Characterizations of Generalized Butterfly Networks

Nadav Golbandi and Ami Litman
Computer Science Department, Technion
Haifa 32000, ISRAEL
email: {golbandi, litman}@cs.technion.ac.il

Abstract

A multistage network is a Generalized Butterfly Network (GBN) if there are $2^n$ end-to-end paths, $(P_0, P_1, \cdots, P_{2^n-1})$, that cover all the vertices and are vertex disjoint and for any two consecutive stages $k$ and $k+1$ there is an integer $\ell$ s.t. the $k$-th vertex of $P_i$ is adjacent to the $(k+1)$-th vertex of $P_j$ if the binary presentations of $i$ and $j$ differ at most at the $\ell$-th bit. The GBN family contains most of the important interconnection networks including the Butterfly networks, the Benes networks and the Batcher bitonic sorting networks.

This work establishes several characterizations and properties of the GBN family that were previously unknown even for the special case of the Butterfly. One characterization is based on automorphism as follows: A multistage network $G$ is a GBN if it is $2$-regular and for any two end-to-end paths, $P'$ and $P''$, there is an automorphism of $G$ that swaps $P'$ and $P''$. Another one is based on the existence of paths that shortcut or bypass a certain path and yet another is based on a certain symmetry of the distance function. These results, and all the other results of this work, are founded on the Layered Cross Product (LCP) introduced by Even and Litman.

It is sometimes useful to assign labels to the vertices or edges of a network. A notable example is the min/max labeling of the edges of a comparator network. To facilitate the study of labeled multistage networks we introduce the Labeled LCP which is an associative and commutative variant of the LCP that operates on vertex-labeled or edge-labeled multistage networks. We show that, under certain conditions, this operator commutes with the concatenation operator. Building on that, we show that the Batcher bitonic sorting network with its min/max labeling is a product of very simple labeled networks – bamboos or rosaries. Moreover, this network, as well as any network which is a product of labeled rosaries under a certain algebra of labels, satisfies a variant of the above path swapping property. Namely, for any two end-to-end paths having identical labels at each stage, there is a label-preserving automorphism that swaps these paths.

1 Introduction

1.1 Preliminaries

Multistage networks. In this work, a multistage network\(^1\) (a stage net) is an undirected\(^2\) simple\(^3\) graph $G$ whose vertices and edges are arranged in stages. Namely, $G = (V^G_1, E^G_1, V^G_2, \cdots, E^G_{\delta^G-1}, V^G_{\delta^G})$, where $\delta^G \geq 1$ is the depth of $G$, $V^G_i$ is the $i$-th stage of vertices, and $E^G_i$, the $i$-th edge-stage, is a set of edges connecting a vertex of $V^G_i$ with a vertex of $V^G_{i+1}$. The set of all vertices is denoted $V^G$ and

\(^1\)Multistage networks are also called multistage interconnection networks, leveled networks and layered networks.

\(^2\)Multistage networks are usually considered to be directed; in this work, however, it is more convenient to consider them to be undirected graphs.

\(^3\)That is, at most one edge connects any two vertices.
that of all edges $E$. We use the superscript $G$ to denote that the object in question (e.g., $S^G$, $V^G$) relates to the stagenet $G$; this superscript is usually omitted when no confusion arises.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{The 8-input Butterfly: (a) and (b) are drawings of this stagenet; (c) and (d) are products of bamboos and of rosaries that yield the same stagenet.}
\end{figure}

This work follows the approach of mathematical logic in which the concept of isomorphism is specific to the type of the structures at hand in order to preserve all the relevant details of these structures. In our case, an isomorphism of one stagenet onto another is required to preserve, not only the adjacency relation, but also the membership relations of the stages (and it is required that the two stagenets have the same depth). Two stagenets, $G$ and $Q$, are isomorphic, denoted $G \cong Q$, if there is an isomorphism of one onto the other. In this work we are interested only in properties of stagenets that are invariant under isomorphism; thus the identity of the vertices is irrelevant. (We do not assume, for example, that the vertices are integers, or pairs of integers, etc.) Therefore, we rarely distinguish between isomorphic stagenets and usually say that two such structures are identical.

**Some terminology.** The vertices of the first (last) stage of a stagenet are called inputs (outputs). A backward (forward) neighbor of a vertex $x$ is a neighbor of $x$ in the preceding (following) stage; the backward (forward) degree of $x$ is the number of the backward (forward) neighbors of $x$. A stagenet is $k$-regular if the backward degree of every non-input vertex and the forward degree of every non-output vertex are $k$; it is regular if it is $k$-regular for some $k > 0$; in this case all the stages have the same cardinality.

In our drawings the stages of a stagenet are arranged in horizontal rows; the first stage occupies the uppermost row, the second stage occupies the next row, etc. See Figures 1(a) and 1(b).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{The 16-input rearrangeable Beneš network: (a) and (b) are drawings of this stagenet; (c) and (d) are products of bamboos and of rosaries that yield the core of this stagenet.}
\end{figure}

**Generalized Butterfly networks.** An end-to-end path is a path that visits each stage exactly once. A stagenet $G$ is a Generalized Butterfly Network (GBN) if there is an integer $n$ and a sequence $S = \langle S_i : i \in [0, 2^n] \rangle$ of $2^n$ end-to-end paths such that:

1. The paths are vertex-disjoint and cover all the vertices of $G$.

2. Let $S_{i,k}$ denote the $k$-th vertex of $S_i$. For any stage $k < \delta^G$ there is an integer $\beta(k) \in [0, n)$ s.t. for any $i \neq j$: $S_{i,k}$ and $S_{j,k+1}$ are neighbors iff the binary presentations of $i$ and $j$ differ exactly
This definition directly implies that a GBN is 2-regular and the cardinality of the stages is a power of two. The sequence $S$ naturally arranges the vertices of each stage in a sequence; thus it is called a *vertex-sequencing* of $G$. We refer to requirement (2) as the *beta-property* of $S$ and to $\beta$ as the *beta-function* of $S$.

A stagenet $G$ is a *Butterfly* if it has a vertex-sequencing whose beta-function is one-to-one and onto the relevant bit positions. It is a simple matter to check that there is exactly one Butterfly of any given depth.

A GBN $G$ is usually drawn in such a way that the paths of a vertex-sequencing of $G$ are laid out as vertical lines, one next to the other in the order of the sequence. Drawings (a) and (b) of Figures 1 to 3 are in this familiar *Butterfly style*. Since a GBN usually has several vertex-sequenceings, it usually has several dissimilar drawings in the Butterfly style. (And, of course, it has highly dissimilar drawings in other styles [23, 18].)

Figure 3: The rearrangeable 16-input Parker network: (a) and (b) are drawings of this stagenet; (c) and (d) are products of bamboos and of rosaries that yield the core of this stagenet.

Some authors classify the edges of a GBN as either “straight” or “cross” edges. Such a classification relates either to a specific drawing of the graph or to a specific identity of the vertices; it has nothing to do with the inherent properties of the edges in question. In fact, one of our characterizations of the GBN family, Theorem 1.4, is based on the inherent property of a GBN that all the edges of an edge-stage are indistinguishable.

The GBN family$^4$ includes most of the important interconnection networks. As said, it includes the well-known Butterfly family, a member of which is depicted in Figure 1. It includes the core$^5$ of the rearrangeable$^6$ Beneš networks [4, 5] displayed in Figure 2. It includes the core of the rearrangeable Parker networks [21] presented in Figure 3. (Such a network is a tandem of three Butterflies and the first two stages of a forth one. It is yet an open question whether a tandem of two Butterflies is rearrangeable.) Another notable sub-family of the GBN family are the bitonic sorting networks of Batcher [2], a member of which is depicted in Figure 8.

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$^4$A family (of stagenets) is a class of stagenets that is closed under isomorphism.

$^5$That is, the stagenet without its input and output stages.

$^6$A stagenet is rearrangeable if for every one-to-one function, $f: V_1 \to V_5$, there is a set $P$ of end-to-end paths s.t. the members of $P$ are edge-disjoint and for any $v \in V_1$ there is a path in $P$ leading from $v$ to $f(v)$. 

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1.2 Our results

This work establishes novel characterizations and properties of the GBN family. Most of our results were previously unknown even for the special case of the Butterfly. We list here our main theorems in a natural order that somewhat differs from the order of their proofs; hence, we provide references from these theorems to their counterparts in the rest of the paper.

**Edge-sequencing.** Our first characterization of the GBN family is a dual of the above definition, in which the edges play the role of the vertices. An *edge-sequencing* of a stagenet $G$ is a sequence $S = (S_i : i \in [0, 2^n])$ of $2^n$ end-to-end paths such that:

1. The paths are edge-disjoint and cover all the edges and vertices of $G$.
2. Let $S_{i,k}$ denote the $k$-th vertex of $S_i$. For any stage $k$ there is an integer $\beta(k) \in [0, n)$ s.t. for any $i \neq j$: $S_{i,k} = S_{j,k}$ iff the binary presentations of $i$ and $j$ differ exactly at the $\beta(k)$-th bit.

We refer to requirement (2) as the *beta-property* of $S$ and to $\beta$ as the *beta-function* of $S$.

**Theorem 1.1 (7.8)** A stagenet is a GBN iff it has an edge-sequencing.

An edge-sequencing of a stagenet $G$ leads to its presentation by a Knuth’s diagram [17, Figure 44] as in Figure 4. In such a diagram, the paths of an edge-sequencing are laid out as vertical lines, one next to the other in the order of the sequence; a vertex is denoted by a horizontal line segment terminating at the two paths that cross this vertex. Unfortunately, such a diagram obscures the clustering of the vertices into stages. Since a GBN usually has several edge-sequencings, it usually has several dissimilar Knuth’s diagrams.

**The path-symmetric property.** A stagenet is *normal* if it has at least one vertex and every vertex is on some end-to-end path. A stagenet $G$ is *weakly path-symmetric* if $G$ is normal and for any two end-to-end paths, $P_1$ and $P_2$, there is an automorphism of $G$ that maps $P_1$ onto $P_2$; $G$ is *(strongly)* *path-symmetric* if, in addition, there is an automorphism that maps $P_1$ onto $P_2$ and $P_2$ onto $P_1$.

A stagenet $G$ is *properly-sized* if $|V^G|$ is a power of two. Such a requirement appears in several of our characterization theorems; it is redundant, however, when the stagenet in question is connected.

**Theorem 1.2 (4.6)** A stagenet is a GBN iff it is 2-regular, path-symmetric and properly-sized.

As shown in Section 2, the stronger form of the path-symmetric property is mandatory in Theorem 1.2.

**Banyan.** A stagenet has the *at-most-one-path* (*at-least-one-path*) property if any input and output are connected by at most (at least) one end-to-end path; it is *banyan* [14] if it has both of these properties. It is well-known that a stagenet is a Butterfly iff it is a banyan GBN. It is also known that a stagenet is a Butterfly iff it is a connected GBN having the at-most-one-path property. Via...
Theorem 1.3 A stagenet is a Butterfly iff it is 2-regular, path-symmetric, connected and has the at-most-one-path property.

The same-stage edge-symmetric property. A stagenet $G$ is same-stage edge-symmetric if it is normal and for any two edges, $e_1$ and $e_2$, of the same edge-stage there is an automorphism of $G$ that maps $e_1$ to $e_2$ and $e_2$ to $e_1$.

Theorem 1.4 (4.6) A stagenet is a GBN iff it is 2-regular, same-stage edge-symmetric and properly-sized.

Theorem 1.5 A stagenet is a Butterfly iff it is 2-regular, same-stage edge-symmetric, connected and has the at-most-one-path property.

More terminology. We use the following terms and notations in the context of a given stagenet: $(x, y)$ denotes that $x$ and $y$ are two vertices of the same stage; $(x \sim y)$ denotes the edge whose endpoints are $x$ and $y$. A straight path or a linear path is a path that visits each stage at most once. Two, or more, vertices and edges are collinear if there is a linear path that contains all of them; $x \sim y$ denotes that $x$ and $y$ are collinear; $x \overset{P}{\sim} y$ denotes that $P$ is a straight path whose endpoints are $x$ and $y$. Finally, a construction like $(w \sim x \overset{P}{\sim} y \overset{Q}{\sim} z)$ denotes a compound path with the obvious meaning.

Diamonds and shortcuts. A diamond is a simple cycle of the form $(x \overset{P_1}{\sim} y \overset{P_2}{\sim} x)$; the vertices $x$ and $y$ are the endpoints of the diamond. A stagenet has the diamond property if the following holds. Let $(x \overset{P_1}{\sim} y \overset{P_2}{\sim} x)$ be a diamond and let $x' \overset{P_1'}{\sim} y'$ be a straight path with $(x', y')$. Then there is a straight path $P_2'$, called a bypass of $P_2$, s.t. $(x' \overset{P_1'}{\sim} y' \overset{P_2'}{\sim} x')$ is a diamond; see Figure 5(a).

![Figure 5](image)

Figure 5: (a) The diamond property. (b) The shortcut property.

A stagenet has the shortcut property if for any path $P = (x_1 \overset{P_1}{\sim} y_1 \overset{P_2}{\sim} x_2 \overset{P_3}{\sim} y_2)$ with $(x_1, y_2)$ there is a straight path $P_4$, called a shortcut of $P$, s.t. $x_1 \overset{P_4}{\sim} y_2$. See Figure 5(b). The shortcut property is a generalization of the buddy property; namely, the buddy property [1] is just the shortcut property where all the paths in question are restricted to be single edges.

Theorem 1.6 (4.6) A stagenet is a GBN iff it is 2-regular, properly-sized and has the diamond and the shortcut properties.

Theorem 1.7 A stagenet is a Butterfly iff it is 2-regular, connected and has the shortcut and the at-most-one-path properties.

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*10 A cycle or a path is simple if it passes at most once through any vertex.*
Semiregularity. A stagenet is semiregular if it is normal and the forward degree, as well as the backward one, is uniform in any stage; i.e., if \((x, y)\) then the backward (forward) degree of \(x\) equals the backward (forward) degree of \(y\). A stagenet is \(k\)-semiregular if it is semiregular and the forward and backward degrees of any vertex are at most \(k\).

Slice of a stagenet. Let \(G\) be a stagenet and \(1 \leq i \leq j \leq \delta^G\). The \(i\) to \(j\) slice of \(G\), denoted \(G[i, j]\), is the stagenet of depth \((j + 1 - i)\) composed of stages \(i\) to \(j\) of \(G\) together with the appropriate edge-stages; i.e., \(G[i, j] = (V_i^{G}, E_i^{G}, V_{i+1}^{G}, \ldots, E_{j-1}^{G}, V_j^{G})\). A slice \(G[i, j]\) is an initial (final) slice if \(i = 1\) (\(j = \delta^G\)). Let \(G[i, \infty]\) denote the final slice starting at stage \(i\); i.e., \(G[i, \infty] = G[i, \delta^G]\).

Forward and backward distances. Let \((x, y)\) be two vertices of stage \(i\) of a stagenet \(G\). The forward (backward) distance between \(x\) and \(y\), denoted \(\overrightarrow{d}(x, y)\) (\(\overleftarrow{d}(x, y)\)), is the distance between them in the slice \(G[i, \infty]\) \((G[i, i])\); this distance is infinite (\(\infty\)) when \(x\) and \(y\) are disconnected in the corresponding slice. (The functions \(\overrightarrow{d}\) and \(\overleftarrow{d}\) are undefined for two vertices of different stages.)

The isometric property. A stagenet \(G\) has the isometric property if it is 2-semiregular and the following holds. Let \((x, y)\), let the forward degree of \(x\) and \(y\) be two, and let \(x_1\) and \(x_2\) (\(y_1\) and \(y_2\)) be the two forward neighbors of \(x\) (\(y\)), in an arbitrary order. Then:

1. \(\overrightarrow{d}(x_1, x_2) = \overrightarrow{d}(y_1, y_2),\)
2. \(\overrightarrow{d}(x_1, y_1) = \overrightarrow{d}(x_2, y_2).\)

**Theorem 1.8** (4.6) A stagenet is a GBN iff it is 2-regular, properly-sized and has the isometric property.

Symmetric property. The inverse of a stagenet \(G\), denoted \(\overleftarrow{G}\), is the stagenet derived from \(G\) by reversing the order of the stages; \(G\) is (end for end) symmetric if \(G \cong \overleftarrow{G}\). A property \(\Psi\) of stagenets is symmetric if it is invariant under the inverse operator. The isometric property does not seem to be symmetric while the property of being a GBN and the other properties discussed in this section are clearly symmetric. Theorem 1.8 indicates the surprising fact that the isometric property is actually symmetric.

Bamboo and rosary. A bamboo (rosary) is a 2-semiregular stagenet having exactly two vertices (edges) in each (edge-)stage. These stagenets are depicted in drawings (c) and (d) of Figures 1 to 3.

Sub-stagenet and collinear sub-stagenet. A stagenet \(G'\) is a sub-stagenet of a stagenet \(G\) if \(\delta^{G'} = \delta^G\), \(V_{i}^{G'} \subset V_{i}^{G}\) and \(E_{i}^{G'} \subset E_{i}^{G}\) for every \(i\). (Note that a proper slice of a stagenet is not a sub-stagenet as its depth is inappropriate.) A stagenet \(G'\) is a collinear sub-stagenet of a stagenet \(G\) if \(G'\) is a sub-stagenet of \(G\) and for every \(u, v \in V_{i}^{G'}\): \(u\) and \(v\) are collinear in \(G'\) iff they are collinear in \(G\). For example, the sub-stagenet induced by the dashed edges in Figure 6(a) is collinear while that of Figure 6(b) is not.

**Figure 6**: A collinear (a) and a non-collinear (b) sub-stagenet.

The Layered Cross Product. The proofs of all our theorems are based on the Layered Cross Product (LCP) technique introduced by Even and Litman [12]. The LCP (defined in Section 3) is an
Our work demonstrates that the LCP is also a powerful tool for exploring the properties of interconnection networks. This analytical power stems from two attributes of the LCP: It preserves many important properties of stagenets and it commutes with many important unary operators on these graphs. While the proofs of all our theorems are based on the LCP, the LCP itself appears explicitly only in the next theorem.

**Theorem 1.9** (4.6, 4.9) Let $F$ be the family of either all bamboos or all rosaries. Then:

(a) A stagenet is a GBN iff it is 2-regular and is a product of members of $F$.

