BASE DEPENDENCE OF EXTENSIONS
FOR OPEN DEFAULT THEORIES

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BASE DEPENDENCE OF EXTENSIONS FOR OPEN DEFAULT THEORIES

RESEARCH THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF MASTER OF SCIENCE
IN COMPUTER SCIENCE

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SUBMITTED TO THE SENATE OF
THE TECHNION – ISRAEL INSTITUTE OF TECHNOLOGY

TEVET 5761 HAIFA JANUARY 2001
THE RESEARCH THESIS WAS DONE UNDER THE
SUPERVISION OF ASSOC.PROF. MICHAEL KAMINSKI IN THE
FACULTY OF COMPUTER SCIENCE.

I wish to thank Michael Kaminski for his excellent guidance, experience and
time that he has given me throughout all stages of the work.

THE GENEROUS FINANCIAL HELP OF THE TECHNION IS
GRATEFULLY ACKNOWLEDGED.
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Abstract

In this thesis we compare the semantic and syntactic definitions of extensions for open default theories. We prove that, over monadic languages, the two definitions are equivalent and do not depend on the cardinality of the underlying infinite world. Thus, for monadic languages, the semantic definition of extensions can always be restricted to a countable base.

Next, we present a syntactic definition of extensions for open default theories over a weaker logic. We show that this definition is not equivalent to the definition which was introduced previously. Nevertheless, we prove that it does not depend on the cardinality of the underlying infinite world as well.

Finally, we prove that, for uniterm default theories over finite languages not containing function symbols, the syntactic definition of extensions does not depend on the cardinality of the underlying world. Using this result we show that, in the case of explicitly defined finite domains, the semantic definition of extensions for uniterm default theories over finite languages not containing function symbols can always be restricted to a finite base.
List of Symbols

- **MP**  
  Modus Ponens (rule of inference)

- **GEN**  
  Generalization (rule of inference)

- **Th**  
  Theory

- **T**  
  The set of all terms

- **L**  
  First-order underlying language

- **L_b**  
  First-order underlying language obtained by augmenting the set of constant symbols of \( L \) with all elements of the set \( b \)
Chapter 1

Introduction

Non-monotonic logics are intended to simulate the process of human reasoning by providing a formalism for deriving consistent conclusions from an incomplete description of the world.

Reiter’s default logic ([11]) is one of the widely used non-monotonic formalisms and maybe the only non-monotonic formalism that has a clearly useful contribution to the wider field of computer science through logic programming and database theory. This logic deals with rules of inference called defaults which are expressions of the form

$$\delta(x) = \frac{\alpha(x) : M\beta_1(x), \ldots, M\beta_m(x)}{\gamma(x)},$$

where \(\alpha(x), \beta_1(x), \ldots, \beta_m(x), m \geq 1\), and \(\gamma(x)\) are formulas of first-order logic whose free variables are among \(x = x_1, \ldots, x_n\). A default is closed if none of \(\alpha, \beta_1, \ldots, \beta_m\), and \(\gamma\) contains a free variable. Otherwise it is open. Roughly speaking, the intuitive meaning of a default is as follows. For every \(n\)-tuple of objects \(t = t_1, \ldots, t_n\), if \(\alpha(t)\) is believed, and the \(\beta_i(t)\)'s are consistent with one's beliefs, then one is permitted to deduce \(\gamma(t)\) and add it to the "belief set." Thus an open default can be thought of as a kind of "default scheme", where free variables \(x\) can be replaced by any of the theory’s objects. Various examples of deduction by defaults can be found in [11].

Whereas closed defaults have been quite thoroughly investigated, very little is known about open ones. However, interesting cases of default reasoning usually deal with open defaults, because the intended use of defaults is to determine whether an object possesses a given property, rather than accepting or rejecting a "fixed statement."
It was pointed out in [7] that when applying open defaults one must specify all the objects of the underlying theory. Also, it was argued in [3] that one must distinguish between objects defined explicitly (closed terms) and objects introduced implicitly (by existential formulas, say).

In this thesis we use the semantic definition of extensions for open default theories proposed in [7] and [3], where, in contrast to the syntactic definitions in [10] and [11], free variables are treated as object variables, rather than meta-variables for the closed terms of the theory. The reason for choosing a semantic definition of extensions is that, on the one hand, it provides a complete description of the theory objects, and, on the other hand, it distinguishes between explicitly and implicitly defined objects.

Since the semantic treatment of open default theories allows one to describe all the elements of the domain under consideration, it has no syntactic counterpart within the ordinary first-order default logic, unless the domain is explicitly finite\(^1\) ([5]). In that case, an extension is defined syntactically over the underlying language of default theory, extended with an infinite set of new constant symbols, where each open default is replaced with the set of all its closed instances.

It was shown in [6] that in the case of countable\(^2\) or finite domain, there is a syntactic definition, which is equivalent to the original semantic definition, in first-order logic extended with the Carnap rule of inference. In this thesis we show that for infinite domains, there exists a syntactic definition, which is equivalent to the semantic definition, when the underlying language of default theory is monadic and first-order logic is extended with the Carnap rule of inference. Like in the case of explicitly defined finite domains, this syntactic definition treats an open default as the set of all its closed instances over the underlying language of default theory, extended with an infinite set of new constant symbols. We prove then, that in this syntactic definition, it is sufficient to extend the underlying language with a countable set of new constant symbols. Using this result, we show that the original semantic definition of extension for open default theories over monadic languages can always be restricted to a countable base.

Next, we present a syntactic definition of extensions over a weaker logic, which also treats a open default as the set of all its closed instances over the underlying language, extended with an infinite set of new constant symbols.

\(^1\)By 'explicitly finite' we mean the domain closure assumption, i.e. an axiom \(\forall x \bigwedge^n_{i=1} x = a_i\) for some constant symbols \(a_1, \ldots, a_n\). See also Section 2.6.

\(^2\)In this thesis, 'countable' means infinite countable.
We prove that this syntactic definition does not depend on the cardinality of the set of new constant symbols, as long as it is infinite.

Finally, we prove that, for uniterm default theories over finite languages not containing function symbols, the syntactic definition of extensions does not depend on the cardinality of the set of new constants. We use this result to prove that, under the domain closure assumption, the original semantic definition of extension for uniterm default theories over such languages can always be restricted to a finite base.

The thesis is organized as follows. In Chapter 2 we recall the notation used throughout this thesis. In Chapter 3 we present a syntactic definition of extensions, which is equivalent to the semantic definition for monadic languages. In Chapter 4 we prove that this syntactic definition of extensions does not depend on the cardinality of its infinite base in a general case, and the semantic definition does not depend on the cardinality of its infinite base for monadic languages. In Chapter 5 we propose an alternative syntactic definition of extensions, based on a weaker logic, and prove that it does not depend on the cardinality of its infinite base as well. In Chapter 6 we present a syntactic definition of extensions, which is equivalent to the semantic definition, when the theory domain is explicitly finite (see [5]). We prove that, for uniterm default theories over finite languages not containing function symbols, this syntactic definition of extensions does not depend on the cardinality of its base. Thus, for such languages, the extensions for uniterm default theories over explicitly defined finite domains can be restricted to a finite base. Chapter 7 contains appendices. Finally, Chapter 8 contains the summary, along with suggestions for further research.
Chapter 2

Background

In this chapter we briefly recall the concept of default theories, the Herbrand semantics of first-order logic, the definition of a monadic language and the domain closure assumption. We assume that the reader is acquainted with classical first-order logic\(^1\).

2.1 Default theories

Reiter’s default logic ([11]) deals with rules of inference called defaults which are expressions of the form

\[
\delta(x) = \frac{\alpha(x) : M\beta_1(x), \ldots, M\beta_m(x)}{\gamma(x)},
\]

where \(\alpha(x), \beta_1(x), \ldots, \beta_m(x), m \geq 1\), and \(\gamma(x)\) are formulas of first-order logic whose free variables are among \(x = x_1, \ldots, x_n\). A default is closed if none of \(\alpha, \beta_1, \ldots, \beta_m\), and \(\gamma\) contains a free variable. Otherwise it is open. The formula \(\alpha(x)\) is called the prerequisite of the default rule, the formulas \(\beta_1(x), \ldots, \beta_m(x)\) are called the justifications, and the formula \(\gamma(x)\) is called the conclusion. These formulas are denoted by \(\text{pre}(\delta)\), \(\text{just}(\delta)\), and \(\text{conc}(\delta)\), respectively.

A default theory is a pair \((D, A)\), where \(D\) is a set of defaults and \(A\) is a set of first-order sentences (axioms). A default theory is closed, if all its defaults are closed. Otherwise it is open.

\(^1\)In this thesis, the first-order logic is without equality.
2.2 Extensions for closed default theories

In this section we recall the syntactic and semantic definitions of extensions for closed default theories.

Recall that closed defaults are expressions of the form

\[
\frac{\alpha : M\beta_1, \ldots, M\beta_m}{\gamma},
\]

where \(\alpha, \beta_1, \ldots, \beta_m, m \geq 1\), and \(\gamma\) are closed formulas.

**Definition 1** ([11]) Let \((D, A)\) be a closed default theory. For any set of sentences \(S\) let \(\Gamma_{(D, A)}(S)\) be the smallest set of sentences \(B\) (beliefs) that satisfies the following three properties.

D1. \(A \subseteq B\).

D2. \(Th(B) = B\), i.e., \(B\) is deductively closed.

D3. If \(\frac{\alpha : M\beta_1, \ldots, M\beta_m}{\gamma} \in D\), \(\alpha \in B\), and \(\neg \beta_1, \ldots, \neg \beta_m \not\in S\), then \(\gamma \in B\).

A set of sentences \(E\) is an extension for \((D, A)\) if \(\Gamma_{(D, A)}(E) = E\), i.e., if \(E\) is a fixed point of the operator \(\Gamma_{(D, A)}\).

We shall need an alternative definition of the operator \(\Gamma\) provided by Lemma 1 below.

**Lemma 1** Let \((D, A)\) be a closed default theory and let \(S\) be a set of sentences. Then \(\Gamma_{(D, A)}(S) = Th\left(\bigcup_{i=0}^{\infty} E_i\right)\), where \(E_0 = A\) and

\[
E_{i+1} = \{\gamma \mid \text{for some } \frac{\alpha : M\beta_1, \ldots, M\beta_m}{\gamma} \in D, \quad \bigcup_{k=0}^{i} E_k \vdash \alpha \text{ and } \neg \beta_1, \ldots, \neg \beta_m \not\in S\}\}.
\]

The proof of the lemma is like that of [11, Theorem 2.1] and is omitted.

Next, we present a semantic definition of extension for closed default theories. Here and hereafter, for any class of interpretations \(W\), by \(Th_{\mathcal{L}}(W)\) we mean the set of all closed formulas over \(\mathcal{L}\) satisfied by all elements of \(W\).
Definition 2 ([2]) Let $(D, A)$ be a closed default theory over $\mathcal{L}$. For any class of interpretations $W$, let $\Sigma_{(D,A)}(W)$ be the largest class $V$ of models of $A$ that satisfies the following condition.

If $\alpha : M_{\beta_1}, \ldots, M_{\beta_m} \in D$, $\alpha \in \mathcal{Th}_\mathcal{L}(V)$, and $\neg \beta_1, \ldots, \neg \beta_m \notin \mathcal{Th}_\mathcal{L}(W)$, then $\gamma \in \mathcal{Th}_\mathcal{L}(V)$.

It is known from [2] that the definition of extensions as the theories of the fixed points of the operator $\Sigma$ is equivalent to Reiter’s original definition (Definition 1, p. 7). That is, a set of sentences $E$ is an extension for a closed default theory $(D, A)$ if and only if $E = \mathcal{Th}_\mathcal{L}(W)$ for some fixed point $W$ of $\Sigma_{(D,A)}$.

2.3 Herbrand semantics of first-order logic

In this section we define Herbrand semantics of first-order logic that is the basis of the semantic approach to open default theories.

We denote by $\mathcal{L}$ the language of the underlying first-order logic. Let $b$ be a set that contains no symbols of $\mathcal{L}$. We denote by $\mathcal{L}_b$ the language obtained from $\mathcal{L}$ by augmenting its set of constants with all elements of $b$.

The set of all closed terms of the language $\mathcal{L}_b$, denoted $\mathcal{T}_{\mathcal{L}_b}$, is called the Herbrand universe of $\mathcal{L}_b$. A Herbrand $b$-interpretation is a set of ground (closed) atomic formulas of $\mathcal{L}_b$. Note that closed formulas over $\mathcal{L}_b$ are of the form $\varphi(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n \in \mathcal{T}_{\mathcal{L}_b}$, and $\varphi(x_1, \ldots, x_n)$ is a formula over $\mathcal{L}$ whose free variables are among $x_1, \ldots, x_n$. The set $b$ is called the base of Herbrand $b$-interpretation. Herbrand $b$-interpretations with the empty base are called Herbrand (or term) interpretations.

Let $w$ be a Herbrand $b$-interpretation and let $\varphi$ be a closed formula over $\mathcal{L}_b$. We say that $w$ satisfies $\varphi$, denoted $w \models \varphi$, if the following holds.

- if $\varphi$ is an atomic formula, then $w \models \varphi$ if and only if $\varphi \in w$;
- $w \models \varphi \supset \psi$ if and only if $w \not\models \varphi$ or $w \models \psi$;
- $w \models \neg \varphi$ if and only if $w \not\models \varphi$; and
- $w \models \forall x \varphi(x)$ if and only if for each $t \in \mathcal{T}_{\mathcal{L}_b}$, $w \models \varphi(t)$.

For a Herbrand $b$-interpretation $w$ we define the $\mathcal{L}$-theory ($\mathcal{L}_b$-theory) of $w$, denoted $\mathcal{Th}_\mathcal{L}(w)$ ($\mathcal{Th}_{\mathcal{L}_b}(w)$), as the set of all closed formulas of $\mathcal{L}$

\[ \text{In particular, } \mathcal{L}_b \text{ is } \mathcal{L}. \]
\((\mathcal{L}_1)\) satisfied by \(w\). For a set of Herbrand \(b\)-interpretations \(W\) we define the \(\mathcal{L}\)-theory \((\mathcal{L}_1\text{-}theory)\) of \(W\), denoted \(Th_{\mathcal{L}}(W)\) \((Th_{\mathcal{L}_1}(W))\), as the set of all closed formulas of \(\mathcal{L}\) \((\mathcal{L}_1)\) satisfied by all elements of \(W\). That is, 
\[Th_{\mathcal{L}}(W) \triangleq \bigcap_{w \in W} Th_{\mathcal{L}}(w) \quad (Th_{\mathcal{L}_1}(W) \triangleq \bigcap_{w \in W} Th_{\mathcal{L}_1}(w)).\]

Finally, let \(X\) be a set of closed formulas over \(\mathcal{L}\) \((\mathcal{L}_1)\). We say that \(w\) is a Herbrand \(b\)-model (Herbrand model) of \(X\), denoted by \(w \models X\), if \(X \subseteq Th_{\mathcal{L}}(w)\).

**Remark** It is well-known that for an infinite set of new constant symbols \(b\), Herbrand \(b\)-interpretations are complete and sound for first-order logic. That is, for a set of formulas \(X\) over \(\mathcal{L}\) and a formula \(\varphi\) over \(\mathcal{L}\), \(X \vdash \varphi\) if and only if \(\varphi\) is satisfied by all Herbrand \(b\)-interpretations which satisfy \(X\). In particular, Herbrand \(b\)-interpretations with an infinite base naturally arise in the Henkin proof of the completeness theorem \((9,\ Lemma\ 2.16,\ p.70)\).

### 2.4 Extensions for open default theories

In this section, departing from Definition 2 (p. 8) and following [7] and [3], we define extensions for open default theories. We start with the intuition underlying the definition.

There are two types of objects in the domain of a default theory. One type consists of the fixed built-in objects which belong to \(T_{\mathcal{L}}\) and must be present in any Herbrand interpretation, and the other type consist of implicitly defined unknown objects which may vary from one Herbrand interpretation to other, e.g., objects introduced by existentially quantified formulas. These objects generate other unknown objects by means of the function symbols of \(\mathcal{L}\). Thus, it seems natural to assume that the theory domain is a Herbrand universe of the original language augmented with a set of new (unknown) objects, cf. [8, Chapter 1, §3].

The following definition of extensions for open default theories is a relativization of Definition 2 to Herbrand \(b\)-interpretations with an infinite set of new constant symbols \(b\). The reason for passing to a semantic definition is that, in general, it is impossible to describe a Herbrand universe by means of the standard proof theory. The only exception is the cases when the theory domain is explicitly finite ([5]).

**Definition 3** ([3]) Let \(b\) be a set of new constant symbols and let \((D, A)\) be a default theory. For any set of Herbrand \(b\)-interpretations \(W\) let \(\Delta^b_{(D,A)}(W)\)
be the largest set $V$ of Herbrand $b$-models of $A$ that satisfies the following condition.

For any $\overline{\alpha(x)} : M\beta_1(x), \ldots, M\beta_m(x) \in D$ and any tuple $t$ of elements of $T_{L_i}$ if $\alpha(t) \in Th_{L_i}(V)$ and $\neg\beta_1(t), \ldots, \neg\beta_m(t) \notin Th_{L_i}(W)$, then $\gamma(t) \in Th_{L_i}(V)$.

Let $b$ be an infinite set of new constant symbols. A set of sentences $E$ is called a $b$-extension for $(D, A)$ if $E = Th_{L_i}(W)$ for some fixed point $W$ of $\Delta^b_{(D, A)}$.

We will also refer to the set $b$ as the base of $E$.

The reason for defining the $b$-extension over an infinite base is that for any finite base, the cardinality of the base can be extracted from the $b$-extension. This behavior is undesirable in the general case.

It is known from [3, Theorem 42] that for closed default theories Definition 3 is equivalent to the original Reiter’s definition (Definition 1, p. 7).

### 2.5 Monadic languages

A language $L$ is called monadic if it contains no function symbols and no $n$-place predicate symbols with $n \geq 2$.

**Definition 4** A literal is an atomic formula or its negation. Open literal is an open atomic formula or its negation. If an (open) literal is an (open) atomic formula we say that it is a positive (open) literal, otherwise it is a negative (open) literal.

Let $P_i$, $i = 0, 1, \ldots$ be all the predicate symbols of $L$ and let $x$ be a variable symbol. A basic formula is a formula of one of the two following forms.

1. $P_i(c)$, for some $c \in T_{L_i}$, $i = 0, 1, \ldots$
2. $\exists x(\bigwedge_j L_j(x))$ for some open literals $L_j(x)$

The following lemma presents a well known quality of monadic languages. The proof is straightforward and will be omitted.

**Lemma 2** Each formula over monadic language $L$ is equivalent to a boolean (propositional) combination of basic formulas over $L$. 

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2.6 The domain closure assumption

Let $\mathcal{L}$ be the language of the underlying first-order logic. A formula $\forall x \bigvee_{i=1}^{m} x = a_i$, for some $a_1, \ldots, a_m \in T_{\mathcal{L}}$ is called the domain closure assumption. In particular, if follows from the domain closure assumption that there are at most $m$ pairwise distinct theory objects.

It is well known that if $\mathcal{L}$ is a finite language not containing function symbols, then, under the domain closure assumption, there are only finitely many non-equivalent formulas over $\mathcal{L}$ and every formula over $\mathcal{L}$ is equivalent to a quantifier-free one.

In order to describe syntactically all the theory objects when the domain is infinite (that is, in order to express syntactically a counterpart of the domain closure assumption for an infinite domain), we should extend first-order logic with the following rule of inference.

**Definition 5** The Carnap rule is the infinitary rule of inference $\frac{\{\varphi(t)\}_{t \in T_{\mathcal{L}}}}{\forall x \varphi(x)}$.

Note that $\varphi$ can contain more than one free variable.
Chapter 3

Extensions for open default theories over monadic languages

In this chapter we present a syntactic definition of extensions for open default theories. Following [6], we treat an open default as the set of all its closed instances over the language $L_1$ - the original language $L$ extended with the infinite set $b$ of new constant symbols. Also, in order to describe syntactically all the theory objects of the obtained infinite domain, we extend first-order logic with the Carnap rule. Then we prove that, over monadic languages, our definition is equivalent to the original semantic definition of extensions for open default theories (Definition 3, p. 9). Note that this is not true in the general case.

