3k+2)/4 \pi^2 (2kd-k^2+2d-3k-2)/4 k^{(2kd-2k^2+d-3k-1)/2} d^{k(k-1)/4}}
\]
concluding that
\[
 c_3 \geq \frac{e^{\frac{k}{2}}(d - k + 1)^{\frac{k}{2}}}{(2 \pi)^{\frac{k+1}{2}} k^{\frac{k+1}{2}} d^{\frac{k(k-1)}{4}}} \cdot \left( \frac{\frac{k^2-k+5}{4}}{\frac{k(k+3)}{4} \pi + \frac{k(k+7)}{4}} \right)^{\frac{1}{2-k+1}} \cdot \left( \frac{\frac{d-k+1}{4}}{\frac{k^2-k+5}{4} \pi + \frac{k(k+7)}{4}} \right)^{\frac{1}{2-k+1}}
\]

This completes the proof. \(\square\)

Substituting \(k = 2\) in this bound yields \(\calH_{2,d}(n) = \Omega(d^{1/2+7/(4(d-1))}/n^{2/(d-1)})\), which is a bit weaker than the bound \(\Omega(d^{1-1/(2(d-1))}/n^{2/(d-1)})\) shown in the previous section.
Chapter 3

The Upper Bound

In Section 3.1 we show that for triangles in $d$ dimensions $\mathcal{H}_{2,d}(n) = O(d/n^{2/d})$. For the general case (Section 3.2) we show two bounds with the same asymptotic dependence on $n$ but differing in their dependence on $k$ and $d$: $\mathcal{H}_{k,d}(n) = O(k^{k/d}d^{k/2}/(k!n^{k/d}))$ and $\mathcal{H}_{k,d}(n) = O\left(\frac{2^{k/2}d^{(k+1)/d}k^{k/d}}{\pi^{k/2}n^{k/d}}\right)$. We close this chapter by showing that each of these bounds can be better than the other for specific values of $k$ and $d$.

3.1 Triangles in $d$ Dimensions

We first demonstrate the argument for $k = 2$:

**Theorem 3** $\mathcal{H}_{2,d}(n) = O(d/n^{2/d})$.

**Proof** The upper bound is set by a pigeonhole argument. Cover the $d$-dimensional unit cube with a regular grid whose step is $1/m^{1/d}$, where $m = (n-1)/2$. This divides the $d$-dimensional unit cube into exactly $m$ small grid cubes. Put $n = 2m+1$ points in the unit cube. There is at least one grid cube that contains at least three
points. Since the length of the main diagonal of a grid cube is \( \sqrt{d/m^{1/d}} \), the area of the triangle defined by these three points is \( O(d/m^{2/d}) = O(2^{2/d}d/n^{2/d}) = O(d/n^{2/d}) \).

\[ \square \]

### 3.2 \( k \)-Dimensional Simplices

#### 3.2.1 A pigeonhole argument

We use a similar argument for a general value of \( k \):

**Theorem 4** \( \mathcal{H}_{k,d}(n) = O(k^{k/d}d^{k/2}/(k!n^{k/d})) \).

**Proof** Again, cover the \( d \)-dimensional unit cube with a regular grid whose step is \( 1/m^{1/d} \), where this time \( m = (n-1)/k \). This divides the \( d \)-dimensional unit cube into exactly \( m \) small grid cubes. Put \( n = km + 1 \) points into the unit cube. There is at least one grid cube that contains at least \( k + 1 \) points. Since the length of the main diagonal of a grid cube is \( \sqrt{d/m^{1/d}} \), the volume of the simplex defined by these \( k + 1 \) points is \( O(d^{k/2}/(k!m^{k/d})) = O(k^{k/d}d^{k/2}/(k!n^{k/d})) \). \( \square \)

#### 3.2.2 A graph-based argument

Employing a technique used by Lefmann [10], we show:

**Theorem 5** \( \mathcal{H}_{k,d}(n) = O\left(\frac{2^{3k/2}d^{(1+1/d)k/2}k^{k/d}}{e^{k/2} \pi^{1/2} \Gamma(1+k/2)k!n^{k/d}}\right) \).

