THE GENERAL STRUCTURE OF EDGE-CONNECTIVITY OF A VERTEX SUBSET IN A GRAPH
AND ITS INCREMENTAL MAINTENANCE.

ODD CASE

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ABSTRACT. Let G = (V, E) be an undirected graph, S be a subset of its vertices, ξS be the set of minimum edge-cuts partitioning S, λS be the cardinality of such a cut. We suggest a graph structure, called the connectivity carcass of S, that represents both cuts in ξS and the partition of V by all these cuts; its size is O(min{|E|, λS|V|}). In this paper we present general constructions and study in detail the case λS odd; the specifics of the case λS even are considered elsewhere. For an adequate description of the connectivity carcass we introduce a new type of graphs: locally orientable graphs, which generalize digraphs. The connectivity carcass consists of a locally orientable quotient graph of G, a cactus tree (in case λS odd, just a tree) representing all distinct partitions of S by cuts in ξS, and a mapping connecting them. One can build it in $O(|S|)$ max-flow computations in G. For an arbitrary sequence of u edge insertions not changing λS, the connectivity carcass can be maintained in time $O(|V||E|/\lambda_S|V| + u)$. For two vertices of G, queries asking whether they are separated by a cut in ξS are answered in $O(1)$ worst-case time per query. Another possibility is to maintain the carcass in $O(|S|\min{|E|, \lambda_S|V|} + u)$ time, but to answer the queries in $O(1)$ time only if at least one of the vertices belongs to S.

1. Introduction

Connectivity problems play an important role in applications of graph theory in computer science and have been extensively studied. In particular, much attention has been given to edge connectivity problems. A pair of vertices s and t of an undirected graph $G = (V, E)$ is said to be $k$-edge-connected if there exist k edge-disjoint paths between them (equivalently, there is no $k'$-cut, $k' \leq k - 1$, separating s and t). The k-connectivity is

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an equivalence relation, and its equivalence classes are called \( k \)-connectivity classes. It is known that the system of globally minimum cuts and connectivity classes formed by them in a graph is represented by a cactus tree, i.e., a graph whose blocks are edges and cycles [DKI]; it is just a tree if the size of globally minimum cuts is odd. On the other hand, the lattice of all \((s,t)\)-minimum cuts for fixed vertices \( s \) and \( t \) is represented by all closed sets of some directed acyclic graph (dag) with one source and one sink [PQ]. Both these data structures can be maintained efficiently under edge insertions (see [DW] for the first one and, e.g., [I] for the second). In [DV1] we have suggested a new structure, which is a natural generalization of both the aforementioned representations. This data structure represents all minimum cuts of \( G \) partitioning a fixed vertex subset and the corresponding partition of this subset into its connectivity classes; besides, it represents the partition of \( V \) by all these cuts. Our structure admits efficient incremental maintenance as well.

In this paper we describe general constructions and provide a detailed analysis of the odd case, that is, of the case when the cardinality of the cuts in question is odd. This allows us to introduce all the main ideas while avoiding many technical difficulties. A detailed analysis of the even case is given in [DV3].

A well-known model for pairwise minimum cuts in a graph is the Gomory-Hu tree [GH]. It represents one minimum \((x,y)\)-cut for any pair of vertices \( x, y \), which is far from being sufficient for the purpose of incremental maintenance. Indeed, if the newly inserted edge increases the value of the single minimum \((x,y)\)-cut represented by the Gomory-Hu tree, one cannot tell whether the size of the minimum \((x,y)\)-cut remains the same, or increases.

One of the goals of connectivity studies is to provide tools for fast answering connectivity queries in dynamic graphs, such as: “Are vertices \( u \) and \( v \) \( k \)-connected?” or “Show \( k \)-cuts separating \( u \) and \( v \)”. An important problem of this type is maintaining of the hierarchy of the \( k \)-connectivity classes, \( k \leq l \); the corresponding incremental algorithms (that allow insertions of edges to \( G \)) are presented in [WT] for \( l = 2 \), [GI] for \( l = 3 \), and [DW] for \( l = 4 \). We hope that our data structure can play the crucial role in solving this problem for arbitrary \( l \). Indeed, take for a vertex subset a \((k-1)\)-connectivity class, then its connectivity subclasses are just \( k \)-connectivity classes of \( G \). The totality of these subclasses gives the partition of \( V \) into \( k \)-classes. Thus, it suffices to maintain our data structures for all \((k-1)\)-classes of the graph, \( k \leq l \). Observe that when an edge insertion causes the merging of some \((k-1)\)-classes, their connectivity structures must be merged as well. So, the results of this paper cover the “local” part of the work on the maintenance of the hierarchy of connectivity classes for a general \( l \).

Another application of connectivity structures is augmentation problems

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\(^4\) This fact was discovered independently by A. Karzanov (personal communication, 1977).
of the type: given a graph, add the minimum number of edges so as to increase the connectivity of the whole graph to a prescribed value. The cactus structure of minimum edge-cuts was employed essentially in [NGM, Be] to develop clear and fast schemes and algorithms for graph augmentation. One can suppose that the suggested generalization of this structure to locally minimum cuts will help to improve the results in this area.

Finally, the analysis of the connectivity structure of a vertex subset in a graph can arise itself in an application. For example, let us consider an interconnection network $\mathcal{N}$ whose nodes are terminals and auxiliary intermediate stations. Then it is natural to define the connectivity structure of $\mathcal{N}$ as the connectivity structure of the set of terminals in $\mathcal{N}$. As an illustration, consider the network with the terminals $A$, $B$ and $C$ shown on Fig. 1.1; the 3-cuts shown by dashed lines are $\{A, B, C\}$-mincuts, but there exist also two irrelevant 2-cuts and one 3-cut.

![Fig. 1.1. Minimum cuts in an interconnection network](image)

Let us consider a vertex subset $S$ and the set of all minimum $S$-cuts, that is, cuts of $G$ that partition $S$ and have the minimum cardinality, $\lambda_S$, among such cuts; $\lambda_S$ is said to be the connectivity of $S$. Two such cuts are equivalent if they partition $S$ in the same way. The system of all such partitions, and thus all the $(\lambda_S + 1)$-connectivity classes of $S$, is represented by a cactus tree $H_S$ (the skeleton), similarly to [DKL]. Partitions are defined by cuts of $H_S$ almost bijectively. In the odd case the skeleton is just a tree, and the representation of the partitions by its cuts is a bijection.

Each family of equivalent $S$-cuts is represented, similarly to [PQ], by cuts of a two-terminal dag of a special type, called a strip, which is defined up to the reversal of all its edges. Such a strip is a quotient graph of $G$; we identify its vertices with their preimages in $V$ under the quotient mapping. These strips agree on intersections in the following sense. First, if two nonterminal vertices of two strips intersect (as subsets of $V$), then these vertices coincide. Thus, the total partition of $V$ by all minimum $S$-cuts subdivides only terminal vertices of each strip. The quotient graph by this total partition we denote by $F_S$. Second, let two vertices of two strips coincide, and hence, these vertices have the same set of incident edges. Then the two 2-partitions of this set into incoming and outcoming edges in both strips coincide as well. So, at each vertex there exists a unique distinguished 2-partition of the set
of incident edges (the inherent partition). The graph $F_S$ with the inherent partitions at its vertices we call the flesh.

We introduce and study a special type of graphs: undirected graphs with a 2-partition of the set of incident edges at each vertex; we call them locally orientable graphs. In general such a graph cannot be oriented globally, but still preserves certain properties of digraphs. An analog of an oriented path in a locally orientable graph is a path that agrees with inherent partitions (a coherent path); we consider reachability along coherent paths. For graphs satisfying a natural restriction on cyclic coherent paths one can define strongly connected components and adjust Depth-First Search (DFS) to scanning such graphs and finding these components. The flesh $F_S$ satisfies even a stronger restriction: all its coherent paths are simple; thus, $F_S$ is, in a sense, acyclic.

Finally, to each vertex of $F_S$ we assign the set of families of equivalent $S$-cuts in whose strips this vertex is nonterminal. The corresponding cuts of $H_S$ define a subset of edges in $H_S$ (the projection $\pi_S$ of the vertex), which turns out to be a path. The structure consisting of the skeleton, the flesh, and the projection mapping is called the connectivity carcass of $S$.

The connectivity carcass allows to answer readily several types of queries. The query asking whether two given vertices in $S$ are $(k + 1)$-connected is answered just by checking whether they are mapped into the same node of $H_S$ or whether they are contained in the same vertex of $F_S$. In the latter way one can check also whether two arbitrary vertices of $G$ are separated by an $S$-cut. An $S$-cut separating two vertices of which at least one belongs to $S$ is defined immediately in terms of their projections. The same query in the general case may require, in addition, to combine such a cut with a reachability cone of one of the given vertices. For any two subsets $S_1$, $S_2$ of $S$, the corresponding strip of minimum $S$-cuts separating $S_1$ from $S_2$ is obtained by a contraction of the flesh defined in a direct way by the projection mapping. The orientations of the edges in this strip are induced by the inherent partitions at the vertices of the flesh.

The connectivity carcass can be naturally “glued” from $(s,t)$-strips of type $[PQ]$, $s, t \in S$, $s$ fixed, by a polynomial algorithm, because of their “coincidences on intersections”. The space requirements are linear, since this data structure is based on a quotient of the original graph.

When considering incremental dynamics of the connectivity carcass, we assume that the cardinality of $S$-mincuts does not change under edge insertions; if such a change occurs, we say that the the connectivity carcass vanishes. As a matter of fact, there arises a new system of larger $S$-mincuts, but this transition is beyond the scope of dynamic considerations in this paper.

Upon inserting a new edge into $G$, all the three components of the carcass change accordingly, and all the changes are of a contractive nature (provided $\lambda_S$ does not change). The change of the graph of the flesh con-
sists in contracting several vertices into a single new vertex. In particular, the endpoints $U_1$ and $U_2$ of the image of the new edge in $F_S$, and all the vertices that can be visited by a coherent path between $U_1$ and $U_2$, “fall” into the new vertex. The other flesh vertices that are contracted are defined in terms of projections and reachability. The inherent partitions remain the same, except for the new flesh vertex, where the inherent partition is glued naturally from those of constituting vertices, or becomes trivial in case it was trivial for at least one of the constituting vertices. The change of the skeleton consists in contracting edges and “squeezing” cycles in the path-of-edges-and-cycles connecting the projections of $U_1$ and $U_2$; in the odd case it is just an ordinary path, and all of its edges are contracted. The projection of the new flesh vertex is $\pi_S(U_1) \cap \pi_S(U_2)$. The set of the other flesh vertices for which the projection changes is defined by their reachability from $U_1$ and $U_2$. From the projection of a vertex, certain parts of the sets $\pi_S(U_1) \setminus \pi_S(U_2)$ or $\pi_S(U_2) \setminus \pi_S(U_1)$ are deleted.

The incremental algorithm [DV1] for maintaining the connectivity carcass is built according to the above scheme. In particular, it maintains the reachability cones of units, and thus involves, as a subproblem, incremental maintenance of transitive closures. The complexity of this algorithm is $O(|V| \min\{|E|, \lambda_S|V|\} + u)$, where $u$ is the number of edge insertions (it is assumed that all of them preserve the connectivity of $S$); therefore, we do not exceed the complexity of best known algorithms for maintenance of the transitive closure. Each query asking whether two given vertices are separated by a minimum $S$-cut is answered in $O(1)$ worst-case time; such a cut itself can be shown in $O(|V|)$ amortized time. The dag representation of all minimum $S$-cuts separating any two subsets of $S$ can be obtained in $O(|E|)$ amortized time. (For a comparison, in [GN] such a representation for an arbitrary pair of one-element subsets is constructed in $O(|V| \cdot |E|)$ time.)

The complexity of maintenance can be reduced at the expense of an increase in the reaction time for certain queries: we guarantee the same $O(1)$ worst-case time only in the cases when at least one of the vertices in question belongs to $S$. To avoid the time consuming maintenance of reachability cones, we suggest (see [DV2]) to maintain, instead of the actual flesh, a weaker contraction of the initial flesh, called the preflesh; the actual flesh is obtained from the preflesh by contraction of its strongly connected components. The skeleton, the preflesh, and the projection mapping can be maintained incrementally in time $O(|S| \min\{|E|, \lambda_S|V|\} + u)$ for an arbitrary sequence of $u$ edge insertions preserving the connectivity of $S$. The only increase of the reaction time, to $O(|E|)$ worst case, arises for the separation query in the case when the vertices in question both do not belong to $S$ and have the same projection; the cut query in this situation can be answered within the same $O(|E|)$ time. As a corollary, we can simultaneously maintain the carcasses for all parts of an arbitrary partition of $V$ in $O(|V| \min\{|E|, \lambda_S|V|\} + u\alpha(u, |V|))$ time. The same complexity can be
achieved, thus, for all the local work while maintaining the hierarchy of partitions of \( V \) into the \( k \)-connectivity classes, \( k \leq l \).

The first part of the paper treats the general case (\( \lambda \leq 3 \) arbitrary). In Sect. 2 we give basic definitions and concepts. In Sect. 3 we introduce locally orientable graphs and study their properties. In Sect. 4 we consider a number of simple cases to gain intuition and to prove several basic lemmas. In Sect. 5 we define the flesh (for the general case) and the skeleton (for the odd case only) and state their properties. In the rest of the paper \( \lambda \) is assumed to be odd. In Sect. 6 we define the projection and present further properties of the connectivity carcass. In Sect. 7 we describe the transformations of the connectivity carcass under the insertion of an edge to \( G \). In Sect. 8 we describe algorithms for construction and incremental maintenance of the connectivity carcass and for answering the queries.

Some results of this paper were published in a preliminary form in [DV1, DV2].

2. Basic definitions and properties

In this paper we consider connected undirected graphs \( G = (V, E) \), \( |V| \geq 2 \), without loops and possibly with parallel edges. We assume that these properties are preserved under vertex contractions; that is, we delete the obtained loops, but keep all the parallel edges. As usually, we denote \( |V| \) by \( n \) and \( |E| \) by \( m \). For each ordered 2-partition \( \mathcal{P} = (V_\mathcal{P}, V_{\overline{\mathcal{P}}}) \), \( V_{\overline{\mathcal{P}}} = V \setminus V_{\mathcal{P}} \), of the vertex set \( V \) we define its edge set \( E_{\mathcal{P}} \) as the set of edges whose ends lie in distinct parts of \( \mathcal{P} \). An ordered 2-partition \( \mathcal{C} \) is said to be a cut of \( G \) if no proper subset of \( E_{\mathcal{C}} \) coincides with \( E_{\mathcal{P}} \) for some other 2-partition \( \mathcal{P} \).

The sets \( V_\mathcal{C} \) and \( V_\overline{\mathcal{C}} \) are called the sides of \( \mathcal{C} \); we say that a vertex \( v \in V \) lies inside \( \mathcal{C} \) if \( v \in V_\mathcal{C} \), and outside \( \mathcal{C} \) if \( v \in V_{\overline{\mathcal{C}}} \). For any cut \( \mathcal{C} \), the opposite cut \( \overline{\mathcal{C}} \) is defined by \( V_{\overline{\mathcal{C}}} = V_\mathcal{C} \), \( V_\mathcal{C} = V_{\overline{\mathcal{C}}} \); we say that \( \overline{\mathcal{C}} \) is obtained from \( \mathcal{C} \) by flipping.

Let \( \mathcal{C} \) and \( \mathcal{C}' \) be two cuts such that \( V_\mathcal{C} \subseteq V_{\overline{\mathcal{C}'}}, V_{\overline{\mathcal{C}}} \subseteq V_\mathcal{C}' \); we say that \( \mathcal{C}' \) dominates \( \mathcal{C} \) and write \( \mathcal{C} \preceq \mathcal{C}' \). If the inclusion \( V_\mathcal{C} \subseteq V_{\overline{\mathcal{C}'}}, V_{\overline{\mathcal{C}}} \subseteq V_\mathcal{C}' \) is strict, we say that \( \mathcal{C}' \) strictly dominates \( \mathcal{C} \) and write \( \mathcal{C} \prec \mathcal{C}' \).

For any two cuts \( \mathcal{C} \) and \( \mathcal{C}' \), denote by \( \mathcal{C} \cap \mathcal{C}' \) the 2-partition \( (V_\mathcal{C} \cap V_{\overline{\mathcal{C}'}}), V_{\overline{\mathcal{C}} \cap \overline{\mathcal{C}'}}, V_\mathcal{C} \cup V_{\overline{\mathcal{C}'}}, \overline{\mathcal{C} \cap \overline{\mathcal{C}'}} \), and by \( \mathcal{C} \cup \mathcal{C}' \) the 2-partition \( (V_\mathcal{C} \cup V_{\overline{\mathcal{C}'}}), V_{\overline{\mathcal{C}} \cup \overline{\mathcal{C}'}}, V_\mathcal{C} \cap V_{\overline{\mathcal{C}'}}, \overline{\mathcal{C} \cup \overline{\mathcal{C}'}} \). Observe that the 2-partitions \( \mathcal{C} \cap \mathcal{C}' \) and \( \mathcal{C} \cup \mathcal{C}' \) are not necessarily cuts; however, if they are, then \( \mathcal{C} \cap \mathcal{C}' \preceq \mathcal{C} \preceq \mathcal{C} \cup \mathcal{C}' \) and \( \mathcal{C} \cap \mathcal{C}' \preceq \mathcal{C} \preceq \mathcal{C} \cup \mathcal{C}' \). As usually, the cardinality \( c(\mathcal{C}) = c(V_\mathcal{C}, V_{\overline{\mathcal{C}}}) \) of a cut \( \mathcal{C} \) is defined to be the size of its edge set \( E_{\mathcal{C}} \); clearly, \( c(\overline{\mathcal{C}}) = c(\mathcal{C}) \). The notion of cardinality is extended naturally to arbitrary 2-partitions of \( V \), and moreover, to arbitrary pairs of disjoint subsets of \( V \): if \( X_1, X_2 \subseteq V \), \( X_1 \cap X_2 = \emptyset \), then \( c(X_1, X_2) \) stands for the number of edges with one end in \( X_1 \) and the other in \( X_2 \).

Let \( S \) be a subset of \( V \); we denote by \( \sigma \) the cardinality of \( S \). Any cut \( \mathcal{C} \) defines a 2-partition \( (S_\mathcal{C}, S_{\overline{\mathcal{C}}}) \) of \( S \), with \( S_\mathcal{C} = V_\mathcal{C} \cap S, S_{\overline{\mathcal{C}}} = V_{\overline{\mathcal{C}}} \cap S \). A cut \( \mathcal{C} \) is said to be an \( S \)-cut if both \( S_\mathcal{C} \) and \( S_{\overline{\mathcal{C}}} \) are nonvoid. For \( S = \{a, b\} \), we speak of \((a, b)\)-cuts instead of \( \{a, b\}\)-cuts, to keep the usual notation. It is
easy to see that the minimum cardinality among 2-partitions dividing $S$ is equal to that among $S$-cuts; we denote it by $\lambda_S$. Moreover, each 2-partition of minimum cardinality $\lambda_S$ dividing $S$ is an $S$-cut. In this paper we are interested only in $S$-cuts of cardinality $\lambda_S$; we call them minimum $S$-cuts, or $S$-mincuts. The following basic property of $S$-mincuts (see Fig. 2.1) is used extensively throughout the paper.

![Fig. 2.1. Two intersecting $S$-mincuts](image)

**Lemma 2.1.** Let $C$ and $C'$ be two arbitrary $S$-mincuts such that both $V_C \cap V_{\bar{C}} \cap S$ and $V_{\bar{C}} \cap V_C \cap S$ are nonempty. Then

(i) $c(V_C \cap V_{\bar{C}} \cap V_S) = 0$;

(ii) $c(V_C \cap V_{\bar{C}} \cap V_S) = c(V_C \cap V_{\bar{C}} \cap V_{\bar{C}'})$, $c(V_C \cap V_{\bar{C}} \cap V_{\bar{C}'}) = c(V_{\bar{C}} \cap V_C \cap V_{\bar{C}'})$.

