3D-INTERVAL-FILAMENT GRAPHS

Fanica GAVRIL

Computer Science Dept., Technion, Haifa 32000, Israel, gavril@cs.technion.ac.il

January, 2006

ABSTRACT: Gavril [GA4] defined two new families of intersection graphs: the interval-filament graphs and the subtree-filament graphs. The complements of interval-filament graphs are the cointerval mixed graphs and the complements of subtree-filament graphs are the cochordal mixed graphs. The family of interval-filament graphs contains the families of cocomparability, polygon-circle, circle and chordal graphs.

In the present paper we introduce a new family of intersection graphs, the 3D-interval-filament graphs, which are a generalization of the subtree-filament graphs. We prove that the family of complements of 3D-interval-filament graphs is exactly the family of co(interval-filament) mixed graphs and every 3D-interval-filament graph has an intersection representation by a family of piecewise linear filaments. We define various subfamilies of 3D-interval-filament graphs characterized as complements of families of $G$-mixed graphs.

KEY WORDS: interval-filament graph, subtree-filament graph, polygon-circle graph, intersection graph
1. INTRODUCTION

We consider only finite graphs \( G(V,E) \) with no parallel edges and no self-loops, where \( V \) is the set of vertices and \( E \) the set of edges. For \( U \subseteq V \), \( G(U) \) is the subgraph induced by \( U \). For \( F \subseteq E \), \( G(F) \) is the subgraph on \( V \) with edge-set \( F \). Two vertices connected by an edge are adjacent and we denote this by \( (u,v) \), without regard for the orientation of the edge; \( coG(V,coE) \) is the complement of \( G \) where \( coE=\{(u,v) \mid u \neq v, (u,v) \notin E\} \). A directed edge from \( u \) to \( v \) is denoted \( (u,v) \). We also denote \( N_G(v)=\{u \mid (u,v) \in E\} \) and \( N_G[v]=N_G(v) \cup \{v\} \).

A graph \( G \) is an intersection graph of a family \( S \) of subsets of a set if there is a one-to-one correspondence between the vertices of \( G \) and the subsets in \( S \) such that two vertices are adjacent iff their corresponding subsets in \( S \) have a non-empty intersection; \( S \) is a representation of \( G \). Intersection graphs are of interest in various domains such as computer science, genetics and ecology [RO,PE]. Intersection graphs of intervals on a line, subtrees on a tree and arcs on a circle are called interval, chordal and circular-arc graphs [GA3,GA2], respectively. An oriented graph \( G(V,E) \) is called transitive if it is acyclic and for every three vertices \( u,v,w \in V \), \( (u,v),(v,w) \in E \) implies \( (u,w) \in E \) [GRU,EPL]; its underlying undirected graph is called a comparability graph.

A family \( H \) of graphs is hereditary if \( G(V,E) \in H \) implies \( G(U) \in H \) for every \( U \subseteq V \); the families of intersection graphs are hereditary. We denote \( coH=\{coG \mid G \in H\} \). Consider a hereditary family \( H \) of graphs. A graph \( H(V,E) \) is called \( H \)-mixed [GA4] if its edge set can be partitioned into two disjoint subsets \( E_1, E_2 \) such that \( H(V,E_1) \in H, H(V,E_2) \) is transitive and for every three distinct vertices \( u,v,w \) if \( (u,v) \in E_2 \) and \( (v,w) \in E_1 \) then \( (u,w) \in E_1 \). The letter \( H \) is generic and can be replaced by names of specific families.

Gavril [GA4] defined two new families of intersection graphs: the interval-filament graphs and the subtree-filament graphs. The complements of interval-filament graphs are exactly the cointerval mixed graphs [GA4]. The family of interval-filament graphs contains the family of polygon-circle graphs which includes the circular-arc, the circle trapezoid, the circle and the chordal graphs [GA2,FMW,GA1,GA3,JK]. The complements of subtree-filament graphs are exactly the cochordal mixed graphs [GA4].

In the present paper we introduce a new family of intersection graphs, the 3D-interval-filament graphs, which are a generalization of the subtree-filament graphs. In Section 3 we prove that the family of complements of 3D-interval-filament graphs is
exactly the family of co(interval-filament) mixed graphs and every 3D-interval-filament graph has a representation by a family of piecewise linear filaments. In Section 4, we define in a similar way various families (Table 1) of 3D-filament graphs.

The importance of these families of intersection graphs is that they have various applications; for example the chordal graphs are used in many fields, more recently in analyses of phylogenetic data [PE,BESS]. These families are also important through their description as intersection graphs of 3D-interval-filaments since this description facilitates the discovery of efficient algorithms. For example, the algorithm on interval-filament (and subtree-filament) graphs for a maximum weight independent set [GA4] uses the transitive

<table>
<thead>
<tr>
<th>Family of filament graphs</th>
<th>Corresponding family of complements</th>
<th>Based on intersection graphs in the plane of:</th>
</tr>
</thead>
<tbody>
<tr>
<td>interval-filament graphs</td>
<td>cointerval mixed graphs</td>
<td>intervals on a line</td>
</tr>
<tr>
<td>3D-interval-filament graphs</td>
<td>co(interval-filament) mixed graphs</td>
<td>interval-filaments</td>
</tr>
<tr>
<td>subtree-filament graphs</td>
<td>cochordal mixed graphs</td>
<td>subtrees on a tree</td>
</tr>
<tr>
<td>3D-polygon-circle-filament graphs</td>
<td>co(polygon-circle) mixed graphs</td>
<td>polygonal-filaments</td>
</tr>
<tr>
<td>cocomparability-filament graphs</td>
<td>comparability mixed graphs</td>
<td>curves (functions) with endpoints on two parallel lines</td>
</tr>
<tr>
<td>permutation-filament graphs</td>
<td>permutation mixed graphs</td>
<td>segments with endpoints on two parallel lines</td>
</tr>
<tr>
<td>subtree-cactus-filament graphs</td>
<td>co(subtree-cactus) mixed graphs</td>
<td>subtrees on a cactus</td>
</tr>
<tr>
<td>3D-circular-arc-filament graphs</td>
<td>co(circular-arc-filament) mixed graphs</td>
<td>arc-filaments on a circle C, no two arcs covering C</td>
</tr>
</tbody>
</table>

Table 1: Various families of 3D-filament graphs.

inclusion relation $E_2$, the algorithm for a maximum weight induced matching [CA] uses the property that the union of two intersecting interval-filaments is an interval-filament, and the algorithm for a maximum weight induced path [GA5] uses the property that once inside the region delimited by an interval-filament $a$, an induced path cannot get out without intersecting $a$ again. An algorithm for maximum independent sets in polygon-circle graphs was not known before, but as intersection graphs of interval-filaments, such an algorithm is easy to describe [GA4]. Other polynomial time algorithms in interval-filament and subtree-filament graphs were given for finding maximum weight cliques [GA4] and holes and antiholes of given parity [GA6]. Polynomial time algorithms for maximum independent sets and cliques in subtree overlap graphs were given in [CS].
Gavril [GA4] described a polynomial time algorithm to find maximum weight cliques in $H$-mixed graphs when $H$ has such an algorithm. Thus, we can find maximum weight independent sets in 3D-interval-filament graphs since their complements are co(interval-filament) mixed graphs and interval-filament graphs have such an algorithm [GA4]. In Sections 5, 6 we describe polynomial time algorithms to find maximum weight holes of a given parity in 3D-interval-filament graphs, and antiholes of a given parity in interval-filament and subtree-filament graphs.

The above algorithms on the various families of 3D-interval-filament graphs do not require a representation by intersecting 3D-filaments, but only the representation in the plane, of the intersection graphs (Table 1) on which they are based, and this can be the most economical intersection representation. For example, the algorithms on subtree-filament graphs can use the representation of chordal graphs by subtrees on clique-trees. Some of the properties of the 3D-interval-filament graphs will be described in Section 7, which also includes Open Problems. A preliminary version [GA7] of this paper was written at DIMACS.

2. DEFINITIONS AND NOTATION

Consider a graph $G(V,E)$. By a path $p=(v_1,v_2),..., (v_{k-1},v_k)$ in $G$ we always mean a simple path, denoted $p(v_1,v_k)$; $p$ is an induced path if it has no chords. A hole $h=(v_1,v_2),..., (v_k,v_1)$ is a chordless cycle with four or more vertices, denoted $h(v_1,v_k)$; coh is called an antihole. A subpath of $h$, clockwise from $v_i$ to $v_j$, is denoted $h(v_i,v_j)$. A subset of $V$ is a clique if every two of its vertices are adjacent. Two subsets of a set intersect if they have a non-empty intersection; two subsets overlap if they intersect but none is contained in the other. A leaf of a tree is a vertex adjacent to exactly one other vertex.