3.1 C-proof system

We need a proof system for first-order logic, extended with the Carnap rule. We will refer this proof system as a C-proof system.

Since we use our proof system only for closed formulas, there is no need for the GEN rule. Moreover, we replace the axiom schema (A4)\(^1\) with

$$(A4) \forall x A(x) \supset A(t) \text{ if } A(x) \text{ is a formula over } L \text{ and } t \text{ is a closed term of } L$$

\(^1\)See appendix for the list of the logical axioms of first-order logic.
Definition 6 Let $\Gamma$ be a set of closed formulas over $\mathcal{L}$. A $C$-proof in $\mathcal{L}$ of a closed formula $A$ from $\Gamma$ is a well-ordered list $A_1, A_2, \ldots, A_\alpha, \ldots, A$ of closed formulas over $\mathcal{L}$, such that the last formula of the list is $A$, and for every $\alpha$, at least one of the following conditions holds.

- $A_\alpha$ is an axiom of $\mathcal{L}$.
- $A_\alpha$ is a formula of $\Gamma$.
- there exists $\beta_1, \beta_2 < \alpha$, such that $A_{\beta_1}$ is of the form $\psi \supset \varphi$, $A_{\beta_2}$ is of the form $\psi$ and $A_\alpha$ is of the form $\varphi$. ($A_\alpha$ is obtained by MP).
- $A_\alpha$ is of the form $\forall x \varphi(x)$ and for each term $t \in T_{\mathcal{L}}$, there exists $\beta_t < \alpha$, such that $A_{\beta_t}$ is $\varphi(t)$. ($A_\alpha$ is obtained by the Carnap rule).

We shall write $\Gamma \vdash_C A$ if there exists a $C$-proof of $A$ from $\Gamma$.

Remark Actually, it can be readily seen that for closed formulas, this proof system is equivalent to the proof system of the classical first-order logic, extended with the Carnap rule: obviously, for any language $\mathcal{L}$, the classical first-order logic, extended with the Carnap rule, contains all the axioms and rules of the $C$-proof system. Also, a straightforward induction shows that if a formula $\varphi(x)$ can be proved from $\Gamma$ in the classical first-order logic, extended with the Carnap rule, then for each term $t \in T_{\mathcal{L}}$, $\varphi(t)$ can also be proved from $\Gamma$. It follows that GEN is admissible in the classical first-order logic, extended with the Carnap rule and, thus, for closed formulas, the $C$-proof system contains all the axioms and rules of that proof system.

For a set of closed formulas $\Gamma$, we denote by $\text{Th}^C(\Gamma)$ the set of all closed formulas having a $C$-proof from $\Gamma$. That is, $\text{Th}^C(\Gamma) \overset{\Delta}{=} \{ \varphi \mid \Gamma \vdash_C \varphi \}$. We say that a set of closed formulas $\Gamma$ is $C$-consistent if $\text{Th}^C(\Gamma)$ is consistent in the usual first-order sense. We say that a set of closed formulas $\Gamma$ is $C$-deductively closed if $\Gamma = \text{Th}^C(\Gamma)$.

Note that the same proof system was used by [6] without being explicitly defined.

3.2 The completeness theorem

In this section we prove the completeness theorem for first-order logic with the Carnap rule over monadic languages.
Theorem 1 Let $\mathcal{L}$ be a monadic language. If $\Gamma$ is a $C$-consistent theory, then $\Gamma$ has a Herbrand model.

In order to prove the Theorem 1 we shall prove the following lemmas.

Lemma 3 (C-Deduction Theorem) If $\Gamma$ is a set of closed formulas and $A, B$ are closed formulas, such that $\Gamma, A \vdash_C B$, then $\Gamma \vdash_C A \supset B$.

Proof The proof is similar to that of the "classical" deduction theorem. Let $B_1, \ldots, B_\alpha, \ldots, B$ be a $C$-proof of $B$ from $\Gamma \cup \{A\}$. We will show by induction on $\alpha$, that for each $B_\alpha$, there exists a $C$-proof of $A \supset B_\alpha$ from $\Gamma$ (that is, $\Gamma \vdash_C A \supset B_\alpha$).

Induction basis: $B_1$ is either in $\Gamma$, or an axiom of $\mathcal{L}$, or $A$ itself. For the first two cases, $B_1 \supset (A \supset B_1)$ is an instance of the axiom schema (A1) and by virtue of MP, $A \supset B_1$. For the third case, $(A \supset ((A \supset A) \supset A)) \supset ((A \supset (A \supset A)) \supset (A \supset A))$ is an instance of the axiom schema (A2), and $(A \supset (A \supset A)) \supset (A \supset A)$. By MP, $A \supset (A \supset A)$ is an instance of the axiom schema (A1), and by virtue of MP on the last two formulas, $A \supset A$. That is $A \supset B_1$.

Induction step: If $B_\alpha$ is either in $\Gamma$ or an axiom of $\mathcal{L}$ or $A$ itself, then $A \supset B_\alpha$ exactly as in the proof of the induction basis. Otherwise, $B_\alpha$ is introduced by one of the following rules of inference.

1. $B_\alpha$ is a consequence of $B_{\beta_1}$ and $B_{\beta_2}$, $\beta_1, \beta_2 < \alpha$, by MP, i.e. $B_{\beta_1}$ is of the form $C_1 \supset C_2$, $B_{\beta_2}$ is of the form $C_1$ and $B_{\beta}$ is of the form $C_2$. By the induction hypothesis, there exists a $C$-proof of $A \supset (C_1 \supset C_2)$ and a $C$-proof of $A \supset C_1$ from $\Gamma$. Now, by the axiom schema (A2), $(A \supset (C_1 \supset C_2)) \supset ((A \supset C_1) \supset ((A \supset C_2)))$. Applying MP twice, $A \supset C_2$. That is $A \supset B_\alpha$.

2. $B_\alpha$ is a consequence of $\{C(t)_{t \in \mathcal{T}}\}$ by the Carnap rule, i.e. $B_\alpha$ is of the form $\forall x C(x)$ and for each term $t \in \mathcal{T}$, there is $\beta < \alpha$, such that $B_{\beta t}$ is $C(t)$. By the induction hypothesis, there exists a $C$-proof of $A \supset C(t)$ from $\Gamma$ for each $t \in \mathcal{T}$. By the Carnap rule, we get $\forall x (A \supset C(x))$. By the axiom schema (A5), $\forall x (A \supset C(x)) \supset (A \supset \forall x C(x))$ is an axiom. Now, applying MP, $A \supset \forall x C(x)$. That is $A \supset B_\alpha$.

Since each proof that is constructed in this lemma is a concatenation of well-ordered lists of finite linearly ordered sets, it is a well-ordered list as
well. Thus, it is a valid $C$-proof. This completes the proof of the lemma. □

**Lemma 4** Let $\Gamma$ be a $C$-consistent theory. If for some closed formula $A$, $\Gamma \vdash_C A$, then $\Gamma \cup \{\neg A\}$ is $C$-consistent.

**Proof** Assume $\Gamma \cup \{\neg A\}$ is $C$-inconsistent. Then, $\Gamma \cup \{\neg A\} \vdash_C \text{false}$. By Lemma 3 (p. 14), $\Gamma \vdash_C \neg A \supset \text{false}$. Since $(\neg A \supset \text{false}) \supset A$ is a tautology, it can be proven from $\Gamma$ using only axioms $(A1) - (A3)$\(^2\) and MP. Thus, $\Gamma \vdash_C (\neg A \supset \text{false}) \supset A$ and, by MP, $\Gamma \vdash_C A$, in contradiction with our hypothesis. □

**Lemma 5** If $\Gamma$ is a $C$-consistent theory such that $\Gamma \vdash_C \exists x \varphi(x)$, then for some $c \in T_L$, $\Gamma \vdash_C \neg \varphi(c)$.

**Proof** Assume to the contrary, that for each $c \in T_L$, $\Gamma \vdash_C \neg \varphi(c)$. Then by the Carnap rule, $\Gamma \vdash_C \forall x \neg \varphi(x)$ in contradiction with $\Gamma \vdash_C \exists x \varphi(x)$ and the assumption that $\Gamma$ is $C$-consistent. □

Now we are ready to prove Theorem 1. To present the proof we shall need the following definition.

**Definition 7** A set of formulas $\Gamma$ is **basically-complete** if for each closed basic formula $\varphi$, either $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg \varphi$.

**Proof of Theorem 1** Let $\exists x \varphi_0(x), \exists x \varphi_1(x), \ldots, \exists x \varphi_i(x), \ldots$ be an enumeration of all basic formulas of the form 2 (p. 10) over $L$. (Since there are countably many predicate symbols over $L$, there are countably many such formulas.) We construct a sequence of sets of closed formulas $\tilde{\Gamma}_i$ by induction as follows.

$\tilde{\Gamma}_0 = \Gamma$.

If $\tilde{\Gamma}_i \vdash_C \exists x \varphi_i(x)$, then $\tilde{\Gamma}_{i+1} = \tilde{\Gamma}_i \cup \{\varphi_i(c)\}$, where $c$ is as in Lemma 5 (p. 15). Otherwise, $\tilde{\Gamma}_{i+1} = \tilde{\Gamma}_i \cup \{\neg \exists x \varphi_i(x)\}$.

In both cases $\tilde{\Gamma}_{i+1}$ is $C$-consistent: in the first case, by Lemmas 4 (p. 15) and 5 (p. 15) and in the second case, by Lemma 4 (p. 15).

We define $\bar{\Gamma} = \bigcup_{i=0}^{\infty} \tilde{\Gamma}_i$. $\bar{\Gamma}$ is consistent since all $\tilde{\Gamma}_i$ are consistent in the usual first-order sense.

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\(^2\)See appendix for the list of the logical axioms of first-order logic.
We build now a consistent and basically-complete extension of $\hat{\Gamma}$.

Let $P_0, P_1, \ldots, P_\alpha, \ldots$ be a well-ordered list of all closed atomic formulas over $\mathcal{L}$, i.e. basic formulas of the form 1 (p. 10). The order in which we list them is immaterial, as long as the list associates in one-one fashion an ordinal number with each formula. Define a well-ordered list $\hat{\Gamma}_0, \hat{\Gamma}_1, \ldots, \hat{\Gamma}_\alpha, \ldots$ of sets of formulas by induction in the following way.

If $\hat{\Gamma}_\alpha \not\vdash -P_\alpha$, then $\hat{\Gamma}_{\alpha+1} = \hat{\Gamma}_\alpha \cup \{P_\alpha\}$. Otherwise, $\hat{\Gamma}_{\alpha+1} = \hat{\Gamma}_\alpha$. For a limit ordinal $\alpha$, $\hat{\Gamma}_\alpha = \bigcup_{\beta < \alpha} \hat{\Gamma}_\beta$.

We define $\hat{\Gamma} = \bigcup_{\alpha} \hat{\Gamma}_\alpha$. To show that $\hat{\Gamma}$ is consistent, it suffices to prove that all the $\hat{\Gamma}_\alpha$ are consistent. We prove the consistency of $\hat{\Gamma}_\alpha$'s by induction on $\alpha$, $\hat{\Gamma}_0 = \hat{\Gamma}$ is consistent. Assume that $\hat{\Gamma}_\beta$ is consistent for all $\beta < \alpha$. For a successor ordinal $\alpha$, if $\hat{\Gamma}_{\alpha-1} \not\vdash -P_{\alpha-1}$, then by [9, Lemma 2.12, p. 67], $\hat{\Gamma}_\alpha = \hat{\Gamma}_{\alpha-1} \cup \{P_{\alpha-1}\}$ is consistent. Otherwise, $\hat{\Gamma}_\alpha = \hat{\Gamma}_{\alpha-1}$, and it is consistent by the induction hypothesis. For a limit ordinal $\alpha$, $\bigcup_{\beta < \alpha} \hat{\Gamma}_\beta$ is consistent as it is a union of consistent sets.

Let us show now that $\hat{\Gamma}$ is a basically-complete extension of $\Gamma$, that is, for each closed basic formula $\varphi$, either $\hat{\Gamma} \vdash \varphi$ or $\hat{\Gamma} \vdash -\varphi$. There are the two following cases.

1. $\varphi$ is a basic formula of the form 1 (p. 10), i.e. $\varphi$ is a closed atomic formula. Then, $\varphi = P_\alpha$ for some $\alpha$. Now, either $\Gamma_\alpha \not\vdash -P_\alpha$, and then, by the definition of $\hat{\Gamma}_{\alpha+1}$, $P_\alpha \in \hat{\Gamma}_{\alpha+1}$, implying $\hat{\Gamma} \vdash P_\alpha$, or $\hat{\Gamma}_\alpha \vdash -P_\alpha$, and then, $\hat{\Gamma} \vdash -P_\alpha$.

2. $\varphi$ is a basic formula of the form 2 (p. 10). Then, $\varphi = \exists x \varphi_i(x)$ for some $i$. Now, either $\hat{\Gamma}_i \not\vdash \exists x \varphi_i(x)$ and then, by the definition of $\hat{\Gamma}_{i+1}$, $\hat{\Gamma}_{i+1} \vdash \exists x \varphi_i(x)$, or $\hat{\Gamma}_i \not\vdash \exists x \varphi_i(x)$, and then, $\hat{\Gamma}_i \vdash \exists x \varphi_i(x)$.

We define a Herbrand interpretation of $\mathcal{L}$, $M$ by $M = \{P(c) \mid \hat{\Gamma} \vdash P(c)\}$.

We shall prove now that $M$ is a Herbrand model for $\hat{\Gamma}$. It will follow then, that $M$ is a Herbrand model for $\Gamma$, because $\Gamma \subseteq \hat{\Gamma}$.

First, we show that for each basic formula $\varphi$,

$$M \models \varphi \text{ if and only if } \hat{\Gamma} \vdash \varphi. \quad (*)$$

Even though the set of predicate symbols is countable, the set of constant symbols may be uncountable.
If $\varphi$ is a basic formula of the form 1 (p. 10), that is, $\varphi$ is a closed atomic formula, it follows immediately from the definition of $M$.

Let $\varphi$ be a basic formula of the form 2 (p. 10), that is, $\varphi$ is of the form $\exists x(L_1(x) \land \ldots \land L_n(x))$ for some open literals $L_1(x), \ldots, L_n(x)$ over $\mathcal{L}$. For the “only if” part, assume that $M \models \varphi$. Then there exists $d \in T_\mathcal{L}$, such that $M \models L_1(d) \land \ldots \land L_n(d)$. It follows that $M \models L_i(d)$, $1 \leq i \leq n$. Now, if $L_i(d)$ is a positive literal, then, by the definition of $M$, $\Gamma \vdash L_i(d)$. Assume $L_i(d)$ is a negative literal. Then, by the definition of $M$, $\Gamma \not\vdash \neg L_i(d)$. Since $\Gamma$ is basically-complete, $\Gamma \vdash L_i(d)$ also in this case. It follows that $\Gamma \vdash L_1(d) \land \ldots \land L_n(d)$ and thus, $\Gamma \vdash \exists x(L_1(x) \land \ldots \land L_n(x))$, i.e., $\Gamma \not\vdash \varphi$.

For the “if” part, assume that $\Gamma \vdash \varphi$. Since $\varphi$ is of the form $\exists x \varphi_i(x)$ for some $i$, and since $\Gamma$ is consistent, $\Gamma \not\vdash \neg \exists x \varphi_i(x)$, implying $\neg \exists x \varphi_i(x) \not\in \Gamma$.

Since $\Gamma \subseteq \Gamma$, $\neg \exists x \varphi_i(x) \not\in \Gamma$. By the definition of $\Gamma$, there exists $c \in T_\mathcal{L}$, such that $\varphi_i(c) \in \Gamma$, and thus, $\Gamma \vdash \varphi_i(c)$. That is $\Gamma \vdash L_1(c) \land \ldots \land L_n(c)$. It follows that $\Gamma \vdash L_i(c)$, $1 \leq i \leq n$. By the definition of $M$, $M \models L_i(c)$, $1 \leq i \leq n$. Thus, $M \models L_1(c) \land \ldots \land L_n(c)$ and therefore, $M \models \exists x(L_1(x) \land \ldots \land L_n(x))$, i.e., $M \models \varphi$.

In the general case, by Lemma 2 (p. 10), each formula of $\Gamma$ is equivalent to a propositional combination of basic formulas. Let $\psi \in \Gamma$ be equivalent to a formula of the form $\bigwedge_i \bigvee_j \psi_{i,j}$, where $\psi_{i,j}$ is either a basic formula or a negation of a basic formula. We will prove that $M \models \bigwedge_i \bigvee_j \psi_{i,j}$. For if not, $M \not\models \bigvee_i \psi_{i,j}$ for some $i$. That is, $M \not\models \psi_{i,j}$ for each $j$. If $\psi_{i,j}$ is a basic formula, then by (*) $\Gamma \not\vdash \psi_{i,j}$ and since $\Gamma$ is basically-complete, $\Gamma \vdash \neg \psi_{i,j}$. Otherwise, $\neg \psi_{i,j}$ is a basic formula, and $M \models \neg \psi_{i,j}$. By (*), $\Gamma \vdash \neg \psi_{i,j}$ also in this case. Since $\Gamma$ is consistent, $\Gamma \not\models \bigvee_i \psi_{i,j}$ and therefore, $\Gamma \not\vdash \bigwedge_i \bigvee_j \psi_{i,j}$. Since $\psi$ is equivalent to $\bigwedge_i \bigvee_j \psi_{i,j}$, $\Gamma \not\vdash \psi$, which contradict the assumption that $\psi \in \Gamma$.

It follows that $M$ is a model of $\Gamma$. This completes the proof of the theorem. □

Theorem 1 has the following corollary.

**Corollary** Let $\mathcal{L}$ be a monadic language and let $S$ be a $C$-deductively closed set of closed formulas over $\mathcal{L}_1$. Then $S = \text{Th}_{\mathcal{L}_1}(V)$, where $V$ is the set of all Herbrand models of $S$.

**Proof** We have to prove that $\varphi \in S$ if and only if $\varphi \in \text{Th}_{\mathcal{L}_1}(V)$ for each closed formula $\varphi$ over $\mathcal{L}_1$. Since $V$ is a set of Herbrand models of $S$, $\varphi \in S$ implies that $\varphi \in \text{Th}_{\mathcal{L}_1}(V)$. Now, if $\varphi \not\in S$, then, since $S$ is $C$-deductively

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closed \( S = Th^C(S) \), \( \varphi \not\in Th^C(S) \), that is, \( S \not\models_C \varphi \). It follows that 
\( S \) is consistent. Since \( S = Th^C(S) \), \( Th^C(S) \) is consistent, that is, \( S \) is \( C \)-consistent. By Lemma 4 (p. 15), \( S \cup \{ \neg \varphi \} \) is \( C \)-consistent and by Theorem 1 (p. 14), there exists a Herbrand model of \( S \cup \{ \neg \varphi \} \). It follows that, there exists a Herbrand model of \( S \) which is not a model of \( \varphi \), and since \( V \) is the set of all Herbrand models of \( S \), \( \varphi \not\in Th_{L_1}(V) \). □

3.3 Syntactic definition of extensions

In this section we give a syntactic definition of extensions for open default theories.

First, we recall the “ground substitution” definition of extensions for open default theories, that treats an open default as an abbreviation of all its closed instances.

**Definition 8** ([10]) A ground (or closed) \( L_1 \) instance of an open default 
\[
\delta(x) = \frac{\alpha(x) : \beta_1(x), \ldots, \beta_m(x)}{\gamma(x)}
\]
\[
\delta(t) = \frac{\alpha(t) : \beta_1(t), \ldots, \beta_m(t)}{\gamma(t)}
\]

where \( t = t_1, \ldots, t_n \) is a tuple of ground terms of \( L_1 \). For an open default \( \delta \), the set of all ground \( L_1 \) instances of \( \delta \) is denoted by \( \delta_{L_1} \), and for a set of defaults \( D \), \( \tilde{D}_{L_1} = \bigcup_{\delta \in D} \delta_{L_1} \) is the set of all ground \( L_1 \) instances of all defaults of \( D \).

Definition 9 below is a relativization of Definition 1 (p. 7) to first-order logic with the Carnap rule.