**Proof.** (i) We assume that $C \neq C'$, since otherwise the assertion is trivial. Let $s$ and $t$ be two arbitrary vertices in $V_C \cap V_{\bar{C}} \cap S$ and $V_{\bar{C}} \cap V_C \cap S$, respectively. Then both $C$ and $C'$ are $(s,t)$-mincuts, and hence, by [FF, Ch. 1, Corollary 5.3], every edge $e$ between $V_C \cap V_{\bar{C}}$ and $V_{\bar{C}} \cap V_C$ must be both free of any maximal $(s,t)$-flow and saturated by it.

(ii) By (i), for any maximal $(s,t)$-flow $c(V_C \cap V_{\bar{C}} \cap V_S)$ is the flow entering $V_C \cap V_{\bar{C}}$, and $c(V_C \cap V_{\bar{C}} \cap V_{\bar{C}'})$ is the flow leaving $V_C \cap V_{\bar{C}} \cap V' \cap F$. These two quantities are equal by the flow conservation law. The second equality is proved in the same way. □

The following observation, though trivial, is important: for any two $S$-mincuts $C_1$ and $C_2$, at least one of the pairs $C_1, C_2$ and $C_1, \bar{C}_2$ satisfies the conditions of Lemma 2.1.

Two $S$-cuts are said to be $S$-equivalent if they define the same 2-partition of $S$. Equivalence classes of $S$-mincuts are called bunches. Evidently, if one replaces each cut of a bunch by the opposite cut, then one again gets a bunch, which is said to be opposite to the initial one. The following statement is an immediate corollary of Lemma 2.1.

**Fact 2.2.** Let $C$ and $C'$ be two $S$-mincuts satisfying the conditions of Lemma 2.1. Then:

(i) Both $C \cap C'$ and $C \cup C'$ are $S$-mincuts.

(ii) Let $C''$ be an $S$-mincut $S$-equivalent to $C'$, then $C \cap C''$ and $C \cup C''$ are $S$-mincuts $S$-equivalent to $C \cap C'$ and $C \cup C'$, respectively.
In particular, the intersection and the union of two $S$-equivalent $S$-mincuts are cuts from the same bunch as well.

Let $S_1$ and $S_2$ be two disjoint subsets of $S$. An $S$-mincut is said to be an $(S_1, S_2)$-mincut if it keeps $S_1$ and $S_2$ on different sides. It follows immediately from Fact 2.2(i) that the intersection of all $(S_1, S_2)$-mincuts having $S_1$ inside is again an $(S_1, S_2)$-mincut. We call it the $S_1$-tight $(S_1, S_2)$-mincut.

Two vertices of $G$ are said to be $S$-equivalent if they are not separated by any $S$-mincut. Equivalence classes of vertices are called $S$-units, or simply units. Evidently, the unit containing a given vertex is just the inner side of the intersection of all $S$-mincuts having this vertex inside. We denote by $F_S$ the quotient graph obtained from $G$ by contracting all the vertices of the same unit. The vertices of $F_S$ are naturally called units. Each $S$-mincut induces naturally a cut of $F_S$ separating the same units; in what follows we do not distinguish between these two cuts.

Observe that, in general, $S$ is partitioned by the set of all $S$-mincuts into subsets $S_1, S_2, \ldots, S_r$, and $r$ can be less than $\sigma$. In this case $F_S$ coincides with $F_{\{v_1, v_2, \ldots, v_r\}}$, where $v_i$ is an arbitrary vertex in $S_i$, $1 \leq i \leq r$.

Let $f: V \to M$ be a mapping to an arbitrary set $M$. For any $W \subseteq V$, we denote by $f(W)$ the set $\{f(w) : w \in W\}$. We say that a 2-partition $P$ of $M$ $f$-induces a cut $C$ if $V_C = f^{-1}(M_P)$. Similarly, given a mapping $g: S \to M$, we say that $P$ $g$-induces a bunch of $S$-cuts if $S_C = g^{-1}(M_P)$ for any cut $C$ in this bunch.

To illustrate the above notions, let us consider the graph $G$ presented on Fig. 2.2a. Large black circles denote vertices in $S$; thus $\lambda_S = 2$. All the ten $S$-mincuts are shown by dashed lines. They fall into four bunches consisting of 1, 2, 2, and 5 cuts, respectively, marked by braces. The vertices of $G$ fall into eight units marked by numbers. The quotient graph $F_S$ is presented on Fig. 2.2b. The quotient mapping $f$ takes all the vertices of a unit $i$ to the vertex $i$ of $F_S$. Each $S$-mincut is $f$-induced by a unique cut of $F_S$ (for example, see cuts $C_1$ and $C_2$ in Figs. 2.2a,b). Let us consider also a four-element set $M = \{a, b, c, d\}$ and the mapping $g: S \to M$ that takes the vertices of $S$ belonging to the units 2, 3, 4, 8 to $a, b, c, d$, respectively. The four 2-partitions of $M$ shown on Fig. 2.2c $g$-induce the four bunches of $S$-mincuts in $G$. 
3. Locally orientable graphs

3.1. General locally orientable graphs. Let \( v \) be an arbitrary vertex of an undirected graph. The star of \( v \) is the set of edges incident to it. The star of a vertex is said to be orientable if some its unordered 2-partition is fixed; the parts of this 2-partition are called the sides of the star. Such a vertex is called stretched if the partition of its star is nontrivial (i.e., both sides are nonempty) and a terminal otherwise. A graph is said to be locally orientable if the stars of all its vertices are orientable; we say that it overlays the initial undirected graph. A locally orientable graph is called balanced if for any stretched vertex the two sides of its star have equal cardinalities.

In particular, for any vertex \( v \) of an arbitrary directed graph, the partition of arcs incident to \( v \) into incoming and outcoming defines a 2-partition of the star of \( v \) in the underlying undirected graph. Thus, for any digraph, there exists a canonically defined locally orientable graph, which underlies this digraph and overlays its underlying undirected graph. A locally orientable graph is called globally orientable if it underlies a directed graph. Figure 3.1 provides two examples of locally orientable graphs of which one is globally orientable, while the other is not. Here and in what follows stretched vertices are denoted by rectangles to visualize the corresponding 2-partitions; terminals are denoted by circles. Globally orientable graphs are essentially equivalent to directed graphs in the following sense.

![Fig. 3.1. Local and Global Orientability](image-url)
Lemma 3.1. Given a connected globally orientable graph, the overlying directed graph is restored uniquely, up to the global reversal of arcs.

Proof. Let $\tilde{G}_1$ and $\tilde{G}_2$ be two digraphs overlying a given locally orientable graph $G$. Let $V^+$ be the subset of vertices at which the ordered 2-partitions of the star into incoming and outgoing arcs in $\tilde{G}_1$ and $\tilde{G}_2$ coincide, and $V^-$ be the subset of vertices at which they are opposite. Evidently, $(V^+, V^-)$ is a partition of $V$. Assume that both its parts are nonempty. Since $G$ is connected, it contains an edge between vertices $v^+ \in V^+$ and $v^- \in V^-$. Assume, w.l.o.g., that the corresponding arc in $\tilde{G}_1$ leaves $v^+$ and enters $v^-$. Then the corresponding arc in $\tilde{G}_2$ must leave both of them, a contradiction. Thus either $V^- = \emptyset$ and $\tilde{G}_1 \cong \tilde{G}_2$, or $V^+ = \emptyset$ and $\tilde{G}_2$ is opposite to $\tilde{G}_1$. $\square$

The following construction provides a criterion for the global orientability of a locally orientable graph $G$ (we do not use this criterion in what follows). Let us split each vertex $v$ of $G$ and its star into the two vertices $v_1$ and $v_2$ whose stars are exactly the sides at $v$ and add the auxiliary edge $(v_1, v_2)$; the obtained graph we denote $G'$.

Proposition 3.2. A locally orientable graph $G$ is globally orientable if and only if $G'$ is bipartite.

Proof. Let $G'$ be globally orientable and $\tilde{G}'$ be its overlying digraph. Let us transform $\tilde{G}'$ into $\tilde{G}''$ similarly to the above transformation; evidently, $\tilde{G}''$ overlies $G'$. The partition ($\{v_1 : v \in V\}, \{v_2 : v \in V\}$) shows that $G'$ is bipartite.

Let now $G'$ be bipartite with the partition $(V^*, V^{**})$. Let us direct all edges from $V^*$ to $V^{**}$. The contraction of all auxiliary edges results in a digraph overlying $G$. It is consistent with the 2-partitions at vertices since for each $v \in V$ the corresponding vertices $v_1$ and $v_2$ belong to different parts, i.e., arcs only leave $v_1$ and only enter $v_2$, or vice versa. $\square$

According to Lemma 3.1, locally orientable graphs are a generalization of directed graphs. Some concepts of digraphs have their natural analogs for this generalization. A path $(v_0, e_1, v_1, e_2, \ldots, e_r, v_r)$ in a locally orientable graph is said to be coherent if for any of its inner vertices $v_i$, $1 \leq i \leq r - 1$, the edges $e_i$ and $e_{i+1}$ belong to the opposite sides of the star of $v_i$. Clearly, the inverses of any coherent path, any single edge path, and any single vertex path are coherent paths. Observe that each coherent path can be extended beyond its endpoint, provided it is not a terminal. A cyclic coherent path (for which $v_0 = v_r$ and $r > 1$) is called a coherent cycle if the edges $e_r$ and $e_1$ belong to the opposite sides at $v_0$ (see Fig. 3.2). Coherent paths and cycles are natural analogs of directed paths and cycles in digraphs. In particular, a coherent path in a globally orientable graph corresponds to a directed path or the inverse of such a path.
We call a 2-partition $\mathcal{P}$ of the vertex set of a locally orientable graph (in particular, its cut) *transversal* if any coherent path intersects the corresponding edge set $E_P$ at most by one edge. Thus, if both endpoints of a coherent path lie on the same side of a transversal cut, then all the path lies on the same side of this cut. Observe that the edge set of a transversal 2-partition intersects at most one side of the star of any vertex. Moreover, the following statement can be proved easily.

**Fact 3.3.** Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two transversal 2-partitions of the vertex set of a locally orientable graph such that $V_{\mathcal{P}_1} \cap V_{\mathcal{P}_2}$ is a single stretched vertex $v$. Then the intersections of the star of $v$ with the edge sets of $\mathcal{P}_1$ and $\mathcal{P}_2$ are exactly the two sides of this star.

A locally orientable graph is called *coherent* if each its cyclic coherent path is a coherent cycle. For a coherent locally orientable graph, all vertices and edges of coherent paths beginning at a vertex $v$ are called *reachable* from $v$. All these vertices and edges form the reachability subgraph of $v$. Observe that the reachability relation on vertices is symmetric (though not transitive; see vertices $x, y, z$ in Fig. 3.1a). Therefore, the notion of *mutual reachability* of two vertices is well defined.

**Lemma 3.4.** The reachability subgraph of any vertex of a coherent locally orientable graph is globally orientable.

*Proof.* To construct a global orientation of the reachability subgraph of $v$ we proceed as follows. Let us choose a side of the star of $v$. Moving along any coherent path starting at $v$ and leaving $v$ from this side, we direct all its edges from $v$. Similarly, moving along any coherent path starting at $v$ and leaving $v$ from the opposite side, we direct all its edges to $v$.

Let us prove the following statement: if two paths $P_1$ and $P_2$ from $v$ to an arbitrary vertex $u$ leave $v$ from the same side (from the opposite sides), then they enter $u$ at the same side (at the opposite sides). Assume to the contrary that $P_1$ and $P_2$ leave $v$ from the same side and enter $u$ at the opposite sides (see Fig. 3.3a). Then the cyclic coherent path composed of $P_1$ and the inverse of $P_2$ is not a coherent cycle. Similarly, if $P_1$ and $P_2$ leave $v$ from the opposite sides and enter $u$ at the same side, then the cyclic coherent path composed of the inverse of $P_2$ and $P_1$ is not a coherent cycle (see Fig. 3.3b).
Assume now that in our construction some edge $e$ acquires two opposite orientations via paths $P_1$ and $P_2$. Evidently, the behavior of these paths at an arbitrary endpoint of $e$ distinct from $v$ contradicts to the above statement. Thus, the orientation of edges is well defined. Finally, to prove its consistency with the initial orientability structure we have to ensure that if two edges $e_1, e_2$ incident to a vertex $u$ are oriented to and from $u$, respectively, then they are at the opposite sides of $u$. For $u = v$ it is evident. For $u \neq v$, we have to consider four cases depending on the types of the paths $P_1$ and $P_2$ that defined the orientations of the edges $e_1, e_2$, respectively. For example, let both $P_1$ and $P_2$ leave $v$ from the side opposite to the one chosen in the construction and let $e_1$ and $e_2$ be oriented as in Fig. 3.3c. Then $P_2$ enters $u$ via $e_2$ while $P_1$ enters $u$ via an edge preceding $e_1$ in $P_1$. Thus $P_2$ and $P_1$ enter $u$ at the opposite sides, a contradiction with the above statement. The three other cases are handled in the same way.

According to Lemmas 3.1, 3.4, one can construct in an obvious way a version of the Depth-First Search (DFS) that starts from an arbitrary vertex $v$ and scans all the vertices and edges reachable from $v$ in the same direction (i.e., by paths leaving $v$ from the same side).

For a coherent locally orientable graph, the relation “two vertices belong to the same coherent cycle” is symmetric and transitive; the subgraphs induced by its equivalence classes are naturally called strongly connected components. It is easy to see that strongly connected components intersecting a reachability subgraph lie entirely in this subgraph and correspond bijectively to the strongly connected components of its overlying digraph. Hence, we can execute any DFS-based linear algorithm for finding strongly connected components on any reachability subgraph.

An analog of a dag is an acyclic locally orientable graph, the one without cyclic coherent paths; thus, it is coherent. In particular, the underlying graph of a dag is acyclic in the above sense. As in the proof of Lemma 3.4, it follows from the acyclicity that for any two coherent paths joining $v_1$ and $v_2$, their initial edges leave $v_1$ from the same side. Thus, the set of vertices reachable from $v_1$ splits into two reachability cones corresponding to the sides of its star (if $v_1$ is a terminal, one of the cones is trivial). It is easy to see that if $v_2$ belongs to a reachability cone $R$ of $v_1$, then one of the reachability cones of $v_2$ contains $v_1$ while the other lies strictly inside $R$ (this property may be regarded as a weak analog of transitivity). Acyclicity implies that each coherent path can be extended beyond each of its endpoints.
up to a terminal. As a corollary, each nontrivial reachability cone of a vertex contains at least one terminal distinct from this vertex.

### 3.2. Strips

In this subsection we consider acyclic two-terminal locally orientable graphs.

**Lemma 3.5.** Any acyclic two-terminal locally orientable graph is globally orientable.

**Proof.** By Lemma 3.4, it is enough to prove that the reachability subgraph of a terminal in such a graph coincides with the entire graph. Each vertex or edge is a coherent path; therefore, it can be extended to a coherent path between two terminals. By the acyclicity, these terminals are distinct, and the result follows. □

According to this Lemma, properties of an acyclic two-terminal locally orientable graph $G$ are similar to those of its underlying dag $\overline{G}$. In particular, the following statements are true.

**Fact 3.6.**

(i) The transversal cuts of $G$ underlie the cuts of $\overline{G}$ in which all arcs are directed from the part containing the source to the part containing the sink.

(ii) A transversal 2-partition of $G$ is a (transversal) cut separating the terminals.

(iii) A reachability cone of a vertex in $G$ defines a transversal cut; it is the intersection of all transversal cuts such that both this vertex and the terminal that belongs to this cone lie inside them.

An acyclic two-terminal globally orientable graph is called a *strip* (of width $w$) if all its transversal cuts are of the same cardinality $w$. Observe that the degree of a terminal in a strip equals its width (since the 2-partition isolating this terminal is a transversal cut).

Let $s$ and $t$ be the terminals and $v$ be a vertex of a strip. We label the sides of the 2-partition and the cones at $v$ by $s$ and $t$; the cones are denoted $R_s(v)$ and $R_t(v)$. Thus, any two coherent paths $(u_1, \ldots, u_3) \in R_x(u_1)$ and $(u_2, \ldots, u_3) \in R_x(u_2)$, $x = s$ or $t$, can be concatenated into a coherent path $(u_1, \ldots, u_2, \ldots, u_3) \in R_x(u_1)$.

**Strip Lemma.**

(i) An acyclic two-terminal locally orientable graph is a strip if and only if it is balanced.

(ii) There exists at most one strip for a fixed underlying graph and a fixed pair of terminals.

**Proof.** (i) Let $G$ be a balanced acyclic locally orientable graph with two terminals $s$ and $t$. By Lemma 3.5, $G$ is globally orientable. Let $\overline{G}$ be the underlying digraph of $G$ in which $s$ is the source and $C$ be a transversal cut of $G$ such that $s \in V_C$. We consider the difference $\Delta$ between the sum of
outdegrees (in $\overline{G}$) for all vertices in $V_C$ and the sum of their indegrees. On one hand, since $G$ is balanced and $t \notin V_C$, $\Delta$ equals the outdegree of $s$. On the other hand, since each internal arc of $V_C$ contributes 1 to both sums and all external arcs are directed outside (by Fact 3.6(i)), $\Delta$ equals the cardinality of $E_C$. Thus $e(C)$ equals the outdegree of $s$ for any transversal cut $C$.

Let now $G$ be a strip and $\overline{G}$ be its overlying dag with the source $s$ and the sink $t$. Let $v$ be an arbitrary nonterminal vertex and $R$ be the reachability cone of $v$ in $\overline{G}$. Both 2-partitions $(R, V \setminus R)$ and $(R \setminus v, (V \setminus R) \cup v)$ are transversal since any directed path can only enter $R$ (or $R \setminus v$) but not leave it. By Fact 3.6(ii), they are cuts. Their cardinalities differ by the difference between the indegree and the outdegree of $v$. Since both cardinalities are equal, the indegree and outdegree at $v$ are equal as well.

(ii) The statement is evident for graphs with two vertices; hence, we assume that there exists a vertex $v \neq s, t$. Let us consider a strip $G$ with a source $s$ and the sink $t$. It is easy to see that the deletion of $s$ from $G$ yields new terminals. Indeed, the one-vertex coherent path $v$ can be extended up to terminals in both directions; these terminals are distinct since the graph is acyclic. As it was shown in the proof of Lemma 3.4, all the deleted edges incident to an arbitrary vertex are at the same side. Hence, for any new terminal exactly a half of incident edges is deleted, and for any other vertex strictly less than a half (a characterization in terms of the underlying graph). Let us prove that the contraction of the new terminals implies a strip. The obtained graph is evidently balanced and two-terminal. Assume that there exists a cyclic coherent path in it. Evidently, it starts and ends at the new terminal and thus can be extended to a cyclic coherent path starting and ending at $s$ in the initial strip, a contradiction. Finally, the initial strip can be restored uniquely from the strip obtained.

Assume to the contrary that assertion (ii) is not valid. Then there exists a minimal counterexample: a graph with the minimal number of vertices giving rise to two distinct strips. Let us execute with these strips the above procedure. According to the characterization mentioned, the resulting undirected graphs coincide. Thus, by the minimality of the counterexample, the resulting strips coincide as well. By the remark at the end of the previous paragraph, the same holds for the initial strips, a contradiction. \( \square \)

The concept of a strip allows us to reformulate and refine the main result of [PQ] on the representation of all minimum cuts between two vertices of a graph. Let us consider an undirected graph $G$ and an arbitrary pair of its vertices $a, b$. Assume $f$ be a maximal flow from $a$ to $b$ in $G$, and $G_f$ be the corresponding residual graph. The result of the contraction of each strongly connected component of $G_f$ to a new supervertex is a dag $D_{a,b}$ not depending on $f$. The corresponding quotient mapping is denoted by $\delta_{a,b}$, and the underlying locally orientable graph of this dag by $\mathcal{W}_{a,b}$ (see Fig. 3.4).
**Theorem 3.7.** Let \( G \) be an undirected graph and \( a, b \) be a pair of its vertices. Then the graph \( W_{a,b} \) overlies \( F_{\{a,b\}} \), is a strip, and its transversal cuts correspond bijectively to the minimum \((a,b)\)-cuts of \( G \) via the \( \delta_{a,b} \)-inducing.