In the 3D Euclidean space, consider the line $L$ defined by $y=z=0$ and the plane $PL$ defined by $z=0$; we assume that $L$ is drawn from left to right on a page representing $PL$. For an interval $[l,r] \subset L$, we define an interval-curve $c$ in $PL$ as a continuous function $c: [l,r] \rightarrow \mathbb{R}^+$ having $c(l)=c(r)=0$; $c(l)$ and $c(r)$ are the endpoints of $c$. Clearly, an interval-curve $c$ starts and ends at the endpoints of the interval $[l,r]$, and is contained between them. Let $\mathcal{f}$ be a family of interval-curves fulfilling that $\bigcup
olimits_{c \in \mathcal{f}} c$ is continuous: $a_{\mathcal{f}}=\bigcup
olimits_{c \in \mathcal{f}} c$ is called an interval-filament. A point $x \in L$ is called an endpoint of an interval-filament $a_{\mathcal{f}}$ if and only if it is an endpoint of exactly one interval-curve in $\mathcal{f}$. An interval-filament can be a
piece of (contained in) another interval-filament, for example $a_1 \subset a_2$ when $c_1 \cap c_2 \neq \emptyset$, $a_1 = c_1$ and $a_2 = c_1 \cup c_2$. By now, an interval-filament $a$ is continuous and its endpoints are in $L$, but its extreme points in $L$ may not be its endpoints, being contained in two of its interval-curves.

Consider a family $FI$ of interval-filaments in $PL$ above $L$. In Algorithm $ADJUST(FI)$ we show how to transform $FI$, without changing the intersection and containment relationships, into a family of interval-filaments fulfilling that no point $x \in L$ is an endpoint of two interval-filaments and the two extreme points of an interval-filament are also two of its endpoints.

**Algorithm $ADJUST(FI)$.** Let $H(V,F)$ be the graph defined by $V = \{ v | a(v) \in FI \}$ and $F = \{ (u,v) | a(u) \subset a(v) \}$. The graph $H(V,F)$ is transitive: we label a source vertex by $v_1$, eliminate it and continue labeling the remaining vertices with $v_2$ to $v_n$ by eliminating sources. We obtain an ordering of $V$ in which $(v_i,v_j) \in F$ implies $i < j$. Consider a point $x \in L$ which is an endpoint of more than one interval-filament, or is an extreme point of an interval-filament, without being one of its endpoints. Let $FI_x = \{ a(v) | a(v) \in FI, x \in a(v) \cap L \}$. On $L$, we denote $|FI_x|$ points $\{ y_i \}$ at the left of $x$, $|FI_x|$ points $\{ z_i \}$ at the right of $x$, at small distances from $x$ (such that the segments $[y_i,x]$, $[x,z_i]$ do not intersect interval-filaments in $FI-FI_x$), every pair $y_i, z_i$ corresponding to an interval-filament $a(v) \in FI_x$, the points ordered from $x$ to the left and right by the ordering of $V$. For each interval-filament $a(v) \in FI_x$ we add to $a(v)$ the two segments $[y_i,x]$, $[x,z_i]$; in Figure 1(a), on plane $PL$, we can see a family of interval-filaments adjusted in this way. By the construction, no new interval-filament intersections are introduced and the containment relationship does not change, since $(v_i,v_j) \in F$ implies $i < j$ and $[x_i,y_j] \subset [x_j,y_j]$. The newly defined family of interval-filaments fulfills that no two interval-filaments have a common endpoint, and every interval-filament is contained in $PL$ between two of its endpoints. Note that if every interval-filament has $O(|FI|)$ interval-curves, after $ADJUST(FI)$ it also has $O(|FI|)$ interval-curves.

Gavril [GA4] defined a new family of intersection graphs: Consider a family $I$ of intervals on a line $L$ and let $V = \{ v | i(v) \in I \}$ be a vertex set. In the plane $PL$ containing $L$, above $L$, we consider for each interval $i(v) \in I$ an interval-filament $a(v)$ with $l(i(v)), r(i(v))$ as extreme points on $L$ (plane $PL$ in Figure 1(a), and Figure 2(a)); $FI = \{ a(v) | i(v) \in I \}$ is a family of interval-filaments and its intersection graph $GA(V,E)$ is an interval-filament graph. Clearly, if two intervals are disjoint, their interval-filaments do not intersect. The complements of interval-filament graphs are exactly the cointerval mixed graphs [GA4].
Gavril [GA4] also defined the subtree-filament graphs which can be described as follows: Consider a tree $T$ in a plane and let $GA(V)$ be the intersection graph of a family $FI=\{a(v)|\ v\in V\}$ of subtrees of $T$; $GA$ is chordal [GA3]. The tree $T$ can be drawn as an interval-filament in the plane $PL$ with all its vertices on $L$, without intersections outside $L$ [JK,GA4], its edges being drawn as semicircles. The family of subtrees $FI$ becomes a family of interval-filaments in $PL$. We apply Algorithm $ADJUST(FI)$ to obtain a new family of subtrees with the same intersection and containment relationships, fulfilling that every subtree $a(v)$ is contained between two of its leaves (two of its endpoints on $L$) and no point of $L$ is a leaf of two subtrees. Let $PP$ be the continuous surface perpendicular to $PL$ such that $PP\cap PL=T$. For every $a(v)\in FI$, let $PP(a(v))$ denote the subsurface of $PP$ whose intersection with $PL$ is exactly $a(v)$. In $PP(a(v))$, above $PL$, we connect all the leaves of $a(v)$ with a function $f(v):a(v)\to \mathbb{R}^+$ called subtree-filament, such that if $a(u),a(v)$ overlap the two subtree-filaments $f(u),f(v)$ intersect, if $a(u),a(v)$ are disjoint, the two subtree-filaments $f(u),f(v)$ do not intersect, and if $a(u)\subset a(v)$, the two subtree-filaments $f(u),f(v)$ may or may not intersect. The intersection graph $G(V,E)$ of a family $FF=\{f(v)|\ v\in V\}$ of subtree-filaments is called a subtree-filament graph. The complements of subtree-filament graphs are exactly the cochordal mixed graphs [GA4].

We introduce now the 3D-interval-filament graphs, which are a generalization of the subtree-filament graphs. Let $GA(V)$ be the intersection graph of a family of interval-filaments $FI=\{a(v)|\ v\in V\}$ on a family of intervals $I$ on the line $L$ in the plane $PL$. We apply Algorithm $ADJUST(FI)$ to obtain a new family of interval-filaments with the same intersection and containment relationships, fulfilling that every interval-filament $a(v)$ is contained between two of its endpoints on $L$ and no point of $L$ is an endpoint of two interval-filaments. Let $PP$ be the family of surfaces perpendicular to $PL$ whose intersection with $PL$ is exactly the family of interval-filaments $FI$; if $\cup_{a\in FI} a$ is continuous, then $PP$ is a continuous surface. For every interval-filament $a\in FI$, let $PP(a)$ denote the continuous subsurface of $PP$ whose intersection with $PL$ is exactly $a$. In $PP(a(v))$, above $PL$, we connect all the endpoints of $a(v)$ by a function $f(v):a(v)\to \mathbb{R}^+$ called a 3D-interval-filament, such that if $a(u),a(v)$ overlap the two filaments $f(u),f(v)$ intersect, if $a(u),a(v)$ are disjoint, the two filaments $f(u),f(v)$ do not intersect, and if $a(u)\subset a(v)$, the two filaments $f(u),f(v)$ may or may not intersect (see Figure 1 for an example). Note that a 3D-interval-filament $f(v)$
does not include $a(v)$, but only its endpoints. The intersection graph $G(V,E)$ of a family $\mathcal{F} = \{f(v) | v \in \mathcal{V}\}$ of 3D-interval-filaments on $\mathcal{F}_1$, $\mathcal{L}$ is called a 3D-interval-filament graph. To every 3D-interval-filament $f(v)$ corresponds the region $s(v)$ in $PP(a(v))$ delimited by $f(v) \cup a(v)$. Therefore, two vertices $u, v$ in $G(V,E)$ are adjacent iff the regions $s(u), s(v)$ overlap, i.e., $PP(a(u)), PP(a(v))$ overlap or say $PP(a(u)) \subset PP(a(v))$ and the 3D-interval-filaments $f(u), f(v)$ intersect in $PP(a(u))$ (see $f(r)$ and $f(t)$ in Figure 1). The edge set $coE = \{(u,v) | f(u) \cap f(v) = \emptyset\}$ of $coG(V, coE)$ can be partitioned into two disjoint subsets $E_1 = \{(u,v) | a(u) \cap a(v) = \emptyset\}$ and $E_2 = \{(u,v) | a(u) \subset a(v) and f(u) \cap f(v) = \emptyset\}$. Clearly, $(u,v) \in E_1$ iff $s(u) \cap s(v) = \emptyset$, while $(u,v) \in E_2$ iff $s(u) \subset s(v)$. The graph $coG(V, E_2)$ is transitive since $(u,v), (v,w) \in E_2$ implies $a(u) \subset a(v) \subset a(w)$ and $f(u) \cap f(v) = \emptyset, f(v) \cap f(w) = \emptyset$, i.e., $s(u) \subset s(v) \subset s(w)$, hence $f(u) \cap f(w) = \emptyset$ and $(u,w) \in E_2$. The complement of $coG(V, E_1)$ is exactly $GA$ since