**Definition 9** ([6]) Let \((D,A)\) be a closed default theory. For any set of sentences \( S \) let \( \Gamma^C_{(D,A)}(S) \) be the smallest set of sentences \( B \) (beliefs) that satisfies the following three properties.

CD1. \( A \subseteq B \).

CD2. \( Th^C(B) = B \), i.e. \( B \) is \( C \)-deductively closed.

CD3. If \( \alpha : \beta_1, \ldots, \beta_m \in D, \alpha \in B \) and \( \neg \beta_1, \ldots, \neg \beta_m \not\in S \), then 
\[
\gamma \in B
\]

A set of sentences \( E \) is a \( C \)-extension for \((D,A)\) if \( \Gamma^C_{(D,A)}(E) = E \), i.e. if \( E \) is a fixed point of the operator \( \Gamma^C_{(D,A)} \).
For an open default theory \((D, A)\) and a \(C\)-extension \(E\) for \((\bar{D}_{\mathcal{L}_1}, A)\), we will refer the set \(b\) as the base of \(E\).

**Lemma 6** Let \(\mathcal{L}\) be a monadic language and let \(b\) be a set of new constant symbols. Then \(E\) is a \(C\)-extension for \((\bar{D}_{\mathcal{L}_1}, A)\) if and only if there is a fixed point \(W\) of \(\Delta^b_{(D, A)}\) such that \(E = \text{Th}_{\mathcal{L}_1}(W)\).

**Remark** Note that this lemma does not hold in the general case, see [6, Example 6.9].

The proof of Lemma 6 is based on Lemmas 7 and 8 below.

**Lemma 7** Let \(\mathcal{L}\) be a monadic language, \(b\) be a set of new constant symbols, \(W\) be a set of Herbrand \(b\)-interpretations and let \(E = \text{Th}_{\mathcal{L}_1}(W)\). If \(W\) is a fixed point of operator \(\Delta^b_{(D, A)}\), then \(\Gamma^C_{(\bar{D}_{\mathcal{L}_1}, A)}(E) = E\).

**Proof** Let \(\Gamma^C_{(\bar{D}_{\mathcal{L}_1}, A)}(E) = B\). By clause CD2 of the definition of \(\Gamma^C_{(\bar{D}_{\mathcal{L}_1}, A)}(E)\) (p. 18) and, by the corollary to Theorem 1 (p. 17) with \(S = B\), \(B = \text{Th}_{\mathcal{L}_1}(V)\), where \(V\) is the set of all Herbrand models of \(B\). By clause CD1 of the definition, \(A \subseteq \text{Th}_{\mathcal{L}_1}(V)\), i.e., every member of \(V\) is a model of \(A\).

By clause CD3 of the definition, if \(\frac{\alpha : M \beta_1, \ldots, M \beta_m}{\gamma} \in \bar{D}_{\mathcal{L}_1}\), \(\alpha \in \text{Th}_{\mathcal{L}_1}(V)\) and \(\neg \beta_1, \ldots, \neg \beta_m \notin \text{Th}_{\mathcal{L}_1}(W)\), then \(\gamma \in \text{Th}_{\mathcal{L}_1}(V)\). Since \(W\) is a fixed point of operator \(\Delta^b_{(D, A)}\), by Definition 3 (p. 9), it is a largest set of Herbrand \(b\)-models of \(A\) which satisfy this property. It follows that \(V \subseteq W\). Since operator \(\text{Th}_{\mathcal{L}_1}\) is monotone decreasing,

\[
\text{Th}_{\mathcal{L}_1}(W) \subseteq \text{Th}_{\mathcal{L}_1}(V)
\]

Also, since \(W\) is a fixed point of \(\Delta^b_{(D, A)}\), \(\text{Th}_{\mathcal{L}_1}(W)\) is a set of formulas such that

1. \(A \subseteq \text{Th}_{\mathcal{L}_1}(W)\),
2. \(\Gamma^C_{\mathcal{L}_1}(\text{Th}_{\mathcal{L}_1}(W)) = \text{Th}_{\mathcal{L}_1}(W)\), because \(\text{Th}_{\mathcal{L}_1}(W)\) is deductively closed and \(W\) is a set of Herbrand \(b\)-interpretations over \(\mathcal{L}_1\). Thus, \(\text{Th}_{\mathcal{L}_1}(W)\) is also \(C\)-deductively closed.
3. For any \(\frac{\alpha(x) : M \beta_1(x), \ldots, M \beta_m(x)}{\gamma(x)} \in D\) and any \(t \in T_{\mathcal{L}_1}\), if \(\alpha(t) \in \text{Th}_{\mathcal{L}_1}(W)\) and \(\neg \beta_1(t), \ldots, \neg \beta_m(t) \notin \text{Th}_{\mathcal{L}_1}(W)\), then \(\gamma(t) \in \text{Th}_{\mathcal{L}_1}(W)\).
From point 3 follows that for any \( \alpha : M\beta_1, \ldots, M\beta_m / \gamma \in \bar{D}_{\mathcal{L}_1} \), if \( \alpha \in T_h_{\mathcal{L}_1}(W) \) and \( -\beta_1, \ldots, -\beta_m \not\in T_h_{\mathcal{L}_1}(W) \), then \( \gamma \in T_h_{\mathcal{L}_1}(W) \). Since \( B \) is the smallest set of formulas, which satisfy those three properties, \( B \subseteq T_h_{\mathcal{L}_1}(W) \). That is, \( T_h_{\mathcal{L}_1}(V) \subseteq T_h_{\mathcal{L}_1}(W) \). By \( (*) \), \( T_h_{\mathcal{L}_1}(V) = T_h_{\mathcal{L}_1}(W) \). Therefore, \( \Gamma^C_t(D_{\mathcal{L}_1}, A)(E) = E \), which completes the proof of the lemma. \( \square \)

**Lemma 8** Let \( \mathcal{L} \) be a monadic language, \( b \) be a set of new constant symbols, \( E \) be a set of closed formulas over \( \mathcal{L}_1 \) and let \( W \) be the set of all Herbrand models of \( E \). If \( E \) is a fixed point of operator \( \Gamma^C_t(D_{\mathcal{L}_1}, A) \), then \( \Delta^b_t(D, A)(W) = W \).

**Proof** Since \( E \) is a fixed point of operator \( \Gamma^C_t(D_{\mathcal{L}_1}, A) \), it is C-deductively closed and, by the corollary to Theorem 1 (p. 17) with \( S = E \) and \( V = W \), \( E = T_h_{\mathcal{L}_1}(W) \). By the definition of \( \Delta^b_t(D, A) \), \( \Delta^b_t(D, A)(W) \) is the largest set \( V \) of Herbrand b-models of \( A \), such that for any \( \alpha(x) : M\beta_1(x), \ldots, M\beta_m(x) / \gamma(x) \in D \) and any tuple \( t \in T_{\mathcal{L}_1} \), if \( \alpha(t) \in T_h_{\mathcal{L}_1}(V) \) and \( -\beta_1(t), \ldots, -\beta_m(t) \not\in T_h_{\mathcal{L}_1}(W) \), then \( \gamma(t) \in T_h_{\mathcal{L}_1}(W) \).

Since \( V \) is a set of Herbrand b-models of \( A \), \( A \subseteq T_h_{\mathcal{L}_1}(V) \). Also, \( T_h_{\mathcal{L}_1}(V) \) is a C-deductively closed set of formulas, because all models in \( V \) are Herbrand interpretations over \( \mathcal{L}_1 \). Moreover, for any \( \alpha : M\beta_1, \ldots, M\beta_m / \gamma \in \bar{D}_{\mathcal{L}_1} \), if \( \alpha \in T_h_{\mathcal{L}_1}(V) \) and \( -\beta_1, \ldots, -\beta_m \not\in T_h_{\mathcal{L}_1}(W) \), then \( \gamma \in T_h_{\mathcal{L}_1}(V) \). Since \( E \) is a fixed point of operator \( \Gamma^C_t(D_{\mathcal{L}_1}, A) \), it is the smallest set of formulas which satisfy those three properties, and thus, \( E \subseteq T_h_{\mathcal{L}_1}(V) \). Since \( E = T_h_{\mathcal{L}_1}(W) \) and operator \( T_h_{\mathcal{L}_1} \) is monotone decreasing,

\[
V \subseteq W
\]

\( (*) \)

Also, since \( E = T_h_{\mathcal{L}_1}(W) \) is a fixed point of \( \Gamma^C_t(D_{\mathcal{L}_1}, A) \), Definition 9 (p. 18) implies that, \( W \) is a set of models of \( A \) such that for any \( \alpha : M\beta_1, \ldots, M\beta_m / \gamma \in \bar{D}_{\mathcal{L}_1} \), if \( \alpha \in T_h_{\mathcal{L}_1}(W) \) and \( -\beta_1, \ldots, -\beta_m \not\in T_h_{\mathcal{L}_1}(W) \), then \( \gamma \in T_h_{\mathcal{L}_1}(W) \).

That is, for any \( \alpha(x) : M\beta_1(x), \ldots, M\beta_m(x) / \gamma(x) \in D \), and any tuple \( t \in T_{\mathcal{L}_1} \), if \( \alpha(t) \in T_h_{\mathcal{L}_1}(W) \) and \( -\beta_1(t), \ldots, -\beta_m(t) \not\in T_h_{\mathcal{L}_1}(W) \), then \( \gamma(t) \in T_h_{\mathcal{L}_1}(W) \). By the definition of \( V \), \( V \) is the largest set of Herbrand b-models
of $A$ which satisfy this property, and thus, $W \subseteq V$. By (*), $V = W$, i.e.,
$\Delta^b_{(D, A)}(W) = W$. This completes the proof of the lemma. \(\square\)

**Proof of Lemma 6** The “if” part of the lemma follows from Lemma 7 (p. 19). For the “only if” part, $E$ is a $C$-extension for $(\tilde{D}_{\mathcal{L}}, A)$, i.e., it is
a fixed point of operator $\Gamma^C_{(\tilde{D}_{\mathcal{L}}, A)}$. Thus, it is a $C$-deductively closed set
of closed formulas, and by the corollary to Theorem 1 (p. 17) with $S = E$
and $V = W$, $E = \text{Th}_{\mathcal{L}}(W)$, where $W$ is the set of all Herbrand models of
$E$. Now, by Lemma 8 (p. 20), $W$ is a fixed point of operator $\Delta^b_{(D, A)}$. This
completes the proof of the lemma. \(\square\)

Theorem 2 below immediately follows from Lemma 6.

**Theorem 2** Let $\mathcal{L}$ be a monadic language, $b$ be a set of new constant sym-
bols and let $(D, A)$ be an open default theory over $\mathcal{L}$. Then $E$ is a $b$-extension
for $(D, A)$ if and only if there is a $C$-extension $\tilde{E}$ for $(\tilde{D}_{\mathcal{L}}, A)$ such that
$E = \tilde{E}\mid_{\mathcal{L}}$. 
Chapter 4

Base independence of C-extensions

In Chapter 3 we proved that, for monadic languages, the semantic definition of extensions (Definition 3, p. 9) is equivalent to the syntactic definition (Definition 9, p. 18), in which the underlying language \( \mathcal{L} \) is extended with an infinite set of new constant symbols and first-order logic is extended with the Carnap rule. In this chapter we prove that when defining the extension syntactically, we can restrict ourself to a countable set of new constant symbols. Thus, in the case of monadic languages, the original definition of extensions for open default theories (Definition 3, p. 9) can be restricted to a countable base.

4.1 C,R-proof system

In this section we define an extension of the C-proof system, called C,R-proof system.

**Definition 10** The \( \mathcal{R}_\mathcal{L} \)-set is a set of expressions of the form \( \frac{\psi}{\varphi} \), where \( \psi \) and \( \varphi \) are formulas over \( \mathcal{L} \). The R-rule is the rule of inference of the form \( \frac{\psi}{\varphi} \), where \( R \) is an \( \mathcal{R}_\mathcal{L} \)-set and \( \frac{\psi}{\varphi} \in R \).

**Definition 11** Let \( \Gamma \) be a set of closed formulas over \( \mathcal{L} \) and \( R \) be an \( \mathcal{R}_\mathcal{L} \)-set. By a C,R-proof system we mean the C-proof system extended by the set of R-rules from R. A C,R-proof in \( \mathcal{L} \) of a closed formula \( A \) from \( \Gamma \) is a well-ordered list \( A_1, A_2, \ldots, A_\alpha, \ldots \), \( A \) of closed formulas over \( \mathcal{L} \), such that
the last formula of the list is \( A \), and for every \( \alpha \), at least one of the following conditions holds.

- \( A_\alpha \) is an axiom of \( \mathcal{L} \).
- \( A_\alpha \) is a formula of \( \Gamma \).
- there exists \( \beta_1, \beta_2 < \alpha \), such that \( A_{\beta_1} \) is of the form \( \psi \supset \varphi \), \( A_{\beta_2} \) is of the form \( \psi \) and \( A_\alpha \) is of the form \( \varphi \). (\( A_\alpha \) is obtained by MP).
- there exists \( \beta < \alpha \), such that \( \frac{A_\beta}{A_\alpha} \in R \). (\( A_\alpha \) is obtained by the R-rule).
- \( A_\alpha \) is of the form \( \forall x \varphi(x) \) and for each term \( t \in T_{\mathcal{L}} \), there exists \( \beta_t < \alpha \), such that \( A_{\beta_t} \) is \( \varphi(t) \). (\( A_\alpha \) is obtained by the Carnap rule).

We shall write \( \Gamma \vdash_{C,R} A \) if there exists a \( C,R \)-proof of \( A \) from \( \Gamma \).

We denote by \( Th^{C,R}(\Gamma) \) the set of all closed formulas having a \( C,R \)-proof from \( \Gamma \). That is, \( Th^{C,R}(\Gamma) \triangleq \{ A \mid \Gamma \vdash_{C,R} A \} \).

Note that (A4) is replaced by its closed version (p. 12) also in this case.

**Definition 12** Let \( \mathcal{L}_\infty \) and \( \mathcal{L}_e \) be languages with the same predicate symbols and the same \( n \)-place function symbols, \( n > 0 \), such that \( T_{\mathcal{L}_\infty} \subseteq T_{\mathcal{L}_e} \). Let \( R \) be an \( \mathcal{R}_{\mathcal{L}_e} \)-set, \( X_1 \) be a set of closed formulas over \( \mathcal{L}_\infty \) and let \( X_2 \) be a set of closed formulas over \( \mathcal{L}_e \). We say that \( X_2 \) is a \( C,R \)-conservative extension of \( X_1 \) if for any closed formula \( \varphi \) over \( \mathcal{L}_\infty \), \( X_1 \vdash_{C,R} \varphi \) if and only if \( X_2 \vdash_{C,R} \varphi \), where \( R_1 \) is the restriction of \( R \) to \( \mathcal{L}_\infty \) (\( R_1 \triangleq R|_{\mathcal{L}_\infty} \), i.e. \( R_1 = \{ \frac{\psi}{\varphi} \in R \mid \text{ both } \psi \text{ and } \varphi \text{ are formulas over } \mathcal{L}_\infty \} \).

**Lemma 9** Let \( b \) be an infinite set of new constant symbols, \( R \) be an \( \mathcal{R}_{\mathcal{L}_1} \)-set and let \( X \) be a set of closed formulas over \( \mathcal{L}_1 \). There exists a countable subset \( \theta \) of \( b \), such that \( X \) is a \( C,R \)-conservative extension of \( X|_{\mathcal{L}_1} \).

**Proof** Let \( b^\theta \) be any countable subset of \( b \). First, we define a function \( f \) from \( \mathcal{L}_1^\theta \) to a countable subset of \( b \) as follows. Let \( \mathcal{F}_1^\theta \) be the set of all closed formulas over \( \mathcal{L}_1^\theta \). This set is countable, because \( T_{\mathcal{L}_1^\theta} \) is countable and the sets of predicate and function symbols are countable. For each \( \varphi \in \mathcal{F}_1^\theta \), we define a finite set \( b_\varphi \) in the following way.

If \( X \vdash_{C,R} \varphi \), then we fix a particular proof of \( \varphi \) from \( X \). In this proof one of the following holds.
1. $\varphi$ is either an axiom of $\mathcal{L}'_{\nu}$ or a formula of $X|_{\mathcal{L}_{\nu}}$. Then we define $b_{\varphi} = \emptyset$.

2. $\varphi$ is deduced from $\psi$ and $\psi \supset \varphi$ by virtue of MP. Then we define $b_{\varphi} = \{ \text{all constants which appear in } \psi \}$.

3. $\varphi$ is deduced from $\psi$ by virtue of the $R$-rule $\psi \overline{\varphi}$. Then we define $b_{\varphi} = \{ \text{all constants which appear in } \psi \}$.

4. $\varphi_i$ is deduced by virtue of the Carnap rule. Then we define $b_{\varphi} = \emptyset$.

If $X \vdash_{C,R} \varphi$, two possible cases may occur.

1. $\varphi$ is of the form $\forall x \psi(x)$, that is, $X \not\vdash_{C,R} \forall x \psi(x)$. Then for some $t \in T_{\mathcal{L}_{\nu}}$, $X \not\vdash_{C,R} \psi(t)$ (for if not, $X \vdash_{C,R} \forall x \psi(x)$, by the Carnap rule) and we define $b_{\varphi} = \{ \text{all constants which appear in } t \}$.

2. $\varphi$ is not of the form $\forall x \psi(x)$. Then we define $b_{\varphi} = \emptyset$.

Now, we define $f(\mathcal{L}'_{\nu}) = T_{\mathcal{L}_{\nu}} \cup \{ \varphi \mid \varphi \in \mathcal{F}'_{\nu} \}$. Since $f(\mathcal{L}'_{\nu})$ is a union of a countable set and a countable union of finite sets, it is countable.

We define $b' = \bigcup_{i=1}^{\infty} b_i$, where $b_i$ is defined by induction in the following way. $b_1 = f(\mathcal{L})$ and $b_{i+1} = f(\mathcal{L}_{b_i})$.

It can be easily shown by induction on $i$ that each $b_i$ is countable, and thus, $b'$ is a countable, as it is a countable union of countable sets.

Define $X' = X|_{\mathcal{L}_{\nu}}$. We shall prove now that $X$ is a $C,R$-conservative extension of $X'$. That is, for any closed formula $\varphi$ over $\mathcal{L}'_{\nu}$, $X' \vdash_{C,R} \varphi$ if and only if $X \vdash_{C,R} \varphi$, where $R' = R|_{\mathcal{L}_{\nu}}$.

For the "only if" part, assume that $X' \vdash_{C,R} \varphi$. We proceed by induction on the length of the $C,R'$-proof of $\varphi$ from $X'$.

Induction basis: if $\varphi$ is an axiom of $\mathcal{L}'_{\nu}$, then it is also an axiom of $\mathcal{L}_{\nu}$ and $X' \vdash_{C,R} \varphi$. If $\varphi$ is a formula of $X'$, then, since $X' \subseteq X$, $X \vdash_{C,R} \varphi$.

Induction step: if $\varphi$ is either an axiom of $\mathcal{L}'_{\nu}$ or a formula of $X'$, then $X' \vdash_{C,R} \varphi$ exactly as in the proof of the induction basis. Otherwise, there are the three following cases:

1. $\varphi$ is deduced from $\psi$ and $\psi \supset \varphi$ by virtue of MP. That is, $X' \vdash_{C,R} \psi$ and $X' \vdash_{C,R} \psi \supset \varphi$. Both $C,R'$-proofs of $\psi$ and $\psi \supset \varphi$ appear in the proof of $\varphi$, and thus, they are shorter than that of $\varphi$. It follows that the induction hypothesis is applicable to those $C,R'$-proofs, yielding $X \vdash_{C,R} \psi$ and $X \vdash_{C,R} \psi \supset \varphi$. Thus, $X \vdash_{C,R} \varphi$ by virtue of MP.
2. $\varphi$ is deduced from $\psi$ by virtue of the $R'$-rule $\frac{\psi}{\varphi}$. That is, $X' \vdash_{C,R'} \psi$.

The $C,R'$-proof of $\psi$ appears in the proof of $\varphi$, and thus, it is shorter than that of $\varphi$. It follows that the induction hypothesis is applicable to this $C,R'$-proof, yielding $X \vdash_{C,R} \psi$. Since $R' \subseteq R$, $X \vdash_{C,R} \varphi$ by virtue of the same $R'$-rule.