(b) The factorization of a GBN into members of $F$ is unique (up to a permutation and isomorphism of the factors).

(c) $A Q \in F$ appears in the above factorization of a GBN $G$ iff $G$ has a collinear sub-stagenet isomorphic to $Q$.

Figure 6 illustrates that the collinear requirement is mandatory in Theorem 1.9(c).

**Almost a GBN.** The last theorem concludes our characterizations of the GBN family. An interesting and relating subject is the border separating properties that characterize the GBN family from those that are too weak to do so. This work sheds some light on this subject, showing for example that a property concerning only slices of depth $\delta-1$ (or less) cannot characterize the GBN family. The next two theorems present families of stagenets whose members are very close to a GBN without actually being so. These theorems resemble a theorem of Hotzel [16] presented soon (Theorem 1.15).

We indicate similarity between stagenets as follows. Two stagenets are $k$-isomorphic if they have the same depth and any slice of depth $k$, or less, of one is isomorphic to the corresponding slice of the other. Let a double-butterfly denote a stagenet composed of two disjoint sub-stagenets both isomorphic to the same Butterfly. (Such a stagenet is, of course, a GBN.)

**Theorem 1.10** (5.3) For any $\delta \geq 2$ there is a 2-regular, symmetric, connected stagenet of depth $\delta$ having the at-most-one-path property which is not a GBN but is $(\delta-1)$-isomorphic to a double-butterfly.

(Moreover, as shown in [20], the stagenet of the above theorem is unique.)

**Theorem 1.11** (5.6) For any $\delta \geq 4$ there is a symmetric, 2-regular, banyan stagenet of depth $\delta$ which is not a Butterfly but is $(\delta-3)$-isomorphic to one.

The construction of the above stagenets is done via the LCP, demonstrating that this operator is a convenient tool, not only for constructing highly regular interconnection networks, but also for constructing ‘custom-made’ networks having particular properties.

**Adjacent stages.** The next theorem establishes a criterion for two GBN graphs to be isomorphic. A diamond of a stagenet $G$ is collinear if it constitutes a collinear sub-stagenet of $G$; see Figure 7. Two stages, $i$ and $j$, of a stagenet $G$ are adjacent, denoted $A^G(i,j)$, if there is a collinear diamond whose endpoints are in $V_i$ and $V_j$. The $A^G$ relation essentially determines the GBN $G$ as follows.

**Theorem 1.12** (4.10) Two GBN graphs, $G$ and $Q$, are isomorphic iff $\delta^G = \delta^Q$, $|V_1^G| = |V_1^Q|$ and $A^G = A^Q$.

11 For products of general graphs (rather than stagenets) see for example [15].
From an isometry to an automorphism. A mapping $\pi$ of a stage $V_i$ of a stagenet $G$ is an isometry of $V_i$ if $\pi$ is a permutation of $V_i$ that preserves the forward distance; i.e., $d(x, y) = d(\pi(x), \pi(y))$ for any $x, y \in V_i$. The following result was previously unknown even for the special case of the Butterfly.

**Theorem 1.13** (2.8, 4.6) Any isometry of the first stage of a GBN $G$ can be extended into an automorphism of $G$. Moreover, if $G$ is connected then it can be extended into an automorphism of $G$ which is the identity function on the last stage.

Labeled stagenets and labeled LCP. In certain applications it is beneficial, or even crucial, to furnish the edges or vertices of a stagenet with additional details. A notable example is a comparator network where it is crucial to specify which of the two edges exiting a comparator carries the minimal value and which the maximal one. When such a network is a sorting network, it is usually desirable to specify the order of the output edges. Such specifications can be accomplished by assigning a label to each edge or vertex, producing a labeled stagenet. Another advantage of labeled stagenets relates to the construction process rather than to the final product. An important construction technique is the concatenation of two networks into a larger one. Most sorting networks, including the bitonic one, are constructed this way. By specifying the input/output matching, labeling converts this concatenation from an indefinite operation to a definite operator.

![Diagram of a labeled stagenet](image-url)

Figure 8: The 16-input Batcher bitonic sorting network.

To facilitate the construction and the analysis of labeled stagenets, we introduce the Labeled LCP which is an associative and commutative variant\(^{12}\) of the LCP that operates on labeled stagenets.

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\(^{12}\)Another variant of the LCP with different aims and capabilities has been introduced by Even and Kupershtok [11].
Theorems 1.1 and 1.9 are proved via that operator. We show that, under certain conditions, this operator commutes with the concatenation operator. Building on that, we show that the Batcher bitonic sorting network [2] with its min/max labeling (as well as with the outputs’ ordinals) is a product of very simple labeled networks – bamboos or rosaries. These products are presented in Figure 8 whose features are explained in Section 10. Moreover, Theorem 8.1 establishes that this network, as well as any network which is a product of labeled rosaries under a certain algebra of labels, has the following variant of the path-symmetric property. For any two end-to-end paths having identical labels at each stage, there is a label-preserving automorphism that swaps these two paths.

A weaker result about the bitonic sorting network, that its underlying (unlabeled) stagenet is a GBN, is already known; a figure in Knuth’s book [17, Figure 56] indicates that this stagenet has an edge-sequencing and thus, by Theorem 1.1, is a GBN. Also, Becker [3] has found that this stagenet is a product of bamboos.

1.3 Related work

The Butterfly. The networks that are now commonly called Butterflies [19] are the most studied members of the GBN family. The majority of the prior work related to ours is about the Butterfly family.

The earliest manifestation of the Butterfly is apparently in the form of the Fast Fourier Transform [10] as the Butterfly is the data dependency graph of this algorithm. Hence, the Butterfly is also called the FFT graph.

Two other notable early inventions, of the sixties, make good use of the Butterfly. The well-known rearrangeable Beneš network [4, 5] is a back-to-back concatenation of two Butterflies as shown in Figure 2. The well-known bitonic sorting network of Batcher [2] is composed of bitonic sorters whose underlying graph is the Butterfly. In fact, Batcher himself wrote “Readers may recognize the similarity between the topologies of the bitonic sorter and the fast-fourier-transform.”

The Butterfly network has been reinvented several times during the seventies, each having a distinct name and drawing of the very same network. Wu and Feng [23] and then Kruskal and Snir [18] have revealed that those different-looking networks are actually identical.

Kruskal and Snir [18] have presented a characterization of the Butterfly that is based on a 4-value labeling of its edges, having a certain property related to routing.

Bermond, Fourneau and Jean-Marie [6, 7] have established a characterization of the Butterfly which is based on the number of the connected components of its slices as follows.

Theorem 1.14 A stagenet $G$ is a Butterfly iff it is 2-regular, banyan and any initial or final slice $S$ of it has $2^{5^S} - 2^S$ connected components.

This theorem leads to a linear-time algorithm that resolves whether a given stagenet is a Butterfly [6, 7].

The Butterfly network and its automorphism group were further investigated by Hotzel [16] who has established additional characterizations of this network. His following conjecture, which is yet open, captures the fact that the Butterfly is exceptionally symmetric and promises to be an outstanding characterization of this family.

Hotzel’s conjecture: A stagenet is a Butterfly iff it is 2-regular and banyan and has at least as many automorphisms as the Butterfly of the same depth.

Bermond, Fourneau and Jean-Marie [6] have constructed a counterexample which shows that a certain weaker variant of the property of Theorem 1.14 fails to characterize the Butterfly. Hotzel [16] has strengthened this result as follows.

Theorem 1.15 For any $\delta \geq 3$ there is a 2-regular, banyan stagenet of depth $\delta$ which is not a Butterfly but is $(\delta - 2)$-isomorphic to one. In fact, there are exactly two such stagenets and each is the inverse of the other.
The last theorem implies that the $\delta - 2$ in Theorem 1.15 is tight and that the 'double-butterfly' in Theorem 1.10 cannot be replaced with 'Butterfly'. Theorem 1.15 implies that the $\delta - 3$ in Theorem 1.11 is tight.

Even and Litman [12] have shown that the Butterfly is the product of two complete binary trees where one tree has its root in the first stage and its leaves in the last one, while the second tree is oriented the other way around. They have also observed that several important interconnection networks are products of bamboos.

Wu and Li [24] have applied the line-graph operator to study the wrapped Butterfly [19] and have established several characterizations of this family. One of them is based on the Heuchenne property, which is analogous to the shortcut property but in the context of general directed graphs (rather than stagenets).

**Unique factorization.** The unique factorizations of GBN graphs provided by Theorem 1.9 are not derived from a unique factorization of every stagenet into a product of primes since, as Etzion and Roth [13] have proved and is shown in Theorem 4.11, the LCP lacks such a unique factorization theorem even for normal stagenets.

The unique factorizations of Theorem 1.9 actually follow from the following result of Paz [22]: A stagenet has at most one factorization into a product of beads. (These beads are very simple primes defined in Sub-section 4.1 and depicted in Figure 18.) Paz has presented a polynomial-time algorithm that resolves whether a given stagenet is a product of beads and, if so, factorizes it into this unique product. This provides a polynomial-time algorithm that resolves whether two given stagenets which are products of beads are isomorphic.

A recent work of Litman [20] shows that any path-symmetric stagenet has a unique factorization into primes and that those primes are path-symmetric. It presents a linear-time algorithm that resolves whether a given stagenet is path-symmetric and, if so, factorizes it into primes. This provides a linear-time algorithm that resolves whether two given path-symmetric stagenets are isomorphic.

**Independent connections or an affine labeling scheme.** The earliest reference to the GBN family is apparently due to the concept of independent connections introduced by Bermond and Fourneau [8]. Following Hotzel [16], this concept can be expressed via affine transformations as follows. An affine transformation of a vector space to itself is a composite of a regular linear transformation and a translation; i.e., it is a mapping of the form $x \mapsto t(x) + z$ where $t$ is a one-to-one linear transformation and $z$ is a fixed vector. An affine pair is an unordered pair $\{f, g\}$ of affine transformations s.t. $f - g$ is a constant non-zero vector. Let $\mathcal{U}$ be a vector space over the field $\mathbb{Z}_2$. A mapping $\lambda : V^G \rightarrow \mathcal{U}$ is an affine labeling scheme of the stagenet $G$ over $\mathcal{U}$ if $G$ is 2-regular and there is a sequence $(F_1, F_2, \cdots, F_{g-1})$ of affine pairs such that:

1. For any stage $i$, $\lambda |_{V_i}$ (the restriction of $\lambda$ to $V_i$) is one-to-one (but not necessarily onto $\mathcal{U}$).

2. For any $v \in V_i$ and $u \in V_{i+1}$: $(v \rightarrow u) \in E_i$ iff $f(\lambda(v)) = \lambda(u)$ for some $f \in F_i$.

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13 A stagenet $G$ is prime (with respect to the LCP) if $G$ is not the product of two stagenets both distinct from $G$, and it is not the identity element of the LCP.
We refer to such a \( \langle F_i \rangle \) as an **affine sequence of** \( \lambda \). A stagenet \( G \) is **affine** if it has an affine labeling scheme over such a vector space. By definition, any GBN is affine. It is not hard to show that any affine stagenet is path-symmetric. This and Theorem 1.2 imply:

**Theorem 1.17** A stagenet is a GBN iff it is affine and properly-sized.

The last theorem implies the following theorem of Bermond and Fourneau [8].

**Theorem 1.18** A stagenet is a Butterfly iff it is affine and banyan.

**Permuting labeling scheme.** A restricted type of an affine labeling scheme is defined as follows. A mapping \( \lambda : V^G \to \mathcal{U} \) is a **permuting labeling scheme** if there is a sequence \( \langle F_i \rangle \) s.t. \( G, \lambda \) and \( \langle F_i \rangle \) satisfy the above conditions and furthermore:

(a) For any stage \( i, \lambda |_{V_i} \) is onto \( \mathcal{U} \).

(b) There is a basis \( B \) of \( \mathcal{U} \) s.t. for every edge-stage \( i, F_i = \{ t, t+b \} \) where \( t \) is a linear transformation induced by a permutation of \( B \) and \( b \in B \).

In this case we say that \( \lambda \) is a permuting labeling scheme with regard to the basis \( B \) as above. Note that a vertex-sequencing of a stagenet \( G \) is actually a permuting labeling scheme of \( G \) in which all the linear transformations of (b) are the identity function. The next two theorems are derived from this observation and Theorems 1.17 and 1.18. The second theorem has been established by Bermond and Fourneau [8] in a somewhat different form.

**Theorem 1.19** A stagenet is a GBN iff it has a permuting labeling scheme.

**Theorem 1.20** A stagenet is a Butterfly iff it is banyan and has a permuting labeling scheme.

Bermond and Fourneau [8] have noted that Theorem 1.20 is a convenient tool for revealing the identity of a Butterfly in disguise; all the networks found by Wu and Feng [23] to be isomorphic to the Butterfly are constructed (explicitly or implicitly) via labeling schemes over the vector space \( \mathcal{U} = (\mathbb{Z}_2)^n \) and all these labeling schemes are, not only affine, but also permuting labeling schemes, and this with respect to the standard basis of \( (\mathbb{Z}_2)^n \).

Bermond and Fourneau [8] and Hotzel [16] have confined their study of affine stagenets to the Butterfly family. Chang, Hwang and Tong [9] have studied permuting labeling schemes\(^1\) of the GBN family and have established an isomorphism criterion in terms of affine sequences of such schemes.

Wu, Bao, Li and Jia [25] have shown:

**Theorem 1.21** A stagenet is a GBN iff it is 2-regular and a product of beads.

Furthermore, they have established other characterizations of this family in terms of the number of connected components of the slices, in the style of Theorem 1.14.

**Organization.** The rest of the paper is organized as follows. Section 2 contains partial results that are established without using the LCP. Section 3 presents the LCP and its main attributes. Section 4 establishes most of our theorems via a factorization of the GBN graphs. The “almost GBN” families are constructed in Section 5. Section 6 presents the labeled LCP. Vertex and edge sequencings are studied in Section 7. Section 8 presents a labeled variant of the path-symmetric property. The concatenation operator is presented in Section 9. Finally, Section 10 studies the bitonic sorting network.

\(^1\)Due to this aspect of the GBN networks, they call them **Bit Permutation Networks**.
2. Elementary Constructions

All our theorems are proved via the Layered Cross Product [12] technique. This section, however, contains partial results that are established by elementary techniques, without using our main tool. It considers several properties of stagenets and shows that, among these properties, the path-symmetric one is the strongest and the isometric property is the weakest. It then constructs certain automorphisms of stagenets having the isometric property. In a later section the relevant stagenets are factorized via these automorphisms and that will yield that all the properties in question are actually equivalent.

Restrictions and extensions of functions. For a function \( f : D \to R \) and \( D' \subseteq D \), let \( f|_{D'} \), denote the restriction of \( f \) to \( D' \). We extend such a function \( f \), in a natural manner, to be defined for any compound object composed of elements of \( D \). For example, if \( \overrightarrow{x} = (x_1, x_2, \cdots, x_n) \) is a sequence of elements of \( D \) then \( f(\overrightarrow{x}) \triangleq (f(x_1), f(x_2), \cdots, f(x_n)) \); if \( D' \subseteq D \) then \( f(D') \triangleq \{ f(x) \mid x \in D' \} \).

2.1 Some properties are more equal than others

**Lemma 2.1** Any path-symmetric \( G \) has the following properties:

(a) The same-stage edge-symmetric property.

(b) The diamond property.

(c) The shortcut property.

(d) If \( G \) is 2-semiregular then \( G \) has the isometric property.

**Proof:** Let \( G \) be path-symmetric. Since \( G \) is normal, each edge is on an end-to-end path. This establish (a). To prove (b), let \((x \ P_1 y \ P_2 \ x')\) be a diamond and let \(x' \ P'_{1} y' \ (x' \ P'_{2} y') \). See Figure 9(a). Since any straight path can be extended to an end-to-end path, there is an automorphism \( \pi \) s.t. \( \pi(P_1) = P'_1 \). Hence, \( P'_1 \) and \( \pi(P_2) \) form the required diamond. (Note that we use here only the weaker form of the path-symmetric property.)

![Figure 9: A path-symmetric stagenet has the diamond property (a) and the shortcut property (b).](image)

To prove (c) let \( P = (x_1 \ P_1 y_1 \ P_2 x_2 \ P_3 y_2) \), \((x_1 \ P'_{1} y_1 \ P'_{2} x_2 \ P'_{3} y_2)\). See Figure 9(b). By the path-symmetric property, there is an automorphism \( \pi \) that swaps \( P_1 \) and \( P_3 \). Hence, \( \pi(P_2) \) is a shortcut of \( P \). (Note that here we apply the full strength of the path-symmetric property.)

We now show that (a) implies (d). Let \( G \) be a 2-semiregular stagenet and let \( x, x_1, x_2, y, y_1 \) and \( y_2 \) be six vertices as in the definition of the isometric property. By (a) there is an automorphism \( \pi \) of \( G \) that swaps the edges \((x \ x_1)\) and \((y \ y_2)\). This \( \pi \) must swap the edges \((x \ x_2)\) and \((y \ y_1)\). Hence, \( \pi(x_1, y_1) = \pi(x_1, \pi(y_1)) = \pi(y_2, x_2) \). Also, \( \pi(x_1, x_2) = \pi(x_1, \pi(x_2)) = \pi(y_2, y_1) \).

The ‘path-symmetric’ requirement of Lemma 2.1 can not be replaced with ‘weakly path-symmetric’ as demonstrated by the depth-two stagenet of Figure 10; this stagenet is weakly path-symmetric but...
lacks the shortcut and the isometric properties. The same stagenet also demonstrates that the stronger form of the path-symmetric property is mandatory in Theorem 1.2.

**The V-property.** A V-path is a path of the form \((x \sim y \sim z)\) with \((\frac{x}{y})\). A stagenet \(G\) has the V-property if any shortest path of a slice \(G[i, \infty]\) whose endpoints are in \(V_i\) is a V-path.

**Lemma 2.2**

(a) **The shortcut property implies the V-property.**

(b) **The isometric property implies the V-property.**

(c) **The isometric property implies that any slice of the stagenet has the V-property.**

**Proof:** Let \(G\) be any stagenet lacking the V-property, let \(P\) be a path of \(G\) which is a counterexample of a minimal length to the V-property and let \(x, z \in V_i\) be the endpoints of \(P\). The path \(P\) must have an additional vertex, \(y\), in \(V_i\). (Otherwise, removing \(x\) and \(z\) from \(P\) produces a shorter counterexample.) Since \(P\) is minimal, its subpaths between \(x\) and \(y\) and between \(y\) and \(z\) are V-paths; so let \(P = (x P_1 u P_2 y P_3 v P_4 z)\); see Figure 11(a).