**Figure 1**: The 3D-interval-filament representation in (a) of the graph $G$ in (b) based on the interval-filament representation of $GA$ in the plane $PL$; $a(e) \subset a(v)$, $PP(a(e)) \subset PP(a(v))$, $a(r) \subset a(t)$, $PP(a(r)) \subset PP(a(t))$. 
(u,v) ∉ E1 iff a(u) ∩ a(v) ≠ ∅ iff s(u) ∩ s(v) ≠ ∅. Let u,v,w be three distinct vertices such that (w,v) ∈ E2 and (u,v) ∈ E1; then, a(w) ⊂ a(v) and a(u) ∩ a(v) = ∅, implying that a(w) ∩ a(u) = ∅ and (w,u) ∈ E1. Thus, coG(V,coE) is a coG-mixed graph where G is the family of interval-filament graphs. Therefore, the complements of 3D-interval-filament graphs are co(interval-filament) mixed graphs. For an intersection graph G(V,E) of a family FF of 3D-interval-filaments, we denote by i(v), a(v) and f(v) the interval, the interval-filament and the 3D-interval-filament corresponding to a vertex v of G.

For a given family G, the families of coG-mixed graphs and of co(G)-mixed) graphs are distinct. For example, when G is the family of cointerval graphs, the cocointerval mixed graphs are the interval mixed graphs, while the co(cointerval mixed) graphs are the interval-filament graphs. When we discuss an H-mixed graph H(V,E1,E2), we assume that the partition E1,E2 is given, and when H(V,E1) is the complement of an intersection graph, we assume that the intersection representation is also given.

3. 3D-INTERVAL-FILAMENT GRAPHS

Consider an interval-filament graph GA and a graph G whose complement coG(V,E1,E2) is a co(interval-filament) mixed graph having co[coG(V,E1)] = GA(V,coE1). In this Section we will prove that G is a 3D-interval-filament graph and will show how to construct for G a representation by a family of piecewise linear 3D-interval-filaments. Not every representation FI of GA can be used as a base to construct a family of 3D-interval-filaments representing G, because in such a representation, if (u,v) ∈ E2 then we must have s(u) ⊂ s(v) and this is possible only if PP(a(u)) ⊂ PP(a(v)) and a(u) ⊂ a(v): if a(u),a(v) in FI intersect, but are not contained one in another, the corresponding 3D-interval-filaments f(u),f(v) cannot be constructed. In Lemma 1 we prove that for each such pair (u,v) ∈ E2, we can replace a(v) by a(u) ∪ a(v) without changing the intersection relationship in FI: after the replacement, PP(a(u)) ⊂ PP(a(v)), a(u) ⊂ a(v) and we will be able (Lemma 2) to construct the 3D-interval-filaments f(v),f(u).

Lemma 1. Let coG(V,E1,E2) be a coG-mixed graph such that GA(V,coE1) = co[coG(V,E1)] is an intersection graph of a family FI={a(v) | v ∈ V} of interval-filaments on a line L.

a) If (u,v) ∈ E2, then a(u) ∩ a(v) = ∅ and every interval-filament in FI which intersects a(u), intersects also a(v).
b) If \((u,v)\in E_2\), then \(a(u) \cup a(v)\) is an interval-filament and we can replace \(a(v)\) in \(FI\) by \(a(u) \cup a(v)\) without changing the intersection relationship.

c) There exists an intersection representation of \(GA(V,coE_1)\) by a family of interval-filaments such that \((u,v)\in E_2\) implies \(a(u) \supseteq a(v)\) and \(PP(a(u)) \subseteq PP(a(v))\).

**Proof.**

a) Assume that \((u,v)\in E_2\) in \(coG(V,E_1,E_2)\); then \((u,v)\in coE_1\), \(u,v\) are adjacent in \(GA(V,coE_1)\) and \(a(u),a(v)\) intersect in \(FI\). Assume that there is an interval-filament \(a(w)\in FI\) which intersects \(a(u)\) but not \(a(v)\), that is, \((w,u)\in coE_1\) and \((w,v)\notin coE_1\). But \((w,v)\in E_1\) and \((u,v)\in E_2\) implies, by the definition of \(coG\)-mixed graphs, that \((w,u)\in E_1\) contradicting the assumption that \((w,u)\in coE_1\). Thus, if \(a(w) \cap a(u) \neq \emptyset\) then \(a(w) \cap a(v) \neq \emptyset\).

b) Since \((u,v)\in E_2\), it follows that \((u,v)\in coE_1\), \(a(u) \cap a(v) \neq \emptyset\) and \(i(u) \cap i(v) \neq \emptyset\), therefore \(a(u) \cup a(v)\) is continuous and is an union of interval-curves within the endpoints of \(i(u) \cup i(v)\), that is, \(a(u) \cup a(v)\) is an interval-filament. Therefore, by (a), we can replace \(a(v)\) in \(FI\) by \(a(u) \cup a(v)\) without the intersection relationship being changed.

c) The graph \(coG(E_2)\) is transitive, and by going from its sources to its sinks and replacing \(a(v)\) by \(a(u) \cup a(v)\) whenever \((u,v)\in E_2\), we obtain an intersection representation of \(GA(V,coE_1)\) in which \((u,v)\in E_2\) implies \(PP(a(u)) \subseteq PP(a(v))\) and \(a(u) \supseteq a(v)\).

Lemma 1 is true also for interval-filaments of specific types which are preserved under union of intersecting elements: the union of two intervals on a line is an interval, the union of two subtrees of a tree is a subtree and the union of two interval-filaments is an interval-filament. The union of two intersecting polygons in a circle may not be a polygon, but since the polygon-circle graphs are equivalent to the spider graphs, the union of two polygons can be replaced by the convex hull of the two polygons, without changing the intersection relationship [CA]. A graph is a polygon-circle graph iff it is an intersection graph of a family of polygonal filaments on a line [GA4], implying that the polygon-circle graphs are interval-filament graphs; using Cameron’s result, a representation by polygonal filaments fulfilling Lemma 1 can be obtained.

We characterize now the interval-filament and the 3D-interval-filament graphs.

**Theorem 2** [GA4]. A graph is an interval-filament graph iff its complement is a cointerval mixed graph. Every interval-filament graph \(G\) has a representation by a family of piecewise linear filaments each filament with at most \(O(|V|)\) linear pieces and intersection points.

**Proof.** The complement of an interval-filament graph is a cointerval mixed graph [GA4].
Conversely, let the complement $coG(V,E1,E2)$ of a graph $G$ be a cointerval mixed graph such that $coG(V,E2)$ is a transitive graph and $coG(V,E1)$ is the complement of an interval graph with an interval representation $I$ on a line $L$. By Lemma 1, we can assume that $(u,v)\in E2$ implies $i(u)\subseteq i(v)$; we apply $ADJUST(I)$ such that $i(u),i(v)$ have no common endpoints.

Now (Figure 2(a)), to each interval $i(u) \in I$ we add, above $L$, a triangular hook $a(u)$ connecting its two endpoints, such that if $i(u) \subseteq i(v)$ the two triangular hooks $a(u),a(v)$ do not intersect. Consider two adjacent vertices $u,v$ of $G$. If $i(u),i(v)$ overlap, then $a(u),a(v)$ intersect. When $i(u) \subseteq i(v)$, to obtain an intersection between $a(u),a(v)$, we pick a point on $a(u)$ and stretch it upwards (perpendicular to $L$), in $PL$, into a spike to reach above and intersect $a(v)$. Also, for every triangular hook $a(w)$ we meet on the way such that $(u,w) \in E2$ (implying $i(u) \subseteq i(w)$) we stretch it also in a spike, such that no intersections between the spikes of $a(u),a(w)$ (and the spikes of two distinct $a(w)$'s) occur; thus, we obtain that $a(v) \cap a(w) \neq \emptyset$. Let us prove that $w,v$ are adjacent in $G$. Assume that the vertices $w,v$ are adjacent in $coG(V,E1,E2)$. Since $i(w) \cap i(w) \neq \emptyset$, it follows that $(w,v) \notin E1$; thus, $(w,v) \in E2$ and $i(v) \subseteq i(w)$ or $i(w) \subseteq i(v)$. By the above construction, the spike of $a(u)$ meets $a(w)$ before $a(v)$ implying that $i(w) \subseteq i(v)$. Thus $(w,v) \in E2$ and by the transitivity of $E2$, $(u,w),(w,v) \in E2$ implies $(u,v) \in E2$, in contradiction to the assumption that $u,v$ are not adjacent in $coG(V,E1,E2)$. By the construction, every filament $a(v)$ is piecewise linear containing at most $O(|V|)$ pieces and intersection points.