3. $\varphi$ is of the form $\forall x \psi(x)$ and it is deduced from $\{\psi(t)\}_{t \in T_{\mathcal{L}_0'}}$ by virtue of the Carnap rule. That is, $X' \vdash_{C,R'} \psi(t)$ for each $t \in T_{\mathcal{L}_0'}$. Also, for each $t \in T_{\mathcal{L}_0'}$, a $C,R'$-proof of $\psi(t)$ appears in the proof of $\varphi$, and thus, it is shorter than that of $\varphi$. It follows that the induction hypothesis is applicable to all those $C,R'$-proofs, yielding $X \vdash_{C,R} \psi(t)$ for each $t \in T_{\mathcal{L}_0'}$. Now, assume to the contrary that $X \not\vdash_{C,R} \forall x \psi(x)$. Since $\forall x \psi(x)$ is a formula over $\mathcal{L}_0'$, by the definition of $b'$, there exists $j$, such that all constants which appear in $\forall x \psi(x)$ are in $b_j$ and thus, $\forall x \psi(x)$ is a formula over $\mathcal{L}_0'$. By the definition of $b_{j+1}$, if $X \not\vdash_{C,R} \forall x \psi(x)$, then for some $t \in b_{j+1}$, $X \not\vdash_{C,R} \psi(t)$. Since $b_{j+1} \subseteq b'$, for some $t \in T_{\mathcal{L}_0'}$, $X \not\vdash_{C,R} \psi(t)$, in contradiction with the induction hypothesis: $X \vdash_{C,R} \psi(t)$ for each $t \in T_{\mathcal{L}_0'}$.

For the “if” part, assume $X \vdash_{C,R} \varphi$. Since $\varphi$ is a formula over $\mathcal{L}_0'$, there exists $b_j$, such that all the constants which appear in $\varphi$ are in $b_j$. In the definition of $b_{j+1}$, we fixed a $C,R$-proof of $\varphi$ from $X$. We shall prove by induction on the length of that proof that $X' \vdash_{C,R'} \varphi$.

Induction basis: if $\varphi$ is an axiom of $\mathcal{L}_0'$, then $X' \vdash_{C,R'} \varphi$. If $\varphi$ is a formula of $X$, then, since $\varphi$ is a formula over $\mathcal{L}_0'$, $\varphi \in X|_{\mathcal{L}_0'}$. That is, $\varphi \in X'$, and thus, $X' \vdash_{C,R'} \varphi$.

Induction step: if $\varphi$ is either an axiom of $\mathcal{L}_0'$ or a formula of $X$, then $X' \vdash_{C,R'} \varphi$ exactly as in the proof of the induction basis. Otherwise, there are the following cases:

1. $\varphi$ was deduced from $\psi$ and $\psi \supset \varphi$ by virtue of MP. That is, $X \vdash_{C,R} \psi$ and $X \vdash_{C,R} \psi \supset \varphi$. By the definition of $b_{j+1}$, all constants which appear in $\psi$ were added to $b_{j+1}$ and since $b_{j+1} \subseteq b'$, both $\psi$ and $\psi \supset \varphi$ are formulas over $\mathcal{L}_0'$. Moreover, both $C,R$-proofs of $\psi$ and $\psi \supset \varphi$ appear in the proof of $\varphi$, and thus, they are shorter than that of $\varphi$. It follows that the induction hypothesis is applicable to those $C,R$-proofs, yielding $X' \vdash_{C,R'} \psi$ and $X' \vdash_{C,R'} \psi \supset \varphi$. Now, $X' \vdash_{C,R'} \varphi$ by virtue of MP.
2. \( \varphi \) was deduced from \( \psi \) by virtue of the \( R \)-rule \( \frac{\psi}{\varphi} \). That is, \( X \vdash_{CR} \psi \). By the definition of \( b_{j+1} \), all constants which appear in \( \psi \) were added to \( b_{j+1} \) and since \( b_{j+1} \subseteq b' \), \( \psi \) is a formula over \( \mathcal{L}'_i \). Moreover, a \( C,R \)-proof of \( \psi \) appears in the proof of \( \varphi \), and thus, it is shorter than that of \( \varphi \). It follows that the induction hypothesis is applicable to this \( C,R \)-proof, yielding \( X' \vdash_{CR} \psi \). Also, since both \( \varphi \) and \( \psi \) are formulas over \( \mathcal{L}'_i \), \( \frac{\psi}{\varphi} \in \mathcal{R}' \) and thus, \( X' \vdash_{CR'} \varphi \) by virtue of same \( R \)-rule.

3. \( \varphi \) was deduced by virtue of the Carnap rule. That is, \( \varphi \) is of the form \( \forall x \psi(x) \) and \( X \vdash_{CR} \psi(t) \) for each \( t \in \mathcal{T}_{\mathcal{L}'_i} \). Since \( b' \subseteq b \), \( X \vdash_{CR} \psi(t) \) for each \( t \in \mathcal{T}_{\mathcal{L}'_i} \). Since \( \forall x \psi(x) \) is a formula over \( \mathcal{L}'_i \), \( \psi(t) \) is a formula over \( \mathcal{L}'_i \) for each \( t \in \mathcal{T}_{\mathcal{L}'_i} \). Moreover, for each \( t \in \mathcal{T}_{\mathcal{L}'_i} \), \( C,R \)-proof of \( \psi(t) \) appears in the proof of \( \varphi \), and thus, it is shorter than that of \( \varphi \). It follows that the induction hypothesis is applicable to all those \( C,R \)-proofs, yielding \( X' \vdash_{CR'} \psi(t) \) for each \( t \in \mathcal{T}_{\mathcal{L}'_i} \). Now, we deduce \( \forall x \psi(x) \) by virtue of the Carnap rule. That is, \( X' \vdash_{CR'} \varphi \).

Since each proof that is constructed in this lemma is a well-ordered concatenation of well-ordered lists, it is a well-ordered list as well. Thus, it is a valid \( C,R \)-proof (\( C,R' \)-proof). This completes the proof of the lemma. \( \square \)

**Corollary** Let \( b \) be an infinite set of new constant symbols, \( \mathcal{R} \) be an \( \mathcal{R}_{\mathcal{L}'_i} \)-set and let \( A \) be a set of closed formulas over \( \mathcal{L} \). There exists a countable subset \( b' \) of \( b \), such that \( \text{Th}^{C,R}(A)|_{\mathcal{L}'_i} = \text{Th}^{C,R'}(A) \), where \( \mathcal{R}' = \mathcal{R}|_{\mathcal{L}'_i} \).

**Proof** By Lemma 9 (p. 23), there exists a countable subset of \( b, b' \), such that \( A \) is a \( C,R \)-conservative extension of \( A|_{\mathcal{L}'_i} \). Since \( A \) is a set of formulas over \( \mathcal{L} \), \( A|_{\mathcal{L}'_i} = A \). It follows that \( A \) is a \( C,R \)-conservative extension of \( A \). That is, for any closed formula \( \varphi \) over \( \mathcal{L}'_i \), \( A \vdash_{CR} \varphi \) if and only if \( A \vdash_{CR'} \varphi \). In other words, \( \text{Th}^{C,R}(A)|_{\mathcal{L}'_i} = \text{Th}^{C,R'}(A) \). \( \square \)

### 4.2 Countable bases

In this section we prove that in the syntactic definition of extension, we can restrict ourselves to a countable set of new constant symbols. Thus, by Theorem 4 given in the end of this section, extensions over monadic languages can be defined semantically over a countable base only.
Theorem 3  Let \((D, A)\) be an open default theory and let \(b\) be an infinite set of new constant symbols. There exists a countable set of new constant symbols \(b'\), such that for any \(C\)-extension \(E\) for \((\bar{D}_C, A)\) there is a \(C\)-extension \(E'\) for \((\bar{D}_C', A)\), such that \(E|_C = E'|_C\).

The proof of Theorem 3 is based on Lemmas 10 and 11 below.

Lemma 10  Let \(L\) and \(L'\) be languages with the same predicate symbols and the same \(n\)-place function symbols, \(n > 0\), such that \(T_L \subseteq T_{L'}\). Let \((D, A)\) be an open default theory over \(L\), \(S\) be a set of closed formulas over \(L'\) and let \(R = \{ \alpha : B_1, \ldots, B_m \in \bar{D}_{C, A} \text{ and } -\beta_1, \ldots, -\beta_m \notin S \} \) be an \(R\)-set. Then \(\Gamma_{\bar{D}_{C, A}}(S) = Th^{C,R}(A)\).

Proof  By the definition of \(\Gamma_{\bar{D}_{C, A}}(S)\), \(\Gamma_{\bar{D}_{C, A}}(S)\) is the smallest \(C\)-deductively closed set of formulas which contains \(A\) and satisfies the following property. For any \(\alpha : B_1, \ldots, B_m \in \bar{D}_{C, A}\) if \(\alpha \in \Gamma_{\bar{D}_{C, A}}(S)\) and \(-\beta_1, \ldots, -\beta_m \notin S\), then \(\gamma \in \Gamma_{\bar{D}_{C, A}}(S)\) is satisfying this property if and only if it is closed under the \(R\)-rule. By the definition of \(Th^{C,R}(A)\), \(Th^{C,R}(A)\) is the set of all formulas having a \(C, R\)-proof from \(A\), that is, it is a \(C\)-deductively closed set of formulas which contains \(A\) and closed under the \(R\)-rule. Since \(\Gamma_{\bar{D}_{C, A}}(S)\) is a smallest set of formulas, which satisfy those properties, \(\Gamma_{\bar{D}_{C, A}}(S) \subseteq Th^{C,R}(A)\).

To prove that \(Th^{C,R}(A) \subseteq \Gamma_{\bar{D}_{C, A}}(S)\), let \(\varphi \in Th^{C,R}(A)\). By the definition of \(Th^{C,R}(A)\), \(\varphi\) has a \(C, R\)-proof from \(A\). We will prove by induction on the length of this \(C, R\)-proof, that \(\varphi \in \Gamma_{\bar{D}_{C, A}}(S)\).

Induction basis: if \(\varphi\) is an axiom of \(L'\), then, since \(\Gamma_{\bar{D}_{C, A}}(S)\) is a deductively closed set of formulas over \(L'\), \(\varphi \in \Gamma_{\bar{D}_{C, A}}(S)\). If \(\varphi\) is a formula of \(A\), then, since \(A \subseteq \Gamma_{\bar{D}_{C, A}}(S)\), \(\varphi \in \Gamma_{\bar{D}_{C, A}}(S)\).

Induction step: if \(\varphi\) is either an axiom of \(L'\) or a formula of \(A\), then \(\varphi \in \Gamma_{\bar{D}_{C, A}}(S)\) exactly as in the proof of the induction basis. Otherwise, there are the three following cases.

1. \(\varphi\) is deduced from \(\psi\) and \(\psi \supseteq \varphi\) by virtue of MP. By the induction hypothesis, \(\psi \in \Gamma_{\bar{D}_{C, A}}(S)\) and \(\psi \supseteq \varphi \in \Gamma_{\bar{D}_{C, A}}(S)\). Since \(\Gamma_{\bar{D}_{C, A}}(S)\) is a deductively closed set of formulas, \(\varphi \in \Gamma_{\bar{D}_{C, A}}(S)\).  

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2. \( \varphi \) is deduced from \( \psi \) by virtue of the \( R \)-rule \( \frac{\psi}{\varphi} \). By the definition of \( R \), there exists a default \( \psi : M_{\beta_1} \ldots , M_{\beta_m} \in \bar{D}_{\mathcal{L}'} \) such that \( -\beta_1 , \ldots , -\beta_m \notin S \). By the induction hypothesis, \( \psi \in \Gamma_{(\bar{D}_{\mathcal{L}'},A)}^{C}(S) \). Thus, from the definition of \( \Gamma_{(\bar{D}_{\mathcal{L}'},A)}^{C}(S) \) follows that \( \varphi \in \Gamma_{(\bar{D}_{\mathcal{L}'},A)}^{C}(S) \).

3. \( \varphi \) is of the form \( \forall x \psi(x) \) and it is deduced from \( \{ \psi(t) \}_{t \in T_{\mathcal{L}''}} \) by virtue of the Carnap rule. By the induction hypothesis, \( \psi(t) \in \Gamma_{(\bar{D}_{\mathcal{L}'},A)}^{C}(S) \) for each \( t \in T_{\mathcal{L}'} \). Thus, \( \Gamma_{(\bar{D}_{\mathcal{L}'},A)}^{C}(S) \vdash_{C,R} \forall x \psi(x) \) by virtue of the Carnap rule. Since \( \Gamma_{(\bar{D}_{\mathcal{L}'},A)}^{C}(S) \) is a \( C \)-deductively closed set of formulas, \( \forall x \psi(x) \in \Gamma_{(\bar{D}_{\mathcal{L}'},A)}^{C}(S) \), i.e., \( \varphi \in \Gamma_{(\bar{D}_{\mathcal{L}'},A)}^{C}(S) \).

This completes the proof of the lemma. \( \square \)

**Lemma 11** Let \((D,A)\) be an open default theory, \( b \) be an infinite set of new constant symbols and let \( E \) be a \( C \)-extension for \((\bar{D}_{\mathcal{L}''},A)\). There exists a countable subset \( b' \) of \( b \) and a \( C \)-extension \( E' \) for \((\bar{D}_{\mathcal{L}'},A)\), such that \( E|_{\mathcal{L}} = E'|_{\mathcal{L}} \).

**Proof** Let an \( R_{\mathcal{L}} \)-set \( R \) be defined by \( R = \{ \frac{\psi}{\alpha : M_{\beta_1} \ldots , M_{\beta_m}} \in \bar{D}_{\mathcal{L}_1} \) and \( -\beta_1 , \ldots , -\beta_m \notin E \} \). By the corollary to Lemma 9 (p. 23), there exists a countable subset \( b' \) of \( b \), such that \( \mathcal{T}h_{C,R}(A)|_{\mathcal{L}_1} = \mathcal{T}h_{C,R}(A) \), where \( R' = R|_{\mathcal{L}_1} \).

We contend that \( E' = E|_{\mathcal{L}_1} \) is a \( C \)-extension for \((\bar{D}_{\mathcal{L}_1},A)\), i.e., \( E' = \Gamma_{(\bar{D}_{\mathcal{L}_1},A)}^{C}(E') \). Since \( E \) is a \( C \)-extension for \((\bar{D}_{\mathcal{L}_1},A)\), \( E = \Gamma_{(\bar{D}_{\mathcal{L}_1},A)}^{C}(E) \). By Lemma 10 (p. 27) with \( S = E \) and \( \mathcal{L}' = \mathcal{L}_1 \), \( \Gamma_{(\bar{D}_{\mathcal{L}_1},A)}^{C}(E) = \mathcal{T}h_{C,R}(A) \) and thus, \( (\Gamma_{(\bar{D}_{\mathcal{L}_1},A)}^{C}(E))|_{\mathcal{L}_1} = \mathcal{T}h_{C,R}(A)|_{\mathcal{L}_1} \). By the definition of \( b' \), \( \mathcal{T}h_{C,R}(A)|_{\mathcal{L}_1} = \mathcal{T}h_{C,R}(A) \).

Therefore, \( E' = E|_{\mathcal{L}_1} = (\Gamma_{(\bar{D}_{\mathcal{L}_1},A)}^{C}(E))|_{\mathcal{L}_1} = \mathcal{T}h_{C,R}(A)|_{\mathcal{L}_1} = \mathcal{T}h_{C,R}(A) \).

Now, by Lemma 10 (p. 27) with \( S = E \) and \( \mathcal{L}' = \mathcal{L}_1 \), \( \Gamma_{(\bar{D}_{\mathcal{L}_1},A)}^{C}(E') = \mathcal{T}h_{C,R}(A) \) where \( R'' = \{ \frac{\psi}{\alpha : M_{\beta_1} \ldots , M_{\beta_m}} \in \bar{D}_{\mathcal{L}_1} \) and \( -\beta_1 , \ldots , -\beta_m \notin E' \} \).

Since \( R' = R|_{\mathcal{L}_1} \) and \( E' = E|_{\mathcal{L}_1} \), \( R'' = R'' \). Therefore, \( E' = \mathcal{T}h_{C,R}(A) = \mathcal{T}h_{C,R}(A) = \Gamma_{(\bar{D}_{\mathcal{L}_1},A)}^{C}(E') \), which completes the proof of the lemma. \( \square \)
**Proof of Theorem 3** Let $\mathcal{B}$ be any countable set of new constant symbols. By Lemma 11 (p. 28), for any $C$-extension $E$ for $(\bar{D}_{\mathcal{L}}, A)$, there exists a countable subset $b^E$ of $b$ and a $C$-extension $E_{b^E}$ for $(\bar{D}_{\mathcal{L}}, A)$, such that $E|_{\mathcal{L}} = E_{b^E}|_{\mathcal{L}}$. Since $\mathcal{B}$ and $b^E$ are of the same cardinality, we can replace all the constants of $b^E$ by those of $\mathcal{B}$. Now, let $E'$ be a set of formulas, which consist of all formulas of $E_{b^E}$ after replacing. Since both $\mathcal{B}$ and $b^E$ are sets of constants which do not appear in $\mathcal{L}$, $E|_{\mathcal{L}} = E_{b^E}|_{\mathcal{L}} = E'|_{\mathcal{L}}$. Since $E'$ is obtained from $E_{b^E}$ by renaming new constants, $E'$ is a $C$-extension $(\bar{D}_{\mathcal{L}'}, A)$. This completes the proof of the theorem. □

Theorem 4 below is an immediate consequence of Lemma 6 (p. 19) and Theorem 3 (p. 26).

**Theorem 4** Let $\mathcal{L}$ be a monadic language, $(D, A)$ be an open default theory over $\mathcal{L}$, $b$ be an infinite set of new constant symbols and let $\mathcal{B}$ be a countable set of new constant symbols. Then $E$ is a $b$-extension for $(D, A)$ if and only if there is a fixed point $W$ of $\Delta_{(D, A)}^{b}$ such that $E = T h_{\mathcal{L}}(W)$. That is, $E$ is a $b$-extension for $(D, A)$ if and only if it is a $b'$-extension for $(D, A)$. 
Chapter 5

$C^w$-extensions and their base independence

In this chapter we define $C^w$-extensions for open default theories. The definition is syntactic: like in Chapter 3, we treat an open default as the set of all its closed instances over the language $L_1$ - the original language $L$ extended with the infinite set $b$ of new constant symbols. We also extend first-order logic with the Carnap rule, but now we define and use a different proof system, called the $C^w$-proof system. We show that this proof system is weaker than the $C$-proof system, introduced in Chapter 3. Then we prove that in this weak proof system, the syntactic definition of extension does not depend on the cardinality of its infinite base.

5.1 $C^w$-proof system

In this section we define a $C^w$-proof system for first-order logic, extended with the Carnap rule. We show that this proof system is weaker than the $C$-proof system, introduced in Chapter 3.

Like the $C$-proof system, $C^w$-proof system is used only for closed formulas. Thus, there is no need for the GEN rule and the axiom schema (A4) is replaced by its closed version (p. 12) also here.

**Definition 13** Let $\Gamma$ be a set of closed formulas over $L$. A $L$-frame is a finite-depth directed tree, whose vertices are of degree 0,1,2 or $|T_L|$ and are labeled with a formulas over $L$. A $C^w$-proof in $L$ of a closed formula $A$ from $\Gamma$ is an $L$-frame with root $A$, which satisfies the following properties.
• The vertices are of degree 0, 2 or \(|T_L|\).

• The label of each leaf (a vertex of degree 0) is either an axiom of \(L\) or a formula of \(\Gamma\).

• The label of each vertex of degree 2 can be obtained from the labels of its descendants by virtue of MP.

• The label of each vertex of degree \(|T_L|\) can be obtained from the labels of its descendants by virtue of the Carnap rule.

We shall write \(\Gamma \vdash_{C^w} A\) if there exists a \(C^w\)-proof of \(A\) from \(\Gamma\).

We will refer the proof system defined here as a \(C^w\)-proof system.