![Figure 11: A weakly path-symmetric stagenet which is not path-symmetric.](image)

To prove (a) assume, for contradiction, that \(G\) has the shortcut property but not the V-property. Let \(P, x, y, \text{etc.}\), be as above. Assume, without loss of generality, that \(P_1\) is not longer than \(P_3\) and let \(w\) be the vertex of \(P_3\) s.t. \((\frac{w}{u})\); see Figure 11(a). The shortcut property implies \(x \sim w\), contradicting the fact that \(P\) is a shortest path.

Statement (b) follows from (c). To prove (c) assume, to the contrary, that \(G\) has the isometric property but a slice \(Q\) of it does not have the V-property. Let \(P, x, y, \text{etc.}\), be as above in the context of the slice \(Q\). Since \(P\) is a shortest path in \(Q\) which is 2-semiregular, the forward degree of \(y\) is two. Let \(x_1, x_2, y_1\) and \(y_2\) be the forward neighbors of \(x\) and \(y\), respectively. Without loss of generality assume \(P = (x P_{1/2} u P_2 y) = (x \sim x_2 \sim u \sim y_2 \sim y)\); see Figure 11(b). Since a shortest path in \(Q\) between \(x_2\) and \(y_2\) is a V-path, we have \(d_Q^{G}(x_2, y_2) = d_Q^{G}(x_2, y_2)\). The isometric property of \(G\) implies \(d_Q^{G}(x_1, y_1) = d_Q^{G}(x_2, y_2)\). Hence, a shortest path of \(G\) connecting \(x_1\) and \(y_1\) must reside entirely in \(Q\), implying \(d_Q^{G}(x_1, y_1) = d_Q^{G}(x_1, y_1) = d_Q^{G}(x_2, y_2) = d_Q^{G}(x_2, y_2)\) and contradicting the fact that \(P\) is a shortest path in \(Q\).

**Remark:** The V-property, even when combined with the inverse V-property and 2-regularity, implies neither the shortcut property nor the isometric property, as demonstrated by the depth-two stagenet of Figure 12.
Lemma 2.3 For a 2-semiregular stagenet $G$, each of the following properties implies the isometric property:

(a) The path-symmetric property.

(b) The same-stage edge-symmetric property.

(c) The shortcut and diamond properties.

Proof: By Lemma 2.1(d) and its proof, each one of (a) and (b) implies the isometric property. To show that (c) implies the isometric property, let $x$, $x_1$, $x_2$, $y$, $y_1$ and $y_2$ be six vertices as in the definition of the isometric property. We need to show that $d(y_1, y_2) = d(x_1, x_2)$ and $d(x_2, y_2) = d(x_1, y_1)$. By symmetry, it suffices to show that $d(y_1, y_2) \leq d(x_1, x_2)$ and $d(x_2, y_2) \leq d(x_1, y_1)$.

Consider the first inequality. It clearly holds when $d(x_1, x_2) = \infty$; so assume otherwise. Due to Lemma 2.2(a), a shortest path $P$ between $x_1$ and $x_2$ is a $V$-path of the form $P = (x_1, y_{\sim} x_2)$. See Figure 13(a). Since $G$ is normal, there is $y_3$ and $S_1$ s.t. $y_1 S_1 y_3$ and $(y_{\sim} y_{\sim})$. By the diamond property, $y_2 S_1 y_3$ for some $S_2$, implying $d(y_1, y_2) \leq d(x_1, x_2)$.

Consider the second inequality, $d(x_2, y_2) \leq d(x_1, y_1)$. It clearly holds when $d(x_1, y_1) = \infty$; so assume otherwise. Let $P$ be a shortest path between $x_1$ and $y_1$. By Lemma 2.2(a), $P$ is of the form $P = (x_1, y_{\sim} x_2, y_{\sim} x_3)$. Since $G$ is normal, there are $x_3$, $y_3$, $P_2$ and $S_2$ s.t. $x_2 P_2 x_3$, $y_2 S_2 y_3$, $(y_{\sim} u)$ and $(y_{\sim} u)$. The shortcut property and $(y_{\sim} y_{\sim})$ imply that $(y_{\sim} x_{\sim})$ for some $P'$. If $P' = (y_{\sim} y_{\sim} x_{\sim})$ then the required inequality holds. So assume that $P' = (y_{\sim} y_{\sim} x_{\sim})$. By symmetry we may assume $x_1 S_3 y_3$ for some $S_3$. By the shortcut property, $(x_{\sim} x_{\sim} y_{\sim} y_{\sim})$ implies $x_{\sim} y_{\sim}$. By symmetry we have $y_{\sim} x_{\sim}$, implying $d(x_2, y_2) \leq d(x_1, y_1)$.

\[\square\]

Figure 13: Establishing the isometric property.

2.2 Construction of automorphisms

In this sub-section we construct certain automorphisms of a stagenet $G$ having the isometric property. The construction is inductive and is based on the fact that any final slice of $G$ has the isometric property. Actually, any slice of $G$ has the isometric property, but we do not know this yet.

Some more definitions. We use the following definitions and notations. Two vertices $x$ and $y$ of a stagenet are backward (forward) buddies if $x \neq y$ and they have a common backward (forward) neighbor. For two neighboring stages, $j$ and $j'$, of a stagenet $G$, let $(x - x') \in V_j - V_{j'}$ denote that
That is, the buddy relation is the identity function.

Let $G$ have the isometric property and fork at $V_j$. Define $\bar{d}(V_{j+1}) = \bar{d}(x', x'')$ where $x', x'' \in V_{j+1}$ are backward buddies; due to the isometric property, $\bar{d}(V_{j+1})$ is independent of $x'$ and $x''$.

Let $\mathbb{N}$ denote the set of non-negative integers, let $\mathbb{N}^+$ denote the set of strictly positive integers and let $\mathbb{N}^+ \Delta \mathbb{N} \cup \{\infty\}$. For an integer $i$ let $\infty + i \triangleq -i \triangleq \infty$. For a stagenet $G$ and $h \in \mathbb{N}^+$ define a binary relation $\Delta^h$ on $V^G$ by:

$$\Delta^h(x, y) \Delta \left( (\frac{x}{y}) \text{ and } \bar{d}(x, y) < h \right).$$

**Lemma 2.4** The following statements hold for any $G$ having the isometric property:

(a) For any $h \in \mathbb{N}^+$, $\Delta^h$ is an equivalence relation.

(b) Let $h \leq h'$ and let $X$ and $X'$ be equivalence classes of $\Delta^h$ and of $\Delta^{h'}$. Then either $X \cap X' = \emptyset$ or $X \subset X'$.

(c) Let $X$ and $Y$ be distinct equivalence classes of $\Delta^h$ that are subsets of the same stage. Then for all $x \in X$ and $y \in Y$, $\bar{d}(x, y)$ is constant.

**Proof:** Consider statement (a). The $\Delta^h$ relation is clearly reflexive and symmetric. To show that $\Delta^h$ is transitive, assume otherwise and let $h$ be the minimal member of $\mathbb{N}^+$ s.t. $\Delta^h$ is not transitive. Clearly $h < \infty$. Let $x$, $y$, and $z$ be three vertices of the same stage s.t. $\Delta^h(x, y)$, $\Delta^h(y, z)$ and not $\Delta^h(x, z)$. Clearly, $x$, $y$, and $z$ are distinct. By Lemma 2.2(b), $G$ has the V-property. This and the isometric property imply that for each forward neighbor of $y$ there is a forward neighbor of $x$ s.t. the forward distance between these neighbors is $\bar{d}(x, y) - 2$. Hence, there are $x'$, $y'$, and $z'$, forward neighbors of $x$, $y$, and $z$, respectively, s.t. $\bar{d}(x', y') = (\bar{d}(x, y) - 2)$ and $\bar{d}(y', z') = (\bar{d}(y, z) - 2)$. Clearly, $\bar{d}(x', z') \geq (\bar{d}(x, z) - 2)$. These inequalities contradict the minimality of $h$.

Statement (b) follows immediately from the definition of $\Delta^h$. To prove (c) let $X$ and $Y$ be two distinct equivalence classes of $\Delta^h$ on the same stage. Pick $x' \in X$ and $y' \in Y$ s.t. $\bar{d}(x', y') = d$ is minimal. Clearly $d \geq h$. The case of $d = \infty$ is easy, so assume $d$ is finite. The vertices $x'$ and $y'$ are in the same $\Delta^{d+1}$ equivalence class $Z$. By (b), $X, Y \subset Z$. Hence, $\bar{d}(x, y) \leq d$ for any $x \in X$ and $y \in Y$. By the minimality of $d$, $\bar{d}(x, y) = d$ for any such vertices.

A binary relation $R$ on a set $D$ is a function if for any $x \in D$ there is exactly one $y \in D$ with $R(x, y)$. The relation $R$ is a perfect matching if it is a function and is symmetric and irreflexive. The relation $R$ is the identity function if it is a function and is reflexive.

A set of vertices $X$ is a buddy of a set of vertices $Y$ if $X$ and $Y$ are subsets of the same stage and there is a vertex having two distinct forward neighbors, one in $X$, the other in $Y$.

**Lemma 2.5** Let $G$ have the isometric property and fork at $V_j$ and let $h \in \mathbb{N}^+$. Then the buddy relation on the set of the $\Delta^h$ equivalence classes of $V_{j+1}$ is either a perfect matching or the identity function.

**Proof:** Assume there is a vertex of $V_j$ whose two forward neighbors are in the same $\Delta^h$ equivalence class of $V_{j+1}$. In this case, by the isometric property, the same condition holds for any vertex of $V_j$. That is, the buddy relation is the identity function.

Assume, otherwise, that the two forward neighbors of any vertex of $V_j$ are in different equivalence classes. In this case the buddy relation is clearly irreflexive and symmetric. To show that this relation is a function, let $Y$ and $Z$ be buddies of $X$; let $y$ have two forward neighbors, $y' \in Y$ and $y'' \in X$, and let $z$ have two forward neighbors, $z' \in Z$ and $z'' \in Y$. See Figure 14. By the isometric property, $\bar{d}(y', z') = \bar{d}(y'', z'')$ and so $Y = Z$. $\blacksquare$
In the special case of \( h = 1 \), Lemma 2.5 implies that \( G \) has the buddy property. The lemma also implies, by induction, that the cardinality of any \( \Delta^h \) equivalence class of a stagenet having the isometric property is a power of two. This implies:

**Lemma 2.6** Any connected stagenet having the isometric property is properly-sized.

**Lemma 2.7** Let \( G \) have the isomorphic property and fork at \( V_1 \), let \( d = d(V_2) \) and let \( (x - x'), (y - y') \in (V_1 - V_2) \) and \( x \neq y \). Then exactly one of the following cases holds, and in each case the associated statement holds.

- **Case 1**, \( \Delta^{d+2}(x, y) \) and not \( \Delta^{d}(x', y') \). In this case \( \bar{d}(x', y') = d \).
- **Case 2**, \( \Delta^{d+2}(x, y) \) and \( \Delta^{d}(x', y') \). In this case \( \bar{d}(x', y') = \bar{d}(x, y) - 2 \).
- **Case 3**, not \( \Delta^{d+2}(x, y) \). In this case \( \bar{d}(x', y') = \bar{d}(x, y) - 2 \).

**Proof:** Clearly, exactly one of the above cases holds. Let \( x'' \) (\( y'' \)) be the other neighbor of \( x \) (\( y \)) in \( V_2 \). By the \( V \)-property,

\[
\bar{d}(x, y) = 2 + \min \left\{ \bar{d}(x^*, y^*) \mid x^* \in \{x', x''\}, \ y^* \in \{y', y''\} \right\}.
\]

Let \( X', X'', Y', Y'' \) be the \( \Delta^d \) equivalence classes of \( x', x'', y' \) and \( y'' \). By the definition of \( d \), \( X' \neq X'' \) and \( Y' \neq Y'' \). (Note that \( d \) may be infinite.) Consider the three cases of our lemma.

- **Case 1.** In this case \( X' = Y'' \) and \( X'' = Y' \); see Figure 15(a). By Lemma 2.4(c), \( \bar{d}(x', y') = \bar{d}(x', x'') = d \).
- **Case 2.** In this case \( X' = Y' \) and \( X'' = Y'' \); see Figure 15(b). This implies \( \bar{d}(x, y) = \bar{d}(x', y') + 2 \).
- **Case 3.** In this case all the classes \( X', Y', X'' \) and \( Y'' \) are distinct. To establish the conclusion of this case it suffices to show the following claim:

For all \( x^* \in \{x', x''\} \) and \( y^* \in \{y', y''\} \): \( \bar{d}(x^*, y^*) \) is constant.

This claim trivially holds when \( d = \infty \), so let us assume that \( d \) is finite. Let \( X \) and \( Y \) be the \( \Delta^{d+1} \) equivalence classes of \( x' \) and \( y' \). By the definition of \( d \) and Lemma 2.4(b), \( X', X'' \subset X \) and \( Y', Y'' \subset Y \). If \( X \neq Y \) then Lemma 2.4(c) implies our claim; see Figure 15(c). If \( X = Y \) then \( \bar{d}(x^*, y^*) = d \) for any \( x^* \) and \( y^* \) of our claim; see Figure 15(d). (Actually, \( X = Y \) never happens, but we do not know this yet.)

**From an isometry to an automorphism.** Recall that a permutation \( \pi \) of \( V_i \) is an *isometry* of \( V_i \) if \( \pi \) preserves the forward distance.
Theorem 2.8 Let $G$ have the isometric property and $\pi'$ be an isometry of $V_1$. Then:

(a) $\pi'$ can be extended into an automorphism $\pi$ of $G$.

(b) Moreover, there is such a $\pi$ in which for any $(x - x') \in \mathcal{E}(V_j - V_{j+1})$: either $x = \pi(x)$ and $x' = \pi(x')$ or $d(x', \pi(x')) = \overline{d}(x, \pi(x)) - 2$.

(Note that when $G$ is connected then the automorphism $\pi$ of (b) is the identity function on $V_{V_1}$.)

Proof: The case of $\delta^G = 1$ is trivial, so assume that $\delta^G > 1$. By induction it suffices to extend $\pi'$ to an automorphism $\pi$ of $G[1, 2]$ s.t. $\pi|_{V_2}$ is an isometry of $V_2$ and statement (b) holds for $(x - x') \in \mathcal{E}(V_1 - V_2)$. This extension is straightforward when $G$ does not fork at $V_1$, so assume that $G$ does fork there.

Let $d \triangleq \overline{d}(V_2)$. This value of $d$ and Lemma 2.5 imply that the buddy relation on the $\Delta^d$ equivalence classes of $V_2$ is a perfect matching. Hence, it is possible to assign a binary color $c(x) \in \{\text{Black}, \text{White}\}$ to every vertex $x \in V_2$ in the following manner: Any $\Delta^d$ equivalence class is monochromatic and $c(x') \neq c(x'')$ whenever $x'$ and $x''$ are backward buddies. Let $c$ be such a coloring.

Let $\pi$ be the unique extension of $\pi'$ to an automorphism of $G[1, 2]$ that preserves the color of the vertices; i.e., for any edge $(x - x') \in \mathcal{E}(V_1 - V_2)$, $\pi(x')$ is the forward neighbor of $\pi(x)$ whose color is $c(x')$. By this construction, the $\Delta^d$ relation is invariant under $\pi|_{V_2}$.

To show that $\pi|_{V_2}$ is an isometry of $V_2$ let $(x - x'), (y - y') \in \mathcal{E}(V_1 - V_2)$. If $x = y$ then $\overline{d}(x', y') = \overline{d}(\pi(x'), \pi(y'))$ by the isometric property. Assume that $x \neq y$. Since $\pi|_{V_1}$ is an isometry of $V_1$ and $\Delta^d$ is invariant under $\pi|_{V_2}$, the conditions of the cases of Lemma 2.7 are invariant under $\pi$. Hence, the same conclusion of the appropriate case holds both for $(x', y')$ and for $(\pi(x'), \pi(y'))$ thus implying $\overline{d}(x', y') = \overline{d}(\pi(x'), \pi(y'))$.

We now show that $\pi$ satisfies (b) for any edge $(x - x') \in \mathcal{E}(V_1 - V_2)$. The hard case is when $\pi(x) \neq x$ in which, by our construction, the two edges $(x - x')$ and $(\pi(x) - \pi(x'))$ fall into either Case 2 or Case 3 of Lemma 2.7. In both cases we have $\overline{d}(x', \pi(x')) = \overline{d}(x, \pi(x)) - 2$. \qed

Lemma 2.9 Let $G$ have the isometric property and fork at $V_1$. Then there is a $k$, $2 \leq k \leq \delta^G$, and an automorphism $\pi$ of $G$ such that:

(a) $\pi|_{V_i}$ is the identity function for every $i \notin [2, k]$.

(b) The binary relation `$\pi(X) = Y$' on the connected components of $G[2, k]$ is a perfect matching.

Proof: By Lemma 2.5, the backward buddy relation on the vertices of $V_2$ is a perfect matching. Let $\pi'$ be the permutation of $V_2$ that swaps each vertex with his backward buddy. By the isometric property, $\pi'$ is an isometry of $V_2$. Let $\pi$ be an automorphism of $G[2, \infty]$ provided by Lemma 2.8(b). We extend $\pi$ over all of $G$ by the identity function of $V_1$. By this construction the extended $\pi$ is an automorphism of $G$.

Let $d = \overline{d}(V_2)$ and define $k \triangleq 1 + \frac{d}{2}$ if $d < \infty$ and $k \triangleq \delta^G$ otherwise. By Lemma 2.8(b), statement (a) holds.