**Proof** Given \( n \) points \( P_1, \ldots, P_n \) in the \( d \)-dimensional unit cube and a real number \( D_0 \), to be specified later, we construct a graph \( G(D_0) = G(V,E) \) whose vertex set is \( V = \{P_i\}_{i=1}^n \) and where \( \{P_i, P_j\} \in E \) if \( |P_i, P_j| \leq D_0 \). Thus, an independent set
$I \subseteq V$ consists of points with pairwise Euclidean distance bigger than $D_0$. The intersection of any ball $B_r(P_i)$ with center $P_i \in [0,1]^d$ and radius $r \leq 1$ with the unit cube is of volume at least $\frac{\text{vol}(B_r(P_i))}{2^d}$, since at least half of the ball is in the interior of the cube in every one of the $d$ dimensions (see Figure 3.1). The balls with radius $D_0/2$ centered at the points $\{P_i\}$ in an independent set are pairwise disjoint, hence the combined volume of all these balls is less than 1. Moreover,

$$\alpha(G) \cdot C_d \cdot \frac{(D_0/2)^d}{2^d} \leq 1,$$

where $\alpha(G)$ is the largest independent set in $G$ and $C_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ is the volume of the $d$-dimensional unit ball. By Turán’s theorem, for any graph $G = (V, E)$ we have the lower bound $\alpha(G) \geq \frac{n}{2t}$, where $t = 2|E|/|V|$ is the average degree of $G$. Thus,

$$\frac{n}{2t} \leq \alpha(G) \leq \frac{4^d}{C_d \cdot D_0^d},$$

that is,

$$t \geq \frac{C_d n D_0^d}{2^{2d+1}}.$$
Fix \( D_0 = \left( \frac{2^{2d+1}k}{C_d n} \right)^{1/d} \). Then,

\[
t \geq \frac{C_d n}{2^{2d+1}} \cdot \frac{2^{2d+1}k}{C_d n} = k.
\]

Hence, there exists a vertex \( P_{i_0} \) with at least \( k \) neighbors \( P_{i_1}, \ldots, P_{i_k} \). The simplex formed by the points corresponding to these vertices has volume at most

\[
D_0^k = \left( \frac{2^{2d+1}k}{C_d n} \right)^{k/d} = O \left( \frac{2^{3k/2} d^{(1+1/d)k/2} k^{k/d}}{e^{k/2} \pi^{(1-1/d)k/2} k! n^{k/d}} \right)
\]

using Stirling’s approximation \( \Gamma(\frac{d}{2} + 1) \sim \sqrt{\pi d} \left( \frac{d}{2e} \right)^{d/2} \). \( \square \)

Note that the ratio between the bound obtained in Theorem 5 and that of Theorem 4 is \( \frac{2^{3k/2} d^{k/(2d)}}{e^{k/2} \pi^{(1-1/d)k/2}} \). Independently, Lefmann [11] obtained recently the upper bound \( O(1/n^{k/d}) \), omitting the dependency of the bound on \( d \) and \( k \). For even \( d \), employing methods first used by Brass [5], he showed \( H_{k,d}(n) = O \left( \frac{1}{n^{k/d+(k-1)/(2d-1)}} \right) \).

### 3.2.3 Comparison of the two methods

We show that depending on \( k \) and \( d \), each of the two bounds shown in the previous subsections can be better than the other.

The ratio between the independent-set bound and the pigeonhole bound is

\[
R(k, d) = \frac{2^{3k/2} d^{k/(2d)}}{e^{k/2} \pi^{(1-1/d)k/2}}.
\]

To compare the two bounds, we wish to find out when

\[
R(k, d) < 1 \iff \log R < 0.
\]

We have

\[
\log R(k, d) = \frac{3k}{2} + \frac{k}{2d} \log d - \frac{k}{2} \log e - \left( 1 - \frac{1}{d} \right) \frac{k}{2} \log \pi.
\]

For \( k = 2 \),

\[
R(2, d) = 3 + \frac{\log d}{d} - \log e - \log \pi + \frac{\log \pi}{d}.
\]
Since $\log e + \log \pi \approx 3.09$, $R(2, d)$ is negative for large-enough values of $d$, and hence the independent-set bound is better. On the other hand, $R(3, 3) \approx 0.2$, in which case the pigeonhole argument is better.
Chapter 4

Conclusion

In this thesis we addressed the general case of Heilbronn’s triangle problem and have shown upper and lower bounds. For triangles in $d$ dimensions we obtained the lower bound $H_{2,d}(n) = \Omega(d^{1-1/(2(d-1))}/n^{2/(d-1)})$ and the upper bound $H_{2,d}(n) = O(d/n^{2/d})$. Note that as $d$ increases the gap between these bounds narrows.