By the depth of a \(C^w\)-proof we mean the depth of the corresponding \(L\)-frame. For a set of closed formulas \(\Gamma\), we denote by \(\text{Th}^{C^w}(\Gamma)\) the set of all closed formulas having a \(C^w\)-proof from \(\Gamma\). That is, \(\text{Th}^{C^w}(\Gamma) \triangleq \{\varphi \mid \Gamma \vdash_{C^w} \varphi\}\). We say that a set of closed formulas \(\Gamma\) is \(C^w\)-deductively closed if \(\Gamma = \text{Th}^{C^w}(\Gamma)\).

The following example shows that \(C^w\)-proof system is weaker then the \(C\)-proof system, previously defined. We present a set of formulas \(\Gamma\) over \(L\) and a formula \(\varphi\) over \(L\), such that \(\Gamma \vdash_C \varphi\), but \(\Gamma \not\vdash_{C^w} \varphi\).

**Example 1** Let \(L\) be a monadic language containing one predicate symbol \(P\) and a countable set of constants \(\{a_i\}, i \geq 1\). Let \(\Gamma\) be the countable set of formulas \(\{P(a_1), P(a_i) \supset P(a_{i+1})\}, i \geq 1\) and let \(\varphi\) be \(\forall x P(x)\).

First, we show by induction on \(i\) that each \(P(a_i)\) has a \(C\)-proof from \(\Gamma\) of the length \(i\) and does not have a \(C\)-proof which is shorter. For the induction basis, \(P(a_1)\) has a \(C\)-proof from \(\Gamma\). Obviously, such \(C\)-proof is of the length 1 (\(P(a_1) \in \Gamma\)) and cannot be shorter. For the induction step, the only way to deduce \(P(a_{i+1})\) is to deduce \(P(a_i)\) first. By the induction hypothesis, \(P(a_i)\) has a \(C\)-proof from \(\Gamma\) of the length \(i\). Since \(P(a_i) \supset P(a_{i+1}) \in \Gamma\), \(P(a_{i+1})\) has a \(C\)-proof from \(\Gamma\) of the length \(i+1\) (by virtue of MP). Since the length of the \(C\)-proof of \(P(a_i)\) from \(\Gamma\) is \(i\) at least, the length of the \(C\)-proof of \(P(a_{i+1})\) from \(\Gamma\) is \(i+1\) at least. Now, \(\varphi\) has a \(C\)-proof since each \(P(a_i)\) has a \(C\)-proof of the length \(i\), and then we can concatenate all those \(C\)-proofs and deduce \(\varphi\) by virtue of the Carnap rule.

It can be shown by a similar induction, that each \(P(a_i)\) has a \(C^w\)-proof from \(\Gamma\) of the depth \(i\) and does not have a \(C^w\)-proof which is shorter. Thus, there is no \(C^w\)-proof for \(\varphi\), because the only possible way to deduce \(\varphi\) is
by virtue of the Carnap rule, but in this case it is impossible to construct a
finite-depth proof tree.

5.2 \( C^w \)-extensions

Following Definition 9 (p. 18), we define a \( C^w \)-extension for open default
theories in the \( C^w \)-proof system.

**Definition 14** Let \((D,A)\) be a closed default theory. For any set of sen-
tences \( S \) let \( \Gamma_{C^w}^{(D,A)}(S) \) be the smallest set of sentences \( B \) (beliefs) that sat-
ifies the following three properties.

WCD1. \( A \subseteq B \).

WCD2. \( Th^{C^w}(B) = B \), i.e. \( B \) is \( C^w \)-deductively closed.

WCD3. If \( \alpha : M_1, \ldots, M_m \in D \), \( \alpha \in B \) and \( \neg \beta_1, \ldots, \neg \beta_m \not\in S \), then
\( \gamma \in B \).

A set of sentences \( E \) is a \( C^w \)-extension for \((D,A)\) if \( \Gamma_{C^w}^{(D,A)}(E) = E \), i.e. if \( E \)
is a fixed point of the operator \( \Gamma_{C^w}^{(D,A)} \).

Let \((D,A)\) be an open default theory and let \( E \) be a \( C^w \)-extension for
\((\tilde{D}, \tilde{A})\). We will refer the set \( b \) as the base of \( E \).

5.3 \( C^w \)-R-proof system

In this section we define an extension of the \( C^w \)-proof system, called \( C^w \)-R-
proof system.

**Definition 15** Let \( \Gamma \) be a set of closed formulas over \( \mathcal{L} \) and \( R \) be an \( R_\mathcal{L} \)-set.
By a \( C^w \)-R-proof system we mean the \( C^w \)-proof system extended by the set
of \( R \)-rules from \( R \). A \( C^w \)-R-proof in \( \mathcal{L} \) of a closed formula \( A \) from \( \Gamma \) is a
\( \mathcal{L} \)-frame with root \( A \), which satisfies the following properties.

- The label of each leaf (a vertex of degree 0) is either an axiom of \( \mathcal{L} \) or
a formula of \( \Gamma \).

- The label of each vertex of degree 1 can be obtained from the label of
its descendant by virtue of the \( R \)-rule.
• The label of each vertex of degree 2 can be obtained from the labels of its descendants by virtue of MP.

• The label of each vertex of degree $|T_L|$ can be obtained from the labels of its descendants by virtue of the Carnap rule.

We shall write $\Gamma \vdash_{C^w, R} A$ if there exists a $C^w, R$-proof of $A$ from $\Gamma$.

By the depth of a $C^w, R$-proof we mean the depth of the corresponding $L$-frame. We denote by $Th^{C^w, R}(\Gamma)$ the set of all closed formulas having a $C^w, R$-proof from $\Gamma$. That is, $Th^{C^w, R}(\Gamma) \triangleq \{ A \mid \Gamma \vdash_{C^w, R} A \}$.

Note that (A4) is replaced by its closed version (p. 12) also in this case.

**Definition 16** Let $L_\infty$ and $L_e$ be languages with the same predicate symbols and the same $n$-place function symbols, $n > 0$, such that $T_{L_\infty} \subseteq T_{L_e}$. Let $R$ be an $R_{L_e}$-set, $X_1$ be a set of closed formulas over $L_\infty$ and let $X_2$ be a set of closed formulas over $L_e$. We say that $X_2$ is a $C^w, R$-conservative extension of $X_1$ if for any closed formula $\varphi$ over $L_\infty$, $X_1 \vdash_{C^w, R_1} \varphi$ if and only if $X_2 \vdash_{C^w, R} \varphi$, where $R_1$ is the restriction of $R$ to $L_\infty$ ($R_1 \triangleq R|_{L_\infty}$, i.e. $R_1 = \{ \psi \in R \mid \text{both } \psi \text{ and } \varphi \text{ are formulas over } L_\infty \}$).

**Lemma 12** Let $b$ be an infinite set of new constant symbols, $R$ be an $R_{L_\infty}$-set, and let $X$ be a set of closed formulas over $L_\infty$. There exists a countable subset $b^\prime$ of $b$, such that $X$ is a $C^w, R$-conservative extension of $X|_{L_\infty}^{b^\prime}$.

The proof of the lemma is similar to that of Lemma 9 (p. 23).

**Proof** Let $b^\prime$ be any countable subset of $b$. First, we define a function $f$ from $L_{|\varphi}$ to a countable subset of $b$ as follows. Let $\mathcal{F}_{b^\prime}$ be the set of all closed formulas over $L_\infty$. This set is countable, because $T_{L_\infty}$ is countable and the sets of predicate and function symbols are countable. For each $\varphi \in \mathcal{F}_{b^\prime}$, we define a set $b_\varphi$ in the following way.

If $X \vdash_{C^w, R} \varphi$ and there exists a $C^w, R$-proof of $\varphi$ from $X$ in which $\varphi$ is obtained by virtue of the Carnap rule, then we fix such $C^w, R$-proof with the smallest depth. If $X \vdash_{C^w, R} \varphi$, but there is no such $C^w, R$-proof, then we fix any $C, R$-proof of $\varphi$ from $X$. In such a proof one of the following holds.

1. the depth of the $C^w, R$-proof of $\varphi$ from $X$ is 1, that is, $\varphi$ is either an axiom of $L_{|\varphi}$ or a formula of $X|_{L_{\prime}}$. Then we define $b_\varphi = \emptyset$. 

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2. \( \varphi \) is obtained from \( \psi \) and \( \psi \supset \varphi \) by virtue of MP. Then we define 
\[ b_{\varphi} = \{ \text{all constants which appear in } \psi \}. \]

3. \( \varphi \) is obtained from \( \psi \) by virtue of the \( R \)-rule \( \frac{\psi}{\varphi} \). Then we define 
\[ b_{\varphi} = \{ \text{all constants which appear in } \psi \}. \]

4. \( \varphi \) is obtained by virtue of the Carnap rule. That is, \( \varphi \) is of the form 
\( \forall x \psi(x) \) and for each \( t \in T_{L^\mu} \), there exists a vertex \( \psi(t) \) in the \( C^w \), \( R \)-proof of \( \varphi \) from \( X \). Let \( \psi(t'), t' \in T_{L^\mu} \), be the vertex of the \( C^w \), \( R \)-proof with the longest path to the most distant leaf. Then we define 
\[ b_{\varphi} = \{ \text{all constants which appear in } t' \}. \]

If \( X \not\vdash_{C^w, R} \varphi \), two possible cases may occur.

1. \( \varphi \) is of the form \( \forall x \psi(x) \), that is, \( X \not\vdash_{C^w, R} \forall x \psi(x) \), which means that there is no \( C^w \), \( R \)-proof of \( \forall x \psi(x) \) from \( X \). There can be two reasons for that.
   (a) for some \( t \in T_{L^\mu} \), \( \psi(t) \) does not have a \( C^w \), \( R \)-proof from \( X \) 
   \( (X \not\vdash_{C^w, R} \psi(t)). \) Then we define \( b_{\varphi} = \{ \text{all constants which appear in } t \}. \)
   (b) each \( t \in T_{L^\mu} \) has a \( C^w \), \( R \)-proof from \( X \), but the depth of those 
   \( C, R \)-proofs is unbounded. Then, there exist countably many 
   constants \( t_1, t_2, \ldots, t_i, \ldots \), such that their \( C^w \), \( R \)-proofs from \( X \) 
   of a minimal depth do not have a uniformly bounded depth, and 
   we define \( b_{\varphi} = \{ \text{all constants which appear in } t_i, i \geq 1 \}. \)

2. \( \varphi \) is not of the form \( \forall x \psi(x) \). Then we define \( b_{\varphi} = \emptyset \).

Now, we define \( f(L^\mu) = T_{L^\mu} \cup \{ \varphi \mid \varphi \in F^* \} \). For each \( \varphi \in F^* \), \( b_{\varphi} \) is 
either finite or countable and, thus, \( f(L^\mu) \) is countable (as it is a union of 
a countable set and a countable union of countable or finite sets).

We define \( b' = \bigcup_{i=1}^{\infty} b_i \), where \( b_i \) is defined by induction in the following way. \( b_1 = f(L) \) and \( b_{i+1} = f(L_{|i}) \).

It can be easily shown by induction on \( i \) that each \( b_i \) is countable. Thus, 
\( b' \) is countable, as it is a countable union of countable sets.

Define \( X' = X|_{L^\mu} \). We shall prove now that \( X \) is a \( C^w, R \)-conservative 
extension of \( X' \). That is, for any closed formula \( \varphi \) over \( L^\mu \), \( X' \vdash_{C^w, R} \varphi \) if 
and only if \( X \vdash_{C^w, R} \varphi \), where \( R' = R|_{L^\mu} \).
For the "only if" part, we prove that for any closed formula $\varphi$ over $\mathcal{L}_1$, if $X_i \vdash_{C^w,R^i} \varphi$ by means of a $C^w,R^i$-proof of depth $i$, then there exists a $C^w,R$-proof of $\varphi$ from $X$ (i.e., $X \vdash_{C^w,R} \varphi$) of depth $i$ at most. We proceed by induction on the depth $i$ of the $C^w,R$-proof of $\varphi$ from $X_i$.

Induction basis: If $\varphi$ is a leaf of the $C^w,R^i$-proof, then it is either an axiom of $\mathcal{L}_1$ or a formula of $X_i$. If $\varphi$ is an axiom of $\mathcal{L}_1$, then it is also an axiom of $\mathcal{L}_1$ and $X \vdash_{C^w,R} \varphi$. If $\varphi$ is a formula of $X_i$, then, since $X_i \subseteq X$, $X \vdash_{C^w,R} \varphi$.

In both cases, the depth of the constructed $C^w,R$-proof of $\varphi$ from $X$ is 1.

Induction step: assume that $X_i \vdash_{C^w,R^i} \varphi$ by a $C^w,R^i$-proof of depth $i + 1$. There are the three following cases.

1. $\varphi$ is obtained from $\psi$ and $\psi \supset \varphi$ by virtue of MP. That is, $X_i \vdash_{C^w,R^i} \psi$ and $X_i \vdash_{C^w,R^i} \psi \supset \varphi$. The depth of one of those $C^w,R^i$-proofs is $i$ and the depth of the other is $i$ at most. By the induction hypothesis, $X \vdash_{C^w,R} \psi$ and $X \vdash_{C^w,R} \psi \supset \varphi$ by $C^w,R$-proofs of depth $i$ at most. Thus, we construct (by MP) a $C^w,R$-proof of $\varphi$ from $X$ of depth $i + 1$ at most.

2. $\varphi$ is obtained from $\psi$ by virtue of the $R^i$-rule $\frac{\psi}{\varphi}$. That is, $X_i \vdash_{C^w,R^i} \psi$ by a $C^w,R^i$-proof of depth $i$. By the induction hypothesis, $X \vdash_{C^w,R} \psi$ by a $C^w,R$-proof of depth $i$ at most. Since $R^i \subseteq R$, we construct (by the same $R^i$-rule) a $C^w,R$-proof of $\varphi$ from $X$ of depth $i + 1$ at most.

3. $\varphi$ is of the form $\forall x \psi(x)$ and it is obtained from $\{\psi(t)\}_{t \in T_{\mathcal{L}_1}}$ by virtue of the Carnap rule. That is, $X_i \vdash_{C^w,R^i} \psi(t)$ for each $t \in T_{\mathcal{L}_1}$ by $C^w,R^i$-proofs of depth $i$ at most. By the induction hypothesis, $X \vdash_{C^w,R} \psi(t)$ for each $t \in T_{\mathcal{L}_1}$ by $C^w,R$-proofs of depth $i$ at most.

Assume to the contrary that $X \not\vdash_{C^w,R} \forall x \psi(x)$. Since $\forall x \psi(x)$ is a formula over $\mathcal{L}_1$, by the definition of $\mathcal{L}_1$, there exists some $b_j$, such that all constants which appear in $\forall x \psi(x)$ are in $b_j$ and thus, $\forall x \psi(x)$ is a formula over $\mathcal{L}_1$. By the definition of $b_{j+1}$, if $X \not\vdash_{C^w,R} \forall x \psi(x)$, then there are two cases.

(a) there exists some $t \in b_{j+1}$, such that $X \not\vdash_{C^w,R} \psi(t)$. Since $b_{j+1} \subseteq b'$, there exists some $t \in T_{\mathcal{L}_1}$, such that $X \not\vdash_{C^w,R} \psi(t)$, which contradict the induction hypothesis.

(b) there exist $t_1,t_2,\ldots,t_i \in b_{j+1}$, such that the depth of their minimal $C^w,R$-proofs from $X$ is not uniformly bounded. Since $b_{j+1} \subseteq b'$, there exist $t_1,t_2,\ldots,t_i \in T_{\mathcal{L}_1}$, such that their
$C^w, R$-proofs from $X$ of minimal depth do not have a uniformly bounded depth, in contradiction with the induction hypothesis that $X \vdash_{C^w, R} \psi(t)$ for each $t \in T_{\mathcal{L}_i}$ by $C^w, R$-proofs of depth $i$ at most.

We have proved that $X \vdash_{C^w, R} \forall x \psi(x)$. It remains to show that there exists such a $C^w, R$-proof with depth $i + 1$ at most.

Clearly, if there exists a $C^w, R$-proof of $\forall x \psi(x)$ from $X$, there exists one, in which $\forall x \psi(x)$ is obtained by virtue of the Carnap rule ($X \vdash_{C^w, R} \psi(t)$ for each $t \in T_{\mathcal{L}_i}$ by (A4), and then, $X \vdash_{C^w, R} \forall x \psi(x)$ by the Carnap rule). Now, in the definition of $b_{j+1}$, we fixed a $C^w, R$-proof of $\forall x \psi(x)$ from $X$ with the smallest depth, in which $\forall x \psi(x)$ is obtained by virtue of the Carnap rule. By the definition of $b_{j+1}$ and $b'$, there exists some $t' \in T_{\mathcal{L}_i}$, such that $\psi(t')$ is the vertex with the longest path to the most distant leaf in this $C^w, R$-proof. Since $X' \vdash_{C^w, R} \psi(t')$ by $C^w, R'$-proofs of depth $i$ at most and by the induction hypothesis, there exists a $C^w, R$-proof of $\psi(t')$ from $X$ of depth $i$ at most. That is, the length of the longest path from $\psi(t')$ to the most distant leaf, in the $C^w, R$-proof of $\forall x \psi(x)$ from $X$ with the smallest depth, is $i$ at most. It follows that there exists a $C^w, R$-proof of $\forall x \psi(x)$ from $X$ with depth $i + 1$ at most.

For the “if” part, assume $X \vdash_{C^w, R} \varphi$. Since $\varphi$ is a formula over $\mathcal{L}_i$, there exists $b_j$, such that all the constants which appear in $\varphi$ are in $b_j$. In the definition of $b_{j+1}$, we fixed a $C^w, R$-proof of $\varphi$ from $X$. We shall use that proof to show by induction on its depth, $i$, that $X' \vdash_{C^w, R'} \varphi$ by a $C^w, R'$-proof of depth $i$ at most.

Induction basis: if $\varphi$ is an axiom of $\mathcal{L}_i$, then $X' \vdash_{C^w, R'} \varphi$. If $\varphi$ is a formula of $X$, then, since $\varphi$ is a formula over $\mathcal{L}_i$, $\varphi \in X|_{\mathcal{L}_i}$. That is, $\varphi \in X'$, and thus, $X' \vdash_{C^w, R'} \varphi$. In both cases the depth of the constructed $C^w, R'$-proofs is 1, like the depth of the $C^w, R$-proof of $\varphi$ from $X$.

Induction step: assume that $X \vdash_{C^w, R} \varphi$ by the $C^w, R$-proof of depth $i + 1$. There were three possible cases:

1. $\varphi$ was obtained from $\psi$ and $\psi \supset \varphi$ by virtue of MP. That is, $X \vdash_{C^w, R} \psi$ and $X \vdash_{C^w, R} \psi \supset \varphi$. The depth of one of those $C^w, R$-proofs is $i$ and the depth of the other is $i$ at most. By the definition of $b_{j+1}$, all constants which appear in $\psi$ were added to $b_{j+1}$ and since $b_{j+1} \subseteq b'$, both $\psi$ and $\psi \supset \varphi$ are formulas over $\mathcal{L}_i$. Thus, by the
induction hypothesis, $X^i \vdash_{C^w,R^i} \psi$ and $X^i \vdash_{C^w,R^i} \psi \supset \varphi$ by $C^w,R^i$-proofs of depth $i$ at most. From two those $C^w,R^i$-proofs, we construct a $C^w,R^i$-proof of $\varphi$ with depth $i+1$ at most by virtue of MP. That is, $X^i \vdash_{C^w,R^i} \varphi$ by the $C^w,R^i$-proof of depth $i+1$ at most.

2. $\varphi$ was obtained from $\psi$ by virtue of the $R$-rule $\frac{\psi}{\varphi}$. That is, $X \vdash_{C^w,R} \psi$ by a $C^w,R$-proof of depth $i$. By the definition of $b_{j+1}$, all constants which appear in $\psi$ were added to $b_{j+1}$ and since $b_{j+1} \subseteq b'$, $\psi$ is a formula over $L_{b'}$. Thus, by the induction hypothesis, $X^i \vdash_{C^w,R^i} \psi$ by a $C^w,R^i$-proof of depth $i$ at most. Also, since both $\varphi$ and $\psi$ are formulas over $L_{b'}$, $\frac{\psi}{\varphi} \in R'$ and thus, we construct a $C^w,R^i$-proof of $\varphi$ by virtue of same $R$-rule. That is, $X^i \vdash_{C^w,R^i} \varphi$ by the $C^w,R^i$-proof of depth $i+1$ at most.