Let block mean a connected component of $G[2, k]$. Since $\pi$ is an automorphism of $G[2, k]$, it maps a block to a block. We show that $\pi(X) \neq X$ and that $\pi^2(X) = X$, for any block $X$. Let $x \in X \cap V_2$. If $X = \pi(X)$ then $x$ and $\pi(x)$ are in the same block $X$; Lemma 2.2(c) for the slice $G[2, k]$ implies $\overline{d}_G^{|V_2|}(x, \pi(x)) < d$ contradicting the definition of $d$. Hence, $X \neq \pi(X)$. The definition of $\pi'$ and $\pi|_{V_2} = \pi'|_{V_2}$ imply $\pi^2(x) = x$. Hence, $\pi^2(X) = X$. In summary, the binary relation `$\pi(X) = Y$' is a perfect matching of the blocks. \qed

3 Layered Cross Product

This section presents the Layered Cross Product and many of its observations are due to [12]. Two notable attributes of the LCP are highlighted – it preserves many important properties of stagewnets.
Homomorphism. A homomorphism of a graph $G = (V^G, E^G)$ into a graph $Q = (V^Q, E^Q)$ is a mapping $\varphi: V^G \rightarrow V^Q$ s.t. $(u - v) \in E^G$ implies $(\varphi(u) - \varphi(v)) \in E^Q$. When $G$ and $Q$ are stagenets, it is also required that $\varphi$ preserves the stages and that the two stagenets have the same depth. Note that if $\varphi$ is a homomorphism of a stagenet $G$ into a stagenet $Q$ and $P$ is a path of $G$ then $\varphi(P)$ is a path of $Q$. This implies the following lemma.

Lemma 3.1 Let $\varphi$ be a homomorphism of a stagenet $G$ into a stagenet $Q$ and let $(x, y)$ be vertices of $G$. Then $d^Q(\varphi(x), \varphi(y)) \leq d^G(x, y)$.

Layered Cross Product. Two stagenets, $G_1$ and $G_2$, are compatible, denoted $G_1 \sim G_2$, if they have the same depth. The Layered Cross Product (LCP) of compatible $G_1$ and $G_2$ is a compatible stagenet denoted $G_1 \bowtie G_2$ and defined by:

1. $V^G_{i \bowtie G_2} = V^G_i \times V^G_2$ for every $i$. (‘$\times$’ is the Cartesian product.)
2. $(u_1, u_2) - (v_1, v_2)$ is an edge of $E^G_{i \bowtie G_2}$ iff $(u_1 - v_1)$ and $(u_2 - v_2)$ are edges of $E^G_i$ and $E^G_2$.

![Figure 16: The LCP operation.](image)

The LCP operation is depicted in Figure 16. As we consider isomorphic stagenets to be identical, we have the following crucial fact.

Lemma 3.2 The LCP operation is associative and commutative.

The stagenet that is a straight path of depth $\delta$, denoted $1^\delta$, is the identity element for stagenets of depth $\delta$; i.e., $G \bowtie 1^\delta \cong G$ for any such $G$.

For $G_1 \sim G_2$, we extend the notation of $\bowtie$ to be defined on vertices and edges of these stagenets as follows: For $u_1 \in V^{G_1}$, $u_2 \in V^{G_2}$ and $(u_1, u_2)$, let $u_1 \bowtie u_2$ denote the vertex $(u_1, u_2)$ of $G_1 \bowtie G_2$. For edges $e_1 = (u_1 - v_1) \in E^{G_1}$ and $e_2 = (u_2 - v_2) \in E^{G_2}$ of the same stages, let $e_1 \bowtie e_2$ denote the edge $((u_1, u_2) - (v_1, v_2))$ of $G_1 \bowtie G_2$. Note that every edge of $G_1 \bowtie G_2$ is of this form. See Figure 16(b). It follows that for any two vertices $u_1$ and $u_2$ of the same stage of $G_1$ and $G_2$, the backward (forward) degree of $u_1 \bowtie u_2$ in $G_1 \bowtie G_2$ is the product of the backward (forward) degrees of $u_1$ and $u_2$ in $G_1$ and $G_2$.

The projections. The projection of $G_1 \bowtie G_2$ into $G_i$, for $i = 0, 1$, is the mapping $\eta_i: V^{G_1 \bowtie G_2} \rightarrow V^{G_i}$ defined by $\eta_i(u_1 \bowtie u_2) \equiv u_i$. Each of the $\eta_i$ projections is a homomorphism of $G_1 \bowtie G_2$ into $G_i$ and if $G_1 \bowtie G_2$ is normal then $\eta_i$ is onto the vertices and onto the edges of $G_i$. (Note that $G_1 \bowtie G_2$ is

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13Homomorphism should not be confused with homeomorphism.
We extend the notation of $\bowtie$ to be defined on straight paths as follows: Let $P_1 = (v_1 \to v_2 - \cdots - v_n)$ and $P_2 = (u_1 \to u_2 - \cdots - u_n)$ be two straight paths of $G_1 \bowtie G_2$, respectively, with $(P_1)$ and $(P_2)$. Define $P_1 \bowtie P_2$ as the following straight path of $G_1 \bowtie G_2$: $P_1 \bowtie P_2 \triangleq ((v_1 \bowtie u_1) \to (v_2 \bowtie u_2) - \cdots - (v_n \bowtie u_n))$. The next lemma directly follows from the definition of the LCP.

**Lemma 3.3** Let $G_1 \sim G_2$ and let $v_1$, $v_2$ and $v$ be vertices of $G_1$, $G_2$ and $G_1 \bowtie G_2$, respectively. Then $v = v_1 \bowtie v_2$ iff $v_1 = \eta_1(v)$ and $v_2 = \eta_2(v)$. Let $P_1$, $P_2$ and $P$ be straight paths of $G_1$, $G_2$ and $G_1 \bowtie G_2$, respectively. Then $P = P_1 \bowtie P_2$ iff $P_1 = \eta_1(P)$ and $P_2 = \eta_2(P)$.

**Preserved properties.** A property $\Psi$ of stagenets is *preserved upward under LCP* if whenever $G_1$ and $G_2$ are compatible and normal and both have the $\Psi$ property then $G_1 \bowtie G_2$ has the $\Psi$ property. A property $\Psi$ is *preserved downward under LCP* if whenever $G_1 \bowtie G_2$ is normal and has the $\Psi$ property then both $G_1$ and $G_2$ have the $\Psi$ property. A property $\Psi$ is *strongly preserved under LCP* if it is preserved upward and downward under LCP. For example, the property of being regular is preserved upward but not downward under LCP. For any fixed $k$, the property of being $k$-semiregular is preserved downward, but not upward, under LCP. The property of being semiregular is strongly preserved under LCP.

**Lemma 3.4** The property of being connected is strongly preserved under LCP.

**Proof:** We use the following fact about a connected normal stagenet $Q$. For any large enough odd integer $k$, any input and output of $Q$ are connected by a path which is a concatenation of $k$ end-to-end paths. (This path is not necessarily simple.)

First we show that the property in question is preserved upward under LCP. Let $G_1$ and $G_2$ be connected normal stagenets. Since $G_1 \bowtie G_2$ is normal, it suffices to show that there is a path between any input and output of $G_1 \bowtie G_2$, say $u_1 \bowtie u_2$ and $v_1 \bowtie v_2$. By the above fact there are an integer $k$ and two paths, $P_1$ and $P_2$, s.t. each $P_i$ is a concatenation of $k$ end-to-end paths, $(S_1', S_2', \cdots, S_k')$, connecting $u_i$ and $v_i$ in $G_i$. By lemma 3.3, the concatenation of the end-to-end paths $(S_1', \bowtie S_2', \bowtie S_3', \cdots, \bowtie S_k')$ is a path connecting $u_1 \bowtie u_2$ and $v_1 \bowtie v_2$ in $G_1 \bowtie G_2$. That is, the property is preserved upward under LCP.

To show that this property is preserved downward, let $G_1 \bowtie G_2$ be a connected normal stagenet. By symmetry it suffices to show that $G_1$ is connected. Since $G_2$ is normal, $\eta_2$ is onto $V^{G_2}$. This fact, combined with the connectivity of $G_1 \bowtie G_2$ and the path-preserving property of $\eta_1$, implies that $G_1$ is connected.

**Banyan.** The next lemma directly follows from Lemma 3.3.

**Lemma 3.5**

(a) The at-most-one-path property is strongly preserved under LCP.

(b) The at-least-one-path property is strongly preserved under LCP.

(c) The banyan property is strongly preserved under LCP.

**Product of isomorphisms.** Let $\pi_1: G_1 \to G_1'$ and $\pi_2: G_2 \to G_2'$ be two isomorphisms and $G_1 \sim G_2$. Define the mapping $(\pi_1 \bowtie \pi_2): V^{G_1 \bowtie G_2} \to V^{G_1' \bowtie G_2'}$ by:

$$(\pi_1 \bowtie \pi_2)(x) \triangleq \pi_1(\eta_1(x)) \bowtie \pi_2(\eta_2(x)).$$

The following observation is straightforward.
Lemma 3.6 Let $\pi_1: G_1 \to G'_1$ and $\pi_2: G_2 \to G'_2$ be two isomorphisms and $G_1 \sim G_2$. Then $\pi_1 \bowtie \pi_2$ is an isomorphism of $G_1 \bowtie G_2$ onto $G'_1 \bowtie G'_2$.

Not every isomorphism of $G_1 \bowtie G_2$ onto $G'_1 \bowtie G'_2$ is a product of two isomorphisms as per the above lemma. Figure 17 shows two compatible stagenets each having two automorphisms, while their product has eight.

![Figure 17: The product has eight automorphisms while each factor has two](image)

Lemma 3.7 The path-symmetric property is preserved upward under LCP.

(Actually, as shown in [20], the path-symmetric property is strongly preserved under LCP, but we do not use this fact.)

Proof: Let $G_1 \sim G_2$ and both have the path-symmetric property and let $P$ and $P'$ be two end-to-end paths of $G_1 \bowtie G_2$. We construct an automorphism of $G_1 \bowtie G_2$ that swaps $P$ and $P'$ as follows. For $i = 1, 2$, let $\varphi_i$ be an automorphism of $G_i$ that swaps $\eta_i(P)$ and $\eta_i(P')$. By Lemma 3.6, $\varphi_1 \bowtie \varphi_2$ is an automorphism of $G_1 \bowtie G_2$. Using the natural extension of $\varphi_1 \bowtie \varphi_2$ to sequences of vertices, we have $(\varphi_1 \bowtie \varphi_2)(P) = \varphi_1(\eta_1(P)) \bowtie \varphi_2(\eta_2(P))$. By the specific property of $\varphi_1$ and $\varphi_2$, $\varphi_1(\eta_1(P)) \bowtie \varphi_2(\eta_2(P)) = \eta_1(P') \bowtie \eta_2(P')$. By Lemma 3.3, $\eta_1(P') \bowtie \eta_2(P') = P'$. Combining the above equalities yields $(\varphi_1 \bowtie \varphi_2)(P) = P$. By symmetry we also have $(\varphi_1 \bowtie \varphi_2)(P') = P$. □

Lemma 3.8 The isometric property is preserved downward under LCP.

Proof: Let $G_1 \bowtie G_2$ have the isometric property. Clearly, $G_1$ and $G_2$ are 2-semiregular. Due to symmetry, it suffices to prove that $G_1$ has the isometric property and for this it is enough to show that for any $x', x'' \in V^{G_1}$ and $z \in V^{G_2}$ of the same stage:

(a) $x'$ and $x''$ are backward buddies (in $G_1$) iff $x' \bowtie z$ and $x'' \bowtie z$ are backward buddies (in $G_1 \bowtie G_2$),

(b) $d^{G_1}(x', x'') = d^{G_1 \bowtie G_2}(x' \bowtie z, x'' \bowtie z)$.

Statement (a) follows from the definition of the LCP. The inequality $d^{G_1}(x', x'') \leq d^{G_1 \bowtie G_2}(x' \bowtie z, x'' \bowtie z)$ follows from Lemma 3.1 and the fact that $\eta_1$ is a homomorphism. Let $S$ be an end-to-end path of $G_2$ having $z$, $G_1 \bowtie S$ is a sub-stagenet of $G_1 \bowtie G_2$ which is isomorphic to $G_1$ by an isomorphism that maps $x'$ and $x''$ to $x' \bowtie z$ and $x'' \bowtie z$. Hence, $d^{G_1}(x', x'') \geq d^{G_1 \bowtie G_2}(x' \bowtie z, x'' \bowtie z)$.

Commuting with LCP. A unary operator $\rho$ on stagenets commutes with LCP if $\rho(G_1 \bowtie G_2) \cong \rho(G_1) \bowtie \rho(G_2)$ for any $G_1 \sim G_2$. The following lemma is straightforward.

Lemma 3.9

(a) For fixed $i \leq j$, the $i$-to-$j$ slice operator, $G \mapsto G[i, j]$, commutes with LCP.

(b) The inverse operator, $G \mapsto \overleftarrow{G}$, commutes with LCP.

The distributive law. For $G_1 \sim G_2$, define $G_1 + G_2$ as the compatible stagenet composed of two disjoint sub-stagenets, one isomorphic to $G_1$, the other to $G_2$. The `$+$' operator is clearly commutative and associative and the following observation is straightforward.
Lemma 3.10 The operators $\cdot$ and $+$ obey the distributive law. That is, $G_1 \cdot (G_2 + G_3) \cong (G_1 \cdot G_2) + (G_1 \cdot G_3)$ for any three compatible stagenets $G_1$, $G_2$ and $G_3$.

Lemmas 3.4 and 3.10 immediately imply the following lemma.

**Lemma 3.11** Let $G_1$ and $G_2$ be compatible normal stagenets having $m_1$ and $m_2$ connected components, respectively. Then $G_1 \cdot G_2$ has $m_1 \cdot m_2$ connected components.

4 Factorization

This section establishes most of our theorems by factoring the relevant stagenets into a product of elementary ones.

4.1 Factorization to beads

This sub-section presents the bead stagenets and shows, via a factorization into beads, that the properties considered in Section 2 are equivalent.

**Bead.** A *bead* is a normal stagenet having exactly two end-to-end paths; see Figure 18. By this definition, each stage of a bead has one or two vertices. The stages of the latter type constitute a non-empty interval that uniquely determines the bead. Let $B^\delta_{i,j}$ denote the bead of depth $\delta$ whose interval is $[i,j]$. The only disconnected bead of depth $\delta$ is $B^\delta_{1,\delta} = 1^\delta + 1^\delta$. A bead is clearly path-symmetric and, by Lemma 3.7, the same is true for a product of beads.

Figure 18: Some beads of depth 5.

**Lemma 4.1** Let $G$ be normal and $G \sim B^\delta_{j,k}$. Then the following statements are equivalent:

(a) $G \cong Q \cdot B^\delta_{j,k}$ for some stagenet $Q$.

(b) There is an automorphism $\pi$ of $G$ such that:

1. $\pi|_V$ is the identity function for every $i \notin [j,k]$.
2. The binary relation $\pi(X) = Y$ on the connected components of $G[j,k]$ is a perfect matching.

**Proof:** By Lemmas 3.4, 3.6, 3.9(a) and 3.10, (a) implies (b). To show the other direction, let block mean a connected component of $G[j,k]$. Since $\pi|_{G[j,k]}$ is an automorphism of $G[j,k]$, it maps a block to a block. Due to (b.2), there is a set $Q'$ of blocks such that for any block $X$: $X \in Q'$ if $\pi(X) \notin Q'$. Let

$$Q' = (\bigcup_{X \in Q} X) \cup (\bigcup_{i \notin [j,k]} V_i)$$

and let $Q$ be the sub-stagenet of $G$ induced by the vertices of $Q'$.

We show that $G \cong Q \cdot B^\delta_{j,k}$. Let $B = B^\delta_{j,k}$ and let $P_1$ and $P_2$ be the two distinct end-to-end paths of $B$. Define the mappings $\varphi, \varphi': V^{Q \cdot B} \to V^G$ by:

$$\varphi(q \cdot b) \triangleq \begin{cases} q & \text{if } b \in P_1 \\ \pi(q) & \text{otherwise} \end{cases} \quad \varphi'(q \cdot b) \triangleq \begin{cases} \pi(q) & \text{if } b \in P_2 \\ q & \text{otherwise} \end{cases}$$
First we show that $\varphi = \varphi'$. Let $q \preceq b \in V_i^{Q\bowtie B}$. If $i \not\in [j,k]$ then $\pi(q) = q$. If $i \in [j,k]$ then $b \in P_1$ iff $b \not\in P_2$. In both cases, $\varphi(q \preceq b) = \varphi'(q \preceq b)$.

We show next that $\varphi$ is an isomorphism of $Q \bowtie B$ onto $G$. By our construction, $|V_i^{Q\bowtie B}| = |V_i^G|$ and $|E_i^{Q\bowtie B}| = |E_i^G|$ for every $i$. Due to these equalities, it suffices to show that $\varphi$ is a homomorphism of $Q \bowtie B$ into $G$ which is onto $V^G$.

To show that $\varphi$ is onto $V^G$ let $g$ be a vertex of $G$. By our construction, either $g \in Q$ or $\pi^{-1}(g) \in Q$ (or both). Hence, for some vertex $b \in B$ either $b \in P_1$ and $\varphi(g \preceq b) = g$ or $b \in P_2$ and $\varphi(\pi^{-1}(g) \preceq b) = \varphi'(\pi^{-1}(g) \preceq b) = g$.

It remains to show that $\varphi$ is an homomorphism. Let $\overline{q} \preceq \overline{b}$ be an edge of $Q \bowtie B$. $\overline{b}$ is either an edge of $P_1$ or an edge of $P_2$ (or both). In the first case $\varphi(\overline{q} \preceq \overline{b}) = \overline{q}$, an edge of $G$. In the second case $\varphi(\overline{q} \preceq \overline{b}) = \varphi'(\overline{q} \preceq \overline{b}) = \pi(\overline{q})$, again an edge of $G$ since $\pi$ is an automorphism. □

The fringe maker. For $k \in \mathbb{N}^+$, a $k$-way input (output) fringe is a depth-2 stagenet s.t., the forward (backward) degree of its $V_1$ ($V_2$) vertices is one and the backward (forward) degree of its $V_2$ ($V_1$) vertices is $k$. The $k$-way input (output) fringe operator, denoted $\phi_k$ ($\bar{\phi}_k$), add a $k$-way fringe to a given stagenet at the input (output) end; e.g., $(\phi_k(G))[2, \infty] = G$ and $(\bar{\phi}_k(G))[1, 2]$ is a $k$-way input fringe. The next lemma is straightforward.

Lemma 4.2

(a) $\phi_{1+k^*}(G^i \bowtie G''^j) = \phi_{k^*}(G^i) \bowtie \phi_{k^*}(G'')$. In particular, the $\phi_1$ operator commutes with LCP.

(b) $\phi_1(B)$ is a bead for any bead $B$.