**Proof.** By [PQ] the \( \delta_{a,b} \)-inducing provides a bijection between the minimum \((a,b)\)-cuts of \( G \) and the 2-partitions of type \{a closed set of \( D_{a,b} \), its complement\}. Recall that a vertex subset \( X \) in an arbitrary dag is said to be closed if no other vertices can be reached from this subset, or, equivalently, if no arc goes from \( X \) to \( \bar{X} \). According to Fact 3.6(i), the 2-partitions of the above type overlie exactly the transversal cuts of \( W_{a,b} \). Since all minimum \((a,b)\)-cuts of \( G \) have the same cardinality, so do all the transversal cuts of \( W_{a,b} \), and hence \( W_{a,b} \) is a strip. It follows from the above discussion that the partition of \( V \) into the vertices of \( W_{a,b} \) is a refinement of the partition into the vertices of \( F_{\{a,b\}} \). To prove that they, in fact, coincide, it is sufficient to show that each vertex \( U \) of \( W_{a,b} \) is separated from all other vertices by transversal cuts. Indeed, by Fact 3.6(iii), these cuts are just the two cuts defined by reachability cones of \( U \).

**Remark.** Observe that natural analogs of the above construction and of Theorem 3.7 are valid for directed graphs.

The strip \( W_{a,b} \) will be referred to as the \((a,b)\)-strip, and its vertices as \((a,b)\)-units, or units of \( W_{a,b} \). Obviously, \( W_{b,a} \) coincides with \( W_{a,b} \) and \( \delta_{b,a} = \delta_{a,b} \). The above constructions and results can be extended easily to the case of minimum cuts separating two disjoint vertex subsets \( A \) and \( B \): it is enough to contract \( A \) and \( B \) into new supervertices. The corresponding \((A,B)\)-strip is denoted by \( W_{A,B} \) and the quotient mapping by \( \delta_{A,B} \).

**Lemma 3.8.** Let \( a \) be a vertex of \( G \) and \( U \neq \delta_{a,b}(b) \) be the corresponding \((a,b)\)-unit. The contraction of the reachability cone of \( U \) in \( W_{a,b} \) containing the terminal \( \delta_{a,b}(a) \) gives the \((\{a,u\},b)\)-strip; its width is equal to the width of the initial strip \( W_{a,b} \).

**Proof (see Fig. 3.5).** It is easy to see that the contraction of a side of a transversal cut in an acyclic locally orientable graph preserves acyclicity. Therefore, by Fact 3.6(iii) and Strip Lemma, the above contraction yields a strip \( W \). Since the terminal \( \delta_{a,b}(b) \) remains untouched, the width of \( W \) equals that of \( W_{a,b} \).
By Strip Lemma(ii), it is enough to prove now that the underlying graph of \( W \) coincides with that of \( W_{\{a,u\},b} \). Since all \((\{a,u\},b)\)-cuts are \((a,b)\)-cuts, we can study them (more exactly, their images) in \( W_{a,b} \). First, by Fact 3.6(iii), the reachability cone \( R \) in question coincides with the \((\{a,u\},b)\)-unit containing \{a,b\}. Second, let \( U' \) and \( U'' \) be two \((a,b)\)-units not belonging to \( R \) and let \( C \) be an \((a,b)\)-cut separating them. Evidently, the union of \( C \) and the cut defined by \( R \) is an \((\{a,u\},b)\)-cut \( C_u \) separating \( U' \) and \( U'' \). □

Observe that for an arbitrary vertex subset \( S \), the set of all minimum \( S \)-cuts is the union of the sets of minimum \((a,b)\)-cuts for certain pairs of vertices \( a,b \in S \). It is natural to ask whether it is possible to glue the corresponding \((a,b)\)-strips together to form a single object. The main finding of this paper is that such an object exists: it is \( F_S \) with a canonical structure of local orientability defined by these \((a,b)\)-strips.

4. Simple cases

To make our ideas more intuitive, let us consider separately several simple cases.

4.1. A two-element subset \( S \). From the statical point of view, the case \( \sigma = 2 \) is covered completely by Theorem 3.7; recall that for \( S = \{a,b\} \) one has \( F_S = W_{a,b} \).

Let us turn to dynamics. In what follows, we put hat over any notation (e.g., \( \hat{R}_c \), or \( \hat{F}_S \)) to denote the corresponding object after edge insertion. Recall that the units of \( F_S \) are defined as the strongly connected components of the graph \( G_f \), where \( f \) is a maximum flow from \( a \) to \( b \) in \( G \). Since we restrict ourselves to the case of a fixed value of \((a,b)\)-mincuts, we may use the same flow \( f \) for the entire dynamic process. Hence, we can maintain the graph \( G_f \) fixing local changes. Namely, the insertion of a new edge \((u_1,u_2)\) implies the addition of two arcs \((u_1,u_2)\) and \((u_2,u_1)\) to \( G_f \). Moreover, one can maintain the dag \( D_{a,b} \) of the strongly connected components of \( G_f \) as follows. Let \( U_1, U_2 \) be the components containing \( u_1, u_2 \), respectively. The new dag \( \hat{D}_{a,b} \) is obtained from the current one by adding arcs \((U_1,U_2)\) and \((U_2,U_1)\) and shrinking the new strongly connected component containing both \( U_1 \) and \( U_2 \). This transformation can be easily extended to the underlying strip \( W_{a,b} \) as follows.
If \( U_1 = U_2 \), then there are no changes in \( W_{a,b} \). Otherwise, \( U_1 \) and \( U_2 \) are contracted to a new unit \( U^{\text{new}} \).

If \( U_1 \) and \( U_2 \) are not mutually reachable in \( W_{a,b} \) (see Fig. 4.1a). Otherwise, assume w.l.o.g. that \( U_2 \in \mathcal{R}_a(U_1) \) and \( U_1 \in \mathcal{R}_a(U_2) \); then \( \mathcal{R}_a(U_1) \cap \mathcal{R}_a(U_2) \) (that is, the set of all the units and edges of all the paths between \( U_1 \) and \( U_2 \)) is contracted into \( U^{\text{new}} \) as well (see Fig. 4.1b).

![Diagram](image.png)

**Fig. 4.1. Dynamics of \( \mathcal{F}_{\{a,b\}} \)**

If both \( U_1 \) and \( U_2 \) are terminals, then the system of \( S \)-mincuts vanishes. If exactly one of them is a terminal, then \( U^{\text{new}} \) is the corresponding terminal in the new strip. If both \( U_1 \) and \( U_2 \) are stretched, then \( U^{\text{new}} \) is stretched as well, and the side of the 2-partition at \( U^{\text{new}} \) labeled by \( a \) (resp., \( b \)) is glued from the sides of the 2-partitions at all the contracted units labeled by the same letter (except for the internal edges of the contracted subgraph).

Observe that the dag of the strongly connected components is exactly the object maintained by algorithms [I, I], with the time complexity \( O(mn) \). Evidently, any such algorithm can be easily modified to maintain a strip, with the same complexity.

The above description covers also a more general situation, when \( \sigma \) is arbitrary, but all the \( S \)-mincuts divide \( S \) in the same way into two subsets \( S_1 \) and \( S_2 \). We then consider these subsets as supervertices \( a \) and \( b \) and arrive at the above situation described completely by the strip \( W_{S_1,S_2} \). Observe that in this case \( \mathcal{F}_S = W_{S_1,S_2} = W_{s_1,s_2} = \mathcal{F}_{\{s_1,s_2\}} \) for any \( s_1 \in S_1, s_2 \in S_2 \).

### 4.2. A three-element subset \( S \); the asymmetric subcase.

The smallest nontrivial case, from the statical point of view, is \( S = \{a, b, c\} \) and any two of these vertices are separated by an \( S \)-mincut. Clearly, from the three possible types of \( S \)-mincuts: \( (a, \{b, c\}), (b, \{a, c\}), \) and \( (c, \{a, b\}) \), at least two must exist. So we have two subcases: the asymmetric, when exactly two types of \( S \)-mincuts are present, and the symmetric, when there are present all the three of them.

Let us assume that there are no \( (b, \{a, c\}) \)-cuts. Thus, every \( S \)-mincut is an \( (a, c) \)-cut, and vice versa. Therefore, the entire family of \( S \)-mincuts is represented by the strip \( W_{a,c} \); in particular, \( (a, c) \)-units coincide with \( S \)-units. We define the structure of local orientability on the graph \( \mathcal{F}_{\{a,b,c\}} \) by keeping that of \( W_{a,c} \) at all the units, except for the unit \( U_b \) containing \( b \), which is now regarded as a terminal. Since \( W_{a,c} \) is acyclic and balanced,
and turning a vertex to a terminal does not extend the set of coherent paths, we see that $F_{\{a,b,c\}}$ is acyclic and balanced as well. Besides, we see that all $S$-cuts of $G$ are transversal cuts in $F_{\{a,b,c\}}$; observe that the converse is not true, see, e.g., the cut that separates $U_b$ from all the other units.

We denote by $(L_b, \overline{L}_b)$ the $\{a,b\}$-tight $(\{a,b\}, c)$-mincut. Similarly, $(R_b, \overline{R}_b)$ is the $\{b,c\}$-tight $(\{b,c\}, a)$-mincut (see Fig. 4.2). Observe that according to Lemma 2.1 (i), there are no edges between $L_b \cap R_b$ and $\overline{L}_b \cap \overline{R}_b$.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig4_2}
\caption{Structure of $F_{\{a,b,c\}}$ in the asymmetric case}
\end{figure}

In our next statement we summarize other properties of the cuts $(L_b, \overline{L}_b)$ and $(R_b, \overline{R}_b)$.

**Lemma 4.1.**

(i) $L_b$ and $R_b$ are the two reachability cones of the unit $U_b$ in $W_{a,c}$, and hence $L_b \cap R_b$ coincides with this unit.

(ii) The sides of the star of $U_b$ in $W_{a,c}$ are induced in $F_{\{a,b,c\}}$ by edge-sets of $(L_b, \overline{L}_b)$ and $(R_b, \overline{R}_b)$.

(iii) The contraction of $L_b$ in $F_{\{a,b,c\}}$ gives $W_{a,b}$, the contraction of $R_b$ in $F_{\{a,b,c\}}$ gives $W_{a,c}$.

**Proof.** (i) Follows from Fact 3.6(iii).

(ii) Follows from (i) and Fact 3.3.

(iii) Follows from Lemma 3.8. 

Evidently, either $U_b$ is a separating vertex of $F_{\{a,b,c\}}$, or $F_{\{a,b,c\}} \setminus U_b$ is connected. In the latter case we build from $F_{\{a,b,c\}}$ a locally orientable graph $W^{ac}$ by deleting all the vertices of $U_b$ and contracting $L_b \cap \overline{R}_b$ to the terminal $a'$ and $\overline{L}_b \cap R_b$ to the terminal $c'$ (see Fig. 4.2). (Informally, $W^{ac}$ is the common part of $W_{a,b}$ and $W_{b,c}$ in $F_{\{a,b,c\}}$.) An $S$-mincut $C$ of $G$ that separates $L_b \cap \overline{R}_b$ from $\overline{L}_b \cap R_b$ and thus generates an $(a', c')$-mincut $C'$ in $W^{ac}$ is called an extension of $C'$.

**Lemma 4.2.**

(i) The graph $W^{ac}$ is a strip of width $\lambda' = \lambda_S \leq \frac{1}{2} \deg U_b < \frac{1}{2} \lambda_S$.

(ii) All its transversal cuts, and only they, are extendable to $S$-mincuts.
(iii) If a 2-partition of $L_b \cup R_b$ enters an extension of some transversal cut of $W^{ac}$, then it enters some extension of any such cut. (Informally: all transversal cuts of $W^{ac}$ are interchangeable in S-mincuts).

Proof. (i) Observe that deletion of vertices does not give rise to new coherent paths, thus, it preserves the acyclicity of a locally orientable graph. For the same reason, any transversal 2-partition induces a transversal 2-partition in the new graph. Since the contraction of a side of a transversal 2-partition preserves acyclicity, $W^{ac}$ is an acyclic two-terminal balanced locally orientable graph. Hence, it is a strip by Strip Lemma (i). It is easy to see, applying Lemma 2.1, that $\lambda_S = c(L_b, \bar{L}_b) = \deg a' + \text{deg } U_b$. Since $\lambda' = \deg a'$, we get $\lambda' = \lambda_S - \frac{1}{2} \deg U_b$. Assume that $\lambda' \geq \frac{1}{2} \lambda_S$. Then $\deg U_b \leq \lambda_S$, which means that $(L_b \cap R_b, L_b \cup \bar{R}_b)$ is a 2-partition of cardinality at most $\lambda_S$ separating $b$ from $\{a, c\}$, a contradiction.

(ii) Let us denote by $W'$ the graph obtained from $F_{\{a,b,c\}}$ by deleting $U_b$. By (i), any transversal cut $(Z, \bar{Z})$ of $W^{ac}$ has cardinality $\lambda'$ and generates a 2-partition $(X, \bar{X})$ of the same cardinality in $W'$. By Lemma 2.1, the edge set of the 2-partition $(X \cup U_b, \bar{X})$ in $F_{\{a,b,c\}}$ has exactly $\frac{1}{2} \deg U_b$ more edges, i.e., $\lambda' + \frac{1}{2} \deg U_b = \lambda_S$ edges. Thus, it is a cut, that is a sought for extension of $(Z, \bar{Z})$.

Conversely, let $(X, \bar{X})$ be an S-mincut separating $L_b \cap \bar{R}_b$ from $\bar{L}_b \cap R_b$. It generates a 2-partition of cardinality $\lambda'$ in $W'$ and a 2-partition of the same cardinality separating $a'$ and $c'$ in $W^{ac}$. Thus, it is a $\lambda'$-cut separating $a'$ and $c'$, i.e., a transversal cut.

(iii) Since there are exactly two such partitions, namely, $(L_b \cap \bar{R}_b) \cup U_b, L_b \cap R_b$ and $(L_b \cap \bar{R}_b, (\bar{L}_b \cap R_b) \cup U_b)$, one can proceed exactly as in the proof of (ii). □

Remark. Observe that Lemma 4.2 remains true for a more general definition of an extension, in which $C$ is required only to separate the ends of all edges between $L_b \cap R_b$ and $\bar{L}_b \cap \bar{R}_b$ that lie in $L_b \cap R_b$ and ends of all edges between $L_b \cap R_b$ and $L_b \cap \bar{R}_b$ that lie in $L_b \cap R_b$. Such a definition would allow, for example, to consider the cuts $C_3$ and $C_4$ on Fig. 4.2b as extensions of $C'$, while according to the current definition, the only extensions of $C'$ are $C_1$ and $C_2$. However, in what follows we do not consider such a generalization of the notion of extension.

Dynamics of $F_{\{a,b,c\}}$ in the asymmetric subcase are induced by those of $F_{\{a,c\}} = W_{ac}$ (see Sect. 4.1) straightforwardly, except for the case when $U_b$, and possibly some other units, is contracted to $U^{new}$. Let us consider this case; it occurs when $U_b \in R_a(U_1) \cap R_a(U_2)$, up to flipping of $U_1$ and $U_2$ (see Fig. 4.4). The only specifics in this case is that $U^{new}$ becomes a new terminal (in fact, $\hat{U}_b$).
a)\[\begin{array}{c}
\bullet_a & U_1 & \bullet_b \\
\text{new} & & \\
U_2 & & U_\ast \\
a & b & c
\end{array}\]
b)\[\begin{array}{c}
\bullet_a = U_1 & \bullet_b \\
U_{\text{new}} & & U_2 \\
& \text{a}\ & c
\end{array}\]

**Fig. 4.3. Dynamics of** $\mathcal{F}_{\{a,b,c\}}$ **in the asymmetric case**

*Remark.* To translate the definition of the contracted set into the terms of $\mathcal{F}_{\{a,b,c\}}$, one has to take into account that a coherent path in $\mathcal{W}_{a,c}$ is either a coherent path in $\mathcal{F}_{\{a,b,c\}}$, or a concatenation of two coherent paths: a path to $U_b$ in $L_b$ and a path from $U_b$ in $R_b$. Under this translation, the above condition on $U_b$ is equivalent to $U_1 \in \mathcal{R}_a(U_b) = L_b$ and $U_2 \in \mathcal{R}_c(U_b) = R_b$.

There is a special situation when not only $U_b$, but $U_a$ as well is contracted to $U_{\text{new}}$; this occurs when $U_1 = U_a$ (see Fig. 4.3b). In this situation the asymmetric three-element case degenerates, and we arrive at the case considered at the end of Sect. 4.1; that is, $\mathcal{F}_{\{a,b,c\}}$ becomes equal to $\mathcal{W}_{\{a,b,c\}} = \mathcal{F}_{b,c} = \mathcal{F}_{a,c}$. Evidently, $U_a$ above can be replaced by $U_c$.

Finally, if $\{U_1, U_2\} = \{U_a, U_b\}$, then the system of $S$-mincuts vanishes.

As we see, the reduction of the three-element case to the two-element case is defined via subsets $L_b$ and $R_b$. Let us consider the dynamics of $L_b$ (those of $R_b$ are similar) in the case when $\mathcal{F}_{\{a,b,c\}}$ does not degenerate.

**Lemma 4.3.** The set $L_b$ is extended if and only if $U_1 \in L_b$ and $U_2 \notin L_b$ (up to flipping of $U_1$ and $U_2$). The new part of $\hat{L}_b$ is the union of units $U^i$ such that $U^i \in \mathcal{R}_a(U_2)$ and $U^i \notin L_b$.

**Proof.** If part of the first statement is trivial, since $U_2$ is added to $L_b$.

Let $U^i$ be a new unit in $\hat{L}_b$ and let $(U^i, \ldots, U_{\text{new}}, \ldots, U_b)$ be a coherent path in $\mathcal{W}_{a,c}$; clearly, it lies entirely in $\hat{\mathcal{R}}_c(U^i)$. Its edges form two coherent paths in $\mathcal{F}_{\{a,b,c\}}$: $(U^i, \ldots, U^*) \in \mathcal{R}_c(U^i)$ and $(U^{**}, \ldots, U_b) \in \mathcal{R}_c(U^{**})$ (see Fig. 4.4). Since $U^i$ is new in $\hat{L}_b$, we have $U^* \neq U^{**}$, and thus the statement holds for the case when $U_{\text{new}}$ contains only $U_1$ and $U_2$. Otherwise we may assume w.l.o.g. that $U^* U^{**} \subset U_{\text{new}} = \mathcal{R}_c(U_1) \cap \mathcal{R}_a(U_2)$. Hence in $\mathcal{W}_{a,c}$ exist coherent paths $(U^*, \ldots, U_2) \in \mathcal{R}_c(U^*)$ and $(U_1, \ldots, U^{**}) \in \mathcal{R}_c(U_1)$. The first of them does not contain $U_b$ by assumption, and thus $(U^i, \ldots, U^*, \ldots, U_2)$ is a coherent path in $\mathcal{F}_{\{a,b,c\}}$. Therefore, $U_2 \in L_b$ would imply $U^i \in L_b$ in a contradiction to the assumption that $U^i$ is new in $\hat{L}_b$. The path $(U_1, \ldots, U^{**}, \ldots, U_b)$ does not contain $U_b$ twice (as a coherent path in $\mathcal{W}_{a,c}$), and thus is a coherent path in $\mathcal{F}_{\{a,b,c\}}$; hence, $U_1 \in L_b$. The assertion follows. □
4.3. A three-element subset $S$: the symmetric subcase. Second, let all types of $\{a, b, c\}$-cuts exist in $G$. Let $C_x = (V_x, \bar{V}_x)$ be the $\{y, z\}$-tight $\{(y, z), x\}$-mincut (Fig. 4.5a); here and in what follows $x, y, z \in \{a, b, c\}$, $x \neq y \neq z$. As usual, these three 2-partitions define the following eight subsets of $V$: $V^\varnothing = V_a \cap V_b \cap V_c$; $V^a = \bar{V}_a \cap V_b \cap V_c$; $V^b = V_a \cap \bar{V}_b \cap V_c$; $V^c = V_a \cap V_b \cap \bar{V}_c$; $V^{ab} = V_a \cap \bar{V}_b \cap \bar{V}_c$; $V^{bc} = V_a \cap V_b \cap \bar{V}_c$; $V^{ac} = V_a \cap \bar{V}_b \cap V_c$; $V^{abc} = V_a \cap \bar{V}_b \cap \bar{V}_c$. These subsets are called cells.