Let us prove that $G$ is the intersection graph of the family of interval-filaments \{a(u)\}, with the spikes. Consider two adjacent vertices $u,v$ of $G$; hence $(u,v) \notin E1 \cup E2$. 

Figure 2: A representation (a) by a family of piecewise linear filaments of the interval-filament graph in (b).
implying \((u,v)\in\text{co}E1\) and \(i(u)\cap i(v)\neq\emptyset\). If \(i(u), i(v)\) overlap, then \(a(u)\cap a(v)\neq\emptyset\). If \(i(u)\subset i(v)\), then, by the above construction \(a(u)\cap a(v)\neq\emptyset\). Consider now two non-adjacent vertices \(u, v\) of \(G\); thus \((u,v)\in E1\cup E2\). If \((u,v)\in E1\), i.e., \(i(u)\cap i(v)=\emptyset\), then by the above construction \(a(u), a(v)\) do not intersect. Otherwise, \((u,v)\in E2\) and \(i(u)\subset i(v)\). Assume that \(a(u)\cap a(v)\neq\emptyset\). This can appear during the construction of the spike of a filament \(a(w)\) to a filament \(a(t)\), when \((w,u)\in E2\) and \(i(w)\subset i(u)\). By the transitivity of \(E2\), we have \((w,v)\in E2\), thus \(a(v)\) was also stretched together with \(a(w), a(u)\) to have a spike above the spikes of \(a(u), a(w)\) to intersect \(a(t)\), implying that \(a(u)\) and \(a(v)\) do not intersect, which is a contradiction. Therefore, \(G\) is the intersection graph of the family \(\{a(u)\}\) of interval-filaments.

**Theorem 3.** A graph is a 3D-interval-filament graph iff its complement is a co(interval-filament) mixed graph. Every 3D-interval-filament graph has a representation by a family of piecewise linear filaments, each filament with \(O(|V|^2)\) linear pieces.

**Proof.** As proved in Section 2, the complement of a 3D-interval-filament graph is a co(interval-filament) mixed graph. Conversely, let the complement \(\text{co}G(V,E1,E2)\) of a graph \(G\) be a co(interval-filament) mixed graph such that \(\text{co}G(V,E2)\) is a transitive graph and \(\text{co}G(V,E1)\) is the complement of an interval-filament graph \(GA\) with a representation as an intersection graph of a family \(FI=\{a(v)| v\in V\}\) of interval-filaments on a line \(L\) in a plane \(PL\). By Theorem 2, we can assume that each interval-filament has \(O(|V|)\) linear pieces and intersection points. By Lemma 1, we add to every \(a(v)\) the interval-filaments \(a(u)\) for which \((u,v)\in E2\); every new \(a(v)\) has \(O(|V|^2)\) linear pieces. Thus, we can assume that \((u,v)\in E2\) implies \(a(u)\subset a(v)\). We apply now \(\text{ADJUST}(FI)\). The result (Figure 1(a), plane \(PL\)) is a family of piecewise linear interval-filaments \(FI\) fulfilling that \((u,v)\in E2\) implies \(a(u)\subset a(v)\), no two interval-filaments have a common endpoint, and every interval-filament is contained in \(PL\) between two of its endpoints. The graph \(\text{co}G(V,E2)\) is transitive: let \(V_i\) be the set of sources of \(\text{co}G(V,E2)\) and recursively let \(V_j\) be the set of sources of \(\text{co}G(V-(V_1 \cup \ldots \cup V_{i-1}),E2)\). The subgraphs \(\text{co}G(V_j,E1,E2)\) have no \(E2\) edges, thus the subgraphs \(G(V_j)\) are interval-filament graphs. Consider two adjacent vertices \(u, v\) in \(G\), such that \(u \in V_i, v \in V_j\) and \(i<j\). The interval-filaments \(a(u), a(v)\) must intersect but their intersection may not contain a segment, needed below to construct the intersection between \(f(u)\) and \(f(v)\); in such a case let \(g \in PL\) be a point in \(a(u)\cap a(v)\) (Figure 1(a), plane \(PL\)). We expand \(g\) into a small segment \(\text{seg}_{u,v}=[g,d]\), in \(PL\) parallel to \(L\), and redefine every interval-filament containing \(g\) by letting it enter \(g\) from its left, as before, continuing on \([g,d]\) and continuing from \(d\) to its right, as it did before from
g; the intersection relationship in $FI$ does not change. Doing this for every such pair $u,v$, we obtain a representation $FI$ fulfilling that for every two adjacent vertices $u,v$ of $G$, when $u\in V_i$, $v\in V_j$ and $i<j$, the interval filaments $a(u),a(v)$ have a segment $seg_{u,v}$ in common.

Let $PP$ be the family of surfaces perpendicular to $PL$ whose intersection with $PL$ is exactly the family of interval-filaments $FI$. For every $V_j$ we construct above $PL$ a plane $PQ_j$ parallel to $PL$, the planes $PQ_j$ ordered from $PL$ by $j$. For every $j$ and every $v\in V_j$, we consider the intersection between $PQ_j$ and $PP(a(v))$ which is in fact the projection $b(v)$ of $a(v)$ on $PQ_j$. We connect in $PP(a(v))$ the endpoints of $b(v)$ to the endpoints of $a(v)$, with linear segments at a slight angle, denoting by $f(v)$ the $3D$-interval-filament composed of $b(v)$ and the connecting segments (Figure 1). By the construction, for two $3D$-interval-filaments $f(u),f(v)$, if $(u,v)\in E_2$, then the plane $PQ_i$ containing $u$ is between the planes $PQ_j$ containing $v$ and $PL$, $a(u)\subset a(v)$ and $f(u)$ is contained in the region $s(v)$ delimited by $f(v)\cup a(v)$.

Now, consider a pair of adjacent vertices $u,v$ of $G$ such that $f(u),f(v)$ do not intersect yet; this happens when $u\in V_i$, $v\in V_j$ and $i<j$ (filaments $f(u)$, $f(v)$ or filaments $f(t)$, $f(v)$ in Figure 1(a)). In this case either $s(u)\subset s(v)$ or $s(u),s(v)$ overlap, and the intersection of $a(u),a(v)$ contains a segment $seg_{u,v}$, but on the common surface $PP(a(u))\cap PP(a(v))$ $3D$-interval-filament $f(v)$ is above $3D$-interval-filament $f(u)$. For every such pair $f(u),f(v)$, one pair at a time, we pick a point in $f(u)$, on $PP(seg_{u,v})$, and stretch it in a spike to reach above and intersect $f(v)$ (filaments $f(u),f(v)$ and $f(t),f(v)$ in Figure 1). Also, for every filament $f(w)$ met on the way such that $(u,w)\in E_2$ (implying $s(u)\subset s(w)$) we stretch it also in a spike (filaments $f(v),f(u),f(w)$ in Figure 3), such that no intersections between the spikes of $f(u),f(w)$ (and the spikes of two distinct $f(w)$'s) occur; thus, we obtain that $f(v)\cap f(w)\neq \emptyset$. Let us prove that $w,v$ are adjacent in $G$. Assume that the vertices $w,v$ are adjacent in $coG(V,E_1,E_2)$. Since $a(w)\cap a(v)\neq \emptyset$, it follows that $(w,v)\notin E_1$; thus, $(w,v)\in E_2$ and $a(v)\subset a(w)$ or $a(w)\subset a(v)$. By the construction, the spike of $f(u)$ meets $f(w)$ before $f(v)$ implying that $a(w)\subset a(v)$. Hence $(w,v)\in E_2$ and by the transitivity of $E_2$, $(u,w),(w,v)\in E_2$ implies $(u,v)\in E_2$, in contradiction to the assumption that $u,v$ are not adjacent in $coG(V,E_1,E_2)$. By the construction, every filament $f(v)$ is piecewise linear containing at most $O(|V|^2)$ pieces.

Let us prove that $G(V,E)$ is the intersection graph of the above family of $3D$-interval-filaments $\{f(u)\}$ with the spikes. Consider two adjacent vertices $u,v$ of $G$; hence $(u,v)\notin E_1\cup E_2$, implying $(u,v)\in coE_1$. Thus, both when $a(u),a(v)$ overlap or $a(u)\subset a(v)$, the $3D$-interval-filaments $f(u),f(v)$ intersect, by the construction of the spikes. Consider now
two non-adjacent vertices \( u, v \) of \( G \); thus \((u,v) \in E1 \cup E2 \). If \((u,v) \in E1 \), i.e., \( a(u) \cap a(v) = \emptyset \), then by the above construction \( f(u), f(v) \) do not intersect. Otherwise, \((u,v) \in E2 \) and \( a(u) \subset a(v) \).