The same statements hold for $\bar{\phi}$.

Lemma 4.3 Any properly-sized $G$ having the isometric property is a product of beads.

Proof: We prove the lemma by induction on the cardinality of $V^G$. Let $G$ be such a stagenet and consider the following four cases that are related to $d_1$, the forward degree of the vertices of $V_1$, and to $d_2$, the backward degree of the vertices of $V_2$, if any.

Case 1, $d_1 = 0$. In this case $\delta = 1$ and since $G$ is properly-sized, $G$ is a product of beads.

Case 2, $d_1 = 1$ and $d_2 = 1$. In this case, $G = \phi_1(G[2, \infty])$. By the induction hypothesis, $G[2, \infty]$ is a product of beads and, by Lemma 4.2, $G$ is a product of beads.

In the following two cases we show that $G = Q \bowtie B$ for some stagenet $Q$ and some bead $B$. This clearly implies that $Q$ is properly-sized and $|V^Q| < |V^G|$. By Lemma 3.8, $Q$ has the isometric property and, by induction, $G$ is a product of beads.

Case 3, $d_1 = 1$ and $d_2 = 2$. By Lemma 4.1, $G \cong Q \bowtie B_{1,1}^\delta$ for some $Q$.

Case 4, $d_1 = 2$. In this case $G$ satisfies the premise of Lemma 2.9. The conclusion of this lemma is a premise of Lemma 4.1 thus implying that $G$ is of the form $Q \bowtie B_{2,k}^\delta$. □

Lemma 4.4 The following statements are equivalent for any properly-sized, 2-semiriugular $G$:

(a) $G$ is a product of beads.

(b) $G$ is path-symmetric.

(c) $G$ is same-stage edge-symmetric.

(d) $G$ has the diamond and the shortcut properties.

(e) $G$ has the isometric property.
4.2 Factorization to bamboos and to rosaries

This sub-section establishes most of our characterization theorems by factoring the GBN graphs into a product of either bamboos or rosaries.

Bamboo. A bamboo is a 2-semiregular stagenet having exactly two vertices in each stage. See subfigure (c) of Figures 1 to 3. By this definition, each slice of depth two of a bamboo is either a perfect matching or a complete bipartite graph. An edge-stage of the former type is called hollow and of the latter type solid. A bamboo is disconnected iff all its edge-stages are hollow.

Rosary. A rosary is a 2-semiregular stagenet having exactly two edges in each edge-stage. See subfigure (d) of Figures 1 to 3. By this definition, a stage of a rosary has either one or two vertices. A stage of the former type is called a singleton. Since our graphs are simple, a rosary does not have two consecutive singletons. A rosary is disconnected if it has no singletons.

All three types of stagenets, the bamboo, the rosary and the bead, are path-symmetric. For any depth \( \delta \), \((1^\delta + 1^\delta)\) is the only disconnected bamboo of depth \( \delta \), as well as the only disconnected rosary and disconnected bead of the same depth.

**Lemma 4.5** Let \( G \) be a 2-regular stagenet. Then the following statements are equivalent:

(a) \( G \) is a product of rosaries.

(b) \( G \) is a product of bamboos.

(c) \( G \) is a product of beads.

**Proof:** Clearly any bamboo and rosary is 2-semiregular, path-symmetric and properly-sized. By Lemma 4.4 each of them is a product of beads. Therefore, each of (a) and (b) implies (c).

To show that (c) implies (a), let \( G \) be a 2-regular stagenet of depth \( \delta \) and let \( G = B_1 \bowtie B_2 \bowtie \ldots \bowtie B_k \) where the \( B_i \) are beads. We partition the set \( W = \{B_1, B_2, \ldots, B_k\} \) into subsets s.t. the product of the beads of each subset is a rosary; see Figure 19. To this end, define a binary relation, \( F \), on \( W \) by:

\[
F(B', B'') \quad \text{iff} \quad \text{one of } \{B', B''\} \text{ forks and the other joins at the same stage.}
\]

Let \( T \) be the transitive closure of \( F \) in \( W \). Since \( F \) is symmetric, \( T \) is an equivalence relation. Since \( G \) is 2-regular, the product of the beads in each of the equivalence classes of \( T \) is a rosary. \( G \) is the product of those rosaries multiplied by the proper power of \( B_{1,\delta}^\delta \), this last factor, \( B_{1,\delta}^\delta = (1^\delta + 1^\delta) \), is also a rosary. This concludes the proof that (c) implies (a). The proof that (c) implies (b) is similar and is omitted; see Figure 20. \( \square \)

![Figure 19: Combining beads to rosaries.](image-url)
Theorem 4.6 below contains most of our characterizations of the GBN family. Its proof is based on Theorem 7.5 which is proved in a later section. Hence, in the current draft of this work it is not crystal-clear that our deductions are not circular. Hopefully, this weakness will be corrected in a later version of this work.

**Theorem 4.6** Let $G$ be a 2-regular stagenet that is either connected or properly-sized. Then the following statements are equivalent:

(a) $G$ is a GBN.

(b) $G$ is a product of bamboos.

(c) $G$ is a product of rosaries.

(d) $G$ is a product of beads.

(e) $G$ is path-symmetric.

(f) $G$ is same-stage edge-symmetric.

(g) $G$ has the diamond and the shortcut properties.

(h) $G$ has the isometric property.

**Proof:** First we show that any connected stagenet satisfying one of the above statements is properly-sized. This clearly holds for statements (a) to (d). By Lemma 2.3, each one of (e), (f) and (g) implies (h) and, by Lemma 2.6, any connected stagenet satisfying (h) is properly-sized.

We henceforth assume that $G$ is properly-sized. By Theorem 7.5, (a) and (b) are equivalent. By Lemma 4.5, statements (b), (c) and (d) are equivalent. By Lemma 4.4, statements (d), (e), (f), (g) and (h) are equivalent.

4.3 Unique factorization

This sub-section shows, via collinear sub-stagenets and diamonds, that a 2-semiregular stagenet has at most one factorization into beads, as well as at most one factorization into rosaries and into bamboos. For two stagenets $Q$ and $G$, let $Q \sqsubset G$ denotes that $G$ has a collinear sub-stagenet isomorphic to $Q$. The next lemma follows directly from Lemma 3.3.

**Lemma 4.7** Let $G_1 \sim G_2$. Then:

(a) If $Q_1 \sqsupset G_1$ and $Q_2 \sqsubset G_2$ then $Q_1 \bowtie Q_2 \sqsubset G_1 \bowtie G_2$.

(b) If $G_2$ is normal then $G_1 \sqsubset G_1 \bowtie G_2$.

Two stagenets $Q'$ and $Q''$ are conflicting if they are compatible, 2-semiregular and $Q' \bowtie Q''$ is not 2-semiregular; i.e., there is a stage at which either both of them fork or both of them join.
Lemma 4.8 Assume \( G \) is a 2-semiregular stagenet having the diamond property, \( Q' \sqsubseteq G \), \( Q'' \sqsubseteq G \), \( Q' \) and \( Q'' \) are conflicting and either both are beads or both are rosaries or both are bambooos. Then \( Q' \equiv Q'' \).

Proof: Consider the case of the rosaries. Since \( Q' \) and \( Q'' \) are conflicting, there is an \( i \) s.t. the \( i \)-th stages of \( Q' \) and \( Q'' \) are singletons. Let \( j \) be the minimal \( j' \) s.t. \( j' > i \) and stage \( j' \) of either \( Q' \) or \( Q'' \) is a singleton. Say that stage \( j \) of \( Q' \) is a singleton. This \( Q' \) has a diamond whose endpoints are in stages \( i \) and \( j \) and therefore the same is true for \( G \). The 2-semiregularity and the diamond property of \( G \), together with \( Q'' \sqsubseteq G \), imply that stage \( j \) of \( Q'' \) is also a singleton. Iteration of this argument and symmetry yield that the same stages are singletons in both \( Q' \) and \( Q'' \), implying \( Q' \equiv Q'' \). The cases of the beads and of the bambooos are similar and are omitted.

\[ \square \]

Theorem 4.9 Let \( F \) be either the bead family or the rosary family or the bamboo family. Let \( f \) be a factorization of a 2-semiregular stagenet \( G \) into members of \( F \) and let \( Q \in F \). Then \( Q \) appears in \( f \) iff \( Q \sqsubseteq G \). Hence, a 2-semiregular \( G \) has at most one factorization into \( F \).

Proof: By Lemma 4.7(b), if \( Q \) appears in a factorization of \( G \) into normal stagenets then \( Q \sqsubseteq G \). This establishes one direction of the iff statement. For the other direction, let \( Q \in F \) and \( Q \sqsubseteq G \). We show that \( Q \) appears in every factorization \( f \) of \( G \) into \( F \). Let \( f \) be such a factorization. By Lemma 4.4, \( G \) has the diamond and the shortcut property. Say first that \( Q \) is disconnected. If \( G \) is connected then, by iteration of the shortcut property, \( G \) has the at-least-one-path property, contradicting \( Q \sqsubseteq G \). Hence, \( G \) is disconnected and \( Q \in f \) by Lemma 3.4.

Say next that \( Q \) is connected. In this case, \( Q \) either forks or joins at some stage \( i \), and the same holds for \( G \). Hence, there is a \( Q' \in f \) that either forks or joins at stage \( i \), the same way as \( Q \) and thus is conflicting with \( Q \). The diamond property of \( G \) and Lemma 4.8 imply that \( Q \equiv Q' \).

The 2-semiregularity of \( G \) implies that it has at most one factorization into \( F \), since the power of any connected member of \( F \) is at most one and the power of \( (1^\delta + 1^\delta) \) is determined by \( |V_1^G| \). □

Note that Theorems 4.6 and 4.9 imply that a GBN has a unique factorization into beads but they do not exclude other factorizations into primes. A stronger result is established in [20] — any path-symmetric stagenet has a unique factorization into primes.

Theorem 4.10 Two GBN graphs, \( G \) and \( Q \), are isomorphic iff \( \delta^G = \delta^Q \), \( |V_1^G| = |V_1^Q| \) and \( A^G = A^Q \).

Proof: The left to right implication is trivial as we consider only properties that are invariant under isomorphism. The other direction is established by showing that the unique factorization of a GBN \( G \) into beads is determined by \( A^G \) and \( |V_1^G| \) as follows. By definition, \( A^G(i, j) \) iff \( G \) has a collinear diamond whose endpoints are in \( V_i \) and \( V_j \). Since \( G \) is normal, \( G \) has such a collinear diamond with \( 0 \leq i < j \leq \delta \) iff \( B_{i,j-1}^\delta \sqsubseteq G \). Hence, by Theorem 4.9, \( A^G \) determines the power of such beads in the factorization. Once the power of the above beads is determined, the power of connected beads of the form \( B_{i,i}^\delta \) is determined by the 2-regularity of \( G \). Finally, \( |V_1^G| \) determines the power of the disconnected bead. □

4.4 Non-unique factorization

The uniqueness of the factorizations of Sub-section 4.3 is not derived from a unique factorization of every stagenet into a product of primes since, as Etzion and Roth [13] have shown, the ICP lacks such a unique factorization theorem even for normal stagenets.

A counterexample, of depth two, is constructed as follows. Let the complete bipartite graph \( K_{i,j} \) be interpreted as a stagenet of depth two with the first stage having the first part of \( i \) vertices and the second stage having the second part of \( j \) vertices. For an integer \( m \) and a stagenet \( G \) define \( m \cdot G \) by:
By the distributive law, Lemma 3.10, we have:

\[(2 \cdot K_{1,1} + K_{1,2} + K_{1,8}) \bowtie (2 \cdot K_{1,1} + K_{1,2})\]

\[= 4 \cdot K_{1,1} + 4 \cdot K_{1,2} + K_{1,4} + 2 \cdot K_{1,8} + K_{1,16}\]

\[= (4 \cdot K_{1,1} + K_{1,4} + K_{1,8}) \bowtie (K_{1,1} + K_{1,2}).\]

It is not hard to check that the four factors in the above equalities are prime. (These factors are disconnected but may be made connected via an additional stage.) This establishes the following theorem.

**Theorem 4.11** Not every normal stagenet has a unique factorization to prime factors.

## 5 Almost a GBN

This section shows that a stagenet can be very close to a GBN without actually being one. It constructs three families of such stagenets, each having distinct properties. The construction is done via the LCP, demonstrating that this operator is a convenient tool, not only for constructing highly regular interconnection networks, but also for constructing 'custom-made' networks having particular properties. To this end we use the following operations.

**Twos-complement and two-completion.** Let \(G\) be a 2-semiregular stagenet. The **twos-complement** of \(G\) is the unique stagenet \(Q_{s,t}\) such that \(Q_{s,t}\) is a product of banyan beads and \(G \bowtie Q_{s,t}\) is 2-regular. (See Figure 21(c) where \(C^*\) is the twos-complement of \(C^t\).) There is exactly one such \(Q_{s,t}\) since the power of each bead in this product is at most one, \(B_{k,1}^2\) appears in the product iff \(G\) does not join at stage \(k+1\), and \(B_{k,1}^2\) appears in the product iff \(G\) does not fork at stage \(k-1\). The **two-completion** of a 2-semiregular stagenet \(G\) is the 2-regular stagenet \(\Upsilon(G) \triangleq G \bowtie Q\) where \(Q\) is the two-complement of \(G\). The operator \(\Upsilon\) is undefined for stagenets which are not 2-semiregular.

A property of stagenets is **preserved under** \(\Upsilon\) if whenever \(G\) is a 2-semiregular stagenet having this property then \(\Upsilon(G)\) has this property. The following lemmas are direct consequences of previous ones.

**Lemma 5.1** The following properties are preserved under \(\Upsilon\):

(a) Being (end-for-end) symmetric.

(b) Being connected.

(c) The at-most-one-path property.

(d) The banyan property.

**Conservative slice.** The **i-to-\(j\) slice of a stagenet \(G\) is a conservative slice** if \(G\) is 2-semiregular and the \(i\)-to-\(j\) slice of its two-complement is connected; i.e., if \(G\) forks in all stages before the \(i\)-th one and joins in all stages after the \(j\)-th one. For example, the 2-to-3 slice of \(C^t\) in Figure 21(c) is conservative while the 1-to-2 and the 3-to-4 ones are not.

**Lemma 5.2** The following operators commute with \(\Upsilon\):

(a) For any fixed \(i\) and \(j\), the \(i\)-to-\(j\) slice operator, under the condition that the slice is conservative.
Conformity and pre-butterflies. For \( G \sim S \), we say that \( G \) conforms to \( S \) on the \( i \)-to-\( j \) slice if \( G \) is 2-semi-regular and \( \Upsilon(G)[i, j] \cong S[i, j] \); we usually abuse this expression and say instead that \( G[i, j] \) conforms to \( S \). A pre-butterfly is a 2-semi-regular stagenet which is a product of banyan beads. By Theorem 4.6 and Lemma 3.5, the two-completion of a pre-butterfly is a Butterfly. (Actually, as shown in [20], only the pre-butterflies have this property.) For stagenets \( G \) and \( G' \) and an integer \( k \), we say that \( G \) is \( k \) times \( G' \) if \( G \cong k \cdot G' \). Lemma 5.2(a,b) implies that if \( G \sim S, S[i, j] \) is \( k \) times a Butterfly, \( G[i, j] \) is a conservative slice of \( G \) and is \( k \) times a pre-butterfly then \( G[i, j] \) conforms to \( S \).

The above infrastructure enables us to reduce the construction of 2-regular networks having particular properties to the construction of 2-semi-regular networks having similar properties. We construct the latter networks via the next simple operator.

**Stretching.** Let \( G \) be a stagenet, \( k \geq 1 \) and \( 1 \leq i < \delta^G \). We say that a stagenet \( G' \) is the (result of) \( k \)-fold stretching of \( G \) at \( i \), denoted \( G' = \zeta^k_i(G) \), if \( G' \) is derived from \( G \) by replacing each edge of \( E_i \) with a straight path of length \( k \) having the same endpoints. The stretched stagenet has \( k-1 \) more stages than the original one. For example, in Figure 21 the stagenet \( C' \) is the 3-fold stretching of \( C_S \) at 1.

**Theorem 5.3** For any \( \delta \geq 2 \) there is a 2-regular, symmetric, connected stagenet of depth \( \delta \) having the at-most-one-path property which is not a GBN but is \((\delta-1)\)-isomorphic to a double-butterfly.

(As shown in [20], the stagenet of the above theorem is unique.)

![Figure 21: Construction of a quasi GBN.](image)

Let us refer to a stagenet satisfying the conditions of the above theorem as a quasi GBN. Let \( C_S \) denote the depth-2 stagenet that is an 8-cycle; see Figure 21(a). It is easy to check that \( C_S \) is a quasi GBN. Moreover, we show that \( \Upsilon(\zeta^k_i(C_S)) \) is a quasi GBN for any \( k > 1 \).

Let \( C' = \zeta^1_i(C_S), Q = \Upsilon(C') \) and \( \delta = \delta^Q \); see Figure 21. By definition, \( Q \) is 2-regular. The stagenet \( C' \) is connected, symmetric and has the at-most-one-path property; Lemma 5.1 implies that the same holds for \( Q \). We have to check that any depth-(\( \delta-1 \)) slice of \( C' \) conforms to the double-butterfly of depth \( \delta \). As shown in Figure 21(b), \( C'[1, \delta-1] \) is a conservative slice of \( C' \) and is four times a pre-butterfly; hence, it conforms to the required stagenet. Due to symmetry, the same holds with the other depth-(\( \delta-1 \)) slice. It remains to show that \( Q \) is not a GBN, for which we use the following observation.

**Lemma 5.4** Let \( Q \) have the shortcut property and \( G \sqsupseteq Q \). Then \( G \) has the shortcut property.

Clearly, \( C_S \) and \( C' \) lack the shortcut property. By Lemmas 4.7(b) and 5.4, any normal stagenet which is a product of \( C' \) lacks the shortcut property and thus is not a GBN. This concludes the proof of Theorem 5.3.
5.1 Almost a Butterfly

Hotzel [16] has shown:

**Theorem 5.5** For any \( \delta \geq 3 \) there is a 2-regular, banyan stagennet of depth \( \delta \) which is not a Butterfly but is \((\delta - 2)\)-isomorphic to one. In fact, there are exactly two such stagennets and each is the inverse of the other.