![Fig. 4.5. Structure of $\mathcal{F}_{\{a,b,c\}}$ in the symmetric case](image)

The following statement is crucial for the rest of our results (see Fig. 4.5b).

**3-Star Lemma.**

(i) $V^{abc} = \emptyset$.

(ii) Vertices of $V^{xy}$ can be adjacent only to vertices of $V^{xy}$, $V^x$, $V^y$, and $c(V^{xy}, V^x) = c(V^{xy}, V^y)$.

**Proof.** Let us consider the cut $C_x^{xy} = C_z \cap \bar{C}_y = (V^x \cup V^{xy}, V \setminus (V^x \cup V^{xy}))$. Since both $C_x$, $C_z$ are minimum $(x, z)$-cuts, the cut $C_x^{xy}$ is a minimum $(x, z)$-cut as well, and hence $c(C_x^{xy}) = \lambda_S$. Similarly, $C_y^{xy} = C_z \cap \bar{C}_y = (V^y \cup V^{xy}, V \setminus (V^y \cup V^{xy}))$ is a minimum $(y, z)$-cut with $c(C_y^{xy}) = \lambda_S$. Therefore, both $C_x^{xy}$ and $C_y^{xy}$ are minimum $(x, y)$-cuts. According to Lemma 2.1(i), there are no edges between $V^{xy}$ and $V \setminus (V^x \cup V^{xy} \cup V^y)$, and moreover, $c(V^{xy}, V^x) = c(V^{xy}, V^y)$, so assertion (ii) is proved.

We know already that there are no edges between $V^{abc}$ and $V^{xy}$. By Lemma 2.1(i) applied to the minimum $(x, y)$-cuts $C_x$ and $C_y$, we get that there are no edges between $V^{xy} \cup V^{abc}$ and $V^z \cup V^\varnothing$. So, there are no external edges incident to $V^{abc}$ at all. Since the initial graph is connected, we thus get $V^{abc} = \emptyset$. □
Properties of the cuts $C_x$ are similar to those of the cuts $(L_b, L_b)$ and $(R_b, R_b)$ (see Sect. 4.1). In particular, the first statement below can be regarded as an analog of Lemma 4.1(iii).

**Lemma 4.4.**

(i) Contraction of the inner side $V_x$ of $C_x$ in $F_{\{a,b,c\}}$ gives the underlying undirected graph of the strip $W_{\{y,z\},x}$.

(ii) Every unit, except for $V^\Phi$, corresponds in this way to a vertex in either one or two strips.

(iii) For every unit $U \neq V^\Phi$, the 2-partitions of its star induced by the 2-partitions at the corresponding vertices in these strips coincide.

**Proof.** (i) Since all $(\{y,z\}, x)$-mincuts are $S$-mincuts and $C_x$ is the $(\{y,z\}, x)$-tight $(\{y,z\}, x)$-mincut, the result of the contraction still represents all $(\{y,z\}, x)$-mincuts. Therefore, the only thing we have to prove is that no $(\{y,z\}, x)$-unit distinct from the contracted one can be divided by an $S$-mincut. Indeed, let $U$, to the contrary, be such a unit and $C$ be an $S$-mincut dividing $U$. Without loss of generality we can assume that $C$ is an $(z, \{x, y\})$-cut containing $z$ inside; recall that $z$ lies also inside $C_x$ (see Fig. 4.6). Then $C_x \cup C$ is a $(\{y, z\}, x)$-mincut dividing $U$, a contradiction.

![Diagram](image)

**Fig. 4.6. To the proof of Lemma 4.4(i)**

(ii) Follows from (i) immediately.

(iii) Assume without loss of generality that $U$ lies in $V^{xy}$. According to (i) and (ii), $U$ is a unit in the strips $W_{\{y,z\},x}$ and $W_{\{x,z\},y}$. By Lemma 3.8, both these strips are contractions of the same strip $W_{x,y}$. Moreover, in both cases $U$ is not a contracted unit, so local orientations at $U$ in both cases are inherited from $W_{x,y}$, and thus coincide. □

Let us assign to every unit, except for $V^\Phi$, the 2-partition of its star defined in Lemma 4.4(iii), and to $V^\Phi$ the trivial 2-partition. Then $F_{\{a,b,c\}}$ becomes a locally orientable graph. Evidently, the contraction of $F_{\{a,b,c\}}$ described in Lemma 4.4(i) yields now the strip $W_{\{y,z\},x}$ itself. One can say, thus, that all the strips $W_{\{y,z\},x}$ “are glued” into the aggregate structure $F_{\{a,b,c\}}$ because they “coincide on intersections”.

**Lemma 4.5.**

(i) The locally orientable graph $F_{\{a,b,c\}}$ is balanced and acyclic.

(ii) Each $S$-mincut is transversal in $F_{\{a,b,c\}}$. 


Proof. (i) Evidently, $\mathcal{F}_{\{a,b,c\}}$ is balanced, since the 2-partition at each non-terminal unit is inherited from a strip. Let us prove its acyclicity. Indeed, let there exist a cyclic coherent path $P$. By the definition, any two units lying on $P$ are separated by an $S$-mincut. However, this cut fails to be transversal, since its edge-set has at least two edges in common with $P$. Thus, (i) follows from (ii).

(ii) First, let us observe that a coherent path in $\mathcal{F}_{\{a,b,c\}}$ that enters $V_x$ cannot leave it. Indeed, the 2-partition $(V_x, \bar{V}_x)$ can be represented as $C^{x,y} \cap C^{x,z}$ (see the proof of 3-Star Lemma), and is thus a minimum $(x, \{y, z\})$-cut. Therefore, a coherent path in $\mathcal{F}_{\{a,b,c\}}$ that enters and leaves $V_x$ yields in $W_x,\{y,z\}$ a coherent path that intersects at least twice the edge-set of the transversal cut $(V_x, \bar{V}_x)$, a contradiction.

Let now $C$ be an $S$-mincut contradicting the claim, and let $P$ be a coherent path intersecting $E_C$ at least twice. Assume w.l.o.g. that $C$ is an $(x, \{y, z\})$-cut, then it is represented as a transversal cut in $W_{x,\{y,z\}}$. If the path $P$ lies entirely in $V_x \cup V_x \cup V_{xz}$, then it is represented as a coherent path in $W_{x,\{y,z\}}$, a contradiction. Thus, by 3-Star Lemma(ii), $P$ must enter either $V_x$, or $V_y$, or $V_z$. However, $V_x$ is a terminal, and a path entering $V_y$ or $V_z$ cannot leave them. Hence, in this case the contraction takes $P$ to a coherent path in $W_{x,\{y,z\}}$, and we are done. □

Therefore, an arbitrary undirected graph with three distinguished vertices in the symmetric case, as well as in the asymmetric one, defines canonically an acyclic balanced locally orientable graph $\mathcal{F}_{\{a,b,c\}}$. Recall that a strip can be considered as a general graph model of a two-ended object of a constant width. In a sense, the locally orientable graph $\mathcal{F}_{\{a,b,c\}}$ with the structure given by Fig. 4.5b represents a general three-ended object of a constant width.

The following statement stems from the proof of Lemma 4.5.

**Fact 4.6.** Any intercell coherent path in $\mathcal{F}_{\{a,b,c\}}$ belongs to one of the following four types: $V_x - V_y$, $V_x - V_y - V_z$, $V_x - V_y$, $V_x - V_z$.

According to Lemma 4.1(i) and (ii), in the asymmetric subcase one can reconstruct the strip $W_{b,c}$ starting from $\mathcal{F}_{\{a,b,c\}}$ and using the cuts $(L_b, L_b)$ and $(R_b, R_b)$. In a similar way, in the symmetric subcase one can obtain each of the three strips $W_{x,y}$ by a contraction of $\mathcal{F}_{\{a,b,c\}}$ defined in terms of the cuts $C_x$.

**Lemma 4.7.** Let us contract $V_y \cap V_z$ in $\mathcal{F}_{\{a,b,c\}}$ and assign to the contracted unit the 2-partition of its star into the edges crossing the cuts $C_y$ and $C_z$, respectively. The result is the strip $W_{y,z}$.

Proof (to visualize the statement and the proof observe that $V_y \cap V_z = V^y \cup V^z$, see Fig. 4.5). Let us denote by $W$ the locally orientable graph defined in the statement of the lemma (it is well defined due to 3-Star Lemma(ii)). We consider the two following ways of constructing the strip
$\mathcal{W}_{(x,z)}$. First, we take the strip $\mathcal{W}_{y,z}$ and, according to Lemma 3.8, contract the cone of $x$ (more exactly, of the unit $U_x$ containing $x$) that contains $z$. Second, we take $\mathcal{F}_{(a,b,c)}$ and contract the inner part $V_y$ of the cut $C_y$ (see Lemma 4.4(i)). Since both ways yield the same result, we conclude that all units $U \in V^y \cup V^{xz} \cup V^{yz}$ in $\mathcal{W}_{y,z}$ and $\mathcal{F}_{(a,b,c)}$ coincide, as well as the 2-partitions at these units. Proceeding in the same way with the strip $\mathcal{W}_{(x,y)}$, we get the coincidence for all units, except for $U_x$ in $\mathcal{W}_{y,z}$ and $V_y \cap V_z$ that is a single unit in $W$; thus, $U_x = V_y \cap V_z$. Observe that, on one hand, the two sides of $U_x$ in $W$ are just the intersections of its star with the edge-sets of the cuts $C_y$ and $C_z$. On the other hand, by Fact 3.6(iii), these two cuts correspond to the cones of $U_x$ in $\mathcal{W}_{y,z}$ via our contraction mapping. Thus, the 2-partitions at $U_x$ in the two graphs coincide as well, and the proof is completed. □

Similarly to the asymmetric case, one can build a locally orientable graph $\mathcal{W}^{xy}$, $x, y \in \{a, b, c\}$, $x \neq y$, by deleting from $\mathcal{F}_{(a,b,c)}$ all the vertices of $V^x \cup V^z \cup V^{xz} \cup V^{yz}$, $z \neq x, y$, and contracting $V^x$ and $V^y$ to terminals. (Informally, $\mathcal{W}^{ab}$, $\mathcal{W}^{bc}$, and $\mathcal{W}^{ac}$ are the common parts of all three strips $\mathcal{W}_{a,b}$, $\mathcal{W}_{b,c}$, and $\mathcal{W}_{a,c}$). An S-mincut $C$ of $G$ that separates $V^x$ from $V^y$, and thus generates a cut $C'$ in $\mathcal{W}^{xy}$ separating its terminals, is called an extension of $C'$. The following statement is an analog of Lemma 4.2.

**Lemma 4.8.**

(i) The graphs $\mathcal{W}^{xy}$ are strips of width $\lambda^{xy} \leq \frac{1}{2} \lambda_S$.

(ii) All their transversal cuts, and only they, are extendable to S-mincuts.

(iii) Each pair of transversal cuts of $\mathcal{W}^{xy}$ and $\mathcal{W}^{xz}$, $z \neq x, y$, has a mutual extension to a minimum $(x, \{y, z\})$-cut.

(iv) If a 2-partition of $V \setminus V^{xy}$ enters an extension of some transversal cut of $\mathcal{W}^{xy}$, then it enters some extension of any such cut. (Informally: all transversal cuts of this strip are interchangeable in S-mincuts).

**Proof.** (i) Observe that, according to Lemma 4.7, one can construct $\mathcal{W}^{xy}$ in a similar way starting from $\mathcal{W}_{(a,y)}$. Therefore, the proof of Lemma 4.2(1) applies with minor changes: the assumption $\lambda^{xy} > \frac{1}{2} \lambda_S$ implies the existence of a 2-partition of cardinality less than $\lambda_S$ separating $z$ from $\{x, y\}$.

(ii) The “all” part of the assertion is proved exactly as in Lemma 4.2. To prove the “only” part we consider an arbitrary cut $C$ separating $V^x$ from $V^y$. Assume, w.l.o.g., that $C$ is an $(x, \{y, z\})$-cut containing $x$ inside. Replacing $C$ by $C \cup C_y$ we arrive to the case considered in the proof of Lemma 4.2(ii).

(iii) It follows easily from the proof of Lemma 4.2(ii) that any transversal cut $C'$ of $\mathcal{W}^{xy}$ can be extended to an $(x, \{y, z\})$-cut $C_1$ of $\mathcal{F}_{(a,b,c)}$. Then $\tilde{C}_1 = C_1 \cap C_z$ is a minimum $(x, z)$-cut, as the intersection of minimum $(x, z)$-cuts. Observe that $\tilde{C}_1$ is an $(x, \{y, z\})$-cut. Similarly, starting from a transversal cut $C'_1$ of $\mathcal{W}^{xz}$ we find an $(x, \{y, z\})$-cut $\tilde{C}_2$. Their union is evidently a mutual extension of $\tilde{C}_1$ and $\tilde{C}_2$. Observe that $\tilde{C}_1 \cup \tilde{C}_2$ is the unique mutual extension
of $C^1_L$ and $C^1_R$, since by definition, each such extension contains $V^x$ inside and $V^\$ \cup V^y \cup V^z$ outside.

(iv) Let $C$ be an extension of some transversal cut of $W^{xy}$ and $C^1_L$ be an arbitrary transversal cut of $W^{xy}$. Assume w.l.o.g. that both $x$ and $V^\$ lie inside $C$. Recall that $C^{y,x,y}$ defined in the proof of 3-Star Lemma is a minimum $(x, y)$-cut. Therefore, $\tilde{C} = C \cap C^{y,x,y}$ is also a minimum $(x, y)$-cut. On the other hand, similarly to the proof of Lemma 4.2(ii), there exists an extension $C_1$ of $C^1_L$ such that $V^x \cup V^z$ lies inside $C_1$ and $V^y \cup V^z \cup V^\$ outside it. The cut $\tilde{C} \cap C_1$ is thus an extension of $C_1$ (since $\tilde{C}$ is an extension of the cut separating the unit $U_x$ in $W^{xy}$), and it partitions $V \setminus V^{xy}$ in the same way as $C$. □

Let us analyze the incremental dynamics of $F_S$ in the symmetric subcase. Since $F_S$ is glued from the strips $W_{x,\{y,z\}}$ (or their extensions $W_{x,y}$), it suffices to study the dynamics of these strips. According to Sect. 4.1, insertion of an edge $(u_1, u_2)$ modifies each of the strips in such a way that all the units, as well as the canonical 2-partitions of their stars, remain the same, except for the units contracted into the new one (containing both $u_1$ and $u_2$). Hence, by Lemma 4.4, the same holds for the locally orientable graph $F_S$. According to the same lemma, if for some strip $\hat{W}_{x,\{y,z\}}$ the single new unit is not the terminal containing $y$ and $z$, then it coincides with the new unit $U^\text{new}$ of $F_S$. The same is true if for some strip $\hat{W}_{x,y}$ the new unit is not the unit containing $z$.

In the description of incremental dynamics of $F_S$ we make use of the partition into the cells $V^x$, $V^{xy}$, $V^\$. Therefore, to be self-contained, we have to describe the changes in the cells; note that it suffices for this to clear up the dynamics of the sets $V_x$.

Below we consider all the nine distinct possibilities to insert a new edge $(u_1, u_2)$, $u_1 \in U_1$, $u_2 \in U_2$, w.r.t. the cell partition.

Case 1. If $U_1 = U_2$ (in particular, $U_1 = U_2 = V^\$), then $F_S$ evidently remains the same.

Case 2. If $U_1$ and $U_2$ belong to the same set $V^{xy}$ (see Fig. 4.7a), we can trace the dynamics in any of $W_{x,\{y,z\}}$, $W_{\{x,y\},z}$, and $W_{x,y}$. Clearly, the definition of the contracted unit $U^\text{new}$ (see Sect. 4.1) can be translated straightforwardly into terms of $F_S$: $U^\text{new}$ results from the contraction of $U_1$, $U_2$, and the units and edges on the coherent paths between $U_1$ and $U_2$ (all of them belong to $V^{xy}$). Since the contracted unit in the modified strip is non-terminal, this contraction is the only change in $F_S$. Note that the changes are localized in $V^{xy}$ and can be traced also in $W^{xy}$.

Case 3. Similarly, if $U_1, U_2 \in V^x$ (see Fig. 4.7b), we can trace the changes in any of the strips $W_{x,\{y,z\}}$, $W_{x,y}$, or $W_{x,z}$ to define $U^\text{new}$ in the same way. It is easy to show that the changes are localized in $V^x$.

In all the above cases all the cuts $C_a$, $C_b$, $C_c$ remain the same, and thus the partition into cells is stable.
Case 4. Let now $U_1 \in V^x, U_2 \in V^{xy}$ (see Fig. 4.8a). As in the previous two cases, the new unit $U^{\text{new}}$ can be defined via $W_{\{x,z\}}, W_{xy},$ or $W_{xz}$. The strip $W_{xy}$ can be used also for establishing the dynamics of the set $V_y$ via Lemma 4.3; in terms of this lemma, it is the set $L_z$. By the same lemma, the new part of $V_y$ is $V' = R_x(U_2) \setminus V_y$. Another way is to use $W_{\{x,z\},y},$ where the set $V_y$ forms the terminal $U_{xz}$ containing $x$ and $z$; then, the new part of $V_y = U_{yz} = R_x(U_2) \setminus U_{yz}$. It follows from Fact 4.6 that $V'$ is contained in $V^{xy}$ and coincides with the cone of $U_2$ in $W^{xy}$ in the direction to the terminal containing $x$, minus this terminal. From the dynamics of $W_{\{x,z\}}$ and $W_{\{x,v\}}$, respectively, it is clear that the sets $V_x$ and $V_z$ do not change. Therefore, all the cells are stable, except for $\hat{V} = V^x \cup V'$ and $\hat{V}^{xy} = V^{xy} \setminus V'$.

Case 5. If $U_1 \in V^{xz}, U_2 \in V^{xy}$ (see Fig. 4.8b), we can use strips $W_{\{x,z\}}, W_{xy}$ and $W_{xz}$ for determining $U^{\text{new}}$. In this case, there is no coherent path between $U_1$ and $U_2$, hence, $U^{\text{new}} = \{U_1, U_2\}$. Using strips $W_{xy}$ and $W_{xz}$, respectively, in the same way as $W_{\{x,z\},y}$ in the previous case, we establish that $\hat{V}_y = V_y \cup R_x(U_2)$ and $\hat{V}_z = V_z \cup R_x(U_1)$. Therefore, $\hat{V}^x = V^x \cup R_x(U_1) \cup R_x(U_2)$, while $V^{xy}$ and $V^{xz}$ are cut down correspondingly.

Case 6. If $U_1 = V^\emptyset, U_2 \in V^{xy}$ (see Fig. 4.9a), we use the strip $W_{x,y}$. Once again $U^{\text{new}} = \{U_1, U_2\}$. Both $V_x$ and $V_y$ grow to $V_x \cup R_x(U_2)$ and $V_y \cup R_y(U_2)$, respectively. Therefore, $\hat{V}^x = V^x \cup R_x(U_2) \setminus U_2$, $\hat{V}^z = V^z \cup R_z(U_2) \setminus U_2$ and $\hat{V}^\emptyset = V^\emptyset \cup U_2$.

Case 7. If $U_1 \in V^x, U_2 = V^\emptyset$ (see Fig. 4.9b), we consider the strip $W_{\{x,y\}}$ with terminals $U_x$, $x \in U_x$, and $U_{yz} = V_x, y, z \in U_{yz}, V^{\emptyset} \subset U_{yz}$. By Sect. 4.1, the cone $R_y(U_1)$ is added to $U_{yz}$, resulting in $\hat{V} = V_x \cup R_y(U_1)$. Evidently, strips $W_{\{x,y\},y}$ and $W_{\{x,y\},z}$ do not change, hence $V_y$
and $V_z$ are stable. Using the sets $\hat{V}_x, \hat{V}_y = V_y$ and $\hat{V}_z = V_z$ in accordance to the definition of a cell, we get $U = V^\emptyset = V^\emptyset \cup (V^x \cap R_{yz}(U_1)), \hat{V}_y = V^y \cup (V^{xz} \cap R_{yz}(U_1))$, $\hat{V}_z = V^z \cup (V^{xy} \cap R_{yz}(U_1))$, and each one of $V^x, V^{xy}, V^{xz}$ is lessened by the corresponding part of $R_{yz}(U_1)$.