For the general case we have $H_{k,d}(n) = \Omega(f(k,d)/n^{k/(d-k+1)})$ and $H_{k,d}(n) = O(k^{k/d}d^{k/2}/(k! n^{k/d}))$. These results appeared in [3]. A week later, Lefmann [11] improved the lower bound, obtaining $H_{k,d} = \Omega((\log n)^{1/(d-k+1)}/n^{k/(d-k+1)})$. (Note that Lefmann uses the symbol “$k$” to denote what we would call “$k + 1$”). He also provided a theorem whose proof is identical to that of Theorem 5, obtaining the same bound.

Heilbronn’s original problem, as well as its generalizations, remains, of course, open. Other questions encountered during the work on this thesis that might be of interest to pursue are generalizations of the problem to triangles on the surface of a sphere and to other geometries with different definitions of area. Optimality proofs for small values of $n$, as in the two-dimensional case for $n = 3, 4, 5, 6$, might
be sought, for example, for \( d = 3 \). Finally, since the best known lower bound for \( \mathcal{H}_{2,2}(n) \) is \( \Omega(\log n/n^2) \), should the largest sum of areas of all the triangles formed by any \( n \) points in the unit square (or at least in the optimal configuration) prove to be \( O(n \log n) \), this would entail \( \mathcal{H}_{2,2}(n) = \Theta(\log n/n^2) \). The max-sum problem could perhaps be approached by different methods than those tried on the classic Heilbronn’s problem.
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יהונתן נאור
גרסאות של d-מימדיות
בעיית המשולש של היילברון

תים בעל מתקר

לשם مليיה חקיק של הדרכים לקבצי נתונים

מענד לתודעה במודעי המחשב

יוונית נואז

הצנה ליאנט הטכנולוגיה - מכון טכנולוגי לישראל
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המחק נועש בהנחיית פרופ' גל ברקת בפקולטה למディיז ומדעי המחשב.

ברצוני להודות наукי לב לפורים; גל ברקת על כל שמותהครบר בעיות מעניינות váת.

על כל מה שלמרדתי ממנון. על תמציתי ארוך ורחום עם קשובים לא פלוס.

תודה מינוהד讓我ים ראשים רוח על אהבתה העדדיה.

אני מודה לטבלה על התרמימה הכסף לתידור מבחרםلون.
התקן עיניים

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רשימת מקורות

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רשימת אודרים מעברית

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ה
רשימת אורות

2.1 ההפיס אח נפרד שני גדולים בעלים מרוכם משוחק 11
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התקציר

בנושאים המתמטיים המתמטים החזות-רמלה של Hans Hailbronn נמדדה באמצעות המ-LASTigueיצס של \( n \) קונדוטה ברובע ההילות, מחקר השטח המקסימלי של פאזה התקרה \( H_{2,2}(n) \)-ב האינדקס המרכז מספרים מעטים בתחילה מחודד והידיד הזה-ידידות. השיטה של המתמטיקאי הלא פורמלית \( H_{2,2}(n) = \Omega(n^2) \) כולם שלולים \( H_{2,2}(n) = O(1/n^2) \) של הראה שטח הסמוך ביותר \( x, x^2 \) : \( x \in \mathbb{Z}_n \), \( H_{2,2}(n) = \Omega(1/n^2) \) של הפתרה \( H_{2,2}(n) = O(1/n^2) \) ב-1982 Heilbronn הפורח את ה-1982 Heilbronn הנמצאות על ישר אבוד ענבר \( n \) הסת.fill

[12] Erdős  \( H_{2,2}(n) = O(1/n^2) \) \( H_{2,2}(n) = \Omega(1/n^2) \) \n[9] Szemerédi - Pintz, Komlós \n[4] Lefmann - Hofmeister, Bertram-Kretzberg \n[8] Lefmann - Hofmeister, Bertram-Kretzberg

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 Shirsh Matroyh ha'sh

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קגורוד בקברט התרסה-h מימיה. של הסיסמקס ה-ד-ה מימיה, באמצעות המכרז

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Brass. H_{d,d}(n) = \Omega(\log n/n^d)-ו [10] Lefmann

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עובר H_{d,d}(n) = O(1/n^1+1/(2d)) זה

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משולש בגריסות-ה מממודי כלאל, מחבר

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, עניקר עבודה זווית ב

H_{k,d}(n) = O\left(\frac{k^d}{d^k} \frac{1}{n^k}\right) -ו H_{k,d}(n) = \Omega\left(\frac{c_{k,d}}{n^{k+k}}\right)