As noted by Hotzel, the counterexample of Bermond et al. [6] is one of the above two stagennets.

We show that the two stagennets of this theorem can be conveniently constructed via the LCP similarly to those of Theorem 5.3. (Another technique is used in [20] to show that there are at most two such stagennets.) Let us refer to a stagennet satisfying the conditions of the above theorem as a quasi Butterfly. Let \( W \) be the depth-3, 2-semiregular, banyan stagennet shown in Figure 22(a). It is easy to check that \( \Upsilon(W) \) is a quasi Butterfly. Moreover, we show that \( \Upsilon(\zeta_k^1(W)) \) is a quasi Butterfly for any \( k > 1 \).

![Figure 22: Construction of a quasi Butterfly.](image)

Let \( W' = \zeta_k^1(W) \), \( Q = \Upsilon(W') \) and \( \delta = \delta^Q \); see Figure 22. Since \( W' \) is banyan, Lemma 5.1(d) implies that the same holds for \( Q \). As shown in Figure 22(b), \( W'[1, \delta - 2] \) is a conservative slice and is four times a pre-butterfly; hence, it conforms to the depth-\( \delta \) Butterfly. Concerning the conformity of its other two depth-\((\delta - 2)\) slices, \( W' \) actually satisfies a stricter condition; as shown in Figure 22(b), \( W'[2, \delta] \) is a conservative slice of \( W' \) and is two times a pre-butterfly; hence, it conforms to the depth-\( \delta \) Butterfly.

The stagennet \( Q \) is not (end-for-end) symmetric. While \( Q[2, \delta] \) is the sum of two Butterflies, \( Q[1, \delta - 1] \) is connected. These facts have been observed by Hotzel, but they are much more visible in the LCP construction. In contrast to this non-symmetry, the next theorem provides a symmetric stagennet.

**Theorem 5.6** For any \( \delta \geq 4 \) there is a symmetric, 2-regular, banyan stagennet of depth \( \delta \) which is not a Butterfly but is \((\delta - 3)\)-isomorphic to one.

Let us refer to a stagennet satisfying the conditions of the above theorem as a symmetric quasi Butterfly. Let \( Z \) be the depth-4, 2-semiregular, banyan stagennet shown in Figure 23(a). It is easy to check that \( Z \bowtie \overline{Z} \) is 2-semiregular and that \( \Upsilon(Z \bowtie \overline{Z}) \) is a symmetric quasi Butterfly. Moreover, we show that \( \Upsilon(\zeta_k^1(Z) \bowtie \overline{\zeta_k^1(Z)}) \) is a symmetric quasi Butterfly for any \( k > 1 \).

![Figure 23: Construction of a symmetric quasi Butterfly.](image)
Let \( Z'' = \xi_1(Z) \), \( Z' = Z'' \bowtie Z'', Q = \Upsilon(Z') \) and \( \delta = \delta_Q \); see Figure 23. Clearly, \( Z'' \) is 2-semiregular. Moreover, the forward degree of any of its vertices, but the inputs, is at most one and the backward degree of the outputs is one; this implies that \( Z' \) is 2-semiregular and \( \Upsilon(Z') \) is defined. Since \( Z' \) is symmetric and banyan, Lemma 5.1 implies that the same holds for \( Q \). As shown in Figure 23(b), \( Z''[2, \delta - 1] \) is two times a pre-butterfly; hence, \( Z'[2, \delta - 1] \) is four times a pre-butterfly; moreover, \( Z''[\delta - 1] \) is a conservative slice of \( Z' \); (in contrast, \( Z''[2, \delta - 1] \) is not a conservative slice of \( Z'' \)); hence, it conforms to the depth-\( \delta \) Butterfly.

It remains to check the conformity of the two depth-(\( \delta - 3 \)) slices of \( Z' \) that are not included in \( Z'[2, \delta - 1] \); due to symmetry, it suffices to check \( Z'[1, \delta - 3] \). As shown in Figure 23(c), \( Z''[1, \delta - 3] \) is four times a pre-butterfly and \( Z''[1, \delta - 3] \) is two times a pre-butterfly; hence, \( Z'[1, \delta - 3] \) is eight times a pre-butterfly; moreover, this slice is a conservative one and thus it conforms to the depth-\( \delta \) Butterfly.

6 Labeled LCP

This section introduces the labeled LCP – an associative and commutative variant of the LCP that operates on labeled stagenets. It starts with the definition of these labeled structures.

**Label-space.** A label-space is an ordered pair \( \Lambda = \langle L, @ \rangle \), where \( L \) is a set of elements called labels and \( @ \) is an associative and commutative binary operator on \( L \) having a neutral element denoted \( 0^\Lambda \).

**Labeled stagenet.** An edge-labeled stagenet, or an el-stagenet in short, (a vertex-labeled stagenet, or a vel-stagenet in short) is a triple \( H = \langle P, \Lambda^H, \lambda^H \rangle \) where \( P \) is a (plain) stagenet called the underlying stagenet of \( H \), \( \Lambda^H \) is a label-space and \( \lambda^H \), call the labeling scheme of \( H \), is a function that assigns a label \( \lambda^H(x) \) to each edge (vertex) \( x \) of \( P \). When no confusion arises, we use the short notation \( \lambda^H(x) \).

As said, we follow the approach that the concept of isomorphism is specific to the type of the structures at hand. An isomorphism \( \pi \) of a labeled stagenet onto another is an isomorphism of the corresponding underlying stagenets that preserves the labels (i.e., \( \pi(\lambda^H(x)) = \lambda^H(\pi(x)) \)). Clearly, the two labeled stagenets are required to be of the same type. That is, they are over the same label-space and either both are edge-labeled or both are vertex-labeled. As in the case of plain stagenets, we are interested only in properties of labeled stagenets that are invariant under isomorphism. Therefore, we rarely distinguish between two isomorphic labeled stagenets and usually say that two such structures are identical.

Some of the terms and notations related to plain stagenets apply verbatim to labeled stagenets and are used this way. These include the slice and the ‘+’ operators, being symmetric, being a sub-stagenet, etc. We add the prefixes ‘el-’ and ‘vel-’ to terms which denote certain plain stagenets to denote the corresponding el-stagenets and vel-stagenets, respectively. For example, we use the terms el-bead, el-bamboo and el-rosary with the obvious meaning. The concept of being compatible is adapted for labeled stagenets as follows. Two such stagenets, \( H_1 \) and \( H_2 \), are compatible, denoted \( H_1 \sim H_2 \), if they have the same type and the same depth.

**Labeled LCP.** The EL-LCP (VEL-LCP) of two compatible el-stagenets (vel-stagenets) \( H_1 \) and \( H_2 \) over the label-space \( \Lambda = \langle L, @ \rangle \) is the compatible labeled stagenet denoted \( H_1 \bowtie H_2 \) and defined as follows:

1. \( H_1 \bowtie H_2 = P_1 \bowtie P_2 \).
2. \( \lambda_1 \bowtie \lambda_2 = \lambda_1 \bowtie \lambda_2 \) for any two edges (vertices) \( x_1 \) and \( x_2 \) in the same stage of \( H_1 \) and \( H_2 \), respectively.

Figure 24 shows the product of two vel-stagenets over the label-space \( \mathcal{M} = \langle \mathbb{N}, + \rangle \) where ‘+’ is the regular addition of natural numbers. All the labeled stagenets in the figures of this section are over
Lemma 6.1 The EL-LCP and the VEL-LCP operators are associative and commutative.

Clearly, for any depth $\delta$ there is a unique el-stagenet (vel-stagenet) which is the identity element of EL-LCP (VEL-LCP) for labeled stagenets of depth $\delta$ over $\Lambda$. The following lemma is straightforward.

Lemma 6.2

(a) For any fixed $i$ and $j$, the $i$-to-$j$ slice operator, $H \mapsto H[i, j]$, commutes with EL-LCP and with VEL-LCP.

(b) The inverse operator, $H \mapsto H^{-1}$, commutes with EL-LCP and with VEL-LCP.

Lemma 6.3

(a) Every el-rosary is a product of el-beads.

(b) Every vel-rosary and every vel-bamboo are products of vel-beads.

Proof: To prove statement (a) let $R$ be such an el-rosary. Its underlying rosary $\overline{R}$ is a product of beads; say, $\overline{R} = B_0 \bowtie B_1 \bowtie \cdots \bowtie B_n$. For each bead $B_j$ let $I_j$ be the interval $I_j = \{i : |E_{ij}^B| = 2\}$. These intervals form a partitioning of the edge-stages of $R$.

We convert each $B_j$ to an el-bead $B'_j$ by labeling its edges as follows. Let $I_j = [i', i'']$. We label the edges outside the edge-stages of this interval by $0^\Lambda$. The two stagenets $\overline{R}[i', i'' + 1]$ and $B_j[i', i'' + 1]$ are isomorphic. We pick such an isomorphism and use it to transfer the labels from the former to the latter. It is easy to see that the product of the resulting el-beads is $R$; see Figure 25(a). This establishes (a). The proof of (b) is similar and is omitted; see Figure 25(b). □

![Image](image.png)

Figure 25: (a) Any el-rosary is a product of el-beads. (b) Any vel-bamboo is a product of vel-beads. (c) An el-bamboo which is not a product of el-beads.

The ‘el-rosary’ in Lemma 6.3(a) cannot be replaced with ‘el-bamboo’, as demonstrated by the el-bamboo of Figure 25(c) which is not a product of el-beads. Such a counterexample exists over any label-space having more than one label because of the following reason. Let a connected, depth-2
el-bamboo be the product of two el-beads, one with the edges \(x_1\) and \(x_2\) and the other with \(y_1\) and \(y_2\). Then the labels of the el-bamboo must satisfy the following equation:
\[
(x_1 \bowtie y_1) \oplus (x_2 \bowtie y_2) = (x_1 \oplus y_1) \oplus (x_2 \oplus y_2) = (x_1 \bowtie y_2) \oplus (x_2 \bowtie y_1).
\]

**Lemma 6.4**

(a) Any 2-regular el-stagenet which is a product of el-rosaries is a product of el-bamboos.

(b) A 2-regular vel-stagenet is a product of vel-bamboos iff it is a product of vel-rosaries.

**Proof:** To prove statement (a) let \(H\) be a 2-regular el-stagenet which is a product of el-rosaries. By Lemma 6.3, \(H\) is a product of el-beads; say, \(H = B'_1 \bowtie \cdots \bowtie B'_m\). By Lemma 4.5, \(\overline{\prod}\) is a product of bamboos which are product of beads. Hence, there is a way to combine the \(B'_j\) into el-bamboos. That is, there is partitioning of the sequence \(B'_1, \ldots, B'_m\) into sub-sequences s.t. the product of each sub-sequence is an el-bamboo. This establishes (a). The proof of (b) is similar and is omitted.

Sets of labels \(L_1, L_2, \ldots, L_n\) are **independent (with respect to \(\oplus\))** if there is at most one way to express any label \(\ell\) as a \(\oplus\)-sum of \(n\) labels, one from each \(L_i\). For two sets of labels, \(L_1 \text{ and } L_2\), define \(L_1 \oplus L_2 = \{\ell_1 \oplus \ell_2 | \ell_1 \in L_1, \ell_2 \in L_2\}\). This operator is clearly associative and commutative.

Let \(X\) be a set of edges (vertices) of an el-stagenet (vel-stagenet) \(H\). Define \(L(X) = \lambda(X)\) (the set of labels associated with members of \(X\)) and \(L(H) = \lambda(E_H)\) (\(L(H) = \lambda(V_H)\)). Such a set \(X\) is **unambiguous** if \(\lambda|_X\) is one-to-one; \(H\) itself is **unambiguous** if each \(E_i (V_i)\) is unambiguous. This unambiguity is preserved downward under the EL-LCP and the VEL-LCP as asserted by the next lemma.

**Lemma 6.5** Let \(H = H_1 \bowtie H_2 \bowtie \cdots \bowtie H_n\) where the factors are **compatible normal el-stagenets (vel-stagenets)** and let ‘U’ stands for ‘E’ (for ‘V’). Then:

(a) For any stage \(j\): \(L(U_j^H) = L(U_{j+1}^{H_1}) \oplus L(U_{j+1}^{H_2}) \oplus \cdots \oplus L(U_{j+1}^{H_n})\).

(b) For any stage \(j\): \(U_j^H\) is unambiguous iff each \(U_{j+1}^{H_i}\) is unambiguous and the sets \(L(U_{j+1}^{H_1})\), \(L(U_{j+1}^{H_2}), \ldots, L(U_{j+1}^{H_n})\) are independent.

**Path-oriented.** A labeled stagenet \(H\) is **path-oriented** if \(\lambda^{-1}(\{\ell\})\) (i.e., the set of objects labeled by \(\ell\)) constitutes an end-to-end path for each \(\ell \in L(H)\). This property is actually a local property as stated by the next lemma whose proof is immediate.

**Lemma 6.6** An el-stagenet (vel-stagenet) is path-oriented iff each of its slices of depth three (two), or less, is path-oriented.

**Lemma 6.7** Let \(H = H_1 \bowtie H_2 \cdots \bowtie H_n\) where the factors are **path-oriented labeled stagenets and \(H\) is unambiguous. Then \(H\) is path-oriented.**

**Proof:** Consider the case of vel-stagenets. By Lemmas 6.2(a) and 6.6, it suffices to prove the lemma for \(\delta \leq 2\). The sub-case of \(\delta = 1\) is trivial, so consider the sub-case of \(\delta = 2\) which we prove by induction on \(n\).

Let \(n = 2\). Denote \(\lambda = \lambda^H\), \(\lambda_1 = \lambda^{H_1}\) and \(\lambda_2 = \lambda^{H_2}\). We need to show that \(\lambda^{-1}(\{\ell\})\) is an edge of \(H\) for any \(\ell \in L(H)\). Let \(\ell\) be such a label. By Lemma 6.5(a), \(\ell = \ell_1 \oplus \ell_2\) where \(\ell_1 \in L(H_1)\) and \(\ell_2 \in L(H_2)\). Since \(H_1\) and \(H_2\) are path-oriented, \(\ell_i = \lambda_i^{-1}(\{\ell_i\})\) is an edge of \(H_i\). The product \(\ell = \ell_1 \bowtie \ell_2\) is an edge of \(H\) whose endpoints are labeled by \(\ell\). Since \(H\) is unambiguous, \(\lambda^{-1}(\{\ell\}) = \ell\).

Let \(n > 2\). Define \(H' = H_1 \bowtie H_2 \cdots \bowtie H_{n-1}\). Since \(H = H' \bowtie H_n\) is unambiguous, Lemma 6.5(b) implies that \(H'\) is unambiguous. By our induction hypothesis, \(H'\) is path-oriented. Reapplying the induction hypothesis yields that \(H = H' \bowtie H_n\) is path-oriented. This concludes the case of vel-stagenets. The case of el-stagenets is similar and is omitted. □
7 Vertex and Edge Sequencings

This section applies the Labeled ICP to establish characterizations of the GBN family in terms of a product of bamboos and of an edge-sequencing.

Naturally path-oriented. Up to now the identities of the labels were irrelevant. In the following discussion we sometimes require that the labels are natural numbers as follows. A path-oriented labeled stagenet $H$ is naturally path-oriented if $L(H) = [0, 2^n)$ for some integer $n$. In this case we denote by $S^H$ the resulting sequence of paths — namely, $S^H = (\lambda^{-1}(i) : i \in [0, 2^n])$.

Sequence-oriented. Recall the definition of vertex-sequencing and edge-sequencing from Section 1. A labeled stagenet $H$ is sequence-oriented if $H$ is naturally path-oriented and $S^H$ is an edge-sequencing or vertex-sequencing of $H$. (To this end, $S^H$ needs only to satisfy the beta-property.) The property (of plain stagenets) of admitting a vertex-sequencing or an edge-sequencing is a global property. In contrast, the property of being sequence-oriented is actually a local property of labeled stagenets as expressed by the next lemma.

**Lemma 7.1** An el-stagenet (vel-stagenet) is sequence-oriented iff each of its slices of depth three (two) or less is sequence-oriented.

Henceforth we work with the label-space $\mathcal{M} = \langle \mathbb{N}, + \rangle$ which somewhat resembles a vector-space spanned by the “basis” $\{2^0, 2^1, 2^2, \ldots \}$ over the field $\mathbb{Z}_2$, as stated by the next lemma.

**Lemma 7.2** The following statements hold in $\mathcal{M}$:

(a) The sets of labels $\{0, 2^0\}, \{0, 2^1\}, \ldots, \{0, 2^n\}$ are independent.

(b) $\{0, 2^0\} + \{0, 2^1\} + \cdots + \{0, 2^n\} = [0, 2^{n+1})$.

(c) Let $x, x' \in \{0, 2^0\} + \cdots + \{0, 2^{k-1}\} + \{0, 2^{k+1}\} + \cdots + \{0, 2^n\}$ and $y, y' \in \{0, 2^k\}$. Then the binary presentations of $x + y$ and $x' + y'$ differ exactly at the $k$-th bit iff $x = x'$ and $y = y'$.

7.1 Vertex-sequencing

Let $G$ and $G'$ be plain stagenets, let $S = \langle S_i : i \in [0, 2^n) \rangle$ and $S' = \langle S'_i : i \in [0, 2^{n'}) \rangle$ be either vertex-sequencings or edge-sequencings of $G$ and $G'$, respectively, and let $\beta$ and $\beta'$ be the beta-functions of $S$ and $S'$. The sequences $S$ and $S'$ are equivalent if $n = n'$ and $\beta = \beta'$. The following lemma is straightforward.

**Lemma 7.3** Two (plain) GBN graphs, $G$ and $G'$, are isomorphic iff there are $S$ and $S'$, vertex-sequencings of $G$ and $G'$, s.t. $S$ and $S'$ are equivalent.

**Lemma 7.4** Let $H = M_0 \not\vee M_1 \not\vee \cdots \not\vee M_n$ where $H$ is a 2-regular vel-stagenet over $\mathcal{M}$ and each of the factors is a path-oriented vel-bamboo with $L(M_i) = \{0, 2^i\}$. Then $H$ is sequence-oriented. Moreover, the beta-function of $S^H$ satisfies $\beta(k) = i$ iff the $k$-th edge-stage of $M_i$ is solid.