A special subcase occurs when $U_1 = U_x$. Then $W_{x,(y,z)}$ vanishes, and we arrive at the asymmetric case for $F_{\{y,z\}}$. In terms of Sect. 4.2, $U_x = V_x \cup V^\emptyset$, $\hat{L}_x = V_x$, $\hat{R}_x = V_y$, $\hat{V}_{yz} = V_{yz}$ and $\hat{W}_{yz} = W_{yz}$.

**Fig. 4.9. Dynamics of $F_{\{a,b,c\}}$ in the symmetric case, III**

Case 8. The case $U_1 \in V^x, U_2 \in V^y$ (see Fig. 4.10a) is equivalent to a composition of the two previous cases: apply the transformation of Case 6 and the transformation of Case 7 with $x$ replaced by $y$.

To see this, assume first that $V^\emptyset \neq \emptyset$ and choose an arbitrary vertex $w \in V^\emptyset$. Then the transformation of $F_{\{a,b,c\}}$ equals the sum of the transformations caused by adding of two edges: $(u_1, w)$ and $(w, u_2)$. Indeed, addition of each one of these edges affects only one of the strips $W_{yz}$ and $W_{x,(y,z)}$, and their dynamics are exactly the same as in the Cases 6 and 7, respectively. The case $V^\emptyset = \emptyset$ can be reduced to the previous one by the following trick. We add to $G$ a new vertex $w$ and three new edges $(a, w), (b, w), (c, w)$. It is easy to see that $S$-mincuts of the new graph $G^w$ correspond bijectively to $S$-mincuts of $G$; more precisely, $(X, \bar{X})$ corresponds to $(X, \bar{X} \cup \{w\})$ if and only if $|X \cap S| = 1$. Hence, all the cells of $F^w_{\{a,b,c\}}$ coincide with the corresponding cells of $F_{\{a,b,c\}}$, except for $(V^w)^\emptyset = \{w\}$. Since $(V^w)^\emptyset$ is now nonvoid, the transformation taking $F^w_{\{a,b,c\}}$ to $F^w_{\{a,b,c\}}$ is exactly as described above; to obtain $\hat{F}^w_{\{a,b,c\}}$ it suffices just to delete $w$.

Case 9. The last case occurs when $U_1 \in V^x, U_2 \in V^y$ (see Fig. 4.10b). It can be considered as a composition of two Cases 7: apply the transformation of Case 7 as described and the same transformation with $x$ replaced by $y$.

In this situation the two transformations are not independent: their joint action forces the set $V^l = V^{xy} \cap R_{yz}(U_1) \cap R_{xz}(U_2)$ to be added to both $V_x$ and $V_y$ and thus to move from $V^{xy}$ to $V^\emptyset$. The validity of the above description can be observed in the same way as in the previous case; the two edges added in this case are the edges between $U_1$ and $V^\emptyset$ and between $V^\emptyset$ and $U_2$. 


The only special subcase not yielded by double application of Case 7 occurs if \( U_1 = U^x, U_2 = U^y \). It results in vanishing of both strips \( \mathcal{W}_{x,y,z} \) and \( \mathcal{W}_{y,x,z} \), and thus leads to degeneration of \( \mathcal{F}_{x,y,z} \) to \( \mathcal{F}_{x,z} = \mathcal{F}_{y,z} \); evidently, \( \mathcal{W}_{x,y,z} \) coincides with \( \mathcal{W}_{x,y,z} \).

4.4. An arbitrary subset \( S \) with the path structure. Comparing the asymmetric and the symmetric cases considered, one can assign to the first of them the structure of the path \( (a, b, c) \), and to the second, of the 3-star with terminals \( a, b, c \) (Fig. 4.11a, b). The 1-cuts of these “skeleton” structures correspond bijectively to distinct partitions of \( \{a, b, c\} \) by minimum \( \{a, b, c\} \)-cuts in the original graph.

Let \( S = \{a = b_0, b_1, \ldots, b_{r-1}, b_r = c\} \); assume that there exist exactly \( r \) distinct partitions of \( S \) by \( S \)-mincuts and each of them is of the form \( \{\{a, b_1, \ldots, b_{i-1}\}, \{b_i, \ldots, b_{r-1}, c\}\} \), \( 1 \leq i \leq r \). This means that the skeleton structure (in the sense defined above) for this case is the path \( (a, b_1, \ldots, b_{r-1}, c) \) (see Fig. 4.11c). Similarly to the case of the path \( (a, b, c) \) (Sect. 4.2), all \( S \)-mincuts are represented by the \( (a, c) \)-strip \( \mathcal{W}_{a,c} \). We define the structure of local orientability on the graph \( \mathcal{F}_S \) by keeping that of \( \mathcal{W}_{a,c} \) at all the units, except for the units \( U_{b_i} \) containing \( b_i \), \( 1 \leq i \leq r - 1 \), which are now regarded as terminals. As in Sect. 4.2, \( \mathcal{F}_S \) is acyclic and balanced and all \( S \)-mincuts of \( G \) are transversal cuts in \( \mathcal{F}_S \).

For each \( i \), \( 0 \leq i \leq r - 1 \), let \( (L_i, \bar{L}_i) \) be the \( \{a, b_1, \ldots, b_i\} \)-tight \( \{a, b_1, \ldots, b_i\}, \{b_{i+1}, \ldots, b_{r-1}, c\}\)-mincut. Similarly, for each \( i \), \( 1 \leq i \leq r \), let \( (R_i, \bar{R}_i) \) be the \( \{c, b_{r-1}, \ldots, b_i\} \)-tight \( \{c, b_{r-1}, \ldots, b_i\}, \{b_{i-1}, \ldots, b_1, a\}\)-mincut. Besides, we define \( L_r \) and \( R_0 \) to be the entire set of units of \( \mathcal{F}_S \). The properties
of the cuts \((L_i, L_i)\) and \((R_i, R_i)\) are similar to those of \((L_b, L_b)\) and \((R_b, R_b)\) (see Lemma 4.1) and of \(C_x\) (see Lemma 4.4(i) and Lemma 4.7). We omit the proof, since it is similar to that of Lemma 4.1; for illustration see Fig. 4.12.

Lemma 4.9.

(i) For any pair \(i, j, 0 \leq i < j \leq r\), one has \(L_i \subseteq L_j\) and \(R_i \subseteq R_j\).

(ii) For any \(i, 0 \leq i \leq r\), \(L_i\) and \(R_i\) are the reachability cones of the unit \(U_{b_i}\) in \(W_{a,c}\), and hence \(L_i \cap R_i\) coincides with this unit.

(iii) The sides of the star of \(U_{b_i}\) in \(W_{a,c}\) are induced in \(\mathcal{F}_S\) by edge-sets of \((L_i, L_i)\) and \((R_i, R_i)\).

(iv) For any pair \(i, j, 0 \leq i < j \leq r\), the contraction of \(L_i\) and \(R_j\) in \(\mathcal{F}_S\) into two terminals gives \(\mathcal{F}_{\{b_i, \ldots, b_j\}}\).

![Fig. 4.12. Cell structure for the path case](image)

By Lemma 4.9(iii), the strip \(W_{a,c}\) can be restored from \(\mathcal{F}_S\). Observe that the sets \(\{b_i, \ldots, b_j\}, i < j\), have the path structure similar to that of \(S\). Therefore, by Lemma 4.9(iii) and (iv), the strips \(W_{b_i,b_j}\) can be restored from \(\mathcal{F}_S\) as well. A unit which is not contracted to a terminal in the construction described in Lemma 4.9(iv) is said to participate in \(W_{b_i,b_j}\).

The two nested families \(\{L_i\}\) and \(\{R_i\}\) define a cell structure on the set of units. Namely, the cell \(Q^{ij}\), \(0 \leq i \leq j \leq r\), is defined as the intersection \((L_j \setminus L_{j-1}) \cap (R_i \setminus R_{i+1})\), where \(L_0 = R_{r+1} = \emptyset\). Observe that \(Q^{ij}\) is exactly the unit containing \(b_j\). Part (iv) of Lemma 4.9 implies that the units of the same cell participate in the same “partial” strips \(W_{b_i,b_j}\). More exactly, the following statement holds.

Lemma 4.10. The units of the cell \(Q^{ij}\) participate in the strip \(W_{b_i,b_j}\) if and only if the paths \((b_i, \ldots, b_j)\) and \((b_p, \ldots, b_q)\) in the skeleton structure have at least one common edge.

Proof. According to (iv) above, to obtain \(W_{b_i,b_j}\) we contract the cells \(Q^{ij}\) with \(j \leq p\) or \(i \geq q\), and the assertion follows. \(\square\)
Edges between cells and, generally, mutual reachability of their units, are subject to the following restriction.

**Lemma 4.11.** Let \( U_1 \) and \( U_2 \) be mutually reachable in \( \mathcal{F}_S \), \( U_1 \in Q^{i_1,j_1} \), \( l = 1, 2 \), and \( i_1 < i_2 \). Then:

(i) \( j_1 < j_2 \);

(ii) if \( U_1 \) is a stretched unit and \( U_3 \) lies in the same reachability cone of \( U_1 \) as \( U_2 \), \( U_3 \in Q^{i_3,j_3} \), then \( i_1 \leq i_3 \) and \( j_1 \leq j_3 \).

**Proof.** (i) Assume that \( i_1 < i_2 \), \( j_2 < j_1 \), and there exists a coherent path \( P_2 \) between \( U_1 \) and \( U_2 \). Evidently, \( U_1 \) is not a terminal. Let us extend \( P_2 \) behind \( U_1 \) up to a terminal \( U_{b_{l_1}} \). Without loss of generality we may assume that \( p > i_1 \) (the case \( p < j_1 \) is treated in the same way). Let us consider the \( S \)-mincut \((X, \bar{X})\) with \( X = L_{j_1} \cap \bar{R}_{i_1+1} \). Evidently, the endpoints \( U_2 \) and \( U_{b_{l_1}} \) of the extended path lie outside this cut, while an intermediate unit \( U_1 \) lies inside this cut. So, the extended path intersects the transversal cut \((X, \bar{X})\) at least twice, a contradiction.

(ii) Let \( U_{b_{l_1}} \) be as above. From (i) we get immediately that either \( p \geq j_1 \), or \( p \leq i_1 \). If \( p \geq j_1 \), then we set \( i = \min\{i_2, j_1\} \); now both \( U_{b_{l_1}} \) and \( U_{i} \) belong to \( R_{i} \), while \( U_1 \) does not (since \( i_1 < i_2 \)), a contradiction. Hence, \( p \leq i_1 \).

Let us now extend a coherent path \( P_3 \) between \( U_1 \) and \( U_{b_{l_1}} \) up to a terminal \( U_{b_{l_2}} \) (since \( U_3 \) is not necessarily a stretched unit, it may happen that \( U_{b_{l_2}} = U_3 \)). Observe that \( P_3 \) can be extended up to \( U_{b_{l_2}} \); we denote the obtained coherent path by \( P_3' \). Once more we get from (i) that either \( q \geq j_1 \), or \( q \leq i_1 \). If \( q \leq i_1 \), then the endpoints \( U_{b_{l_2}} \) and \( U_{b_{l_1}} \) of \( P_3' \) belong to \( L_{i_1} \), while its intermediate unit \( U_1 \) does not, a contradiction. Hence, \( q \geq j_1 \).

Assume now that \( i_3 < i_1 \); in this case \( U_3 \neq U_{b_{l_2}} \). Let us consider the part of \( P_3' \) between \( U_1 \) and \( U_{b_{l_2}} \). Both its endpoints belong to \( R_{i_1} \), while its intermediate unit \( U_3 \) does not, a contradiction.

Assume now that \( i_3 = i_1 \) and \( j_3 < j_1 \); again we have \( U_3 \neq U_{b_{l_2}} \). Let us consider the same path as in the previous case. Both its endpoints do not belong to \( L_{j_3} \), while its intermediate unit \( U_3 \) does, a contradiction.

Therefore, \( i_1 = i_3 \) implies \( j_1 < j_3 \). Finally, \( i_1 < i_3 \) implies the same inequality. \(\square\)

**Remark.** Observe that the first assertion of the above Lemma can be regarded as a generalization of Lemma 2.1(i).

Let us now extend the definition of the cell structure to the edges of \( \mathcal{F}_S \). Let \( e = (U_1, U_2) \) be an edge of \( \mathcal{F}_S \), and assume that \( U_1 \in Q^{i_1,j_1} \) and \( U_2 \in Q^{j_1,j_2} \). We put \( i = \min\{i_1, i_2\} \), \( j = \max\{j_1, j_2\} \), and say that \( e \) belongs to the cell \( Q^{i,j} \). Observe that by Lemma 4.11 there are three types of edges belonging to a cell \( Q^{i,j} \): internal (both endpoints belong to \( Q^{i,j} \)), external (one endpoint belongs to \( Q^{i,j} \) and the other one either to \( Q^{i,j} \), \( i \leq j' < j \), or to \( Q^{i,j} \), \( i < i' \leq j \)), and through (one endpoint belongs to \( Q^{i,j} \), \( i \leq j' < j \), and the other one to \( Q^{i,j} \), \( i < i' \leq j \)); for an illustration see Fig. 4.10.
This definition can be justified as follows. Insert a dummy vertex \( u \) into the edge \( e \), thus dividing it into \( e_1 \) and \( e_2 \). Evidently, if an \( S \)-mincut \((X, \bar{X})\) of the initial graph is crossed by \( e \), then upon insertion of \( u \) it is replaced by two \( S \)-mincuts of the modified graph: \((X \cup \{u\}, \bar{X})\) and \((X, \bar{X} \cup \{u\})\). Conversely, let \((Y, \bar{Y})\), \( u \in \bar{Y} \), be an \( S \)-mincut of the modified graph crossed, say, by \( e_1 \). Then \((Y \cup \{u\}, \bar{Y} \setminus \{u\})\) is another \( S \)-mincut of the modified graph, and it is crossed by \( e_2 \). Finally, if an \( S \)-mincut is not crossed by \( e \), then it remains intact in the modified graph, and vice versa. It follows from the above discussion that the only difference between the initial \( \mathcal{F}_S \) and the modified one is the new unit \( U \) that consists of the single dummy vertex \( u \) and belongs to the cell \( Q^{ij}; U \), together with the incident edges \( e_1 = (U_1, U) \) and \( e_2 = (U_2, U) \), replaces the edge \( e \) in \( \mathcal{F}_S \). This dummy unit, in a sense, represents the edge \( e \), and thus justifies the above definition. Observe that this definition is stable w.r.t. insertion of dummy vertices. Namely, if \( e \) belongs to the cell \( Q^{ij} \), then both \( e_1 \) and \( e_2 \) belong to the same cell.

For any pair \( i, j \), \( 0 \leq i \leq j \leq r \), excluding cases \( i = j = 0 \) and \( i = j = r \), we build from \( \mathcal{F}_S \) a locally orientable graph \( W^{ij} \) with the set of nonterminals \( Q^{ij} \) (see Fig. 4.13). We delete \( \bigcup \{Q^{pq} : p < i \leq j < q \} \) and \( \bigcup \{Q^{pq} : i < p \leq q < j \} \) (observe that by Lemma 2.1 there are no edges between \( Q^{ij} \) and the deleted sets). Next, we contract the sets \( \bigcup \{Q^{pq} : p \leq i, \ q \leq j, \ (p, q) \neq (i, j) \} \) and \( \bigcup \{Q^{pq} : p \geq i, \ q \geq j, \ (p, q) \neq (i, j) \} \) into two terminals \( a' \) and \( c' \). Finally, we delete the edges between \( a' \) and \( c' \) that do not belong to \( Q^{ij} \). Observe that this definition is consistent with the definition of \( W^{0,0} \) given in Sect. 4.2 (\( W^{0,0} \), in our new notation), because all the edges between \( a' \) and \( c' \) in \( \mathcal{F}_{\{a, b, c\}} \) belong to \( Q^{0,0} \). Evidently, all the edges that belong to \( Q^{ij} \) participate in \( W^{ij} \). Besides, each external edge participates in one more such graph, and each through edge in two more such graphs. Extensions of cuts of \( W^{ij} \) to \( S \)-mincuts are defined similarly to the case analyzed in Sect. 4.2. The following statement is an analog of Lemmas 4.2 and 4.8.

Lemma 4.12.
(i) The graph $W^{ij}$ is a strip of width $\lambda^{ij}$; if $j = i$ then $\lambda^{ij} > \lambda_S/2$, while if $j \geq i + 2$ then $\lambda^{ij} < \lambda_S/2$.

(ii) All its transversal cuts, and only they, are extendable to $S$-mincuts.

(iii) For any sequence $i_1 < i_2 < \cdots < i_i \leq j_i < \cdots < j_i < j_k$, any set of $l$ transversal cuts of $W^{ij_1}, \ldots, W^{ij_l}$ has a mutual extension to an $S$-cut. Moreover, if $j_i \geq i_i + 2$ then $\lambda^{ij_1} + \cdots + \lambda^{ij_l} < \lambda_S/2$.

(iv) Let $i_1 < i_2 < \cdots < i_l \leq j_l < \cdots < j_l < j_k$ and let a 2-partition of $V \setminus \{Q^{ij_1} \cup \cdots \cup Q^{ij_l}\}$ enter a mutual extension of a set of transversal cuts of $W^{ij_1}, \ldots, W^{ij_l}$. Then it enters some mutual extension of any set of such cuts. (Informally: all sets of transversal cuts for such a sequence of strips are interchangeable in $S$-mincuts).

**Proof.** (i) The fact that $W^{ij}$ is a strip is proved similarly to Lemma 4.2.(i) taking into account that the sets $X_a = \bigcup\{Q^p : p \leq i, \ q \leq j, \ (p, q) \neq (i, j)\}$ and $X_a = \bigcup\{Q^p : p \geq i, \ q \geq j, \ (p, q) \neq (i, j)\}$ define transversal cuts $C_a = (X_a, \bar{X}_a)$ and $C_c = (X_c, \bar{X}_c)$ (shown by a double line on Fig. 4.13). The inequality for the case $j = i$ follows from the fact that the cardinality of the cut $(U_{ik}, \ V \setminus U_{ik})$ exceeds $\lambda_S$. In the case $j \geq i + 2$ we see that all the edges of $F_S$ incident to $a'$-type terminals in the strips $W^{ij}$ and $W^{i+1,j+1}$ belong to the edge-set of $C_a$. Therefore, $\lambda^{ij} + \lambda^{i+1,j+1} \leq c(C_a) = \lambda_S$. Since $\lambda^{i+1,j+1} > \lambda_S/2$, we are done.

(ii) Let $X_a$ and $X_i$ be defined as above. Given any transversal cut $C' = (a' \cup Z, Z \cup c')$ of $W^{ij}$, let us consider the edge-set of the 2-partition $C = (X_i \cup Z, Z \cup \bar{X}_a)$ of $V$. As compared with the edge-set of $C_a$, it loses the edges between $X_a$ and $Z$ and gets the edges between $Z$ and $\bar{Z} \cup \bar{X}_a$. However, the same holds for the cut $C'$ as compared with the cut of $W^{ij}$ that separates $a'$. Since, by (i), the cuts in the second pair have equal cardinalities, the same is true for the first pair. Thus $C$ is a minimum $S$-cut.

Conversely, let $(X, \bar{X})$ be an $S$-cut separating $X_a$ from $\bar{X}_a$. By the same reasons as above, it generates a 2-partition of cardinality $\lambda^{ij}$ in $W^{ij}$ that is a minimum (i.e., transversal) cut of $W^{ij}$.

(iii) Let $C'_k$ be a transversal cut of $W^{ij_k}$, $1 \leq k \leq l$. Denote by $C_k$ the extension of $C'_k$ defined as in the proof of (ii) and consider the $S$-cut $C = C_1 \cup \cdots \cup C_l$. Inequalities $i_1 < i_2 < \cdots < i_l \leq j_1 < \cdots < j_l$ guarantee that $C$ is an extension for any of the $C'_k$, and hence, is a mutual extension of the whole set $\{C'_1, \ldots, C'_k\}$. The same inequalities guarantee that no edge is counted more than once in the sum $\lambda^{ij_1} + \cdots + \lambda^{ij_l}$; the rest of the proof of the inequality $\lambda^{ij_1} + \cdots + \lambda^{ij_l} < \lambda_S/2$ follows the proof of (i).