Assume that \( f(u) \cap f(v) \neq \emptyset \). This could appear only during the construction of the spike of a filament \( f(w) \) to a filament \( f(t) \), when \((w,u) \in E2 \) and \( a(w) \subset a(u) \). But, by the transitivity of \( E2 \), we have \((w,v) \in E2 \), thus \( f(v) \) was also stretched together with \( f(w) \) and \( f(u) \) to have a spike above the spikes of \( f(u) \) and \( f(w) \) to intersect \( f(t) \), implying that \( f(u) \) and \( f(v) \) do not intersect, which is a contradiction. Therefore, \( G \) is the intersection graph of the family \{\( f(u) \)\} of 3D-interval-filaments.

4. OTHER FAMILIES OF 3D-FILAMENT GRAPHS

In this Section we discuss various families (see Table 1) of intersection graphs of 3D-filaments and characterize them as complements of \( coG \)-mixed graphs.

Reference [GRU] proved without details, that a cocomparability graph is an intersection graph of piecewise linear curves between two parallel lines. In fact this can be seen as follows: Given a transitive graph \( coG \), we label a source vertex by \( v_1 \), eliminate it and continue labeling the remaining vertices with \( v_2 \) to \( v_n \) by eliminating sources. We arrange a family of parallel linear segments between two vertical parallel lines, and label the linear segments from the lowest to the highest with \( a(v_1) \) to \( a(v_n) \). Consider segment \( a(v_j) \). We pick a point on \( a(v_j) \) and stretch it into a spike, above the highest segment labeled \( a(v_i) \) fulfilling \((v_i,v_j) \in G \). Also, for every segment \( a(v_j) \) met on the way, fulfilling \((v_i,v_j) \notin G \), i.e. \((v_i,v_j) \notin coG \), we stretch also \( a(v_j) \) in a spike, such that no intersections between the spikes of \( a(v_i) \) and \( a(v_j) \) occurs. The intersections added are between \( a(v_i) \) and \( a(v_j) \), where \( a(v_i) \) is not stretched with a spike since \((v_i,v_j) \in G \). Assume \((v_k,v_j) \notin G \). Therefore \((v_j,v_k) \in coG \) and \((v_j,v_k) \notin coG \), implying \((v_i,v_k) \in coG \) by transitivity, in contradiction to the assumption that \((v_i,v_k) \notin G \).

Consider a cocomparability graph \( G \) and its representation as a family of curves between two parallel vertical segments \( L1, L2 \) in a plane \( PL \), each curve starting in \( L1 \) and ending in \( L2 \). We define a filament as the union of curves in a family \( ff \) fulfilling that \( \cup_{C \in ff} C \) is continuous; a filament can be a piece of (contained in) another filament. Therefore the union of two intersecting filaments is also a filament. For two intersecting filaments \( a(u), a(v) \), if every filament intersecting \( a(u) \) intersects \( a(v) \), we replace \( a(v) \) by \( a(u) \cup a(v) \); the intersection relationship does not change and still we have a representation by a family \( FI \).
of filaments ($FI$ can be transformed into a family of interval-filaments by connecting two opposite endpoints of $L1$ or $L2$, and straightening $L1,L2$ into a line $L$). In fact, every filament delimitates a continuous region between $L1,L2$ such that $G$ is also the intersection graph of these regions. We apply Algorithm $\text{ADJUST}(FI)$, putting the segments added by $\text{ADJUST}(FI)$ on $L1,L2$. Perpendicular to $PL$, we consider a family of surfaces $PP$ whose intersection with $PL$ is the set of above filaments. For every filament $a(v)$, we connect on the surface $PP(a(v))$ all the endpoints of $a(v)$ by a $3D$-filament $f(v)$ such that if $a(u),a(v)$ overlap the two filaments intersect, if $a(u),a(v)$ are disjoint, the two filaments do not intersect and if $a(u) \subset a(v)$, the two filaments $f(u),f(v)$ may or may not intersect (the proof of Theorem 3 shows how to obtain this). These filaments are called cocomparability-filaments and their intersection graphs are called cocomparability-filament graphs. Similarly for permutation graphs: A graph $G(V,E)$ is a permutation graph [EPL] iff both $G$ and $coG$ are comparability graphs iff $G$ is an intersection graph of linear segments between two vertical parallel segment $L1,L2$. We define a filament as the union of linear segments in a family $ff$ fulfilling that $\cup c \in ff$ is continuous. For two intersecting filaments $a(u),a(v)$, if every filament intersecting $a(u)$ intersects $a(v)$, we replace $a(v)$ by $a(u) \cup a(v)$ without changing the intersection relationship. Note that every such filament delimitates a continuous region between $L1,L2$ such that $G$ is also the intersection graph of these regions. We can define $3D$-filament graphs for permutation graphs, called permutation-filament graphs, in the same way we defined for cocomparability graphs; copermutation graphs are permutation graphs [EPL]. A similar family of $3D$-filaments can be defined using families of polygonal filaments on a line obtained from polygons in a circle [GA4]. Note that every cointerval mixed graph is a comparability mixed graph and every comparability mixed graph is a co(interval-filament) mixed graph.

Consider [GA4] a circular-arc-filament graph $G(V,E)$ of a family $\{a(v)\}$ of arc-filaments based on a family of arcs in a circle such that no two arcs cover the circle and $N_G(u) \subseteq N_G(v)$ implies $a(u) \subset a(v)$. We can define $3D$-circular-arc-filaments and intersection graphs in a similar way to the $3D$-interval-filaments and intersection graphs.

As in Theorem 3, it can be proved that (Table 1):

**Corollary 4.** A graph is a cocomparability-filament (permutation-filament, $3D$-polygon-circle filament, $3D$-circular-arc-filament) graph iff its complement is a comparability (permutation, co(polygon-circle), co(circular-arc-filament), respectively)
mixed graph. Every such graph has a representation by a family of piecewise linear filaments, each filament with at most \( O(|V|^2) \) linear pieces.

Families of 3D-interval-filaments representing subtree-filament graphs can be constructed as in Theorem 3 (according to the model in Section 2), but we give here a more detailed characterization and proof of the construction method described in [GA4].

**Theorem 5.** A graph is a subtree-filament graph iff its complement is a cochordal mixed graph. Every subtree-filament graph \( G(V,E) \) has a representation by a family of piecewise linear filaments, each filament with at most \( O(|V|) \) linear pieces.

**Proof.** As proved in Section 2, the complement of a subtree-filament graph is a cochordal mixed graph.

Conversely, let the complement \( coG(V,E_1,E_2) \) of a graph \( G(V,E) \) be a cochordal mixed graph such that \( coG(V,E_2) \) is a transitive graph and \( coG(V,E_1) \) is the complement of a chordal graph \( GA(V,coE_1) \). A chordal graph \( GA(V,coE_1) \) has at most \(|V|\) maximal cliques. Also [GA3], there exists a tree \( T \) in a plane \( PL \), whose vertices are in a one-to one correspondence with the maximal cliques of \( GA(V,coE_1) \), such that for every \( v \in V \), the family \( C_v \) of maximal cliques containing \( v \) defines a subtree \( a(v) \) of \( T \); \( GA(V,coE_1) \) is the intersection graph of \( FI=\{a(v) \mid v \in V\} \). By the definition of \( T \), if \((u,v)\in E_2\) then \( C_u \subseteq C_v \), by Lemma 1(a) and \( a(u) \subseteq a(v) \). We add to every vertex \( x \) of \( T \) a branch \((x,y)\) and for every subtree \( a(v) \) containing \( x \) we insert a point \( y_v \) between \( x \) and \( y \), such that if \( a(u) \subseteq a(v) \) or \( a(u)=a(v) \), then \( y_u \) is closer to \( x \) than \( y_v \). We add to \( T \) the path from \( x \) defined by the points \( y_v \) on \((x,y)\) and we add to every \( a(v) \) the subpath from \( x \) to \( y_v \); the leaves of every subtree \( a(v) \in FI \) are now the points \( \{y_v,i\} \) added above. In this way, we obtain a family \( FI \) of subtrees of a tree \( T \) whose intersection graph is \( GA \), such that: a) no two subtrees have a common leaf; b) if \((u,v)\in E_2\) then \( a(u) \subseteq a(v) \); c) the intersection of the subtrees corresponding to a maximal clique \( C \) of \( GA(V,coE_1) \) contains an edge \( seg_C \) of \( T \).