**Proof:** By Lemmas 6.2(a) and 7.1, it suffices to prove the lemma for $\delta \leq 2$. By Lemmas 6.5 and 7.2, $H$ is unambiguous and $L(H) = [0, 2^{n+1})$. The conditions of Lemma 6.7 hold, implying that $H$ is path-oriented and therefore naturally path-oriented. It remains to show that the sequence $S^H$ has the beta-property. This clearly holds when $\delta = 1$, so assume $\delta = 2$. Let $u \in V_1^H$ and $v \in V_2^H$ be two vertices with $\hat{u} \neq \hat{v}$. Since $H$ is 2-regular, there is exactly one $k$ s.t. $M_k$ is connected. By Lemma 7.2(c), $(u - v)$ is an edge of $H$ iff $\hat{u}$ and $\hat{v}$ differ exactly in the $k$-th bit.

**Theorem 7.5** A (plain) stagenet $G$ is a GBN iff it is 2-regular and is a product of bamboos.
To prove the left-to-right direction, let $G$ be a GBN. We construct a product of bamboos isomorphic to $G$ as follows. Let $S$ be a vertex-sequencing of $G$, let $\beta$ be the beta-function of $S$ and let $2^n = |V_1^G|$. Define a sequence $(M_0, M_1, \ldots, M_{n-1})$ of vel-bamboos of depth $\delta^G$ over $\mathcal{M}$ as follows. Each $M_i$ is path-oriented with $L(M_i) = \{0, 2^i\}$ and the $k$-th edge-stage of $M_i$ is solid iff $\beta(k) = i$.

Let $H = M_0 \uplus M_1 \uplus \cdots \uplus M_{n-1}$. By our construction, $H$ is 2-regular. By Lemma 7.4, $H$ is sequence-oriented and its $S^H$ sequence is equivalent to $S$. By Lemma 7.3, $\overline{H}$ is isomorphic to $G$. That is, $G$ is a product of bamboos.

To prove the other direction, let $G$ be a 2-regular stagenet which is a product of bamboos; say, $G = M_0 \uplus M_1 \uplus \cdots \uplus M_{n-1}$. We convert each bamboo $M_i$ into a vel-bamboo $M'_i$ over $\mathcal{M}$ so that $M_i = M'_i$, $M'_i$ is path-oriented and $L(M'_i) = \{0, 2^i\}$. Let $H = M'_0 \uplus M'_1 \uplus \cdots \uplus M'_{n-1}$. The conditions of Lemma 7.4 hold and therefore $H$ is sequence-oriented. Since $G$ is isomorphic to $\overline{H}$, $G$ has a vertex-sequencing and thus is a GBN.

\section{Edge-sequencing}

\textbf{Lemma 7.6} Two (plain) GBN graphs, $G$ and $G'$, are isomorphic iff there are $S$ and $S'$, edge-sequencings of $G$ and $G'$, s.t. $S$ and $S'$ are equivalent.

\textbf{Proof:} The left to right direction is trivial. Assume, for the other direction, that $S = \langle S_i | i \in [0, 2^n) \rangle$ and $S' = \langle S'_i | i \in [0, 2^n) \rangle$ are such edge-sequencings. Let $\pi : V^G \mapsto V^{G'}$ be the mapping satisfying $\pi(S_i) = S'_i$ for every $i$. Such a mapping exists since $S$ and $S'$ are equivalent. It is easy to see that $\pi$ is an isomorphism of $G$ onto $G'$.

\textbf{Lemma 7.7} Let $H = R_0 \uplus R_1 \uplus \cdots \uplus R_n$ where $H$ is a 2-regular el-stagenet over $\mathcal{M}$ and each of the factors is a path-oriented el-rosary with $L(R_i) = \{0, 2^i\}$. Then $H$ is sequence-oriented. Moreover, the beta-function of $S^H$ satisfies $\beta(k) = i$ iff the $k$-th stage of $R_i$ is a singleton.

The proof of this lemma is similar to that of Lemma 7.4 and is omitted; see Figure 26.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure26.png}
\caption{A sequence-oriented el-stagenet which is a product of path-oriented el-rosaries.}
\end{figure}

\textbf{Theorem 7.8} A stagenet is a GBN iff it has an edge-sequencing.

\textbf{Proof:} Let $G$ be a 2-regular stagenet. By Theorem 7.5, $G$ is a GBN iff $G$ is a product of bamboos and by Lemma 4.5, $G$ is a product of bamboos iff $G$ is a product of rosaries. Thus it suffices to show that $G$ is a product of rosaries iff it has an edge-sequencing. The proof of this statement is similar to that of Theorem 7.5, using Lemmas 7.6 and 7.7 instead of Lemmas 7.3 and 7.4, and is omitted. \hfill \Box

\section{Translation}

This section establishes a labeled variant of the path-symmetric property which is summarized by the following theorem.
Theorem 8.1 Let $\Lambda$ be a vector space over $\mathbb{Z}_2$ and let $H$ be a (vertex or edge) labeled stagenet over $\Lambda$ which is a product of labeled beads. Let $P_1$ and $P_2$ be two end-to-end paths of $H$ s.t. the corresponding edges or vertices have identical labels. Then there is an automorphism of $H$ that swaps $P_1$ and $P_2$.

The automorphism provided by this theorem is not necessarily induced by automorphisms of the factors. The el-stagenet of Figure 27 has, by this theorem, a non-trivial automorphism while all its factors have none.

![Figure 27: An el-stagenet having a non-trivial automorphism while the factors have none.](image)

For the sake of simplicity, we confine our discussion to el-stagenets. For definiteness, let us use the following $\mathbb{R}_0$-dimensional vector space over the field $\mathbb{Z}_2$: $N \triangleq (\mathbb{N}, \triangle)$, where $i \neq j$ is the integer produced by the bit-wise exclusive-or of the binary presentation of $i$ and $j$. (The identities of the members of $N$ are irrelevant in this section but are relevant in a following one.) Henceforth, all the el-stagenets in this section are implicitly over $N$.

Unit. An el-unit is an el-stagenet $I$ s.t. $T = 1^\delta$. The el-units are the only invertible el-stagenets and they satisfy $I \triangleq I = 1^\delta_N$ where $1^\delta_N$ is the depth-$\delta$ identity element under EL-LCP over $N$. In the sequel we implicitly consider the end-to-end paths of an el-stagenet to be el-units. Since the EL-LCP of el-units does not effect the underlying stagenet, let us write $I' \triangleq I''$ instead of $I' \triangleq I''$ for el-units $I' \sim I''$. Let $I_i$ denote the $i$-th edge of an el-unit $I$.

Translation. As said, the automorphism provided by Theorem 8.1 is not necessarily induced by automorphisms of the factors; it is induced, however, by auto-translations of the factor, and these translations are defined as follows. Let $H, W$ and $I$ be el-stagenets. A mapping $\pi$ is an $I$-translation of $H$ onto $W$, denoted $\pi : H \xrightarrow{I} W$, if $H \sim W \sim I$, $I$ is an el-unit, $\pi$ is an isomorphism of $H$ onto $W$ and for any $e \in E^H_i$: $\pi(e) = \hat{e} \triangleq \hat{I}_i$. Note that a $1^\delta_N$-translation is an isomorphism. A mapping $\pi$ is a translation of $H$ onto $W$, denoted $\pi : H \xrightarrow{} W$, if it is an $I$-translation for some $I$; $H$ is congruent to $W$, denoted $H \overset{\sim}{\rightarrow} W$, if $\pi : H \xrightarrow{I} W$ for some $\pi$. For two functions, $f$ and $g$, let $(g \circ f)$ denote the ‘$g$ after $f$’ composition of $g$ and $f$. The following lemma is straightforward and its second statement is an extension of Lemma 3.6.

Lemma 8.2

(a) The binary relation ‘$H \xrightarrow{\sim} W$’ is an equivalence relation. In particular, let $\pi' : H \xrightarrow{I'} H'$ and $\pi'' : H'' \xrightarrow{I''} H''$; then $(\pi'' \circ \pi') : H \xrightarrow{I'} \xrightarrow{I''} H''$.

(b) Let $H' \xrightarrow{\sim} W'$, $H'' \xrightarrow{\sim} W''$ and $H' \sim H''$. Then $(H' \trianglel {I'} W') \xrightarrow{\sim} (W' \trianglel {I''} W'')$. In particular, let $\pi' : H' \xrightarrow{I'} W'$, $\pi'' : H'' \xrightarrow{I''} W''$ and $H' \sim H''$; then $(\pi' \trianglel {I'} \pi'') : (H' \trianglel {I'} H'') \xrightarrow{I''} (W' \trianglel {I''} W'')$.

(c) Assume that $\pi : H \xrightarrow{\sim} W$, $P$ is an end-to-end path of $H$ and $P \cong \pi(P)$. Then $\pi$ is an isomorphism.

Span of a bead. The span of an el-bead $B$ is the el-unit $\gamma(B) \triangleq P_1 \trianglel P_2$ where $P_1$ and $P_2$ are the two end-to-end paths of $B$. Note that an el-bead absorbs its span; that is, $B \cong B \trianglel \gamma(B)$ for any el-bead $B$. The following lemma is immediate.

Lemma 8.3 For any two el-beads $B'$ and $B''$: $B' \trianglel B''$ iff $\overline{B'} \cong \overline{B''}$ and $\gamma(B') \cong \gamma(B'')$. 

Lemma 8.4 The path-translatable property is preserved upward under EL-LCP.

The proof of this lemma is essentially the proof of Lemma 3.7 (that the path-symmetric property is preserved upward under LCP) augmented with Lemma 8.2(b), and is omitted. Lemma 8.4 and the simple fact that el-beads and el-units are path-translatable yield:

Lemma 8.5 Any product of el-beads and el-units is path-translatable.

Lemma 8.2(c) and Lemma 8.5 imply Theorem 8.1. Lemma 8.2(c) and a composition of the appropriate translations imply:

Lemma 8.6 Assume \( A \stackrel{\leftrightarrow}{\Rightarrow} B \), \( H \) is path-translatable, \( P \) and \( Q \) are end-to-end paths of \( H \) and \( W \), respectively, and \( P \equiv Q \). Then \( H \equiv W \).

9 Concatenation

An important advantage of labeled stagenets over plain stagenets is that concatenation is a definite operator on the former ones. This section defines the concatenation operator and shows that, under certain conditions, it commutes with the labeled LCP. The section focuses on el-stagenets since they can naturally represent comparator networks which are discussed in the following section. However, the main result about concatenation holds for vel-stagenets as well. We start with some preliminary concepts.

Stretching an el-stagenet. The stretching operator, \( \zeta_i \), of Section 5 is naturally extended to be defined on el-stagenets by transferring the label from a ‘stretched’ edge to the edges of the path that replaces it; see Figure 28.

Splitting edge-stages. We extend the s-to-t slice operator on el-stagenets to be defined for \( s \) and \( t \) s.t. \( 2s \) and \( 2t \) are integers (rather than even integers) as follows. The extended slice of \( H \), denoted \( H[s,t] \), is defined iff \( 1 \leq s \leq t \leq \delta H \) and is a (regular) slice either of \( H \) or of a stretched variant of \( H \). If \( t \) is an integer then the last stage of \( H[s,t] \) is, as usual, \( V^H_t \). Otherwise, \( H \) is stretched 2-fold at \( [t] \) and the last stage of \( H[s,t] \) is the newly generated stage. We say in the latter case that \( H[s,t] \) splits \( E^H_{[t]} \). The first stage of \( H[s,t] \) is defined analogously and it splits \( E^H_{[s]} \) when \( s \) is not an integer.

See Figure 28. The depth of \( H[s,t] \) is one when \( s = t \) and \( [t] + 1 - |s| \) otherwise. The following two lemmas are straightforward.

Lemma 9.1 For fixed integers \( i \) and \( j \), the extended \((i/2)\)-to-\((j/2)\) slice operator, \( H \mapsto H[i/2,j/2] \), commutes with EL-LCP.

Lemma 9.2 Let \( H_1 \) and \( H_2 \) be compatible el-stagenets and \( E^H_{i} \) be unambiguous. Then \( H_1 \equiv H_2 \) iff \( H_1[1,i+\lceil \frac{1}{2} \rceil] \equiv H_2[1,i+\lceil \frac{1}{2} \rceil] \) and \( H_1[i+\frac{1}{2},\infty] \equiv H_2[i+\frac{1}{2},\infty] \).

Terminal. A terminal is either an input or an output vertex having exactly one neighbor. A stagenet is backward-terminated (forward-terminated) if all its inputs (outputs) are terminals and its depth is at least two. Note that if \( s < t \) and \( H[s,t] \) splits \( E^H_{[s]} \) then \( H[s,t] \) is backward-terminated (forward-terminated).

Concatenation. For three el-stagenets \( H, U \) and \( W \), \( H \) is a concatenation of \( U \) with \( W \) if all are over the same label-space and for some integer \( \delta' \): \( 1 < \delta' + \frac{1}{2} < \delta H \), \( E^H_{\delta'} \) is unambiguous, \( U[1,\delta'+\lceil \frac{1}{2} \rceil] \equiv U \) and \( H[\delta'+\lceil \frac{1}{2} \rceil,\infty] \equiv W \). By Lemma 9.2, there is at most one such \( H \). When such a \( H \) exists, we say
Lemma 9.3 Assume that $U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$, $W = W_1 \oplus W_2 \oplus \cdots \oplus W_n$, all are el-stagenets, each $U_i$ is concatenable with $W_i$, and $E^W_i$ is unambiguous. Then $U$ is concatenable with $W$, and $\langle W_1 \rangle \oplus \cdots \oplus \langle W_n \rangle$ is the concatenation of $U$ with $W$. Let $\delta' = \delta^U - 1$. We have:

$$H[1, \delta' + \frac{1}{2}] = \langle U_1 \rangle \oplus \cdots \oplus \langle U_n \rangle \quad \text{by Lemma 9.1}$$

Similarly we have $H[\delta' + \frac{1}{2}, \infty] = W$. Moreover, since $E^W_i$ is unambiguous, so is $E^H_i$. Hence, $U$ is concatenable with $W$ and the concatenation is the above product. \hfill \Box$

10 The Bitonic Sorting Network

This section shows that the Batcher bitonic sorting network [2] with its min/max labeling, as well as with its output ordinals, is a product of el-bamboos and of el-rosaries over the label-space $\mathcal{N} \triangleq \langle \mathbb{N}, \triangleright \rangle$. In this section, the identity of the members of $\mathcal{N}$ is relevant, and in particular their order.

We represent a multistage comparator network by an el-stagenet $H$ over $\mathcal{N}$ according to the following rules. The inputs and outputs of $H$ represent the input and output ports of the network, while the other vertices represent comparators. Hence, $H$ is backward- and forward-terminated and the forward and backward degree of vertices, other then inputs and outputs, is two. The label of an edge $e$ is composed of two components. The first, denoted $\lambda(e)$, is a 0-1 value that provides the min/max type of the edge; zero means a min edge and one means a max edge, with the obvious
The second component, denoted $\lambda''(e)$, is a natural number called the *ordinal* of the edge. This label is usually meaningless, in which case it is zero, and its only purpose is to sequence certain edge-stages. Which stages are sequenced and the semantics of these sequences relate to the functionality of the network and is described below. For example, when the network in question is a sorting network then only the output edges (those of $E_1$) are sequenced and the exiting keys are sorted according to that sequence. The ordinals sequence an edge-stage $E_i$ as follows: $E_i$ is unambiguous w.r.t. $\lambda''$ and $L''(E_i) \triangleq \lambda''(E_i) = [0, n)$ for some $n$. As said, if the ordinals do not sequence an edge-stage $E_i$ then $L''(E_i) = \{0\}$. Recall that $H$ is formally an el-stagenet over $N'$; the formal label of an edge $e$ is the combined label $\lambda(e) = \lambda'(e) + 2\lambda''(e)$; the labels $\lambda'(e)$ and $\lambda''(e)$ can be restored from $\lambda(e)$.

In our drawings, the min/max type of an edge is denoted by the form of its arrowhead; a hollow one denotes min and a solid one denotes max; an open arrowhead denotes that the type of the edge is either unknown or irrelevant for the current discussion. The ordinal of an edge, when meaningful, is written next to it; the omission of this number denotes that the ordinal is meaningless and equals zero. See the drawing of a sorting network in Figure 29; actually, this network is the 4-input bitonic sorting network and is the product of the two indicated el-rosaries.

![Figure 29: The el-stagenet form of a sorting network.](image)

Henceforth in this section, $n$ is a power of two and $n = 2^k$. Our construction follows that of Batcher [2] and uses the following networks, in el-stagenet form, indexed by $n$ which is the number of keys processed by the network, as well as the number of its input vertices. $BS_n$ is the Batcher bitonic sorter, $BG_n$ is the Batcher bitonic sorting network, and $BN_n$ is the Batcher bitonic sorting network. The functionality and the structure of these networks is described below.

### 10.1 The bitonic sorter

Following Batcher [2], a sequence is *bitonic* if it is a rotation of a concatenation of two monotonic sequences, one ascending, the other descending. A comparator network is a *bitonic sorter* if it sorts all the bitonic sequences of the appropriate size. In this definition, it is implicit that the input edges (as well as the output ones) are organized in a sequence, and the given keys constitute a bitonic sequence w.r.t. this edge sequence. In the el-stagenet form of such a network, the ordinals sequence the input edges, as well as the output ones, and are meaningless in other edge-stages. Batcher has noted that the underlying stagenet of his bitonic sorter, $BS_n$, is a Butterfly. We show that $BS_n$ is a product of el-rosaries. The cornerstone of this bitonic sorter is the following lemma of Batcher [2].

**Lemma 10.1** Let $A = \langle a_i : i \in [0,2n]\rangle$ be a bitonic sequence and let $B = \langle b_i = \min (a_i, a_{i+n}) : i \in [0,n]\rangle$ and $C = \langle c_i = \max (a_i, a_{i+n}) : i \in [0,n]\rangle$. Then $B$ and $C$ are bitonic sequences and $b_i \leq c_j$ for any $i$ and $j$.

**Single and double rosary.** For $j \in N$, a *single-$j$ rosary* is an el-rosary $R$ (over $N'$) s.t. $L''(E_i^R) = \{0, 2^j\}$ and $L''(E_i^{R'}) = \{0\}$ for any other edge-stage $i$; a *double-$j$ rosary* is an el-rosary $R$ s.t. $L''(E_i^R) = L''(E_i^{R'}) = \{0, 2^j\}$ and $L''(E_i^{R''}) = \{0\}$ for any other edge-stage $i$. 