(iv) Assume that $l = 1$. Let $C_0$ be an extension of some transversal cut of $W^{ij_1}$ and $C'$ be an arbitrary transversal cut of $W^{ij_1}$. Take $\bar{X}_c$ as above and put $X_c = X_c \setminus Q^{ij_1}$; then $C = (X_c, \bar{X}_c)$ is an $S$-cut. Now define the cut $C$ of $G$ as in the proof of (ii); then $(C_0 \cap C) \cup C$ is an extension of $C'$ that divides $V \setminus Q^{ij_1}$ in the same way as $C_0$. For $l > 1$ the proof applies.
with evident minor changes.

Dynamics of $F_S$, $S = \{a = b_0, \ldots, b_r = c\}$, in the case of the path structure are induced by those of $F_{\{a, c\}} = W_{a,c}$, in the same way as it was done for $F_{\{a, b, c\}}$ in the asymmetric subcase (see Sect. 4.2). To be more precise, let $U_1 \in Q^{j_1 i_1}$, $U_2 \in Q^{j_2 i_2}$, $i_1 \leq i_2$. Elements of $S$ "fall inside" $U_{\text{new}}$ when $j_1 \leq i_2$; in this case $U_{\text{new}}$ is a terminal and it contains all the $b_k$ such that $j_1 \leq k \leq i_2$ (see Fig. 4.14a).

Concerning the dynamics of the sets $L_k$ (those of $R_k$ are similar), Lemma 4.3 can be generalized as follows.

**Lemma 4.13.** The sets $L_k$, $i_1 \leq k < i_2$, and only them are extended (at least by $U_2$). The new part of $L_k$ is the union of the units $U^{i} \in Q^{i' j'}$ such that $i' > k$ and $U^{i} \in R_a(U_2)$.

The proof of Lemma 4.13 is similar to that of Lemma 4.3 (see Fig. 4.14b).

![Fig. 4.14. Dynamics in the case of the path structure](image)

In the case when $U_{\text{new}}$ is not a terminal, its new cell can be defined via the characterization of the sets $\hat{L}_k$, $\hat{R}_l$ given in Lemma 4.13.

**Corollary 4.14.** Let $i_{\text{min}} = \min\{i_1, i_2\}$, $i_{\text{max}} = \max\{i_1, i_2\}$, $j_{\text{min}} = \min\{j_1, j_2\}$, $j_{\text{max}} = \max\{j_1, j_2\}$. The new unit $U_{\text{new}}$ belongs to the set $Q^{i_{\text{min}} j_{\text{max}}}$.

Let a unit $U^{i} \in Q^{i' j'}$ move to the cell $\hat{Q}^{i^* j^*}$.

(i) If $i > i_{\text{min}}$ and $U^{i} \in R_a(U_{i_{\text{max}}})$ then $i^* = i_{\text{min}}$, otherwise $i^* = i$.

(ii) If $j < j_{\text{max}}$ and $U^{i} \in R_c(U_{j_{\text{min}}})$ then $j^* = j_{\text{max}}$, otherwise $j^* = j$.

5. The flesh and the skeleton of the connectivity carcass

5.1. The system of units, or the flesh of the connectivity carcass. Let us consider an arbitrary bunch $B$ of $S$-cuts, and let $(S_B, \bar{S}_B)$ be the corresponding partition of $S$. In fact, $B$ is the set of all minimum $(S_B, \bar{S}_B)$-cuts; thus the structure of $B$ can be represented by the strip $W_{S_B, \bar{S}_B}$. For brevity, in what follows this strip is denoted by $W_B$, the corresponding quotient mapping by $\delta_B$, and the units of $W_B$ are called $B$-units. Therefore, the following statement holds.

**Fact 5.1.** Any bunch of $S$-cuts $B$ is represented by the strip $W_B$ and the mapping $\delta_B$ so that the $\delta_B$-inducing provides a bijection between the transversal cuts of $W_B$ and the $S$-cuts in $B$. 

Evidently, for the bunch $B$ of cuts opposite to cuts in $B$, the strip $W_B$ coincides with $W_B$.

By the definition, the partition of $V$ into $S$-units is a refinement of its partition into $B$-units. The following statement shows that only the two terminal $B$-units can be refined.

**Lemma 5.2.** Each nonterminal $B$-unit is an $S$-unit.

*Proof.* The proof is similar to that of Lemma 4.4(i). Let $U$ be a nonterminal $B$-unit subdivided by an $S$-mincut $C$. Evidently, there exist vertices $x \in S_B$ and $y \in S_B$ separated by $C$; assume w.l.o.g. that $x$ lies inside $C$. By the definition of $B$-units, the bunch $B$ contains cuts $C_1$ and $C_2$ separating $S_B$ from $U$ and $U$ from $S_B$, respectively. Observe that all the three cuts $C$, $C_1$, and $C_2$ are minimum $(x,y)$-cuts. Let us consider the cut $C \cup (C \cap C_2)$. It contains $S_B$ inside, contains $S_B$ outside, and thus belongs to $B$. However, it subdivides $U$, a contradiction. □

By the definition, the star of any nonterminal vertex of the locally orientable graph $W_B$ is partitioned into two sets of equal cardinality. Thus, by Lemma 5.2, the star of the corresponding vertex of $F_S$ is halved. The crucial observation is that the arising 2-partition does not depend on the choice of $B$ (see Lemma 4.4(iii) for a particular case of the same statement).

**Lemma 5.3.** If an $S$-unit participates in several bunch strips as a nonterminal unit, then the 2-partitions of its star in all these strips coincide.

*Proof.* Let $U$ be an $S$-unit participating in bunches $B$ and $B'$. Without loss of generality, the 2-partition $(S_B, \tilde{S}_B)$ divides $S_B$. Therefore, for any $x \in \tilde{S}_B$ one can choose $y \in S_B$ separated from $x$ by this 2-partition. Similarly to the proof of Lemma 4.4(iii), both strips $W_B$ and $W_{B'}$ are contractions of the same strip $W_{x,y}$ (in this case Lemma 3.8 is used repeatedly). Hence, the 2-partitions at $U$ in these strips are inherited from $W_{x,y}$ and thus coincide. □

![Connectivity carclass of a graph and its contractions to strips](image)

**Fig. 5.1.** Connectivity carclass of a graph and its contractions to strips
A unit that participates at least in one bunch strip is called stretched; the canonical partition of its star indicated in Lemma 5.3 is called inherent. All the other units are called terminal. A unit is called heavy if it intersects $S$. Observe that only terminal units (but, in general, not all of them) can be heavy. To the star of a terminal unit of $F_S$ we assign the trivial inherent partition. All the other units are called terminal. A unit is called heavy if it intersects $S$.

Observe that only terminal units (but, in general, not all of them) can be heavy. To the star of a terminal unit of $F_S$ we assign the trivial inherent partition. Thus, the quotient graph $F_S$ acquires a canonical structure of a locally orientable graph; it is said to be the flesh of the connectivity carcass of $S$ (see Fig. 5.1a,b). Below we prove several basic properties of $F_S$.

According to the definitions above, we build $F_S$ by “gluing together” all the bunch strips. Conversely, each bunch strip can be reconstructed from $F_S$ with the help of contractions. Indeed, let $B$ be an arbitrary bunch of $S$-mincuts. The intersection of all cuts in $B$ is called the tight cut of $B$; the union of all cuts in $B$ is called the loose cut of $B$ (evidently, each of them separates a terminal of $W_B$ from all other units). Observe that the tight cut of $B$ is just the $S_B$-tight ($S_B, \bar{S}_B$)-mincut, while the loose cut of $B$ is the opposite to the $S_B$-tight ($S_B, \bar{S}_B$)-mincut. The following statement is a generalization of Lemma 4.1(iii) and Lemma 4.4(i).

**Lemma 5.4.** For any bunch $B$, the strip $W_B$ is obtained from $F_S$ by contracting all the units inside the tight cut of $B$ and all the units outside the loose cut of $B$ into terminals.

**Proof.** Follows immediately from Lemmas 5.2 and 5.3 (see the proof of Lemma 4.4(i) for details). □

Parts (i) and (ii) of the next statement provide a generalization of Lemma 4.5.

**Theorem 5.5.**

(i) The flesh $F_S$ is a balanced acyclic locally orientable graph.

(ii) Each $S$-mincut is transversal in $F_S$.

(iii) Let $\tilde{n}$ and $\tilde{m}$ be the numbers of vertices and edges of $F_S$, respectively. Then $\tilde{n} \leq n$, $\tilde{m} \leq m$, $\tilde{m} = O(\lambda_S \tilde{n})$.

**Proof.** (i) The fact that $F_S$ is balanced is clear, since inherent partitions at stretched units of $F_S$ come from those in bunch strips (see Lemma 5.3). Acyclicity of $F_S$ follows from part (ii) of this theorem exactly in the same way as in the proof of Lemma 4.5.

(ii) Assume to the contrary that the claim is violated by some pair $G, S$ and the corresponding flesh $F_S$. Then there exists a minimal counterexample, namely, an $S$-mincut $C$ and a coherent path $P$ in $F_S$ such that $P$ intersects the edge set of $C$ at least twice and the number of units in $P$ is minimal among all such pairs $C, P$ (see Fig. 5.2). By minimality, the endpoints of $P$ lie outside $C$, and all the inner units of $P$ lie inside $C$ (up to flipping of $C$). Let $U$ be an arbitrary inner unit of $P$; since $P$ is a coherent path, $U$ is a stretched unit. Therefore, there exists a bunch in which $U$ participates. Let $C_1$ be the $S$-mincut in this bunch defined by a cone $R$ of $U$ in the corresponding strip and $C_2$ be the $S$-mincut in this bunch defined by
\( \mathcal{R} \setminus U \). Up to taking the other cone of \( U \), we may assume that both pairs \( \mathcal{C}_1 \setminus \mathcal{C} \) and \( \mathcal{C}_2 \setminus \mathcal{C} \) satisfy the conditions of Lemma 2.1. Thus, by Fact 2.2, both \( \mathcal{C}_1 \cap \mathcal{C} \) and \( \mathcal{C}_2 \cap \mathcal{C} \) are \( S \)-mincuts; moreover, they belong to the same bunch and their inner parts differ exactly by \( U \).

\[
\begin{array}{c}
\mathcal{C}_2 \\
\mathcal{C}_1 \\
\mathcal{C}_2 \cap \mathcal{C} \\
U \\
P \\
\end{array}
\]

**Fig. 5.2. To the proof of Theorem 5.5**

Let us traverse \( P \) from an endpoint up to \( U \) and take the edge entering \( U \) and the next edge of \( P \). By Fact 3.3, exactly one of the above edges intersects the edge set of \( \mathcal{C}_2 \cap \mathcal{C} \); let \( U' \) be the other endpoint of this edge. Since \( U' \) lies inside \( \mathcal{C}_2 \cap \mathcal{C} \), it lies inside \( \mathcal{C} \) as well, and thus cannot be an endpoint of \( P \). Let \( P' \) be the subpath of \( P \) leaving \( U \) by the edge \((U, U')\) and ending at the corresponding endpoint of \( P \). Since both \( U \) and this endpoint of \( P \) lie outside \( \mathcal{C}_2 \cap \mathcal{C} \), the pair \( \mathcal{C}_2 \cap \mathcal{C}, P' \) contradicts the minimality of the pair \( \mathcal{C}, P \).

(iii) The first two inequalities are evident. Let us show that \( \bar{m} = \lambda_S(\bar{n} - 1) \).

Since any two units are separated by an \( S \)-mincut, the size of a minimum cut between them in \( \mathcal{F}_S \) is at most \( \lambda_S \). Let us consider the Gomory-Hu tree of \( \mathcal{F}_S \) (see [GH]). Recall that its vertices correspond bijectively to the units, and thus any of its edges defines a cut of \( \mathcal{F}_S \). By [GH], this cut is a minimum cut between some two units, since its size is at most \( \lambda_S \); there are \( \bar{n} - 1 \) such cuts. Since each edge of \( \mathcal{F}_S \) takes part in at least one such cut, their total number \( \bar{m} \) does not exceed \( \lambda_S(\bar{n} - 1) \), as required. \( \Box \)

The flesh can be considered, in a sense, as a generalization of a strip. By the way, at early stages we called our structure “branching dag”. The meaning of such a nickname will become more clear in the next subsection.

**5.2. The system of bunches, or the skeleton of the connectivity carcass: odd case.** For the case \( S = V \), the structure of all \( S \)-mincuts is represented by a cactus tree (see [DKL]); it is, in fact, a tree if the graph connectivity is odd. The construction is based on Crossing Lemma [Bi, DKL]. A generalization of this lemma to \( S \)-mincuts and the existence of a cactus tree representation of all 2-partitions of \( S \) by \( S \)-mincuts were stated (without proof) independently in [N, W93] (see also [DV1]). The generalization of the lemma is straightforward (see below); for a proof of the existence of
a cactus tree representation see [DN]. Below we describe this representation for the odd case.

Two $S$-cuts $\mathcal{C}$ and $\mathcal{C}'$ are said to be $S$-crossing if all the four subsets $S_C \cap S_C'$, $S_C \cap S_{C'}$, $S_C \cap S_{C'}$, $S_C \cap S_{C'}$ are nonvoid.

**S-Crossing Lemma.** Let $\mathcal{C}$ and $\mathcal{C}'$ be $S$-crossing $S$-mincuts, then

- (i) $c(V_C \cap V_{C'}, V_C \cap V_{C'}) = c(V_C \cap V_{C'}, V_C \cap V_{C'}) = c(V_{C'} \cap V_{C'}, V_{C'} \cap V_{C'}) = \lambda_S/2$;
- (ii) $c(V_C \cap V_{C'}, V_C \cap V_{C'}) = c(V_C \cap V_{C'}, V_C \cap V_{C'}) = 0$.

Hence, the 2-partitions $\mathcal{C} \cap \mathcal{C}'$, $\mathcal{C} \cap \mathcal{C}'$, $\mathcal{C} \cap \mathcal{C}'$, $\mathcal{C} \cap \mathcal{C}'$ are $S$-mincuts as well.

The proof coincides with that of Crossing Lemma, with an additional observation that in our case all 2-partitions involved are actually $S$-mincuts.

It follows immediately from S-Crossing Lemma(i) that for $\lambda_S$ odd there are no $S$-crossing $S$-mincuts. Whenever this $S$-crossing-free property holds, we can represent the system of bunches of $S$-mincuts by a tree (for an example see Fig. 5.1c). Let $\tilde{\sigma} \leq \sigma$ be the number of $(\lambda_S + 1)$-connectivity classes in $S$.

**Theorem 5.6.** For any $S \subseteq V$ with $S$-crossing-free property there exist a unique tree $H_S$ with the node set $N_S$ and a mapping $\varphi_S : S \rightarrow N_S$ such that $|N_S| = O(\tilde{\sigma})$ and $\varphi_S$-inducing provides a bijection between the cuts of $H_S$ and the pairs of mutually opposite bunches of $S$-mincuts.

**Proof.** It is an immediate corollary of a more general statement concerning tree models for crossing-free systems of 2-partitions of an arbitrary set (see [DW, Sect. 4.1], [DN]). □

The tree $H_S$ is said to be the skeleton of the connectivity carcass of the set $S$. Its edges are called structural edges (to distinguish them from the edges of the graph $G$). Any cut of $H_S$ is defined by a single structural edge.

Observe that the image $\varphi_S(S)$ does not necessarily coincide with $N_S$. The nodes $N$ for which $\varphi_S^{-1}(N) = \emptyset$ are called empty nodes (e.g., node $N$ on Fig. 5.1c). It follows from the construction that the degree of each empty node is at least three. The preimages of the nonempty nodes in $N_S$ are exactly the classes of $(\lambda_S + 1)$-connectivity of $S$.

By Theorem 5.6, for any $S$ with $\lambda_S$ odd the system of bunches of $S$-mincuts is represented by the skeleton $H_S$, which is a tree. For $\lambda_S$ even this may be also the case, if the $S$-crossing-free property holds. In general, an analog of Theorem 5.6 is valid, in which the skeleton $H_S$ is a cactus tree. It means that each pair of mutually opposite bunches is represented by a cut of $H_S$, which is defined either by a structural edge not belonging to any cycle, or by a pair of structural edges belonging to the same cycle. For details see [DV1], [DV3].

In the rest of the paper we assume that the skeleton is a tree, thus completely covering the odd case.
6. Structure and properties of the connectivity carcass

6.1. Tight cuts. Let $\varepsilon$ be an arbitrary structural edge of $\mathcal{H}_S$ and $N$ be one of its endpoints. By Theorem 5.6, this edge defines two mutually opposite bunches of $S$-mincuts; these two bunches are represented by a strip (see Fact 5.1), which we denote $\mathcal{W}(\varepsilon)$. (For example, Figs. 5.1d,e show the strips $\mathcal{W}(N, N_1)$ and $\mathcal{W}(N, N_2)$, respectively.) Consider the cut of $\mathcal{H}_S$ corresponding to $\varepsilon$ and containing $N$ inside. The intersection of all $S$-mincuts of the bunch represented by this cut of $\mathcal{H}_S$ is the tight cut of this bunch. This cut is said to be the $N$-tight cut in direction $\varepsilon$ and is denoted by $C(N, \varepsilon)$ (e.g., the cut $C_3$ on Fig. 5.1a is the $N$-tight cut in direction $(N, N_1)$).

A path in $\mathcal{H}_S$ is an alternating sequence of distinct nodes and structural edges; we consider the opposite sequence as the same path traversed in the opposite direction. The unique path between two arbitrary nodes $M, N$ of the skeleton is denoted by $[M, N]$. For any two structural edges $\varepsilon'$ and $\varepsilon''$ there exists a unique path that starts by one of them and ends by the other one; we denote this path by $[\varepsilon', \varepsilon'']$.

It turns out that tight cuts are monotonous along paths on the skeleton. More precise, the following generalization of Lemma 4.9(i) holds.

**Lemma 6.1.** For any path $P = (N_0, \varepsilon_1, N_1, \ldots, \varepsilon_r, N_r)$ on the skeleton $\mathcal{H}_S$, $r \geq 2$, one has $C(N_0, \varepsilon_1) < C(N_{r-1}, \varepsilon_r)$.

**Proof.** Observe that the sets $S \cap V_{C(N_0, \varepsilon_1)}$ and $S \cap V_{C(N_{r-1}, \varepsilon_r)}$ are nonempty, since the cuts in question are $S$-cuts. Moreover, $S \cap V_{C(N_0, \varepsilon_1)}$ is contained in $S \cap V_{C(N_{r-1}, \varepsilon_r)}$, since both sets are $\varphi_S$-preimages of subtrees of $\mathcal{H}_S$ and the subtree of the former is contained in that of the latter; the set inclusion is strict since the partitions of $S$ by distinct bunches are distinct.

Let $a$ and $b$ be arbitrary vertices in $S \cap V_{C(N_0, \varepsilon_1)}$ and $S \cap V_{C(N_{r-1}, \varepsilon_r)}$, respectively. Both $C(N_0, \varepsilon_1)$ and $C(N_{r-1}, \varepsilon_r)$ are minimum $(a, b)$-cuts, so their intersection $C$ is a minimum $(a, b)$-cut as well. By the set inclusion proved above, the cut $C$ belongs to the bunch defined by $\varepsilon_1$, and thus coincides with $C(N_0, \varepsilon_1)$, since the latter cut is tight. This coincidence, together with the strict inclusion $S \cap V_{C(N_0, \varepsilon_1)} \subset S \cap V_{C(N_{r-1}, \varepsilon_r)}$, implies the assertion of the lemma. $\Box$

**Corollary 6.2.** For any path $P = (N_0, \varepsilon_1, N_1, \ldots, \varepsilon_r, N_r)$ on the skeleton $\mathcal{H}_S$, $r \geq 2$, one has $V_{C(N_0, \varepsilon_1)} \cap V_{C(N_r, \varepsilon_r)} = \emptyset$ and $V_{C(N_1, \varepsilon_1)} \nsubseteq V_{C(N_{r-1}, \varepsilon_r)}$.