Let \( PP \) be a surface perpendicular to \( PL \) whose intersection with \( PL \) is exactly \( T \). The graph \( coG(V,E_2) \) is transitive: let \( V_1 \) be the set of sources of \( coG(V,E_2) \) and recursively let \( V_j \) be the set of sources of \( coG(V-(V_1\cup \ldots \cup V_{j-1}),E_2) \). The subgraphs \( coG(V_j,E_1,E_2) \) have no \( E_2 \) edges, thus the subgraphs \( G(V_j) \) are chordal graphs. Now, for every \( V_j \) we construct above \( PL \) a plane \( PQ_j \) parallel to \( PL \), the planes \( PQ_j \) ordered from \( PL \) by \( j \). For every \( j \) and every \( v \in V_j \), we consider the intersection between \( PQ_j \) and \( PP(a(v)) \) which is in fact the projection \( b(v) \) of \( a(v) \) on \( PQ_j \). We connect in \( PP(a(v)) \) the endpoints of \( b(v) \) to the endpoints of \( a(v) \) with linear segments at a slight angle, denoting by \( f(v) \) the subtree-
filament composed of $b(v)$ and the connecting linear segments (Figure 3). By the construction, for two subtree-filaments $f(u), f(v)$, if $(u,v)\in E_2$, then the plane $PQ_i$ containing $b(u)$ is between the planes $PQ_j$ containing $b(v)$ and $PL$, $a(u) \subset a(v)$ and $f(u)$ is contained in the region $s(v)$ delimited by $f(v) \cup a(v)$.

Now, consider a pair of adjacent vertices $u, v$ of $G$, $u, v \in C$, such that $f(u), f(v)$ do not intersect yet; this can happen when $s(u) \subset s(v)$ or when $s(u), s(v)$ overlap and on their common surface $PP(a(u)) \cap PP(a(v))$, the subtree-filament $f(v)$ passes above the subtree-filament $f(u)$. For every such pair $f(u), f(v)$, one pair at a time, we pick a point in $f(u)$, on $PP(seg_C)$ and stretch it in a spike to reach above and intersect $f(v)$. Also, for every filament $f(w)$ met on the way such that $(u, w) \in E_2$ (implying $s(u) \subset s(w)$) we stretch it also in a spike, such that no intersections between the spikes of $f(u), f(w)$ (and the spikes of two distinct $f(w)$'s) occur; thus, we obtain that $f(v) \cap f(w) \neq \emptyset$. By the construction, every $f(v)$ is piecewise linear containing at most $O(|V|)$ pieces. The proof that $G(V, E)$ is the intersection graph of the family of subtree-filaments $\{f(u)\}$, with the spikes, is similar to the one of Theorem 3. □

![Figure 3](image.jpg)

**Figure 3:** A representation (a) by subtree-filaments of the graph $G$ in (b).

### 5. HOLES OF GIVEN PARITY IN 3D-INTERVAL-FILAMENT GRAPHS
Consider a hereditary family of graphs $G$, having a polynomial time algorithm to find an induced path of a given parity between two vertices; this implies that $G$ also has a polynomial time algorithm to find an induced hole of a given parity. Consider a graph $G(V,E)$ such that $\text{co}G(V,E_1,E_2)$ is a $\text{co}G$-mixed graph; denote by $G(A,V,\text{co}E_1)$ the complement of $\text{co}G(V,E_1)$. When $G$ is a 3D-interval-filament graph, $G(A,V,\text{co}E_1)$ is an interval-filament graph.

**Lemma 7.** Let $p(v_i,v_k)$ be an induced path in $G$ and consider some $i$, $1 \leq i \leq k-2$. If $(v_i,v_k) \in E_2$, then for every $j$, $1 \leq j \leq k-2$, we have $(v_j,v_k) \in E_2$.

**Proof.** Assume that $(v_i,v_k) \in E_2$. By the definition of the $\text{co}G$-mixed graphs, we cannot have $(v_{i+1},v_k) \in E_1$, since this would imply $(v_i,v_{i+1}) \in E_1$. Thus, $(v_{i+1},v_k) \in E_2$; and so on, for every vertex $v_j$, $1 \leq j \leq k-2$, to the right and left of $v_i$ in $p$. \[\square\]

**Lemma 8.** Any hole $h(v_1,v_k)$ of $G$, which is not a hole of $G(A,V,\text{co}E_1)$, has two non-adjacent vertices $v_i,v_j$ such that $(v_i,v_j) \in E_2$ and has no three vertices $v_i,v_j,v_k$ such that $(v_i,v_j),(v_j,v_k) \in E_2$.

**Proof.** If $h$ is not a hole of $G(A,V,\text{co}E_1)$, then $h$ has two non-adjacent vertices $v_i,v_j$ such that $(v_i,v_j) \in E_2$. Assume that $h$ has three vertices $v_i,v_j,v_k$ such that $(v_i,v_j),(v_j,v_k) \in E_2$; w.l.o.g. assume that $i<k<j$. But, Lemma 7 applied to $(v_i,v_j) \in E_2$ and the path $p(v_i,v_j)$ implies that for every $v_r \in p(v_i,v_{j-2})$, we have $(v_r,v_j) \in E_2$, hence $(v_k,v_j) \in E_2$. \[\square\]

**Lemma 9.** Consider a hole $h(v_1,v_k)$ of $G$. Then, either $h$ is a hole of $G(A,V,\text{co}E_1)$ or $h$ has a vertex $v_i$ such that for every $v_j \in h(v_{i+2},v_{i-2})$, we have $(v_j,v_i) \in E_2$ and $h(v_{i+2},v_{i-2})$ is an induced path of $G(A,V,\text{co}E_1)$.

**Proof.** Assume that $h$ is not a hole of $G(A,V,\text{co}E_1)$; hence $h$ has two non-adjacent vertices $v_i,v_j$ such that $(v_i,v_j) \in E_2$. Thus, by Lemma 7 applied to the path $h(v_{i+2},v_j)$, for every $v_j \in h(v_{i+2},v_{i-2})$, we have $(v_j,v_i) \in E_2$. By Lemma 8, there are no $E_2$ edges between non-adjacent vertices in $h(v_{i+2},v_{i-2})$, thus $h(v_{i+2},v_{i-2})$ is an induced path of $G(A,V,\text{co}E_1)$. \[\square\]

**Lemma 10.** A hole $h$ of $G(A,V,\text{co}E_1)$ has no $E_2$ edges and is a hole of $G$. An induced path $p(v_i,v_k)$ of $G(A,V,\text{co}E_1)$ can have only its first and last edges in $E_2$: $(v_i,v_2),(v_k,v_{k-1}) \in E_2$.

**Proof.** Let $h(v_i,v_k)$ be a hole of $G(A,V,\text{co}E_1)$ and assume $(v_i,v_2),(v_k,v_{k-1}) \in E_2$. Since $v_2,v_k$ are not adjacent in $G(A,V,\text{co}E_1)$ it follows that $(v_2,v_k) \in E_1$, and by the definition of $\text{co}G$-mixed graphs it follows that $(v_1,v_2),(v_2,v_k) \in E_1$, contradicting the fact that $h(v_i,v_k)$ is a hole of $G(A,V,\text{co}E_1)$. Therefore $h$ has no $E_2$ edges and is a hole of $G$.\[\square\]
Let \( p(v_1, v_k) \) be an induced path of \( GA(V, coE_1) \). If \((v_i, v_{i+1}) \in E_2, 2 \leq i \leq k-1\), then \((v_i, v_{i+1}) \in E_1\), since \( v_{i-1}, v_{i+1} \) are not adjacent in \( GA(V, coE_1) \), implying \((v_i, v_{i-1}) \in E_1\), contradicting the fact that \( p(v_1, v_k) \) is an induced path of \( GA(V, coE_1) \). Similarly, if \((v_{i+1}, v_i) \in E_2, 1 \leq i \leq k-2\). Therefore \( p(v_1, v_k) \) can have only its first and last edges in \( E_2 \), with the orientation \((v_1, v_2), (v_k, v_{k-1}) \in E_2\). □

The algorithm to find a hole \( h(v_1, v_k) \) of a given parity in the complement \( G \) of a \( coG \)-mixed graph \( coG(V, E_1, E_2) \), when the family of \( GA(V, coE_1) \) graphs has a polynomial time algorithm to find an induced path of a given parity between two vertices, works as follows: By Lemmas 9, 10, either \( h \) is a hole of \( GA(V, coE_1) \), to be found directly, or \( h \) has a vertex \( v_i \) such that for every \( v_j \in h(v_{i+2}, v_{i-2}) \), \((v_j, v_i) \in E_2\) and \( h(v_{i+2}, v_{i-2}) \) is an induced path of \( GA(V, coE_1) \). We take every vertex of \( G \) as candidate for \( v_i \) of \( h \), every two non-adjacent vertices in \( NG(v_i) \) as candidates for \( v_{i-1}, v_{i+1} \) and every \( v \in NG(v_{i-1})-NG[v_i] \), \( w \in NG(v_{i+1})-NG[v_i] \) as candidates for \( v_{i-2}, v_{i+2} \). In \( GA((V-NG[v_i, v_{i-1}, v_{i+1}]) \cup \{v_{i-2}, v_{i+2}\}, coE_1) \) we find an induced path of the needed parity from \( v_{i+2} \) to \( v_{i-2} \) requesting that its first and last edges \((v_{i+2}, v_{i+3})\) and \((v_{i-3}, v_{i-2})\) are in \( G \) and not in \( E_2 \).