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The eraser. The \( i \)-eraser, denoted \( \xi^i \), is the unary operator on el-stagenets that erases the labels of the \( i \)-th edge-stage, replacing them with the zero label. Note that \( \xi^j(\xi^i(T^k_j)) = T^{k+1}_j \) for any \( j < k \).

**Lemma 10.2** For any \( k > 0 \), \( BS_{2n} = T^k_0 \ast T^k_1 \ast \cdots \ast T^k_{2^{k}-1} \).

![Figure 30](image)

**Proof:** We prove the lemma by induction on \( n = 2^k \). The case of \( n = 2 \) is trivial. Assume that the lemma holds for \( n \) and let \( Q = \xi^i(\xi^j(\text{BS}_n)) \ast T^{k+1}_i \); see Figure 30(a). By Lemma 10.1, \( Q \) is a 2\( n \)-input bitonic sorter, since the first stage of comparators produces the bitonic sequences \( B \) and \( C \) of the lemma and sends each of them to the appropriate \( n \)-input bitonic sorter. Moreover, this recursive construction is identical to that of Batcher; hence, \( Q = \text{BS}_{2n} \).

By the induction hypothesis, \( \text{BS}_n = \prod_{i=0}^{k-1} T^i_k \). The \( \xi^2 \) and \( \xi^1 \) operators commute with EL-LCP and \( \xi^2(\xi^j(T^k_i)) = T^{k+1}_i \). Hence,

\[
\text{BS}_{2n} = \xi^2(\xi^j(\text{BS}_n)) \ast T^{k+1}_k = \xi^2(\xi^j(\prod_{i=0}^{k-1} T^i_k)) \ast T^{k+1}_k = \prod_{i=0}^{k} T^i_k.
\]

\[
\Box
\]

### 10.2 The bitonic generator

An \( n \)-input bitonic generator is an \( n \)-input comparator network that transforms any given combination of keys into a bitonic sequence. In this definition, it is implicit that the output edges are organized in a sequence. In the el-stagenet form of such a network the ordinals sequence the output edges and are meaningless otherwise. As said, \( BG_n \) is the bitonic generator of Batcher in el-stagenet form.

There are two natural procedures to construct a 2\( n \)-input bitonic generator out of two copies of an \( n \)-input sorting network \( Q \). In both procedures, one copy of \( Q \), say \( Q' \), is kept intact while the other copy, \( Q'' \), is slightly modified. In the straightforward procedure used by Batcher, the ordinals (of the implicit sequence) of the output edges of \( Q'' \) are transformed from \( \langle 0, 1, 2, \ldots, n-1 \rangle \) to \( \langle 2n-1, 2n-2, \ldots, n \rangle \). In the second procedure, the min/max types of all the edges are flipped. This causes \( Q'' \) to sort its keys, but in the reversed order. In order that the keys exiting \( Q' \) and \( Q'' \) constitute a bitonic sequence, the ordinals of the output edges of \( Q'' \) are transformed from \( \langle 0, 1, 2, \ldots, n-1 \rangle \) to \( \langle n, n+1, n+2, \ldots, 2n-1 \rangle \).

When \( Q \) is a multistage network in el-stagenet form, both procedures can be accomplished by multiplying \( Q \) with a disconnected el-rosary. See Figure 31. Recall that \( n = 2^k \) and let \( U^j_k \) and \( U''_k \) denote the el-rosaries of the first and second procedure, respectively. These rosaries are of the form \( U^j_k = 1^j_N + I^j_k \) and \( U''_k = 1^j_N + I''_k \) where \( \delta = 2^j \) and \( I^j_k \) and \( I''_k \) are el-units. As shown in Figure 31, all edges of \( I^j_k \) are of type min and of zero ordinal, except for the output edge whose ordinal is \( 2n - 1 \). In
contrast, all edges of \( P'_k \), but the first one, are of type max and the ordinal of the last edge is \( n \) and is zero otherwise. In the general case, these two procedures produce different networks; however, the following lemma shows that, for \( Q = BN_n \), the two procedures produce identical networks.

**Lemma 10.3** Let \( Q \) be a \( 2^k \)-input sorting network in \( el \)-stagenet form which is a product of \( el \)-rosaries. Then \( Q \overset{\delta}{\bowtie} U''_k \cong Q \overset{\delta}{\bowtie} U'_k \).

**Proof:** It suffices to show that \( H'' = Q \overset{\delta}{\bowtie} I''_k \) is isomorphic to \( H' = Q \overset{\delta}{\bowtie} I'_k \). Any \( el \)-stagenet is congruent to itself, and any compatible el-units are congruent. This and Lemma 8.2(b) imply that \( H'' \) is congruent to \( H' \). Since \( Q \) is a product of \( el \)-rosaries, it is a product of \( el \)-beads; thus, by Lemma 8.5, \( H'' \) is path-translatable. Due to Lemma 8.6, it remains to establish two end-to-end paths, \( P'' \) of \( H'' \) and \( P' \) of \( H' \), having identical labels. Except for their non-standard ordinals, \( H'' \) and \( H' \) are sorting networks and, in both networks, the maximal key exits through the edge of ordinal \( n \). Hence, the paths traversed by the maximal keys in both networks, under an arbitrary combination of incoming keys, are the desired isomorphic el-units \( P'' \) and \( P' \).

We now show that under general conditions, which hold for \( Q = BN_n \) and for \( U^* \) that is somewhat similar to \( U'' \), the product \( Q \overset{\delta}{\bowtie} U^* \) is highly independent of \( U^* \). Let a minimum-selector (maximum-selector) be a comparator network having an output edge that always transmits the minimal (maximal) key. In contrast to our usual practice, we impose no restriction on the ordinals of an \( el \)-stagenet form of such a network. Note that flipping the min/max labels transforms a minimum-selector to a maximum-selector and vice versa.

**Lemma 10.4** Let \( Q \) be an \( n \)-input comparator network in \( el \)-stagenet form s.t. \( Q \) is a product of \( el \)-rosaries, the ordinals of its non-output edges are zero, and \( Q[j + \frac{1}{2}, \infty] \) is a minimum-selector. Let \( U^* \) satisfy \( \bigwedge \delta = \overline{U''}_k \), \( U^*[j + 1, \infty] \cong U''_k[j + 1, \infty] \), the ordinal of non-output edges is zero, and the type of the input edges is min. Then \( Q \overset{\delta}{\bowtie} U^* \cong Q \overset{\delta}{\bowtie} U''_k \).

The proof of this lemma is almost identical to that of Lemma 10.3 and is omitted. Arguments relating to the 0-1 principle [17] imply that any bitonic sorter is a minimum-selector. It turns out that \( BN_2k[\delta - k - \frac{1}{2}, \infty] \), with appropriate input ordinals, is a bitonic sorter. Building on that and on Lemma 10.4, we define \( U_k \) to be compatible with \( BN_2k \) and derived from \( U''_k \) by clearing the min/max labels of the edges before the \( \delta-k \) edge-stage. This \( U_k \) provides a third procedure to construct the Batcher bitonic generator, as summarized in the following lemma.

**Lemma 10.5** Assume that \( BN_2k \) is a sorting network which is a product of rosaries and \( BN_2k[\delta - k - \frac{1}{2}, \infty] \) is a minimum-selector. Then \( BG_2k+1 \cong BN_2k \overset{\delta}{\bowtie} U_k \) and \( BG_2k+1 \) is a bitonic generator.
10.3 The bitonic sorting network

A sorting network is a comparator network that sorts any given combination of keys. The n-input bitonic sorting network, BN_n, is constructed, of course, by concatenating BG_n with BS_n. However, we have a technical difficulty. By our definition, these two es-stagenets are not concatenable as \( L(E_{\infty}^{\text{BG}_n}) \neq L(E_{1}^{\text{BS}_n}) \). Due to this difficulty, we use the following variant of concatenation over \( \mathcal{N} \) as follows. In the ad hoc concatenation of \( H \) with \( W \), denoted \( \langle H \rangle_2 \langle W \rangle_1 \), the \( \lambda'' \) component of the label establishes the input/output matching. (This works in the case at hand since \( L'(E_{\infty}^{\text{BG}_n}) = L''(E_{1}^{\text{BS}_n}) = [0, n] \) and the two edge-stages are unambiguous w.r.t. \( \lambda'' \).) As an additional twist, the ad hoc operator clears the \( \lambda'' \) of the resulting edges and sets their \( \lambda' \) to that of the edge of \( H \) which they replace. (In our case, the min/max label of \( E_{\infty}^{H} \) is meaningful while that of \( E_{1}^{W} \) is meaningless.) We summarize this construction into the following lemma.

Lemma 10.6 For \( n > 1 \), if \( \text{BG}_n \) is a bitonic generator then \( \text{BN}_n = \langle \text{BG}_n \rangle^* \langle \text{BS}_n \rangle^* \) is a sorting network.

As a special case, \( \text{BN}_1 \) is a single edge; by our rules, its type is min and its ordinal is zero; i.e., \( \text{BN}_1 = 1_X \).

Lemma 10.7 For \( k > 0 \), \( \text{BN}_2^k \) is a sorting network and is of the form \( \text{BN}_{2^k} = R_0^k \triangleright R_1^k \triangleright \cdots \triangleright R_{k-1}^k \), where each \( R_j^k \) is the single-\( j \) rosary that is the ad hoc concatenation of \( U_j; T_j^{j+1}, T_j^{j+2}, \ldots, T_j^k \), in this order.

These rosaries are shown in Figures 8 (for \( k = 4 \)) and 32 (for \( k = 3 \)).

Proof: We prove the lemma by induction on \( n = 2^k \). The case of \( n = 2 \) is trivial. Assume that the lemma holds for \( n \). By Lemma 10.6, \( \text{BN}_{2n} = \langle \text{BG}_{2n} \rangle^* \langle \text{BS}_{2n} \rangle^* \). By Lemma 10.5 and the induction hypothesis,

\[
\text{BG}_{2n} = \text{BN}_n \triangleright U_k = R_0^k \triangleright R_1^k \triangleright \cdots \triangleright R_{k-1}^k \triangleright U_k,
\]

where each \( R_j^k \) is the single-\( j \) rosary which is the concatenation of the indicated es-rosaries. By Lemma 10.2,

\[
\text{BS}_{2n} = T_0^{k+1} \triangleright T_1^{k+1} \triangleright \cdots \triangleright T_k^{k+1}
\]

and each \( T_j^{j+1} \) is a double-\( j \) rosary.

Each factor of the upper product is (ad hoc) concatenable with the corresponding factor of the lower product. Adapting Lemma 9.3 for the ad hoc operator yields that \( \text{BN}_{2n} \) is the product of the ad hoc concatenation of the corresponding rosaries. This concatenation of \( R_j^k \) with \( T_j^{j+1} \) (or of \( U_k \) with \( T_k^{k+1} \)) is the single-\( j \) (or the single-\( k \)) rosary that is the concatenation of the requested es-rosaries. \( \Box \)

Earing. An earing is a (plain) stagenet of the form \( \phi_2(\phi_2(1)) \); i.e., it is an end-to-end path augmented with 2-way input and output fringes. In other words, the depth-\( \delta \) earing is \( B_{1,1}^\delta \triangleright B_{\delta-1,\delta-1}^\delta \).

Theorem 10.8 The Batcher bitonic sorting network is a product of es-rosaries and also a product of el-bamboos and an el-earing over \( \mathcal{N} \).

Proof: Lemma 10.7 establishes the factorization of \( \text{BN}_n \) to es-rosaries. The slice \( \text{BN}_n[2, \delta - 1] \) is 2-regular and thus, by Lemma 6.4, is a product of el-bamboos. The complete network, \( \text{BN}_n \), is \( \text{BN}_n[2, \delta - 1] \) augmented with input and output 2-way fringes having uniform labels and these fringes are producible by a multiplication with an el-earing. \( \Box \)

Since \( \text{BN}_n \) is a product of el-beads, Theorem 8.1 implies that for any two end-to-end paths having identical labels there is an automorphism of \( \text{BN}_n \) that swaps them. Moreover, any sorting network \( Q \) which is a product of el-beads has a stronger property: For any two end-to-end paths having identical
min/max labels there is an automorphism of $Q$ that swaps them. This is due to the facts that a variant of $Q$ without the ordinals is also a product of el-beads and that the ordinals are redundant in a sorting network – they are determined by the min/max labels.

### 10.4 To the drawing-board

We describe here the form of the factors of the bitonic sorting network and some ways to draw the network in the Butterfly style, as in Figures 8 and 32. The recursive construction of Batcher partitions the stages of $BN_{2^k}$, except for the input and output ones, into the slices $S_1, S_2, \ldots, S_k$, in this order, of depth $1, 2, \ldots, k$, as shown in the figures. The connected components of $S_i$ are $2^i$-input bitonic sorters without their input and output terminals. Each of the $T_j$ of Lemma 10.7, without its inputs and outputs, resides in the slice $S_i$. The edge-stages connecting adjacent slices are the interfaces of the concatenations specified in Lemmas 10.6 and 10.7. We refer to these edge-stages and their edges as *inter-slice* ones, while the others are *intra-slice* ones.

![Figure 32: The 8-input bitonic sorting network, BN_8.](image)

The form of the el-rosary factors of $BN_{2^k}$ is as follows. An el-rosary $R_j^k$ has at most one singleton in each slice; its first singleton is the first stage of $S_{j+1}$ and henceforth it has a singleton at the $i$-th stage of $S_{j+i}$, for $i \geq 1$. The labels of $R_j^k$ are as follows. Being a single-$j$ rosary, all its ordinals are zero except for a single output edge of ordinal $2^j$. Concerning the min/max labels, $R_j^k$ has a path of max edges leading from the first stage of $S_j$ to the first stage of $S_{j+1}$ ($R_0^0$ lacks this path) and it has a single max edge emanating from each singleton; moreover, the max edge emanating from the last singleton is collinear with the output of ordinal $2^j$. All the other edges of $R_j^k$ are of type min.

The unique factorization of $BN_{2^k}$ to beads is as follows (see Figure 32). For any consecutive slices $S_j$ and $S_{j+1}$ and any stage $i$ of the first slice, there is a bead that forksootnote{Recall that a (plain) 2-semiregular stagenet *forks (joins)* at a certain stage if the forward (backward) degree of vertices in this stage is two.} at the $i$-th stage of the first slice and joins at the $(i+1)$-stage of the second one. In addition, for every slice there is a bead that joins at the first stage of the slice but never forks, and for any stage of the last slice, $S_k$, there is a bead that forks there and never joins. There are many ways to furnish these beads with labels to get a factorization of $BN_{2^k}$. We use the straightforward way applied in the proof of Lemma 6.3 that factorizes an el-rosary into el-beads.

These el-beads are combined into an el-earing and $k-1$ el-bamboos, $X_1^k, X_2^k, \ldots, X_{k-1}^k$. The bamboo $X_j^k$ forks at the $j$-th stage of $S_j, S_{j+1}, \ldots, S_k$ and in no other stage. (Hence, it has $k+1-j$ solid edge-stages.) This uniformity is derived as follows. Assume that a bamboo $X$ in the factorization forks at the $i$-th stage of $S_j$. This implies that a factor of $X$ is a bead which forks at the same stage. If $j < k$, then this bead joins at the $i+1$ stage of $S_{j+1}$ and the same holds for $X$. Since $X$ is a bamboo,
The labels of $X^k_j$ are as follows. All the ordinals are zero except for a single output edge of ordinal $2^k-j$. The min/max labels are determined by the following property of $X^k_j$. It has two vertex-disjoint end-to-end paths, $P^0_i$ and $P^1_i$, that cover all the vertices. All labels of $P^0_i$ (including ordinals) are zero. The following are the max edges of $X^k_j$; intra-slice edges of $S_j$ and edges of intra-slice solid edge-edges; all other edges of $P^1_i$ are min ones. In addition, any two distinct edges of $X^k_j$ emanating from the same vertex are of distinct type.

As usual, we draw $BN_n$ in the Butterfly style according to some vertex-sequencing of this stagenet (or, more precisely, of $BN_n[2, \delta - 1]$). We apply the method of Theorem 7.5 to create such a vertex-sequencing via an auxiliary vertex-labeling of the bamboos; the vertices of $P^0_i$ are labeled with zero and those of $P^1_i$ with $2^m$, where $m+1$ is the position of the bamboo $X^k_j$ in the left slice in some arbitrary sequence of the bamboos. The left drawing in Figures 8 and 32 is derived from the sequence $(X^k_{k-1}, \cdots, X^k_{2}, X^k_{1})$ while the right one is derived from $(X^k_{1}, X^k_{2}, \cdots, X^k_{k-1})$.

Here is another approach that produces the same drawings. Let $P$ be an end-to-end path which is a member of the above vertex-sequencing. (Such a $P$ is a vertical line in the drawing.) This $P$, as an el-unit, is a product of $k$ el-units, one from each of the sets $\{P^0_i, P^1_i\}$, $\{P^2_i, P^2_i\}$, $\cdots$, $\{P^{k-1}_i, P^{k-1}_i\}$. Since $F^0_i \cong 1_{X^k}$, $P$ is of the form $P \cong \prod_{s \in s} P^s_i$ for some $s \subset [1, k-1]$; moreover, any subset $s$ is associated with a unique path $P$ as above. In our drawings, these subsets are written above the corresponding vertical lines.

Let $P_s$ denote the path associated with $s$. The set $s$ determines the min/max labels of $P_s$ as follows. All the inter-slice edges are min edges. The $i$-th intra-edge of $P_s$ in the slice $S_j$ is of type $\max$ iff $|s \cap \{i, j\}| = 1$. Moreover, $P_s$ and $P'_s$ are connected by “cross edges” emanating from the $i$-th stage of $S_j$ iff $s \cup s' = \{i\}$. (‘$\cup$’ is the symmetric difference.) The min/max labels of the cross edges are determined by the fact that two edges emanating from the same vertex have distinct labels. The above rules determine the drawing except for a single issue - the relative position of the $P_s$ lines. This issue is determined, as per Theorem 7.5, by an arbitrary sequencing of the bamboos. All the drawings produced this way are symmetric via a reflection w.r.t. a vertical line, except that in the last slice the reflection flips the type of the edges and transforms the ordinals from $i$ to $n-1-i$. This reflection is actually an auto-translation of the el-stagenet.

Acknowledgements
We wish to thank Shimon Even for helpful discussions.

References