**Remark.** Let $\varepsilon'$ and $\varepsilon''$ be two arbitrary structural edges and $C'$ and $C''$ be two tight cuts in directions $\varepsilon'$ and $\varepsilon''$, respectively. Observe that Lemma 6.1 and Corollary 6.2 cover all the possible choices for $C'$ and $C''$.

6.2. Reconstruction of pairwise strips. According to Lemma 5.4, one can obtain the strip representation for any bunch of $S$-mincuts just by a contraction of $\mathcal{F}_S$. This fact holds also in a more general situation, when one is interested in the strip representation $\mathcal{W}_{s_1, s_2}$ for an arbitrary pair of
vertices $s_1, s_2 \in S$, or, more generally, in the strip $W_{S_1,S_2}$ for an arbitrary pair of disjoint subsets $S_1, S_2 \subset S$, provided the cardinality of $(S_1, S_2)$-mincuts equals $\lambda_S$. Observe that in this case the $(S_1, S_2)$-mincuts, which are represented by $W_{S_1,S_2}$, are exactly the $S$-mincuts separating $S_1$ from $S_2$. In this subsection we describe the contractions of $F_S$ that give the corresponding strips. The reader can use for illustration Fig. 5.1d ($S_1 = \{s_1\}$, $S_2 = \{s_2, s'_2, s_3\}$), Fig. 5.1e ($S_1 = \{s_3\}$, $S_2 = \{s_1, s_2\}$), Fig. 5.1f ($S_1 = \{s_2\}$, $S_2 = \{s_3\}$).

Let $T_1$ and $T_2$ be two edge-disjoint subtrees of $H_S$. The link of $T_1$ and $T_2$ is defined as the unique path with one endpoint in $T_1$ and the other one in $T_2$ and having no edges in common with $T_1$ and $T_2$. We denote the link of $T_1$ and $T_2$ by $L(T_1, T_2)$. Observe that the link of two distinct nodes $M$ and $N$ is just the path $[M, N]$, the link of two distinct edges $e'$ and $e''$ extended by these edges themselves gives exactly the path $[e', e'']$, and the link of two subtrees intersecting by a node is this very node.

Let $S_1$ and $S_2$ be disjoint subsets of $S$. By Theorem 5.6, a bunch of $S$-mincuts separates $S_1$ from $S_2$ if and only if the deletion of the corresponding edge of $H_S$ produces two subtrees containing $\varphi_S(S_1)$ and $\varphi_S(S_2)$, respectively. Let $T(S_1)$ and $T(S_2)$ be the minimal subtrees of the tree $H_S$ containing $\varphi_S(S_1)$ and $\varphi_S(S_2)$, respectively. For brevity, we write $L(S_1, S_2)$ instead of $L(T(S_1), T(S_2))$. The following statement follows easily from Theorem 5.6 and Lemma 6.1.

**Fact 6.3.**

(i) If $T(S_1) \cap T(S_2) \neq \emptyset$, then there are no $S$-mincuts separating $S_1$ from $S_2$.

(ii) Otherwise, the edges of $H_S$ corresponding to the bunches of $S$-mincuts separating $S_1$ from $S_2$ form the link $L(S_1, S_2)$.

(iii) Under the assumptions of (ii), let $N_1$ be the endpoint of $L(S_1, S_2)$ lying in $T(S_1)$ and $e_1$ be the edge of $L(S_1, S_2)$ incident to $N_1$. Then the $S_1$-tight $(S_1, S_2)$-mincut coincides with the $N_1$-tight cut in direction $e_1$.

Now we can prove the main result of this subsection, which is a generalization of Lemma 4.1(i),(ii), Lemma 4.7, and Lemma 4.9 (ii)-(iv). For an illustration see Fig. 6.1.

![Fig. 6.1. To the proof of Theorem 6.4](image_url)
Theorem 6.4. Let $S_1$ and $S_2$ be two subsets of $S$ such that $\mathcal{T}(S_1) \cap \mathcal{T}(S_2) = \emptyset$. The strip $W_{S_1,S_2}$ is obtained as follows.

The underlying graph of $W_{S_1,S_2}$ is obtained from $\mathcal{F}_S$ by contractions of certain subsets of units to a single heavy unit. These subsets correspond bijectively to the nodes of the link $\mathcal{L}(S_1,S_2)$. The subset $U_N$ that corresponds to a node $N$ is equal to the intersection of the inner sides of $N$-tight cuts in directions of the edges of $\mathcal{L}(S_1,S_2)$ incident to $N$.

The inherent partition at any noncontracted unit remains the same. The inherent partition at a new unit $U_N$ is induced by the edge-sets of the $N$-tight cuts defining $U_N$.

Proof. Let us delete all the edges of the path $\mathcal{L}(S_1,S_2)$. The trees of the forest thus obtained correspond bijectively to the nodes of $\mathcal{L}(S_1,S_2)$: a node $N$ corresponds to the tree $T_N$ that contains $N$. Denote by $\Sigma_N$ the inverse image of the set of all the nodes of $T_N$ under the mapping $\varphi_S$. Observe that $\Sigma_N$ is a nonempty subset of $S$. Indeed, if $N$ is an endpoint of the link, then $\Sigma_N$ is just the intersection of a side of a certain $S$-mincut with $S$, and is thus nonempty. If $N$ is an inner node of the link, then $\Sigma_N = \emptyset$ would imply that the cuts of $\mathcal{H}_S$ defined by the two structural edges incident to $N$ $\varphi_S$-induce the same 2-partition of $S$, in a contradiction to Theorem 5.6 (cp. the proof of Lemma 6.1). Besides, by Fact 6.3(ii), the $S$-mincuts separating $S_1$ from $S_2$ do not divide $\Sigma_N$. Therefore, in any case all the set $\Sigma_N$ lies in a single heavy unit of the strip $W_{S_1,S_2}$, which we denote by $U^H_N$. Observe that if $N$ is an endpoint of the link, then $U^H_N$ is a terminal of $W_{S_1,S_2}$ (since $\Sigma_N$ contains one of $S_1$ and $S_2$).

Let now $N_1$ and $N_2$ be two neighboring nodes in the link such that $N_1$ and $\varphi_S(S_1)$ lie on the same side of the edge $(N_1,N_2)$ in $\mathcal{H}_S$. Let us consider the set of all $(S_1 \cup \Sigma_{N_1}, S_2 \cup \Sigma_{N_2})$-mincuts (all of them are represented in $W_{S_1,S_2}$). It is easy to see that the subtrees $T(S_1 \cup \Sigma_{N_1})$ and $T(S_2 \cup \Sigma_{N_2})$ contain nodes $N_1$ and $N_2$, respectively. By Fact 6.3(ii), this set of cuts is represented by the strip $W(N_1,N_2)$ defined by the structural edge $(N_1,N_2)$. Therefore, by Lemma 3.8, to get the strip $W(N_1,N_2)$, it is sufficient to contract the corresponding cones of the nodes $U^H_{N_1}$ and $U^H_{N_2}$ in $W_{S_1,S_2}$.

By Facts 3.6(iii) and 6.3(ii) and Lemma 6.1, these cones are just the inner sides of the tight cuts $\mathcal{C}(N_1,(N_1,N_2))$ and $\mathcal{C}(N_2,(N_1,N_2))$, respectively. Since the intersection of the two opposite cones of any unit in a strip coincides with this unit, we thus get that the units $U^H_N$ constructed in the proof coincide with the sets $U_N$ defined in the formulation of the theorem.

Observe that, by Lemma 5.4, one can obtain $W(N_1,N_2)$ directly from $\mathcal{F}_S$ by contracting the inner sides of the same tight cuts $\mathcal{C}(N_1,(N_1,N_2))$ and $\mathcal{C}(N_2,(N_1,N_2))$. Therefore, there is a bijection between the units of $\mathcal{F}_S$ and $W_{S_1,S_2}$ that lie between pairs of the above described tight cuts; moreover, this bijection preserves the inherent partitions. Let us consider the set of units in $W_{S_1,S_2}$ that do not participate in that bijection. One can deduce
from Lemma 6.1 and Corollary 6.2 that each such unit $U$ lies either inside the tight cut $C(N, (N, M))$, where $N$ is an endpoint of the link and $(N, M)$ belongs to the link, or in the intersection of the inner sides of the two tight cuts $C(N, (N, N_1))$ and $C(N, (N, N_2))$, where $N$ is an inner node of the link and $N_1$ and $N_2$ are the neighbors of $N$ in the link. In the first case, $U$ is one of the terminal units of $W_{S_1, S_2}$ (by Fact 6.3(iii)). In the second case, $U$ lies in the intersection of the two opposite cones of the unit $U_N$ in $W_{S_1, S_2}$, and thus coincides with $U_N$. The claim concerning the inherent partition at $U_N$ follows immediately from the above construction and Fact 3.3. □

Remarks. 1. Let us distinguish a subset of coherent paths in $\mathcal{F}_S$ by preventing their inner units from “falling inside” any unit $U_N$, $N \in \mathcal{L}(S_1, S_2)$. By the construction above, these paths correspond naturally to the coherent paths of $W_{S_1, S_2}$ without inner units $U_N$, $N \in \mathcal{L}(S_1, S_2)$.

2. Observe that for any path $P = (N_0, \varepsilon_1, \ldots, \varepsilon_r, N_r)$ in $\mathcal{H}_S$, the link of the subsets of $S$ lying inside $C(N_0, \varepsilon_1)$ and $C(N_r, \varepsilon_r)$ is exactly $P$, since there are no empty leaves in $\mathcal{H}_S$. The strip corresponding to these two subsets we denote by $W(P)$.

6.3. The projection mapping. Let $U$ be an arbitrary unit of the flesh $\mathcal{F}_S$. We say that $U$ is projected to an edge $\varepsilon$ of the skeleton $\mathcal{H}_S$ if $U$ is a nonterminal unit of $W(\varepsilon)$ (for example, unit $U_2$ on Fig. 5.1b is a stretched unit of the strip $W(N_1, N)$ shown on Fig. 5.1d, and thus $U_2$ is projected to the edge $(N_1, N)$). In other words, $U$ is projected to $\varepsilon = (N_1, N_2)$ if it lies outside the $N_1$-tight and $N_2$-tight cuts in direction $\varepsilon$.

Theorem 6.5.

(i) For any stretched unit $U$ of the flesh, the set $\pi_S(U)$ of structural edges to which $U$ is projected is nonempty and is the edge set of a path in the skeleton.

(ii) For any terminal unit $U$ of the flesh, the set of structural edges to which $U$ is projected is empty, and there exists a unique node $\pi_S(U)$ of the skeleton such that $U$ belongs to all $\pi_S(U)$-tight cuts.

Proof. (i) Let us show that there exists a path on the skeleton containing all the structural edges in $\pi_S(U)$. If there is only one such edge, then the statement is trivial; thus in what follows we assume that $|\pi_S(U)| \geq 2$. Let us take a path $P = (N_0, \varepsilon_1, N_1, \ldots, N_{r-1}, \varepsilon_r, N_r)$ that is inclusion–maximal among all paths with both end-edges in $\pi_S(U)$. Suppose, to the contrary, that there exists an edge $\varepsilon \in \pi_S(U)$ that does not belong to $P$ (see Fig. 6.2a). Then the link of $\varepsilon$ and $P$ meets $P$ in an inner node $N_i$ (otherwise $P$ could be extended). We denote by $\varepsilon'$ the edge of this link incident to $N_i$. By Lemma 6.1, $U$ lies outside the cuts $C(N_i, \varepsilon)$, $C(N_{i+1}, \varepsilon_{i+1})$, and $C(N_{i+1}, \varepsilon')$. However, this contradicts to the following generalization of 3-Star Lemma(i).
Lemma 6.6. Let $N$ be a node of $H_S$, $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be structural edges incident to $N$, $C_i = C(N, \varepsilon_i)$, $i \in \{1, 2, 3\}$. Then $\bigcap C_i = \emptyset$.

Proof. Let us choose a vertex $s_i \in S$ in such a way that $s_i \in \hat{C}_i$, $i = 1, 2, 3$. Vertices $s_i$ are distinct, since they belong to inverse images (under $\varphi_S$) of disjoint subtrees of $H_S$. Observe that $T(\{s_j, s_k\})$ is a path containing both $\varepsilon_j$ and $\varepsilon_k$, but not containing $\varepsilon_i$. Thus, by Fact 6.3(iii), the $\{s_j, s_k\}$-tight $(\{s_j, s_k\}, s_i)$-minicut coincides with $C_i$. Therefore, $C_i$ is exactly the cut $C_{s_i}$ as defined in Sect. 4.3, and the assertion of the lemma follows from 3-Star Lemma(i). □

Remark. Clearly, the remaining assertions of 3-Star Lemma can be generalized to our case as well.

It remains to show that each edge $(N_{i-1}, \varepsilon_i, N_i)$ in $P$ belongs to $\pi_S(U)$. Indeed, $U$ lies outside both $C(N_1, \varepsilon_1)$ and $C(N_{r-1}, \varepsilon_r)$ and thus, by Lemma 6.1, outside both $C(N_i, \varepsilon_i)$ and $C(N_{i-1}, \varepsilon_i)$, as required.

(ii) Let now $U$ be a terminal unit of the flesh. Then, for each structural edge $\varepsilon$, $U$ lies inside exactly one of the two tight cuts in direction $\varepsilon$. Let us consider the collection $I$ of all tight cuts that contain $U$ inside and have inclusion-minimum inner sides. Let $C'$ and $C''$ be two arbitrary cuts in $I$, $\varepsilon'$ and $\varepsilon''$ be the corresponding structural edges, and let $[\varepsilon', \varepsilon''] = (N_0, \varepsilon', N_1, \ldots, N_{r-1}, \varepsilon'', N_r)$. Then, by Lemma 6.1 and the minimality of the cuts in question, one of the following possibilities holds: either $C' = C(N_0, \varepsilon')$ and $C'' = C(N_r, \varepsilon'')$, or $C' = C(N_1, \varepsilon')$ and $C'' = C(N_{r-1}, \varepsilon'')$.

In the first case, the tight cut $C(N_1, \varepsilon')$ contains $U$ inside (by Lemma 6.1), and thus both tight cuts in direction $\varepsilon'$ contain $U$ inside, a contradiction. In the second case, let us assume that there exists an edge $\varepsilon \in P(\varepsilon', \varepsilon'')$ distinct from $\varepsilon'$ and $\varepsilon''$. Then, by Lemma 6.1 and the minimality of $C'$ and $C''$, both tight cuts in direction $\varepsilon$ do not contain a terminal $U$ inside, a contradiction. Therefore, $C'$ and $C''$ are tight cuts of the same node $N$ ($N = N_1 = N_{r-1}$).

One can prove easily that any other cut in $I$ is also an $N$-tight cut.

Moreover, any $N$-tight cut belongs to $I$. Indeed, assume that $C$ is the $N$-tight cut in direction $\varepsilon$ and $C \notin I$ (see Fig. 6.2b). Denote by $C^*$ the other tight cut in direction $\varepsilon$. Then $U$ lies either inside $C$, or inside $C^*$. In the first
Thus, the projection mapping $\pi_S$ is defined, assigning to each unit $U$ the path of Theorem 6.5 (observe that a single node is itself a path); the endpoints of this path are called the coordinates of $U$ in $H_S$. For an example see Fig. 5.1b,c: the coordinates of the unit $U_1$ are $N$ and $N_3$, those of $U_2$ are $N_1$ and $N_3$, and those of $U$ are $N$ and $N$.

The triple $(F_S, H_S, \pi_S)$ is called the connectivity caricase of $S$. By Theorems 5.5(iii) and 5.6 its size is $O(\min\{m, \lambda_S n\}) = O(\min\{m, \lambda_S n\})$.

Let us consider a cut of the skeleton $H_S$ defined by a structural edge and representing a bunch $B$. If we delete this edge from the skeleton, it falls into two connected components $H_S(B)$ and $H_S(B)$ containing $\varphi_S(S_B)$ and $\varphi_S(\tilde{S}_B)$, respectively.

**Theorem 6.7.** The set of vertices lying inside the tight (resp. outside the loose) cut of $B$ is the union of all units $U$ of $F_S$ such that $\pi_S(U) \subseteq H_S(B)$ (resp. $\pi_S(U) \subseteq H_S(B)$).

**Proof.** Let us denote by $\varepsilon$ the structural edge corresponding to $B$, and by $N$ and $N$ its endpoints lying in $H_S(B)$ and $H_S(B)$, respectively. Then the tight cut of $B$ is $C(N, \varepsilon)$, and the loose cut of $B$ is $C(N, \varepsilon)$. Let now $U$ be an arbitrary unit of $F_S$. If $\varepsilon \in \pi_S(U)$, then, by definition, $U$ lies outside both $C(N, \varepsilon)$ and $C(N, \varepsilon)$. Otherwise, $\varepsilon \notin \pi_S(U)$; by Theorem 6.5 we may assume w.l.o.g. that $\pi_S(U) \subseteq H_S(B)$. Let us prove that $U$ lies inside $C(N, \varepsilon)$.

Assume to the contrary that this is not the case; then $U$ lies inside $C(N, \varepsilon)$, since otherwise the projection of $U$ would contain $\varepsilon$. If $U$ is a stretched unit, then there exists a structural edge $e' \in \pi_S(U) \subseteq H_S(B)$.

Let $[e', \varepsilon] = (N', \varepsilon, N', \ldots, N, \varepsilon, N)$. Then $U$ lies outside $C(N', \varepsilon)$ and inside $C(N, \varepsilon)$, in a contradiction to Lemma 6.1. Let now $U$ be a terminal unit, and $N' = \pi_S(U) \subseteq H_S(B)$ be its projection. We consider the link $(N', \varepsilon, \ldots, N)$ of $N'$ and $\varepsilon$ in $H_S$. By Theorem 6.5(ii), $U$ lies inside $C(N', \varepsilon)$, which, by Lemma 6.1, contradicts to the fact that $U$ lies outside $C(N, \varepsilon)$. $\Box$

It follows readily from Theorem 6.7 that the projections of distinct terminal units are distinct (it suffices to take for $B$ the bunch of an arbitrary cut separating one of these units from the other). Hence, $\pi_S$ provides an injection of the set of terminal units into the set of nodes of $H_S$.

**Theorem 6.7** provides another point of view on the construction of Theorem 6.4.

**Corollary 6.8.** A unit $U_N$ is the union of all units $U$ of $F_S$ such that $\pi_S(U) \subseteq T_N$. A unit $U$ of $F_S$ does not fall inside any unit $U_N$, $N \in$
\(L(S_1, S_2)\), if \(\pi_S(U) \cap L(S_1, S_2) \neq \emptyset\). A unit \(U_N\) is reachable from \(U\) if \(N\) is not inner for \(\pi_S(U) \cap L(S_1, S_2)\).

The following statement reveals an intimate relation between the projection of a stretched unit and its reachability cones. Let \(U\) be a stretched unit, \(R_1\) and \(R_2\) be the reachability cones of \(U\), \(N_1\) and \(N_2\) be the sets of nodes that are the projections of the terminals belonging to \(R_1\) and \(R_2\), respectively, \(T_1\) and \(T_2\) be the minimum subtrees of \(H_S\) containing \(N_1\) and \(N_2\), respectively. Observe that each reachability cone of \(U\) contains at least one terminal, and hence both \(T_1\) and \(T_2\) are nonempty.

**Theorem 6.9.** The projection of \(U\) coincides with the link of \(T_1\) and \(T_2\).

**Proof.** Let \(\epsilon\) be an arbitrary structural edge in the projection \(\pi_S(U)\) and \(B\) be one of the two opposite bunches represented by \(\epsilon\). It is easy to see that all the terminals that belong to \(R_1\) lie inside the tight cut of \(B\), while those in \(R_2\) lie outside the loose cut of \(B\), up to flipping of \(B\). Hence, by Theorem 6.7, \(T_1 \subseteq H_S(B)\) and \(T_2 \subseteq H_S(B)\). It follows immediately that the trees \(T_1\) and \(T_2\) are disjoint, and that \(\pi_S(U) \subseteq L(T_1, T_2)\).