The above algorithm can be adjusted to find maximum weight holes of a given parity, when the family \( GA(V, coE_1) \) has a polynomial time algorithm to find a maximum weight induced path of a given parity between two vertices. The algorithms work in time \( O(|V|^5 |V|^c) \) where \( O(|V|^c) \) is the time needed to find a maximum weight induced path of a given parity in \( GA(V, coE_1) \).

When \( G \) is a 3D-interval-filament graph, \( GA(V, coE_1) \) is an interval-filament graph. By Lemmas 9, 10, a hole \( h(v_1, v_k) \) of \( G \), either is a hole of \( GA(V, coE_1) \), to be found directly, or \( h \) has a vertex \( v_i \) such that for every \( v_j \in h(v_{i+2}, v_{i-2}) \), \((v_j, v_i) \in E_2 \) (that is \( f(v_j) \cap f(v_i) = \emptyset \), \( a(v_j) \subset a(v_i) \)) and \( h(v_{i+2}, v_{i-2}) \) is an induced path of \( GA(V, coE_1) \) with no first and last edge in \( E_2 \). Since the family of interval-filament graphs has an \( O(|V|^{12}) \) polynomial time algorithm to find a maximum weight induced path of a given parity between two vertices [GA5], the algorithm finds a maximum weight hole of a given parity in 3D-interval-filament graphs in \( O(|V|^7) \) time.

When \( G \) is an interval-filament graph, \( GA(V, coE_1) \) is an interval graph; interval graphs have no holes. Thus, by Lemmas 9, 10 (see also [GA6]), a hole \( h(v_1, v_k) \) of \( G \) has a vertex \( v_i \) such that for every \( v_j \in h(v_{i+2}, v_{i-2}) \) there exists \( a(v_j), a(v_i) = \emptyset \), \( i(v_j), i(v_i) \subset i(v_j) \), and \( h(v_{i+2}, v_{i-2}) \) is an induced path of \( GA(V, coE_1) \) with \( i(v_{i+2}) \subset i(v_{i+3}) \) and \( i(v_{i-2}) \subset i(v_{i-3}) \). A maximum induced path of a given parity in an interval graph can be found in \( O(|V|) \) time.
[GA6], hence a maximum weight induced hole of a given parity in $G$ can be found in $O(|V|^6)$ time. A similar $O(|V|^6)$ time algorithm exists for subtree-filament graphs $G$, since $GA(V, coE1)$ is an intersection graph of subtrees in a tree, i.e., it is a chordal graph having no holes, and a maximum induced path of a given parity between $u, v \in GA$ is an induced path in the interval graph defined by the intersections of the subtrees with the unique path in $T$ from $a(u)$ to $a(v)$.

6. ANTIHOLES OF GIVEN PARITY IN INTERVAL-FILAMENT AND SUBTREE-FILAMENT GRAPHS

We consider now the problem of finding a hole $h(v_1,v_k)$ of a given parity in the complement $coG(V,E1,E2)$ of a 3D-interval-filament graph $G(V,E)$. The co(interval-filament) graph $coG(E1)$ has a partition of its edge set $E1$ into $E11$ corresponding to pairs of non-intersecting interval-filaments having disjoint intervals and $E12$ corresponding to pairs of non-intersecting interval-filaments having intervals contained one in another. If $h$ has an edge $(v_i,v_k) \in E11$, then $i(v_i) \cap i(v_k) = \emptyset$ and for a point $x \in L$ between $i(v_i), i(v_k)$, every 3D-interval-filament $f(v_j)$, $2 < i < k - 1$, intersects both $f(v_i), f(v_k)$ (since $(v_i,v_j) , (v_k,v_j) \in E$) having endpoints on both sides of $x$; let $V_x = \{ v \in V, x \in i(v) \}$, $W_x = V_x \cap NC(v_i) \cap NC(v_k)$. Let $s(a(v))$ denote the region in $PL$ bounded by $a(v) \cup i(v)$. The subgraph $G(V_x)$ is a cocomparability-filament graph: for every two non-intersecting interval-filaments $a(u), a(v)$, $u,v \in V_x$, we have $x \in i(u) \cap i(v) \neq \emptyset$, thus $s(a(u)) \subset s(a(v))$ or $s(a(v)) \subset s(a(u))$ which is a transitive relation. Hence $h(v_i,v_k)$ is an induced path in the comparability mixed graph $coG(W_x)$. If $h(v_i,v_k)$ has no $E11$ edges, then every two intervals $i(v_i), i(v_j)$, $1 \leq i,j \leq k$, intersect and by the Helly property there is a point $x$ in $L$ contained in all of them. Hence $h$ is an induced hole in $coG(V_x)$. Unfortunately, there is no known algorithm for finding an induced hole or path of a given parity between two vertices in a comparability mixed graph.

When $G$ is an interval-filament graph or a subtree-filament graph, $coG(V_x)$ is a comparability graph [GA4]. Thus, a hole $h(v_i,v_k)$ of $coG$ is either a hole of a comparability graph $coG(V_x)$, or for some $i$, $h(v_i,v_k)$ is an induced path in a comparability graph $coG(W_x)$. The holes of a comparability graph are all even. The algorithm to find an induced path of a given parity from a vertex $v_1$ to any other vertex in a comparability graph $H(V,F)$ oriented as a transitive graph, works as follows: The directions of the edges in an induced path of $H(V,F)$ are alternating, otherwise a chord appears in the path. We label with path
parity even every vertex \( v_2 \) adjacent to \( v_1 \) by an edge \((v_1,v_2)\) or \((v_2,v_1)\) and insert a pointer from \( v_2 \) to \( v_1 \). We continue labeling the vertices not labeled with both parities. Assume that we found an induced path \( p \) of a given parity from \( v_1 \) to a vertex \( v \) and we have pointers from \( v \) along \( p \) to \( v_1 \), to recover \( p \). If the last edge on \( p \) is \((x,v)\), we label every \( w \) having \((w,v)\) with the parity of \(|p|+1\). If the last edge on \( p \) is \((v,x)\), we label every \( w \) having \((v,w)\) with the parity of \(|p|+1\). From \( v \) we backtrack on \( p \) to find the vertex \( u \) closest to \( v_1 \) and adjacent to \( w \). In the first case, the edge between \( u \) and \( w \) is oriented \((w,u)\), because \((u,w)\) and \((w,v)\) would imply \((u,v)\). Similarly, in the second case, the edge between \( u \) and \( w \) is oriented \((u,w)\). Thus, the path \( q \) obtained by going on \( p \) from \( v_1 \) to \( u \) and then directly to \( w \) has the same parity as \(|p|+1\), since both end in backward edges or in forward edges; we insert a pointer from \( w \) to \( u \). The algorithm requires \( O(|V|^4) \) time. This gives an \( O(|V|^6) \) time algorithm to find antiholes of a given parity in interval-filament and subtree-filament graphs. Note that finding a maximum induced path in a comparability (even a bipartite) graph is NP-hard.

7. ADDITIONAL OBSERVATIONS

In this Section we describe a number of properties of 3D-interval-filament graphs, useful in constructing new algorithms. We conclude with some open problems.

1. The union of two filaments is a filament. Let \( G(V,E) \) be the intersection graph of a family of 3D-interval-filaments \( FF=\{f(v)\mid v\in V\} \). Using the observation that the union of two intersecting interval-filaments is an interval-filament, Cameron [CA] proved that a maximum induced matching in an interval-filament graph can be found in polynomial time. We obtain similar results by changing slightly the definition of the 3D-interval-filaments: we allow the union of intersecting 3D-interval-filaments to also be 3D-interval-filaments. As in Theorem 3, it can be proved that the complements of the newly defined 3D-interval-filament graphs are co(interval-filament) mixed graphs. In the 3D-interval-filament graphs (with the new definition) we can find maximum weight induced subgraphs having connected components with defined properties and bounded size \( k \), by finding a maximum weight independent set [GA4] in the 3D-interval-filament graph represented by filaments which are unions of \( k \) (or less) intersecting filaments. We can obtain a maximum induced matching when \( k=2 \), the dissociation number [BCL] when \( k\leq 2 \), a maximum number of
non-adjacent triangles when \( k=3 \) and a maximum induced forest with trees having at most \( k \) vertices.