3. $\varphi$ was obtained by virtue of the Carnap rule. That is, $\varphi$ is of the form $\forall x \psi(x)$ and $X \vdash_{C^w,R} \psi(t)$ for each $t \in T_{L_b}$ by $C^w,R$-proofs of depth $i$ at most. Since $b' \subseteq b$, $X \vdash_{C^w,R} \psi(t)$ for each $t \in T_{L_{b'}}$ and therefore, by the induction hypothesis, $X^i \vdash_{C^w,R^i} \psi(t)$ for each $t \in T_{L_{b'}}$ by $C^w,R^i$-proofs of depth $i$ at most. Thus, we construct a $C^w,R^i$-proof of $\forall x \psi(x)$ with depth $i+1$ at most by virtue of the Carnap rule. That is, $X^i \vdash_{C^w,R^i} \varphi$ by the $C^w,R^i$-proof of depth $i+1$ at most.

This completes the proof of the lemma. □

**Corollary** Let $b$ be an infinite set of new constant symbols, $R$ be an $R_{L_b}$-set and let $A$ be a set of closed formulas over $L$. There exists a countable subset $b'$ of $b$, such that $Th^{C^w,R}(A)|_{L_{b'}} = Th^{C^w,R'}(A)$, where $R' = R|_{L_{b'}}$.

**Proof** By Lemma 12 (p. 33), there exists a countable subset of $b$, $b'$, such that $A$ is a $C^w,R$-conservative extension of $A|_{L_{b'}}$. Since $A$ is a set of formulas over $L$, $A|_{L_{b'}} = A$. It follows that $A$ is a $C^w,R$-conservative extension of $A$. That is, for any closed formula $\varphi$ over $L_{b'}$, $A \vdash_{C^w,R} \varphi$ if and only if $A \vdash_{C^w,R'} \varphi$. In other words, $Th^{C^w,R}(A)|_{L_{b'}} = Th^{C^w,R'}(A)$. □

### 5.4 Countable bases

In this section we prove that in the $C^w$-proof system, like in the $C$-proof system, the syntactic definition of extension does not depend on the cardinality of an infinite set of new constant symbols.
Theorem 5 Let \((D, A)\) be an open default theory and let \(b\) be an infinite set of new constant symbols. There exists a countable set of new constant symbols \(b', \) such that for any \(C^w\)-extension \(E\) for \((\bar{D}_\mathcal{L}, A)\) there is a \(C^w\)-extension \(E'\) for \((\bar{D}_{\mathcal{L}'}, A)\), such that \(E|_\mathcal{L} = E'|_\mathcal{L}\).

Similarly to the proof of Theorem 3 (p. 26), that is based on Lemmas 10 (p. 27) and 11 (p. 28), the proof of Theorem 5 is based on Lemmas 13 and 14 below.

Lemma 13 Let \(\mathcal{L}\) and \(\mathcal{L}'\) be languages with the same predicate symbols and the same \(n\)-place function symbols, \(n > 0\), such that \(T_\mathcal{L} \subseteq T_{\mathcal{L}'}\). Let \((D, A)\) be an open default theory over \(\mathcal{L}\), \(S\) be a set of closed formulas over \(\mathcal{L}'\) and let
\[
R = \{ \alpha \mid \alpha := M_{\beta_1}, \ldots, M_{\beta_m} \in \bar{D}_\mathcal{L} \text{ and } -\beta_1, \ldots, -\beta_m \notin S \}
\]
be an \(R_{\mathcal{L}'}\)-set. Then \(\Gamma_D^{C^w(\bar{D}_{\mathcal{L}'}, A)}(S) = \Theta^{C^w(\mathcal{L})}(A)\).

Lemma 14 Let \((D, A)\) be an open default theory, \(b\) be an infinite set of new constant symbols and let \(E\) be a \(C^w\)-extension for \((\bar{D}_\mathcal{L}, A)\). There exists a countable subset \(b'\) of \(b\) and a \(C^w\)-extension \(E'\) for \((\bar{D}_{\mathcal{L}'}, A)\), such that \(E|_\mathcal{L} = E'|_\mathcal{L}\).

The proofs of Lemmas 13 and 14 are obtained from the proofs of Lemmas 10 (p. 27) and 11 (p. 28) by replacing \(C\) with \(C^w\) and will be omitted.

Proof of Theorem 5 Let \(b'\) be any countable set of new constant symbols. By Lemma 14 (p. 38), for any \(C^w\)-extension \(E\) for \((\bar{D}_\mathcal{L}, A)\), there exists a countable subset \(b^E\) of \(b\) and a \(C^w\)-extension \(E_{b^E}\) for \((\bar{D}_{\mathcal{L}'}, A)\), such that \(E|_\mathcal{L} = E_{b^E}|_\mathcal{L}\). Since \(b'\) and \(b^E\) are of the same cardinality, we can replace all the constants of \(b^E\) by those of \(b'\). Now, let \(E'\) be a set of formulas, which consist of all formulas of \(E_{b^E}\) after replacing. Since both \(b'\) and \(b^E\) are sets of constants which do not appear in \(\mathcal{L}\), \(E|_\mathcal{L} = E_{b^E}|_\mathcal{L} = E'|_\mathcal{L}\). Since \(E'\) is obtained from \(E_{b^E}\) by renaming the new constants, \(E'\) is a \(C^w\)-extension \((\bar{D}_{\mathcal{L}'}, A)\). This completes the proof of the theorem. □
Chapter 6

Uniterm default theories under the domain closure assumption

This chapter deals with the definition of uniterm default theories defined in Section 6.1. We shall restrict ourselves only to finite languages not containing function symbols. We prove that for uniterm default theories under the domain closure assumption\(^1\), the syntactic definition of extensions for open default theories does not depend on the cardinality of its base.

6.1 Uniterm default theories

**Definition 17** ([1, Definitions 6,10]) Let \( P \) be an \( n \)-place predicate symbol and \( t \) be a term. \( P(t, \ldots, t) \) is called an *atomic uniterm formula over* \( t \). A propositional (boolean) combination of atomic uniterm formulas over the same term \( t \) is called a *uniterm formula over* \( t \).

A default theory \((D, A)\) is *uniterm* if the underlying language contains finitely many predicate and constant symbols, \( A \) is a set of closed formulas, and for every default \( \delta \in D \), \( \text{pre}(\delta) \), \( \text{just}(\delta) \), and \( \text{conc}(\delta) \), are uniterm formulas over the same variable.

Next, we define the similarity relation between constant symbols.

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\(^1\)See Section 2.6, p. 11.
**Definition 18** Let \((D, A)\) be a unitern default theory, \(b\) be a set of new constant symbols and let \(E\) be an extension for \((\bar{D}_{\mathcal{L}_i}, A)\). Two constant \(b_i, b_j \in b\) are called \(E\)-similar, denoted by \(b_i \sim_E b_j\), if for any default \(\frac{\alpha(x) : M\beta_1(x), \ldots, M\beta_n(x)}{\gamma(x)}\) in \(D\), \(\gamma(b_i) \in E\) if and only if \(\gamma(b_j) \in E\).

We denote by \(b^E\) a (finite) subset of \(b\) which contains exactly one constant from each class of similar constants.

A function \(\eta : b \rightarrow b^E\) is defined as follows. For each \(b_i \in b\), \(\eta(b_i) = b_j\) if and only if \(b_i \sim_E b_j\) and \(b_j \in b^E\).

**Remark** Since \(D\) is finite, say, of cardinality \(k\), there are at most \(2^k\) non \(E\)-similar constants. It follows that \(b^E\) is finite for each \(E\).

**Lemma 15** ([5, Lemma 2]) Let \((D, A)\) be an open default theory such that for some \(a_1, \ldots, a_m \in T_{\mathcal{L}_i}\), \(A \vdash \forall x \bigvee_{i=1}^m x = a_i\) and let \(b\) be a set of new constant symbols. Then \(E\) is an extension for \((\bar{D}_{\mathcal{L}_i}, A)\) if and only if there is a fixed point \(W\) of \(\Delta_b^h\) such that \(E = Th_{\mathcal{L}_i}(W)\).

Let \((D, A)\) be an open default theory and let \(E\) be an extension for \((\bar{D}_{\mathcal{L}_i}, A)\). We will refer the set \(b\) as the base of \(E\).

### 6.2 Finite bases

In this section we prove that the syntactic definitions of extension for unitern default theories does not depend on the cardinality of its base. It follows then, that under the domain closure assumption, the semantic definitions of extension for unitern default theories does not depend on the cardinality of its base either.

**Theorem 6** Let \((D, A)\) be a unitern default theory and let \(b\) be an infinite set of new constant symbols. There exist a finite set of new constant symbols \(\mathcal{B}\), such that there is an extension \(E\) for \((\bar{D}_{\mathcal{L}_i}, A)\) if and only if there is an extension \(E'\) for \((\bar{D}_{\mathcal{L}_i}, A)\), such that \(E|_{\mathcal{L}_i} = E'|_{\mathcal{L}_i}\).

**Theorem 7** Let \((D, A)\) be a unitern default theory, such that for some \(a_1, \ldots, a_m \in T_{\mathcal{L}_i}\), \(A \vdash \forall x \bigvee_{i=1}^m x = a_i\) and let \(b\) be an infinite set of new constant symbols. There exist a finite set of new constant symbols \(\mathcal{B}\), such
that $E$ is a $b$-extension for $(D, A)$ if and only if there is a fixed point $W$ of $\Delta^b_{(D, A)}$, such that $E = Th_W(L)$. That is, $E$ is a $b$-extension for $(D, A)$ if and only if it is a $b'$-extension for $(D, A)$.

As an immediate corollary to Theorem 7 we obtain that extensions for uniterm default theories are efficiently computable.

In order to prove Theorems 6 and 7 we shall prove the following lemmas.

**Lemma 16** Let $\Gamma$ be a set of formulas over $L$ and let $\varphi, \gamma_1(x_1), \ldots, \gamma_n(x_n)$, $n \geq 0$ be formulas over $L$ such that none of $x_1, \ldots, x_n$ are free in $\varphi$, and $\Gamma, \gamma_1(x_1), \ldots, \gamma_n(x_n) \vdash \varphi$. If there was no application of GEN to $x_1, \ldots, x_n$ in this proof, then $\Gamma, \exists x_1 \gamma_1(x_1), \ldots, \exists x_n \gamma_n(x_n) \vdash \varphi$.

**Proof** By the conditions of the lemma, the deduction theorem ([9, Proposition 2.6, p. 59]) is applicable to the proof of $\varphi$ from $\Gamma \cup \{\gamma_1(x_1), \ldots, \gamma_n(x_n)\}$, yielding $\Gamma, \gamma_1(x_1), \ldots, \gamma_n-1(x_n-1) \vdash \gamma_n(x_n) \supset \varphi$.

By GEN, $\Gamma, \gamma_1(x_1), \ldots, \gamma_n-1(x_n-1) \vdash \forall x_n (\gamma_n(x_n) \supset \varphi)$.

By ([9, Exercise 2.30(d), p. 62]) $\Gamma, \gamma_1(x_1), \ldots, \gamma_n-1(x_n-1) \vdash \exists x_n \gamma_n(x_n) \supset \varphi$.

Thus, $\Gamma, \gamma_1(x_1), \ldots, \gamma_n-1(x_n-1), \exists x_n \gamma_n(x_n) \vdash \varphi$.

Repeating this argument, for $\gamma_1(x_1), \ldots, \gamma_n-1(x_n-1)$ one after the other, we obtain that $\Gamma, \exists x_1 \gamma_1(x_1), \ldots, \exists x_n \gamma_n(x_n) \vdash \varphi$. □

**Lemma 17** Let $(D, A)$ be a uniterm default theory, $b$ be a set of new constant symbols, $E$ be an extension for $(\tilde{D}_L, A)$, and let $\gamma(x)$ be a conclusion of a default from $D$. Define $E_i, i \geq 0$ as in Lemma 1 (p. 7) with $D = \tilde{D}_L$. Then, for any $b_j, b_k \in b$, such that $b_j \sim_E b_k$, $\gamma(b_j) \in E_i$ if and only if $\gamma(b_k) \in E_i, i \geq 0$.

**Proof** We prove the lemma by induction on $i$. The basis is immediate, because $E_0 = A$ and $b$ is a set of new constant symbols. For the induction step, assume, without loss of generality, that $\gamma(b_j) \in E_{i+1} \setminus E_i$. Then for some $\delta(b_j) = \frac{\alpha(b_j) : M \beta_1(b_j), \ldots, M \beta_m(b_j)}{\gamma(b_j)} \in \tilde{D}_L$, $\bigcup_{k=0}^i E_k \vdash \alpha(b_j)$ and $\neg \beta_1(b_j), \ldots, \neg \beta_m(b_j) \notin E$. Since $\bigcup_{k=0}^i E_k \vdash \alpha(b_j)$, there exist uniterm formulas
\( \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b_j), \ldots, \gamma_q(b_j), \gamma_{q+1}(b_k), \ldots, \gamma_r(b_k) \in \bigcup_{k=1}^{i} E_k, \)

\( 0 \leq p \leq q \leq r, b_1, \ldots, b_p \in b \) (not necessarily different), such that \( b_i \neq b_j, b_i \neq b_k \) for \( 1 \leq i \leq p \) and

\( A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b_j), \ldots, \gamma_q(b_j), \gamma_{q+1}(b_k), \ldots, \gamma_r(b_k) \vdash \alpha(b_j). \)

Simultaneously replacing each \( b_j \) with \( b_k \) and \( b_k \) with \( b_j \) in this proof, we obtain a proof

\( A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b_k), \ldots, \gamma_q(b_k), \gamma_{q+1}(b_j), \ldots, \gamma_r(b_j) \vdash \alpha(b_k). \)

By the induction hypothesis, \( \gamma_{p+1}(b_k), \ldots, \gamma_q(b_k), \gamma_{q+1}(b_j), \ldots, \gamma_r(b_j) \in \bigcup_{k=1}^{i} E_k, \)

and thus, \( \bigcup_{k=0}^{i} E_k \vdash \alpha(b_k). \)

We shall prove now that \( -\beta_1(b_k), \ldots, -\beta_m(b_k) \notin E. \) Assume to the contrary that \( -\beta_{m'}(b_k) \in E \) for some \( m', 1 \leq m' \leq m. \) Then, there exist unitary formulas

\( \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b_k), \ldots, \gamma_q(b_k), \gamma_{q+1}(b_j), \ldots, \gamma_r(b_j) \in E, \)

\( 0 \leq p \leq q \leq r, b_1, \ldots, b_p \in b \) (not necessarily different), such that \( b_i \neq b_j, b_i \neq b_k \) for \( 1 \leq i \leq p \) and

\( A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b_k), \ldots, \gamma_q(b_k), \gamma_{q+1}(b_j), \ldots, \gamma_r(b_j) \vdash -\beta_{m'}(b_k). \)

Simultaneously replacing each \( b_k \) with \( b_j \) and \( b_j \) with \( b_k \) in this proof, we obtain a proof

\( A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b_j), \ldots, \gamma_q(b_j), \gamma_{q+1}(b_k), \ldots, \gamma_r(b_k) \vdash -\beta_{m'}(b_j). \)

Now, since \( \gamma_{p+1}(b_k), \ldots, \gamma_q(b_k), \gamma_{q+1}(b_j), \ldots, \gamma_r(b_j) \in E, \) by the definition of similar constants, \( \gamma_{p+1}(b_j), \ldots, \gamma_q(b_j), \gamma_{q+1}(b_k), \ldots, \gamma_r(b_k) \in E. \) Thus, \( E \vdash -\beta_{m'}(b_j). \) Since \( E \) is deductively closed, \( -\beta_{m'}(b_j) \in E, \) in contradiction to the assumption that \( -\beta_1(b_j), \ldots, -\beta_m(b_j) \notin E. \)

It follows that \( \gamma(b_k) \in E_{i+1} \) by the default rule \( \delta(b_k) \in D_{\mathcal{L}_i}, \) which completes the proof of the lemma. \( \square \)

The “only if” part of Theorem 6 will follow from the next lemma.

**Lemma 18** Let \( (D, A) \) be a unitary default theory, \( b \) be a set of new constant symbols and let \( E \) be an extension for \( (D_{\mathcal{L}_i}, A). \) Let \( b^E \) be as in Definition 18 (p. 39) and let \( b' \) be a finite subset of \( b \) such that \( b^E \subseteq b'. \) Then, there exists an extension \( E' \) for \( (D_{\mathcal{L}_i'}, A) \) such that \( E|_{\mathcal{L}} = E'\big|_{\mathcal{L}}. \)

**Proof** Define \( E_i, i \geq 0, \) as in Lemma 1 (p. 7) with \( D = D_{\mathcal{L}_i}. \) Since \( E \) is an extension for \( (D_{\mathcal{L}_i}, A), E = \Gamma(D_{\mathcal{L}_i}, A)(E) \) and thus, by Lemma 1 (p. 7), \( E = Th(\bigcup_{i=0}^{\infty} E_i). \) We define \( E'_i = E_i|_{\mathcal{L}_i'}, i \geq 0, \) and \( E' = Th(\bigcup_{i=0}^{\infty} E'_i). \)
First, we prove that $E|_{\mathcal{L}'} = E'$ (and thus, $E|_{\mathcal{L}} = E'|_{\mathcal{L}}$). By the definition of $E'_i$, $i \geq 0$, $\bigcup_{i=0}^{\infty} E'_i \subseteq \bigcup_{i=0}^{\infty} E_i$ and thus, $\mathcal{T} \cup \bigcup_{i=0}^{\infty} E'_i \subseteq \mathcal{T} \cup \bigcup_{i=0}^{\infty} E_i$. Therefore, $E' \subseteq E$. Since $E'$ is a set of formulas over $\mathcal{L}'$, $E' \subseteq E|_{\mathcal{L}'}$.

Conversely, let $\varphi(t) \in E|_{\mathcal{L}'}$ for some $t = t_1, \ldots, t_n \in T_{\mathcal{L}'}$. Then, there exist uniterm formulas $\gamma_1(b_1), \ldots, \gamma_p(b_p) \in \bigcup_{i=1}^{\infty} E$, $p \geq 0$, $b_1, \ldots, b_p \in b$ (not necessarily different) which do not appear in $t$, and there exists a subset $G$ of $\bigcup_{i=1}^{\infty} E_i$.

$G = \{ \gamma(t_j) \in \bigcup_{i=1}^{\infty} E_i \mid \text{a constant } t_j \text{ appears in } t \}$, such that,

$A, G, \gamma_1(b_1), \ldots, \gamma_p(b_p) \vdash \varphi(t)$. After replacing in this proof each $b_i$ with some new variable $x_i$ that does not appear in the proof, we obtain a proof $A, G, \gamma_1(x_1), \ldots, \gamma_p(x_p) \vdash \varphi(t)$. By Lemma 16 (p. 41),

$A, G, \exists x_1 \gamma_1(x_1), \ldots, \exists x_p \gamma_p(x_p) \vdash \varphi(t)$. \quad (*)

Now, by the definition of $\eta$, $b_i \sim_{E'} \eta(b_i)$, $0 \leq i \leq p$ and thus, by Lemma 17 (p. 41), $\gamma_1(\eta(b'_1)), \ldots, \gamma_p(\eta(b'_p)) \in \bigcup_{i=1}^{\infty} E'_i$. From the definition of $\eta$ follows that $\gamma_i(\eta(b_i))$, $0 \leq i \leq p$, are formulas over $\mathcal{L}'$ and since $b^E \subseteq b'$, $\gamma_i(\eta(b_i))$, $0 \leq i \leq p$, are formulas over $\mathcal{L}'$. Thus, $\gamma_1(\eta(b'_1)), \ldots, \gamma_p(\eta(b'_p)) \in \bigcup_{i=1}^{\infty} E'_i$.

We obtain that $\bigcup_{i=0}^{\infty} E'_i \vdash \exists x_1 \gamma_1(x_1), \ldots, \exists x_p \gamma_p(x_p)$. Also, since $A, G$ are sets of formulas over $\mathcal{L}'$, $A, G \subseteq \bigcup_{i=0}^{\infty} E'_i$. Thus, $\bigcup_{i=0}^{\infty} E'_i \vdash \varphi(t)$ by (*), and by the definition of $E'$, $\varphi(t) \in E'$.

We shall prove now that $E'$ is an extension for $(D_{\mathcal{L}'}, A)$, i.e., $\Gamma((\mathcal{L}_i, \mathcal{A})(E')) = E'$. By the definition of $E'$, $A \subseteq E'$ and $E'$ is deductively closed. Let $\alpha(x) : M \beta_1(x), \ldots, M \beta_m(x) \gamma(x) \in D$ and $t \in T_{\mathcal{L}_i}$, be such that $\alpha(t) \in E'$ and $\neg \beta_1(t), \ldots, \neg \beta_m(t), \gamma(t) \notin E'$. Since $E' \subseteq E$, $\alpha(t) \in E$. Since $E' = E|_{\mathcal{L}_i}$, and $\neg \beta_i(t), i = 1, \ldots, m$ are formulas over $\mathcal{L}'$, $\neg \beta_i(t), \gamma(t)$ is a formula over $\mathcal{L}'$, $\gamma(t) \in E'$. That is, $\Gamma((\mathcal{L}_i, \mathcal{A})(E')) \subseteq E'$.
For the proof of inclusion $E' \subseteq \Gamma(D_{\mathcal{L}_i}, A)(E')$ we proceed as follows. By Lemma 1 (p. 7) with $D = \tilde{D}_{\mathcal{L}_i}$, $\Gamma(D_{\mathcal{L}_i}, A)(E') = \mathcal{T}h \left( \bigcup_{i=0}^{\infty} \tilde{E}_i \right)$, where $\tilde{E}_0 = A$ and for $i > 0$,

$$
\tilde{E}_{i+1} = \{ \gamma(t) \mid \text{for some } \frac{\alpha(t)}{\gamma(t)} : M\beta_1(t), \ldots, M\beta_m(t) \in \tilde{D}_{\mathcal{L}_i}, \bigcup_{k=0}^{i} \tilde{E}_k \vdash \alpha(t) \text{ and } -\beta_1(t), \ldots, -\beta_m(t) \not\in E' \}.
$$

We shall prove by induction on $i$ that $E'_i \subseteq \tilde{E}_{i+1}$, $i \geq 0$. The induction basis follows from the definition of $E'_0$ and $\tilde{E}_0$ ($= A$). For the induction step, let $\gamma(t) \in E'_{i+1}$ for some $t \in \mathcal{T}_{\mathcal{L}_i}$. That is, there exists

$$
\delta(t) = \frac{\alpha(t)}{\gamma(t)} : M\beta_1(t), \ldots, M\beta_m(t) \in \tilde{D}_{\mathcal{L}_i}, \text{ such that } \bigcup_{k=0}^{i} E_k \vdash \alpha(t) \text{ and } -\beta_1(t), \ldots, -\beta_m(t) \not\in E.
$$

Since $\bigcup_{k=0}^{i} E_k \vdash \alpha(t)$, there exist unterm formulas $\gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(t), \ldots, \gamma_q(t) \in \bigcup_{k=0}^{i} E_k$, $0 \leq p \leq q$, $b_1, \ldots, b_p \in b$ (not necessarily different), such that $b_i \neq t$, $1 \leq i \leq p$ and $A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(t), \ldots, \gamma_q(t) \vdash \alpha(t)$.

After replacing in this proof each $b_i$, $0 \leq i \leq p$, with some new variable $x_i$ that does not appear in the proof, we obtain a proof

$$
A, \exists x_1 \gamma_1(x_1), \ldots, \exists x_p \gamma_p(x_p), \gamma_{p+1}(t), \ldots, \gamma_q(t) \vdash \alpha(t).
$$

(***)

Now, by the definition of $\eta$, $b_i \sim_E \eta(b_i)$, $0 \leq i \leq p$, and thus, by Lemma 17 (p. 41), $\gamma_1(\eta(b_1)), \ldots, \gamma_p(\eta(b_p)) \in \bigcup_{k=0}^{i} E_k$. Also, from the definition of $\eta$ follows that $\gamma_i(\eta(b_i)), 0 \leq i \leq p$, are formulas over $\mathcal{L}_E$ and since $b^E \subseteq b'$, $\gamma_i(\eta(b_i)), 0 \leq i \leq p$, are formulas over $\mathcal{L}_i'$. Thus,

$$
\gamma_1(\eta(b_1)), \ldots, \gamma_p(\eta(b_p)), \gamma_{p+1}(t), \ldots, \gamma_q(t) \in \bigcup_{k=0}^{i} E'_k.
$$

By the induction hypothesis, $\gamma_1(\eta(b_1)), \ldots, \gamma_p(\eta(b_p)), \gamma_{p+1}(t), \ldots, \gamma_q(t) \in \bigcup_{k=0}^{i} E'_k$. Therefore, $\bigcup_{k=0}^{i} E'_k \vdash \exists x_1 \gamma_1(x_1), \ldots, \exists x_p \gamma_p(x_p)$. Since $\tilde{E}_0 = A, A \subseteq \bigcup_{k=0}^{i} E'_k$ and thus, $\bigcup_{k=0}^{i} E'_k \vdash \alpha(t)$ by (***)

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Now, since $\neg \beta_1(t), \ldots, \neg \beta_m(t) \notin E, E' = E|_{\mathcal{L}'_i}$ and $\neg \beta_i(t), i = 1, \ldots, m$ are formulas over $\mathcal{L}'_i, \neg \beta_1(t), \ldots, \neg \beta_m(t) \notin E'$ and thus, $\gamma(t) \in E'_{i+1}$ by the same default rule $\delta(t)$.

It follows that $E' = \mathcal{T} \sum_{i=0}^{\infty} E'_i \subseteq \mathcal{T} \sum_{i=0}^{\infty} \tilde{E}'_i = \Gamma_{(\tilde{D}_{\mathcal{L}'_i}, A)}(E')$, which completes the proof of the lemma. $\square$

The next two lemmas are used to prove the “if” part of Theorem 6. The proof will follow from Lemma 21, which is based on those lemmas.

**Lemma 19** Let $(D, A)$ be a uniterm default theory, $b$ be a set of new constant symbols, $E$ be an extension for $(\tilde{D}_{\mathcal{L}_i}, A)$ and let $c$ be a constant that does not appear in $b$. Then there exists an extension $E^c$ for $(\tilde{D}_{\mathcal{L}_i \cup \{c\}}, A)$ such that $E = E^c|_{\mathcal{L}_i}$.

**Proof** Define $E_i, i \geq 0$, as in Lemma 1 (p. 7) with $D = \tilde{D}_{\mathcal{L}_i}$. Since $E$ is an extension for $(\tilde{D}_{\mathcal{L}_i}, A)$, $E = \Gamma_{(\tilde{D}_{\mathcal{L}_i}, A)}(E)$ and thus, by Lemma 1 (p. 7), $E = \mathcal{T} \sum_{i=0}^{\infty} E_i$. We choose some $b^* \in b$ and define $E^c = \mathcal{T} \sum_{i=0}^{\infty} E^c_i$, where $E^c_i = E_i \cup \{\gamma(c) \mid \gamma(b^*) \in E_i\}, i \geq 0$.

First, we prove that $E = E^c|_{\mathcal{L}_i}$. By the definition of $E^c_i, i \geq 0$, $\sum_{i=0}^{\infty} E^c_i \subseteq \sum_{i=0}^{\infty} E^c_i$ and thus, $\mathcal{T} \sum_{i=0}^{\infty} E^c_i \subseteq \mathcal{T} \sum_{i=0}^{\infty} E^c_i$, that is $E \subseteq E^c$. Since $E$ is a set of formulas over $\mathcal{L}_i, E \subseteq E^c|_{\mathcal{L}_i}$. Conversely, let $\varphi(t) \in E^c|_{\mathcal{L}_i}$ for some $t = t_1, \ldots, t_n \in \mathcal{T}_{\mathcal{L}_i}$. Then there exist uniterm formulas

$\gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(c), \ldots, \gamma_q(c) \in \sum_{i=1}^{\infty} E^c_i, 0 \leq p \leq q, b_1, \ldots, b_p \in b$ (not necessarily different) which do not appear in $t$, $b_i \neq c$ for $0 \leq i \leq p$, and there exists a subset $G$ of $\sum_{i=1}^{\infty} E^c_i$,

$G = \{\gamma(t_j) \in \sum_{i=1}^{\infty} E^c_i \mid a$ constant $t_j$ appears in $t\}$, such that $A, G, \gamma_1(b_1), \ldots, \gamma_{p}(b_p), \gamma_{p+1}(c), \ldots, \gamma_q(c) \vdash \varphi(t)$. After replacing in this proof each $c$ with some new variable $x$ that does not appear in the proof, we obtain a proof $A, G, \gamma_1(b_1), \ldots, \gamma_{p}(b_p), \gamma_{p+1}(x), \ldots, \gamma_q(x) \vdash \varphi(t)$ and then, by Lemma 16 (p. 41),

$A, G, \gamma_1(b_1), \ldots, \gamma_{p}(b_p), \exists x \gamma_{p+1}(x), \ldots, \exists x \gamma_q(x) \vdash \varphi(t)$. (*)

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Now, by the definition of \( E^c \), if \( \gamma_{p+1}(c), \ldots, \gamma_q(c) \in \bigcup_{i=1}^{\infty} E_i^c \) then
\[
\gamma_{p+1}(b^*), \ldots, \gamma_q(b^*) \in \bigcup_{i=1}^{\infty} E_i, \text{ and thus, } \bigcup_{i=1}^{\infty} E_i \vdash \exists x \gamma_{p+1}(x), \ldots, \exists x \gamma_q(x).
\]
Since \( A, G \) are sets of formulas over \( L_1 \), \( A, G \subseteq \bigcup_{i=0}^{\infty} E_i \). Also, by the definition of \( E_i^c, i \geq 0 \), \( E_i = E_i^c \cap L_1 \), thus, since \( \gamma_1(b_1), \ldots, \gamma_p(b_p) \) are formulas over \( L_1 \), \( \gamma_1(b_1), \ldots, \gamma_p(b_p) \in \bigcup_{i=1}^{\infty} E_i \). It follows that \( \bigcup_{i=0}^{\infty} E_i \vdash \varphi(t) \) by (\(^*\)), and by the definition of \( E, \varphi(t) \in E \).

We shall prove now that \( E^c \) is an extension for \( (\tilde{D}_{L_1 \cup \{\ell\}}, A) \), i.e.,
\[
\Gamma_{(\tilde{D}_{L_1 \cup \{\ell\}}, A)}(E^c) = E^c.
\]
By the definition of \( E^c \), \( A \subseteq E^c \) and \( E^c \) is deductively closed. Let \( \frac{\alpha(x) : M \beta_1(x), \ldots, M \beta_m(x)}{\gamma(x)} \in D \) and \( t \in b \cup \{c\} \) be such that \( \alpha(t) \in E^c \) and \( \neg \beta_1(t), \ldots, \neg \beta_m(t) \not\in E^c \). There are two possible cases.

1. \( t \neq c \). Since \( \alpha(t) \) is a formula over \( L_1 \) and since \( E^c \mid_{L_1} = E, \alpha(t) \in E \). Since \( E \subseteq E^c \), \( \neg \beta_i(t) \not\in E, i = 1, \ldots, m \). Thus, \( \gamma(t) \in E \) by the definition of \( E \) and, since \( E \subseteq E^c \), \( \gamma(t) \in E^c \).

2. \( t = c \). Then there exist uniterm formulas
\[
\gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b^*), \ldots, \gamma_q(b^*), \gamma_{q+1}(c), \ldots, \gamma_r(c) \in \bigcup_{i=1}^{\infty} E_i^c, 0 \leq p \leq q \leq r, b_1, \ldots, b_p \in b \ (\text{not necessarily different}), \text{ such that } b_i \neq c, b_i \neq b^* \text{ for } 1 \leq i \leq p \text{ and}
\]
\[
A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b^*), \ldots, \gamma_q(b^*), \gamma_{q+1}(c), \ldots, \gamma_r(c) \vdash \alpha(c).
\]
After replacing in this proof \( c \) with some new variable \( x \) that does not appear in the proof, we obtain a proof
\[
A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b^*), \ldots, \gamma_q(b^*), \gamma_{q+1}(x), \ldots, \gamma_r(x) \vdash \alpha(x).
\]
Since there is no application of GEN to \( x \) in the above proof, by the deduction theorem \([9, \text{Proposition 2.4, p. 59}]\),
\[
A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b^*), \ldots, \gamma_q(b^*) \vdash (\gamma_{q+1}(x) \land \ldots \land \gamma_r(x)) \supset \alpha(x).
\]
Now, by GEN,
\[
A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b^*), \ldots, \gamma_q(b^*) \vdash \forall x ((\gamma_{q+1}(x) \land \ldots \land \gamma_r(x)) \supset \alpha(x))
\]
and thus,
\[
A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b^*), \ldots, \gamma_q(b^*) \vdash (\gamma_{q+1}(b^*) \land \ldots \land \gamma_r(b^*)) \supset \alpha(b^*).\]
It follows that,
\[
A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b^*), \ldots, \gamma_q(b^*), \gamma_{q+1}(b^*), \ldots, \gamma_r(b^*) \vdash \alpha(b^*).\]
By the definition of \( E^c_i \), \( i \geq 0 \), \( E_i = E^c_i \big|_{\mathcal{L}_i} \) and thus,
\[ \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b^*), \ldots, \gamma_q(b^*) \in \bigcup_{i=1}^{\infty} E_i. \]

Also, since \( \gamma_{q+1}(c), \ldots, \gamma_r(c) \in \bigcup_{i=1}^{\infty} E^c_i \) and, by the definition of \( E^c_i \),
\[ i \geq 0, \quad \gamma_{q+1}(b^*), \ldots, \gamma_r(c) \in \bigcup_{i=1}^{\infty} E_i. \]
It follows that \( \bigcup_{i=0}^{\infty} E_i \vdash \alpha(b^*) \) by (**), and thus, by the definition of \( E \), \( \alpha(b^*) \in E \).

Now, assume to the contrary, that \( \neg \beta_{m'}(b^*) \in E \) for some \( m' \), \( 1 \leq m' \leq m \). Then there exist uniterm formulas
\[ \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b^*), \ldots, \gamma_q(b^*) \in \bigcup_{i=1}^{\infty} E_i, \quad 0 \leq p \leq q, \quad b_1, \ldots, b_p \in b \) (not necessarily different), such that \( b_i \neq b^* \) for \( 1 \leq i \leq p \) and
\[ A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b^*), \ldots, \gamma_q(b^*) \vdash \neg \beta_{m'}(b^*). \]
After replacing in this proof \( b^* \) with \( c \) we obtain a proof
\[ A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(c), \ldots, \gamma_q(c) \vdash \neg \beta_{m'}(c). \quad (***) \]

By the definition of \( E^c_i \), \( \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(c), \ldots, \gamma_q(c) \in \bigcup_{i=1}^{\infty} E^c_i \),
and thus, \( \bigcup_{i=0}^{\infty} E^c_i \vdash \beta_{m'}(c) \) by (***) in contradiction to the assumption
that \( \neg \beta_1(c), \ldots, \neg \beta_m(c) \notin E^c \).

From the definition of \( E \) follows that, \( \gamma(b^*) \in E \). Now, an argument similar to one previously used shows that, in this case, \( \gamma(c) \in E^c \).

We have proved that \( \Gamma_{(\mathcal{D}_{\mathcal{L}_i \cup \{A\}})}(E^c) \subseteq E^c \).

For the proof of inclusion \( E^c \subseteq \Gamma_{(\mathcal{D}_{\mathcal{L}_i \cup \{A\}})}(E^c) \) we proceed as follows.

By Lemma 1 (p. 7) with \( D = \mathcal{D}_{\mathcal{L}_i \cup \{A\}} \), \( \Gamma_{(\mathcal{D}_{\mathcal{L}_i \cup \{A\}})}(E^c) = TH \left( \bigcup_{i=0}^{\infty} \tilde{E}_i^c \right) \),
where \( \tilde{E}_0^c = A \) and for \( i > 0 \),
\[ \tilde{E}_{i+1}^c = \{ \gamma(t) \mid \text{for some} \quad \frac{\alpha(t): M \beta_1(t), \ldots, M \beta_m(t)}{\gamma(t)} \in \mathcal{D}_{\mathcal{L}_i \cup \{A\}}, \quad \bigcup_{k=0}^{i} \tilde{E}_k^c \vdash \alpha(t) \text{ and } \neg \beta_1(t), \ldots, \neg \beta_m(t) \notin E^c \}. \]

We shall prove by induction on \( i \) that \( E^c_i \subseteq \tilde{E}_i^c \), \( i \geq 0 \). The induction basis follows from the definition of \( E_0^c \) and \( \tilde{E}_0^c (= A) \). For the induction step, let \( \gamma(t) \in E^c_i \). There are two possible cases which follow from the definition of \( E^c_{i+1} \).
1. $t \neq c$. Then for some $\alpha(x) : M\beta_1(x), \ldots, M\beta_m(x) / \gamma(x) \in D$, $\bigcup_{k=0}^i E_k \vdash \alpha(t)$ and $-\beta_1(t), \ldots, -\beta_m(t) \not\in E$. Since $E_k \subseteq E^c_k$ for $k \geq 0$, $\bigcup_{k=0}^i E^c_k \vdash \alpha(t)$.

By the induction hypothesis, $\bigcup_{k=0}^i E^c_k \subseteq \bigcup_{k=0}^i \bar{E}^c_k$ and thus, $\bigcup_{k=0}^i \bar{E}^c_k \vdash \alpha(t)$. Since $E = E^c |_{\mathcal{L}_1}$ and $-\beta_i(t)$, $i = 1, \ldots, m$, are formulas over $\mathcal{L}_1$, $-\beta_1(t), \ldots, -\beta_m(t) \not\in E^c$. Thus, by the definition of $\bar{E}^c_{i+1}$, $\gamma(t) \in \bar{E}^c_{i+1}$.

2. $t = c$. Then, by the definition of $E^c_{i+1}$, $\gamma(c) \in E^c_{i+1}$. It follows that for some $\alpha(x) : M\beta_1(x), \ldots, M\beta_m(x) / \gamma(x) \in D$, $\bigcup_{k=0}^i E_k \vdash \alpha(b^*)$ and $-\beta_1(b^*), \ldots, -\beta_m(b^*) \not\in E$. Since $\bigcup_{k=0}^i E_k \vdash \alpha(b^*)$, there exist uniterm formulas $\gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b^*), \ldots, \gamma_q(b^*) \in \bigcup_{k=0}^i E_k$, $0 \leq p \leq q$, $b_1, \ldots, b_p \in b$ (not necessarily different), such that $b_i \neq b^*$, $1 \leq i \leq p$ and $A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(b^*), \ldots, \gamma_q(b^*) \vdash \alpha(b^*)$. After replacing $b^*$ with $c$ in this proof we obtain a proof $A, \gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(c), \ldots, \gamma_q(c) \vdash \alpha(c)$.

By the definition of $E^c_k$, $k \geq 0$, $\gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(c), \ldots, \gamma_q(c) \in \bigcup_{k=0}^i E^c_k$, and by the induction hypothesis, $\gamma_1(b_1), \ldots, \gamma_p(b_p), \gamma_{p+1}(c), \ldots, \gamma_q(c) \in \bigcup_{k=0}^i \bar{E}^c_k$. Thus, $\bigcup_{k=0}^i \bar{E}^c_k \vdash \alpha(c)$.

We shall prove now that $-\beta_1(c), \ldots, -\beta_m(c) \not\in E^c$. Assume to the contrary, that $-\beta_{m'}(c) \in E^c$ for some $1 \leq m' \leq m$. The proof similar to one used above shows that in this case $-\beta_{m'}(b^*) \in E$, in contradiction to the assumption that $-\beta_{m'}(b^*) \not\in E$.

We obtain that $\bigcup_{k=0}^i \bar{E}^c_k \vdash \alpha(c)$ and $-\beta_1(c), \ldots, -\beta_m(c) \not\in E^c$. By the definition of $\bar{E}^c_{i+1}$, $\gamma(c) \in \bar{E}^c_{i+1}$ also in this case.