2. Every maximum clique is contained in a cocomparability-filament subgraph. Consider a maximum clique \( C \) of \( G \). For every two vertices \( u,v \in C \) the 3D-interval-filaments \( f(u), f(v) \) intersect, thus \( s(u) \cap s(v) \neq \emptyset \) and \( i(u) \cap i(v) \neq \emptyset \). By the Helly property, there is a point \( x \) on \( L \) contained in all the intervals \( i(v), v \in C \), and a plane \( PPP \) through \( x \), perpendicular to \( L \) and \( PL \), such that every \( f(v), v \in C \), has endpoints on both sides of \( x \). The subgraph \( G(V_x) \), \( V_x=\{v | v \in V, x \in i(v)\} \), is a cocomparability-filament graph (Section 6). Therefore, a maximum clique of \( G \) is contained in a cocomparability-filament graph and could be found if a polynomial time algorithm to find a maximum clique in a cocomparability-filament graph were known. As proved in [GA4], when \( G \) is the intersection graph of a family \( FF \) of subtree-filaments based on a family of subtrees \( FI \) on a tree \( T \), for every point \( x \in T \), the subgraph \( G(V_x) \), \( V_x=\{v | v \in V, x \in i(v)\} \), is a cocomparability graph and a maximum clique can be found in polynomial time.

3. Decompositions by cocomparability-filament graphs. For every \( x \in L \), the subgraph \( G(V_x) \), \( V_x=\{v | v \in V, x \in i(v)\} \), is a cocomparability-filament graph. The number of segments on \( L \) defined by the consecutive extreme endpoints of the 3D-interval-filaments is \( O(|V|) \). By taking a point \( x \) in every such segment, we obtain a decomposition of \( G \) into cocomparability-filament graphs. For subtree-filament graphs, this decomposition is by cocomparability graphs.

4. Decompositions by star cut-sets. Let \( G(V,E) \) be the intersection graph of a family of 3D-interval-filaments \( FF=\{f(v) | v \in V\} \); every \( N_G[v] \) is a star cut-set separating between the two vertex sets \( X_v=\{u | a(u) \subset a(v) \text{ and } f(u) \cap f(v) = \emptyset \} \) and \( Y_v=\{u | a(u) \cap a(v) = \emptyset \} \); \( X_v \) corresponds to the filaments contained in \( s(v) \) and \( Y_v \) corresponds to the filaments containing \( s(v) \) or disjoint from \( s(v) \). Therefore, an induced path of \( G \) which contains a vertex in \( X_v \), cannot get out of \( s(v) \) without intersecting \( f(v) \). This property is used in [GA5] to find a maximum induced path of a given parity, and in Section 5, to find a maximum hole of a given parity.

Consider a star cut-set decomposition \( N_G[v] \cup X_v, N_G[v] \cup Y_v \). By decomposing \( G \) using star cut-sets until no more possible, we obtain a family of 3D-interval-filament subgraphs \( G(Z) \) with no star cut-sets. Let \( ZI \) be the subset of \( Z \) containing the vertices \( v \) for which there is a vertex \( u_v \in Z \) fulfilling \( a(u_v) \subset a(v) \text{ and } f(u_v) \cap f(v) = \emptyset \); \( G(Z-ZI) \) is an interval-filament graph. If \( ZI = \emptyset \), then \( G(Z) \) is an interval-filament graph. Assume that \( ZI \neq \emptyset \).
contains no two vertices \( v, w \) having \( f(v) \cap f(w) = \emptyset \), and \( a(v) \subseteq a(w) \) or \( a(v) \cap a(w) = \emptyset \), since \( N_G[v] \) would be a star cut-set separating between \( u \) and \( w \). Therefore \( ZI \) is a clique. In conclusion, every \( Z \) can be partitioned into two subsets \( ZI \) and \( Z-ZI \) such that \( G(Z-ZI) \) is an interval-filament graph and \( G(ZI) \) is a clique. If \( G \) is an interval-filament graph, then \( G(Z-ZI) \) is an interval graph and \( G(ZI) \) is a clique. This observation can be generalized to the complements of \( H \)-mixed graphs [GA7]. Unfortunately, a sequence of decompositions by star cut-sets until no more possible, may be exponential.

5. Alternative definition of 3D-interval-filaments. The 3D-interval-filaments can also be defined in a different way: Instead of applying \( ADJUST(FL) \) to the interval-filaments on which they are based, we drop from the 3D-interval-filaments \( f(v) \) their endpoints (which by Section 2 are endpoints of only one interval-curve of \( a(v) \)). The resulting 3D-interval-filaments are still continuous, but two 3D-interval-filaments with only endpoints in common are non-intersecting. We still must correct the situation that for some \( a(v) \in FI \) one of its extreme points \( x \in L \) is not one of its endpoints: we expand \( x \) into a small segment \([x, y]\), in \( L \), and redefine every interval-filament containing \( x \) by letting it enter \( x \) from its left, as before, continuing on \([x, y]\) and continuing from \( y \) to its right, as it did before from \( x \). With the new definition, the 3D-interval-filaments fulfill everything needed by Theorem 3.

6. Recursive definition of iD-interval-filament graphs. We can define recursively the iD-interval-filament graphs as the complements of the co[\( (i-1)D \)-interval-filament] mixed graphs. None of the families contains all the graphs, as we can see from the following examples. The bipartite graph \( G \) composed of a hole \( h(v_1, v_3) \) and two vertices \( w_1 \) adjacent to \( v_1, v_3 \), and \( w_2 \) adjacent to \( v_3, v_7 \), is not an interval-filament graph: Assume that \( G \) is an interval-filament graph. By Lemma 9, \( h \) contains a vertex \( v \) such that for every \( u \in h-N_G[v] \), we have \( (u, v) \in E2 \) and for every two non-adjacent vertices \( u, x \in h-N_G[v] \) we have \( (u, x) \in E1 \). Assume that \( v = v_1 \). In the hole \( (w_2, v_7), (v_7, v_8), (v_8, v_1), (v_1, v_2), (v_2, v_3), (v_3, v_2) \), we also have \( (w_2, v_1) \in E2 \). In the hole \( h1 = (w_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_7), (v_7, w_2) \), \( w_2 \) must be the vertex required by Lemma 9 to have incoming \( (u, w_2) \in E2 \) edges in \( coG \) from vertices \( u \in h1-N_G[w_2] \). But then, by Lemma 7 applied to the induced path \( (w_2, v_7), (v_7, v_8), (v_8, v_1), (v_1, v_2), (v_2, v_3), (v_3, w_2) \) we obtain that \( (v_1, w_2) \in E2 \), contradicting \( (w_2, v_1) \in E2 \). Assume that \( v = v_2 \). In the hole \( (w_2, v_7), (v_7, v_8), (v_8, v_1), (v_1, v_2), (v_2, v_3), (v_3, w_2) \) we also have \( (w_2, v_2) \in E2 \). In the hole \( h1 = (w_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_7), (v_7, w_2) \), the vertex required by Lemma 9 to have \( E2 \) incoming edges in \( coG \) from its other non-adjacent
vertices of $hI$ is $w_2$ or $v_3$; if it is $w_2$, we obtain the contradiction $(v_2, w_2) \in E_2$, as above, and if it is $v_3$, by symmetry, we are in the same situation as $v = v_1$. Therefore $G$ is not an interval-filament graph. The bipartite graph $H(V)$ composed of two copies of the above graph $G_i(h_i(v_{1,i}, v_{8,i}) \cup (w_{1,i}, w_{2,i})), i = 1, 2$, and four additional vertices $z_1, z_2, z_5, z_7$, every $z_j$ adjacent to $v_{j,i}$ and $v_{j,2}$, is not a 3D-interval-filament graph: Assume that $H$ is a 3D-interval-filament graph. The graph $G_i$ is not an interval-filament graph, hence $coG_i$ must contain an $E_2$ edge $(v_{j,1}, v_{s,1}) \in E_2$. By deleting $NH[v_{s,1}]$ from $H$ we obtain a connected graph, hence every $u \in V - NH[v_{s,1}]$ has $(u, v_{s,1}) \in E_2$. Also, through every two non-adjacent vertices $v_{i,2}, v_{j,2}$ of $G_2$ and $v_{s,1}$ there is a hole of $H$ implying that $(v_{i,2}, v_{j,2}) \in E_1$. Therefore $coG_2$ has only $E_1$ edges and must be an interval-filament graph, contradicting the previous example. In a similar way, by taking $i-1$ copies of $G_i$ and connecting every two with $z$'s, we obtain bipartite graphs which are not 3D-interval-filament graphs.

7. Open Problems. The new families of filament graphs (Table 1) present a number of open problems:

a) Are there polynomial time algorithms for their recognition? Such algorithms are known for circle graphs and circular-arc graphs, but not for polygon-circle graphs.

b) Are there polynomial time algorithms for additional problems which are NP-complete in general, but are polynomial for circle graphs and circular-arc graphs?

c) Are there additional problems having polynomial time algorithms for interval, chordal or any hereditary family $G$ of graphs, which may be solvable for interval-filament graphs, 3D-interval-filament graphs or $coG$-mixed graphs.

ACKNOWLEDGEMENTS: I am grateful to Fred Roberts at DIMACS and Peter Hammer at RUTCOR for support and hospitality during my Sabbatical when this research was done. I thank the referees for their observations which helped improve the paper.

REFERENCES