It follows that $E^c = TH(\bigcup_{i=0}^\infty E^c_i) \subseteq TH(\bigcup_{i=0}^\infty \bar{E}^c_i) = \Gamma(\bar{E}_{\mathcal{L}_1 \cup \{t\} \cup A})(E^c)$.

This completes the proof of the lemma. □

**Lemma 20** Let $(D, A)$ be an open default theory and let $b$ be an infinite set of new constant symbols. Let $b_0 \subset b_1 \subset \ldots \subset b_\alpha \subset \ldots$ be an infinite well-ordered list of sets of constant symbols, such that $b = \bigcup_\alpha b_\alpha$ and let
$E^0 \subseteq E^1 \subseteq \ldots \subseteq E^\alpha \subseteq \ldots$ be an infinite well-ordered list of sets of closed formulas, such that each $E^\alpha$ is an extension for $(\bar{D}_{a}, A)$. If for each limit ordinal $\alpha$, $E^\alpha = \bigcup_{\delta < \alpha} E^{\delta}$ and for each successor ordinal $\alpha$, $E^\alpha|_{L_{a, \infty}} = E^{\alpha-1}$, then $E = \bigcup_{\alpha} E^\alpha$ is an extension for $(\bar{D}_{a}, A)$.

**Proof** First, we prove that $E|_{L_{a}} = E^\alpha$ for each $\alpha$. By the definition of $E$, $E^\alpha \subseteq E$ and since $E^\alpha$ is a set of formulas over $L_{a}$, $E^\alpha \subseteq E|_{L_{a}}$. Conversely, let $\varphi \in E|_{L_{a}}$. By the definition of $E$, $\varphi \in E^\beta$ for some $\beta$. If $\beta \leq \alpha$, then, by the inclusion $E^\beta \subseteq E^{\beta+1} \subseteq \ldots \subseteq E^\alpha$, $\varphi \in E^\alpha$. Let $\beta > \alpha$ and assume to the contrary that $\varphi \notin E^\alpha$. We will prove by transfinite induction of $\delta$ that in this case, $\varphi \notin E^\delta$ for each $\alpha \leq \delta \leq \beta$, in contradiction to the assumption that $\varphi \in E^\beta$.

Induction basis: $\delta = \alpha$. By the assumption, $\varphi \notin E^\alpha$.

Induction step: assume that for all $\delta'$, $\alpha \leq \delta' < \delta$, $\varphi \notin E^{\delta'}$. For the successor ordinal $\delta$, by the induction hypothesis $\varphi \notin E^{\delta-1}$. Since $\alpha \leq \delta - 1$, $b_{\alpha} \subseteq b_{\delta-1}$ and thus, $\varphi$ is a formula over $L_{a, \infty}$. Since $E^{\delta-1} = E^{\delta}|_{L_{a, \infty}}$, $\varphi \notin E^{\delta}$. For a limit ordinal $\delta$, $\varphi \notin E^\delta$ because $E^\delta = \bigcup_{\delta < \delta} E^\delta$.

That is, if $\varphi \notin E^\alpha$ then $\varphi \notin E^\beta$ for each $\alpha \leq \delta \leq \beta$, in contradiction to the assumption that $\varphi \in E^\beta$. It follows that $\varphi \in E^\alpha$ and, consequently, $E|_{L_{a}} = E^\alpha$ for each $\alpha$.

Now, we shall prove that $E$ is an extension for $(\bar{D}_{a}, A)$, that is, $\Gamma(\bar{D}_{a}, A)(E) = E$. By the definition of $E$, $A \subseteq E$. To prove that $E$ is deductively closed we shall prove that $\mathit{Th}(E) \subseteq E$ (the inclusion $E \subseteq \mathit{Th}(E)$ is by the definition of operator $\mathit{Th}$). Let $\varphi \in \mathit{Th}(E)$, that is, there exist $\psi_{1}, \ldots, \psi_{n} \in E$ such that $\psi_{1}, \ldots, \psi_{n} \vdash \varphi$. By the definition of $E$, $\psi_{1}, \ldots, \psi_{n} \in \bigcup E^\alpha$. Since $E^\alpha \subseteq E^{\alpha+1}$ for each $\alpha$, there exists $\beta$, such that $\psi_{1}, \ldots, \psi_{n} \in E^\beta$. Since $E^\beta$ is an extension for $(\bar{D}_{a}, A)$, $E^\beta$ is deductively closed and thus, $\varphi \in E^\beta$.

By the definition of $E$, $\varphi \in E$.

Now, let $\frac{\alpha(x)}{\gamma(x)} : M \beta^{1}(x), \ldots, M \beta^{m}(x) \in D$ and $t = t_{1}, \ldots, t_{n} \in T_{a}$ be such that $\alpha(t) \in E$ and $\neg \beta_{1}(t), \ldots, \neg \beta_{m}(t) \notin E$. Since $E = \bigcup E^\alpha$, there exists $E^\beta$, such that $\alpha(t) \in E^\beta$. Since $\neg \beta_{1}(t), \ldots, \neg \beta_{m}(t) \notin E$, by the definition of $E$, $\neg \beta_{1}(t), \ldots, \neg \beta_{m}(t) \notin E^\beta$. Thus, $\gamma(t) \in E^\beta$ and, by the definition of $E$, $\gamma(t) \in E$. That is, $\Gamma(\bar{D}_{a}, A)(E) \subseteq E$. 

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For the proof of inclusion $E \subseteq \Gamma_{(\mathcal{D}_{L_a})}(E)$ proceed as follows. By 
Lemma 1 (p. 7) with $D = \tilde{D}_{L_1}$, $\Gamma_{(\mathcal{D}_{L_1})}(E) = \mathcal{T} h\left(\bigcup_{i=0}^{\infty} E_i\right)$, where $E_0 = A$ and for $i \geq 0$,
\[ E_{i+1} = \{ \gamma(t) \mid \text{for some } \frac{\alpha(t)}{\gamma(t)} : M \beta_1(t), \ldots, M \beta_m(t) \in \tilde{D}_{L_1}, \]
\[ \bigcup_{k=0}^{i} E_k \vdash \alpha(t) \text{ and } \neg \beta_1(t), \ldots, \neg \beta_m(t) \not\in E \}.
\]
Also, by the same lemma, with $D = \tilde{D}_{L_a}$, for each $\alpha$, $E^\alpha = \Gamma_{(\mathcal{D}_{L_a})}(E^\alpha) = \mathcal{T} h\left(\bigcup_{i=0}^{\infty} E^\alpha_i\right)$, where $E_0^\alpha = A$ and for $i \geq 0$,
\[ E^\alpha_{i+1} = \{ \gamma(t) \mid \text{for some } \frac{\alpha(t)}{\gamma(t)} : M \beta_1(t), \ldots, M \beta_m(t) \in \tilde{D}_{L_a}, \]
\[ \bigcup_{k=0}^{i} E^\alpha_k \vdash \alpha(t) \text{ and } \neg \beta_1(t), \ldots, \neg \beta_m(t) \not\in E^\alpha \}.
\]
We shall prove by induction on $i$ that $E^\alpha_i \subseteq E^\alpha_i$, $i \geq 0$, for each $\alpha$. The induction basis follows from the definition of $E^\alpha_0$ and $E_0 (= A)$. For the induction step, let $\gamma(x)$ be a conclusion of a default from $D$ and $t \in T_{L_a}$ be such that $\gamma(t) \in E^\alpha_{i+1}$. Then for some $\frac{\alpha(x)}{\gamma(x)} : M \beta_1(x), \ldots, M \beta_m(x) \in D$,
\[ \bigcup_{k=0}^{i} E^\alpha_k \vdash \alpha(t) \text{ and } \neg \beta_1(t), \ldots, \neg \beta_m(t) \not\in E^\alpha. \]
By the induction hypothesis, $\bigcup_{k=0}^{i} E^\alpha_k \subseteq \bigcup_{k=0}^{i} E_k$, and thus, $\bigcup_{k=0}^{i} E_k \vdash \alpha(t)$. Since $E|_{L_a} = E^\alpha$ and $\neg \beta_1(t), \ldots, \neg \beta_m(t)$ are formulas over $L_a$, $\neg \beta_1(t), \ldots, \neg \beta_m(t) \not\in E$. Thus, by the definition of $E^\alpha_{i+1}$, $\gamma(t) \in E^\alpha_{i+1}$.

It follows that $\mathcal{T} h\left(\bigcup_{i=0}^{\infty} E^\alpha_i\right) \subseteq \mathcal{T} h\left(\bigcup_{i=0}^{\infty} E_i\right)$ for each $\alpha$ and thus,
\[ E = \bigcup_{\alpha} E^\alpha = \bigcup_{\alpha} \mathcal{T} h\left(\bigcup_{i=0}^{\infty} E^\alpha_i\right) \subseteq \mathcal{T} h\left(\bigcup_{i=0}^{\infty} E_i\right) = \Gamma_{(\mathcal{D}_{L_a})}(E).
\]
This completes the proof of the lemma. \(\square\)

**Lemma 21** Let $(D, A)$ be a uniform default theory, $b$ be an infinite set of new constant symbols and let $b'$ be a finite subset of $b$. Then for any extension $E'$ for $(\tilde{D}_{L_a}, A)$ there is an extension $E$ for $(D_{L_a}, A)$ such that $E|_{L} = E'|_{L}$.

**Proof** Let $E'$ be an extension for $(\tilde{D}_{L_a}, A)$ and let $b'' = b \setminus b'$. We arrange all the constants in $b''$ in a well-ordered list $c_0, c_1, \ldots, c_\alpha, \ldots$. The order
in which they are arranged is immaterial, as long as the list associates in
one-one fashion an ordinal number with each constant. We define an infinite
well-ordered list of sets of constants \( b_0 \subset b_1 \subset \ldots \subset b_\alpha \subset \ldots \) in the following
way. \( b_0 = \emptyset \). \( b_{\alpha+1} = b_\alpha \cup \{c_\alpha\} \). For a limit ordinal \( \alpha \), \( b_\alpha = \bigcup_{\beta < \alpha} b_\beta \).

Then, we build an infinite well-ordered list of sets of closed formulas
\( E^0 \subseteq E^1 \subseteq \ldots \subseteq E^\alpha \subseteq \ldots \), where each \( E^\alpha \) is an extension for \((\tilde{D}_{L_\alpha}, A)\)
and \( E^\alpha|_{L_{\alpha-\infty}} = E^{\alpha-1} \), in the following way.
\( E^0 = E' \). By the definition, \( E^0 \) is an extension for \((\tilde{D}_{L_\alpha}, A)\).
Assume that \( E^\beta, \beta < \alpha \) are already defined.

- for a limit ordinal \( \alpha \), \( E^\alpha = \bigcup_{\beta < \alpha} E^\beta \). \( E^\alpha \) is an extension for \((\tilde{D}_{L_\alpha}, A)\),
  by Lemma 20 (p. 48).

- for a successor ordinal \( \alpha \), by Lemma 19 (p. 45) with \( D_{L_\alpha} = D_{L_{\alpha-\infty}} \)
  and \( c = c_{\alpha-1} \), there exists an extension \( E'' \) for \((\tilde{D}_{L_\alpha}, A)\), such that
  \( E^{\alpha-1} = E''|_{L_{\alpha-\infty}} \) and we define \( E^\alpha = E'' \).

Now, we define \( E = \bigcup_{\alpha} E^\alpha \). By Lemma 20 (p. 48), \( E \) is an extension for
\((\tilde{D}_{L_\alpha}, A)\). This completes the proof of the lemma. \( \square \)

Now, we are ready to prove Theorem 6.

**Proof of Theorem 6** Let \( b' \) be a set of constants with the maximal card-
inality among \( \{b^E \mid E \text{ is an extension for } (\tilde{D}_{L_\alpha}, A)\} \), where \( b^E \) is as in
Definition 18 (p. 39). The “if” part of the theorem follows from Lemma
21 (p. 50). For the “only if” part, let \( E \) be an extension for \((\tilde{D}_{L_\alpha}, A)\) and
let \( b^E \) be a finite set as defined in Definition 18 (p. 39). We build a set
of constants \( \bar{b} \) by renaming all the constants of \( b' \) in the way that \( b^E \subseteq \bar{b} \).
Now, by Lemma 18 (p. 42) with \( \bar{b} = b' \), there exists an extension \( \bar{E} \) for
\((\tilde{D}_{L_\alpha}, A)\), such that \( E|_L = \bar{E}|_L \). Since \( \bar{b} \) and \( b' \) are sets of new constants
having the same cardinality, there exists an extension \( E' \) for \((\tilde{D}_{L_\alpha}, A)\), such that \( E'|_L = \bar{E}'|_L \). It follows that there exist an extension \( E' \) for \((\tilde{D}_{L_\alpha}, A)\),
such that \( E|_L = E'|_L \). This completes the proof of the theorem. \( \square \)

Theorem 7 follows immediately from Theorem 6 (p. 40) and Lemma 15
(p. 40).
Chapter 7

Appendix

7.1 Logical Axioms

In this section we list the logical axioms of the classical first-order logic, as defined in [9]. We need the following definition first.

Definition 19 ([9]) If $A$ is a formula and $t$ is a term, then $t$ is said to be free for $x_i$ in $A$ if no free occurrence of $x_i$ in $A$ lies within the scope of any quantifier $\forall x_j$, where $x_j$ is a variable in $t$.

If $A$, $B$, and $C$ are formulas over $\mathcal{L}$, then the following are logical axioms of $\mathcal{L}$.

(A1) $A \vdash (B \supset A)$

(A2) $(A \vdash (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

(A3) $(\neg B \supset \neg A) \supset ((\neg B \supset A) \supset B)$

(A4) $\forall x A(x) \supset A(t)$ if $A(x)$ is a formula over $\mathcal{L}$ and $t$ is a term of $\mathcal{L}$ that is free for $x$ in $A(x)$\(^1\).

(A5) $(\forall x (A \supset B)) \supset (A \supset \forall x B)$ if $A$ is a formula over $\mathcal{L}$ that contains no free occurrences of $x$.

\(^1\)Here $t$ may be $x$. 

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7.2 Deduction theorem for the $C^w$-proof system

In this section we present a deduction theorem for the $C^w$-proof systems.

**Lemma 22** ($C^w$-Deduction Theorem) If $\Gamma$ is a set of closed formulas and $A, B$ are closed formulas, such that $\Gamma, A \vdash_{C^w} B$, then $\Gamma \vdash_{C^w} A \supset B$.

**Proof** Let $T$ be a $C^w$-proof of $B$ from $\Gamma \cup \{A\}$. We will show by induction on the depth of $T$, $i$, that for each vertex of $T$, say $C$, there exists a $C^w$-proof $\bar{T}_C$ of $A \supset C$ from $\Gamma$ of depth $3i$ at most.

For the induction basis ($i = 1$), let $C$ be a leaf of $T$. Then $C$ either in $\Gamma$, or an axiom of $L$, or $A$ itself. By the axiom schema (A1), $C \supset (A \supset C)$ is an axiom, so we add it as a vertex in $\bar{T}_C$. Also, in the first two cases, we add $C$ as a vertex to $\bar{T}_C$, and hence by virtue of MP, we obtain the following $C^w$-proof $\bar{T}_C$ of $A \supset C$.

The depth of the constructed $C^w$-proof is 2.

For the third case, $(A \supset ((A \supset A) \supset A)) \supset ((A \supset (A \supset A)) \supset (A \supset A))$ is added as a vertex to $\bar{T}_C$ by the instance of the axiom schema (A2), and $(A \supset ((A \supset A) \supset A)) \supset (A \supset A)$ is added by the instance of the axiom schema (A1). By MP, there is a vertex $(A \supset (A \supset A)) \supset (A \supset A)$ in $\bar{T}_C$. Now, $A \supset (A \supset A)$ is also added as a vertex to $\bar{T}_C$ by the instance of the axiom schema (A1), and by virtue of MP on the last two formulas, we obtain the following $C^w$-proof $\bar{T}_C$ of $A \supset C$ ($C$ is $A$).

The depth of the constructed $C^w$-proof is 3.
For the induction step, let \( C \) be a vertex of \( T \) at the depth \( i + 1 \). Then, \( C \) is introduced by one of the following rules of inference.

1. \( C \) is a consequence of \( C' \supset C \) and \( C' \) by MP. That is, there exist \( C^w \)-proofs of \( C' \) and \( C' \supset C \) from \( \Gamma \cup \{ A \} \). The depth of one of those \( C^w \)-proofs is \( i \) and the depth of the other is \( i \) at most. By the induction hypothesis, there exists a \( C^w \)-proof of \( A \supset (C' \supset C) \) and a \( C^w \)-proof of \( A \supset C' \) from \( \Gamma \). The depth of those \( C^w \)-proofs is \( 3i \) at most. Now, by the axiom schema (A2), \( (A \supset (C' \supset C)) \supset ((A \supset C') \supset ((A \supset C') \supset (A \supset C))) \) is an axiom. Hence we obtain the following \( C^w \)-proof of \( A \supset C \).

(By applying MP twice.) The depth of the constructed \( C^w \)-proof is \( 3i + 2 \) at most, which is smaller than \( 3(i + 1) \).

2. \( C \) is of the form \( \forall x C'(x) \) and it is a consequence of \( C'(t) \), \( t \in T_L \) by virtue of the Carnap rule. That is, for each \( t \in T_L \) there exist a \( C^w \)-proof of \( C'(t) \) from \( \Gamma \) of depth \( i \) at most. By the induction hypothesis, for each \( t \in T_L \), there exists a \( C^w \)-proof of \( A \supset C'(t) \) from \( \Gamma \) of depth \( 3i \) at most. We construct a \( C^w \)-proof of \( \forall x (A \supset C'(x)) \) by virtue of the Carnap rule as follows.
(By the axiom schema (A5), $\forall x (A \supset C'(x)) \vdash (A \supset \forall x C'(x))$. Now apply MP.) The depth of the constructed $C^w$-proof is $3i + 2$ at most, which is smaller than $3(i + 1)$.

This completes the proof of the lemma. $\square$
Chapter 8

Summary and further research

In our work we proposed a syntactic definition of extension for open default theories based on a proof system for first-order logic extended with the Carnap rule of inference. We proved that, over monadic languages, our definition is equivalent to the original definition of extension for open default theories and does not depend on the cardinality of its infinite base. Consequently, in this case, the original semantic definition of extensions does not depend on the cardinality of its infinite base.

We also studied a syntactic definition of extension for open default theories based on a weaker proof system. We proved that this definition does not depend on the cardinality of its infinite base as well.

Finally, we proved that in the case of uniterm default theories over finite languages not containing function symbols, the syntactic definition of extension does not depend on the cardinality of its base. Since under the domain closure assumption, the syntactic definition of extension is equivalent to the semantic definition, we concluded that in this case, the original semantic definition of extensions for uniterm default theories over languages, which do not contain function symbols, does not depend on the cardinality of its base as well, and thus, can always be restricted to a finite base. A possible practical value of the result above is that the extensions for uniterm default theories in this case are efficiently computable.

There is some room for future research on the semantic definition of extensions for open default theories under the domain closure assumption. Following the results of Chapter 6, it might be possible that, for a general
class of open default theories (not only uniterm default theories), under the domain closure assumption, the semantic definition of extensions of open default theories does not depend on the cardinality of its base.
Bibliography


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ה.closePath בכסף של החובות
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חת療 על מחקר

לשם מילוי חלקי של הדרישותוכלבלת התוכן
נגישר למעריכים מבית מהתוכן

והנה מוכן

ארון לסנטה הטכניון - מכון טכנולוגי לישראל
המחנה
2001
ה乜חכרים עליה בחינתן פורמי/ית מיכאל קמנסקי
בקולותה למדעי המחשב.

אני מחוות למחאות קמנסקי על הנחיית המעורטים,
הנسياس והזמנ שואת הקדיש לי במשך כל שלבי העבורה.

אני מודע ל trầnינו על התוכנה הסכיפה הנדרשת בحسبמלתי.
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
Carnap rule
Carnap rule

uniterm default theories
uniterm default theories

Technion - Computer Science Department - M.Sc. Thesis MSC-2001-03 - 2001

Carnap rule

uniterm default theories
uniterm default theories

הわか האל-ר וה-רコミュニケーションנעני-לון קולגנוזא סביסיםフィシナル כנספי מסלול של תלות

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