Assume now that \(\pi_S(U) \neq L(T_1, T_2)\); then there exists a subpath \(P = (N', \epsilon', N, \epsilon'', N'')\) of the link such that \(\epsilon'\) belongs to \(\pi_S(U)\), while \(\epsilon''\) does not. By Corollary 6.8, \(U_N\) is reachable from \(U\) in \(W(P)\). Hence, by the remark after Theorem 6.4, there exists a coherent path in \(F_S\) that leads from \(U\) to the intersection of the inner sides of \(L(N, \epsilon')\) and \(L(N, \epsilon'')\). Let us extend this path to a terminal \(U^*\) of \(F_S\) (see Fig. 6.3).

**Fig. 6.3. To the proof of Theorem 6.8**

If the extended path leaves the inner side of the \(N\)-tight cut in direction \(\epsilon\) for some \(\epsilon \neq \epsilon', \epsilon''\), then it cannot return back, and, by Theorem 6.7, the projection of \(U^*\) lies in the subtree of \(H_S\) that is separated from \(N\) by \(\epsilon\). However, this projection belongs either to \(T_1\), or to \(T_2\), and hence one of the edges \(\epsilon'\) and \(\epsilon''\) does not belong to \(L(T_1, T_2)\), a contradiction.

Otherwise, \(U^*\) lies inside all the \(N\)-tight cuts, and thus, by Theorem 6.5, it is projected to \(N\), which gives the same contradiction as above. \(\square\)

By Theorem 6.9, each of \(T_1\) and \(T_2\) contains exactly one coordinate of \(U\). Therefore, we can label the reachability cones of any stretched unit (and
thus the parts of the inherent partition) by its coordinates; the reachability
cone of $U$ that contains the terminals whose projections coincide with or lie
"behind" the coordinate $N$ is denoted by $R_N(U)$. Moreover, we can now
redenote the trees $T_i$: the tree that contains $N$ is denoted by $T_N(U)$.

6.4. The cell structure. For any path on the skeleton with endpoints $M$
and $N$ we define the cell $Q^{MN}$ as the set of units with the coordinates $M, N$;
by definition, the cell $Q^{NM}$ coincides with $Q^{MN}$. A cell of type $Q^{MM}$ is
called terminal; it is either empty or consists of exactly one terminal unit
(with projection $M$).

Let us define the $\pi$-reachability relation on cells. We say that a cell
$Q^{M_2N_2}$ is $\pi$-reachable from a nonterminal cell $Q^{M_1N_1}$ in direction $M_1$ if both
$P_1 = [M_1, N_1]$ and $P_2 = [M_2, N_2]$ are subpaths of the same path $P$ so that
$M_1$ coincides with one of the endpoints of $P$ and at least one of $M_2, N_2$
coincides with the other endpoint of $P$. We denote this path $P$ by $[P_1, P_2]$
(which is consistent with the previous use of notation $\langle \cdot, \cdot \rangle$). Evidently, if
$Q^{M_2N_2}$ is $\pi$-reachable from $Q^{M_1N_1}$ in both directions, then $Q^{M_2N_2} = Q^{M_1N_1}$.

The following statement is a generalization of Lemma 4.11.

**Theorem 6.10.** Let $U_1$ be a stretched unit with coordinates $M_1$ and $N_1$, $U_2 \in R_{N_1}(U_1)$, $Q_1$ and $Q_2$ be the cells containing $U_1$ and $U_2$, respectively. Then $Q_2$ is $\pi$-reachable from $Q_1$ in direction $N_1$.

**Proof.** If $U_1$ is a terminal, then the assertion follows immediately from
Theorem 6.9. Let now $U_2$ be a stretched unit with coordinates $M_2$ and $N_2$, and let w.l.o.g. $U_1 \in R_{M_2}(U_2)$. Since $R_{N_1}(U_2) \subseteq R_{N_1}(U_1)$, we get
$T_{N_1}(U_2) \subseteq T_{N_1}(U_1)$, hence the path $[N_1, N_2]$ lies entirely in $T_{N_1}(U_1)$, and,
by Theorem 6.9, intersects with $[M_1, N_1]$ exactly by $N_1$. Therefore, $N_1$ lies
on $[M_1, N_2]$. Since $U_2$ is stretched, we can prove in a similar way that $M_2$
lies on $[M_1, N_2]$, as required. $\Box$

By Theorem 6.10, when we consider several units lying on the same
coherent path, we may assume that their coordinates are named consistently. In other words, if $\pi_S(U_1) = [M_1, N_1], \pi_S(U_2) = [M_2, N_2]$, and
$U_2 \in R_{N_1}(U_1)$, then we assume that $U_1$ belongs to $R_{M_2}(U_2)$, and hence
$[\pi_S(U_1), \pi_S(U_2)] = [M_1, N_2]$.

**Corollary 6.11.** Let $U$ lie on a coherent path $P$ from $U_1$ to $U_2$, then $M \in [M_1, M_2]$ and $N \in [N_1, N_2]$.

**Remark.** It follows immediately from Corollary 6.11 that when one builds
the strip $W([M_1, N_2])$ (as described in Theorem 6.4), no intermediate unit
of $P$ is subject to contraction.

We now can extend the definition of the projection to the edges of $F_S$,
similarly to what was done in Sect. 4.4. Let $e = (U_1, U_2)$ be an edge of $F_S$,
and assume that $U_1 \in Q^{M_1N_1}, U_2 \in Q^{M_2N_2}$. We the say that the projection
of $e$ equals $[M_1, N_2]$; in other words, we define $\pi_S(U_1, U_2) = \pi_S(U_1), \pi_S(U_2)]$. 
As in Sect. 4.4, this definition can be justified by inserting a dummy vertex in the edge $e$; it is easy to see that the projection of the unit that contains such a vertex is exactly $[M_1, N_2]$. As before, there are three types of edges belonging to a cell $Q^{MN}$: internal (both endpoints belong to $Q^{MN}$), external (exactly one endpoint belongs to $Q^{MN}$), and through (both endpoints do not belong to $Q^{MN}$).

For any proper path $[M, N]$, $M \neq N$, we build from $F_S$ a locally orientable graph $W^{MN}$ with the set of nonterminals $Q^{MN}$ (which may be as well empty). We delete all the cells non-$\pi$-reachable from $Q^{MN}$ (observe that by Theorem 6.10 there are no edges between $Q^{MN}$ and the deleted sets) and contract all the cells $\pi$-reachable from $Q^{MN}$ only in direction $M$ and those $\pi$-reachable only in direction $N$ into two terminals $\tilde{M}$ and $\tilde{N}$, respectively. Finally, we delete all the edges between $\tilde{M}$ and $\tilde{N}$ that do not belong to $Q^{MN}$. An $S$-mincut $\mathcal{C}$ of $G$ that separates the preimages of the terminals and thus generates a $(\tilde{M}, \tilde{N})$-mincut $\mathcal{C}$ in $W^{MN}$ is called an extension of $\mathcal{C}$.

The following result is a generalization of Lemmas 4.2, 4.8 and 4.12.

**Theorem 6.12.**

(i) The graph $W^{MN}$ is a strip; moreover, if $[M, N]$ contains at least two structural edges, then its width $\lambda^{MN}$ does not exceed $\lambda_S/2$.

(ii) All its transversal cuts, and only they, are extendable to $S$-mincuts. Such an extension can be found in any bunch corresponding to an edge in $[M, N]$ and in no other bunch.

(iii) Let $\{Q^{Mi, Ni}\}$ be a set of pairwise non-$\pi$-reachable cells such that all the paths $[M_i, N_i]$ contain at least one common edge. Then any set of transversal cuts of $W^{Mi, Ni}$ has a mutual extension to an $S$-mincut. Moreover, if all the paths $[M_i, N_i]$ contain at least two common edges, then $\sum_i \lambda^{Mi, Ni} \leq \lambda_S/2$.

(iv) If a 2-partition of $V \setminus \bigcup_i Q^{Mi, Ni}$ enters a mutual extension of some sample of transversal cuts, one from each of $W^{Mi, Ni}$, then it enters some mutual extension of any such sample. (Informally: samples of transversal cuts of these strips are interchangeable in $S$-mincuts).

**Proof.** We start from the following general observation. Let $e' = (M', N')$ be an arbitrary structural edge of $[M, N]$ (we assume that $[M, M']$ and $[N, N']$ do not intersect). Let us consider the strip $W(e')$. Its terminals correspond naturally to $M'$ and $N'$. Evidently, any unit of $Q^{MN}$ remains stretched in $W(e')$; its cones and the sides of its star are labeled by $M'$ and $N'$. Let $\mathcal{R}_M$ denote the union over $U \in Q^{MN}$ of the cones $\mathcal{R}_M(U)$ in $W(e')$; $\mathcal{R}_N$ is defined similarly.

Let us delete $\tilde{\mathcal{R}}_M \cap \mathcal{R}_N$ and contract $\mathcal{R}_M \cap \mathcal{R}_N$ and $\mathcal{R}_M \cap \tilde{\mathcal{R}}_N$ into two new terminals $\tilde{M}'$ and $\tilde{N}'$, respectively. It is easy to see that each direct edge between $\tilde{M}'$ and $\tilde{N}'$ corresponds to a through edge in $Q^{MN}$; the converse is not true, since the endpoints of a through edge do not necessarily belong to $\mathcal{R}_M$ or $\mathcal{R}_N$. Thus, the number of direct edges between $\tilde{M}'$ and $\tilde{N}'$ does
not exceed the number of through edges in $Q^{M\cap N}$. If it is strictly less, we add complementary direct edges in order to equalize these quantities. Let us denote the obtained locally orientable graph by $W^{M\cap N}$. We claim that $W^{M\cap N}$ coincides with $W^{M\cap N}$.

First of all, $Q^{M\cap N} = R_M \cap R_N$. Indeed, the inclusion $Q^{M\cap N} \subseteq R_M \cap R_N$ is trivial. On the other hand, if $U \in R_M \cap R_N$, then there exist $U^I, U^{II} \in Q^{M\cap N}$ such that $U$ lies on a coherent path from $U^I$ to $U^{II}$. Hence, by Corollary 6.11, $\pi_S(U) = [M, N]$, and $U \in Q^{M\cap N}$.

Let us consider an arbitrary edge $e = (U, U^*)$ such that $U \in Q^{M\cap N}$ and $U^* \notin Q^{M\cap N}$. While constructing $W^{M\cap N}$ and $W^{M\cap N}$, this edge is not deleted (by Lemma 2.1 and Theorem 6.10, respectively) and goes from $U$ to a terminal. In $W^{M\cap N}$ this terminal corresponds to the label of the side at $U$ in $F_S$ that contains $e$. By Theorems 6.7 and 6.10, the same holds true for $W^{M\cap N}$. Therefore, the stars of the corresponding terminals in $W^{M\cap N}$ and $W^{M\cap N}$ coincide, and thus these locally orientable graphs coincide as well.

(i) By the construction of $W^{M\cap N}$, it is a balanced acyclic two-terminal locally orientable graph, hence, by Strip Lemma (i), it is a strip.

Let $[M, N]$ contain at least two structural edges, and let $N^1, N^2, N^3$ be any three consequent nodes in $[M, N]$. We consider the strip $W([N^1, N^3])$ and define the cones $R_{N^1}$ and $R_{N^3}$ in it and the strip $\tilde{W}^{M\cap N}$ in the same way as above. By Theorem 6.10 and the construction of $W([N^1, N^3])$ in Theorem 6.4, the contracted unit $U_{N^2}$ belongs to $R_{N^1} \cap R_{N^3}$. Since $C = (R_{N^1}, R_{N^3})$ and $C^3 = (R_{N^1}, R_{N^3})$ are $S$-mincuts, there are no edges between $R_{N^1} \cap R_{N^3}$ and $R_{N^1} \cap R_{N^3}$ (by Lemma 2.1). Next, since $C \cap C^3$ is an $S$-mincut, one has $e(R_{N^1} \cap R_{N^3}, R_{N^1} \cap R_{N^3}) = \lambda_S - \lambda^{M\cap N}$; similarly, $e(R_{N^1} \cap R_{N^3}, R_{N^1} \cap R_{N^3}) = \lambda_S - \lambda^{M\cap N}$ (see Fig. 6.4). Hence, $e(R_{N^1} \cap R_{N^3}, R_{N^1} \cap R_{N^3}) = 2(\lambda_S - \lambda^{M\cap N})$. However, since $U_{N^2} \cap S \neq \emptyset$, one has $e(R_{N^1} \cap R_{N^3}, R_{N^1} \cup R_{N^3}) > \lambda_S$. Thus $\lambda^{M\cap N} < \lambda_S / 2$ as required.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig6.4.png}
\caption{To the proof of inequality $\lambda^{M\cap N} < \lambda_S / 2$}
\end{figure}

(ii) Existence of an extension for any $(M', N') \in [M, N]$ is proved in the same way as it was done in Lemma 4.12(ii), with the set $X_a$ replaced by $R_M \cap R_N$, and $X_a$ by $R_N \cap R_M$.

Any extension of a transversal cut of $W^{M\cap N}$ separates the preimages of $M$ from those of $N$, and hence, the set of terminals “behind” $M$ from the
set of terminals “behind” \( N \). By Fact 6.3(ii) and the second remark after Theorem 6.4, the bunch of such a cut must be defined by an edge in \([M, N]\).

(iii) Let \( \varepsilon' = (M', N') \) be a common edge of all the \([M_i, N_i]\). We consider the strip \( W(\varepsilon') \) and build the sets \( \mathcal{R}^i_{M'} \) and \( \mathcal{R}^j_{N'} \) as above. Let \( \mathcal{R}^*_M \) be the union of \( \mathcal{R}^i_{M'} \) over all \( i \), and let \( \mathcal{R}^*_N \) be defined similarly. Let \( \varepsilon = (U, U^*) \) be an edge such that \( U \in Q^{M, N_i} \), \( U^* \notin Q^{M, N_i} \), and \( U^* \notin \mathcal{R}^*_M \), then \( U^* \notin \mathcal{R}^*_N \). Indeed, assume to the contrary that \( U^* \in \mathcal{R}^*_M \), \( j \neq i \), then, by Theorem 6.10, the cell \( Q^{M, N_i} \) is \( \pi \)-reachable from \( Q^{M, N_i} \), a contradiction.

The rest of the proof of the existence of a mutual extension is similar to that of Lemma 4.12(iii), as above. The inequality for \( \sum \lambda^{M, N_i} \) follows immediately from Lemma 4.12(iii) applied to the strip \( W(\cap_i [M_i, N_i]) \). One should note, however, that some nodes of \((\cap_i [M_i, N_i])\) can be empty, and hence the inequality of type \( \lambda_{ji} \geq \lambda \lambda_{S}/2 \) for such a node is replaced by \( \lambda_{ji} \geq \lambda \lambda_{S}/2 \). As a consequence, we get a nonstrict inequality for \( \sum \lambda^{M, N_i} \).

(iv) The proof is similar to that of Lemma 4.12(iv). \( \square \)

6.5. Other properties of the connectivity carcass. Assume that we are interested in pointing out an \( S \)-mincut separating two given vertices of \( G \) that lie in distinct units \( U_1 \) and \( U_2 \). The following statement is obtained easily.

**Lemma 6.13.**

(i) Let \( \pi_S(U_1) \) and \( \pi_S(U_2) \) be distinct nodes of \( \mathcal{H}_S \). Consider any cut of \( \mathcal{H}_S \) separating them, and let \( B \) be any of the corresponding bunches. Then the tight and the loose cuts of \( B \) separate \( U_1 \) from \( U_2 \).

(ii) Let one of the projections, say, \( \pi_S(U_1) \), contain a structural edge \( \varepsilon_1 = (M_1, N_1) \), and let \( M, N \) be the coordinates of \( U_1 \) in the corresponding order. Then one of the sets \( \mathcal{R}_M(U_1) \cup \mathcal{V}^{c(M_1, \varepsilon_1)} \) and \( \mathcal{R}_N(U_1) \cup \mathcal{V}^{c(N_1, \varepsilon_1)} \) defines an \( S \)-mincut separating \( U_1 \) from \( U_2 \).

**Proof.** (i) Follows from Fact 6.3(ii).

(ii) It is clear from the structure of \( W_{\varepsilon_1} \) that both sets mentioned define \( S \)-mincuts; on the other hand, their intersection is exactly \( U_1 \neq U_2 \), and the result follows. \( \square \)

Consider now a pair \( s, t \) of non \((\lambda_{S} + 1)\)-connected vertices in \( S \). Let us decompose the connectivity carcass with respect to this pair.

Let \( U_i = U_{N_i}, 0 \leq i \leq k \), be the heavy units of \( W_{s,t} \), \( s \in U_0, t \in U_k \) (see Theorem 6.4 for details); it is easy to prove that \( U_i \) are numbered according to any topological order of \( W_{s,t} \). For any \( i \) as above we put \( \Sigma_i = S \cap U_i \) and contract the set \( S \) in two different ways: \( S^i \) is obtained by contraction of \( S \setminus \Sigma_i \) into a single vertex \( s^i \), and \( \tilde{S} \) is obtained by contraction of each of \( \Sigma_i \) into a single vertex \( \tilde{s} \). The graphs \( G^i \) and \( \tilde{G} \) are defined similarly as the results of the above two contractions applied to \( G \). Note that the connectivity of \( \tilde{S} \) in \( \tilde{G} \) is exactly \( \lambda_S \), while the connectivity of \( S^i \) in \( G^i \) can be greater than \( \lambda_S \).
It can be shown easily that the connectivity carcass of $S$ in $\tilde{G}$ has the same type as one considered in Sect. 4.4. Its skeleton is the path $[N_0, N_k]$ and its flesh is obtained from $W_{s,t}$ by turning the heavy units $U_i$, $0 \leq i \leq k$, into terminals.

Let $I$ denote the set of indices such that $\lambda_{s^i}(G^i) = \lambda_S$.

**Lemma 6.14.**

(i) The skeleton $H_S$ is obtained from the path $[N_0, N_k] = H_S(\tilde{G})$ and the skeletons $H_{s^i}(G^i)$, $i \in I$, by identifying $N_i$ with the node $N^i = \varphi_{s^i}(s^i)$ of $H_{s^i}(G^i)$.

(ii) The units of the flesh $F_S$ are:
- the units of $F_{s^i}(G^i)$, $i \in I$, except for the unit $U^i$ containing $s^i$;
- the nonheavy units of $W_{s,t}$;
- the terminal units $U^{(i)} = U^i \cap C_{N_i,N_{i+1}} \cap C_{N_i,N_{i-1}} \neq \emptyset$, $i \in I$, and $U^{(i)} = C_{N_i,N_{i-1}} \cap C_{N_{i+1},N_i} \neq \emptyset$, $i \notin I$, (for $i = 0$ and $i = k$ only one of the above two cuts is taken).

Moreover, if a unit other than $U^{(i)}$ participates in a more than one flesh as above, then it participates in $F_S(\tilde{G})$ and at most two of $F_{s^i}(G^i)$.

(iii) The projection of any unit other than $U^{(i)}$ is the concatenation of its projections in the corresponding connectivity carcasses.

**Proof.** Observe that each $S$-mincut of $G$ either partitions one of $\Sigma_i$, $i \in I$, and does not partition its complement in $S$, or separates some of $\Sigma_i$ from the others according to the skeleton $H_S$. More exactly, each bunch of $S$-mincuts of $G$ coincides either with a bunch of $\tilde{S}$-mincuts of $\tilde{G}$, or with a bunch of $S^i$-mincuts of $G^i$ for some $i \in I$. This implies assertion (i).

Let $U$ be a terminal unit of $F_S$ corresponding to a node $N$ of $H_S$. Recall that $U$ is the intersection of all $N$-tight cuts. If $N$ belongs to the subtree $H_i$ of $H_S$ hanging on the path $[N_0, N_k]$ at $N_i$ and $N \neq N_i$, then all these cuts are $N$-tight cuts of the connectivity carcass of $S^i$ in $G^i$. Therefore, $U$ is a terminal unit of this partial connectivity carcass. If $N = N_i$, $i \in I$, then all $N$-tight cuts are of the same type, except for one or two cuts corresponding to the edges of $[N_0, N_k]$ incident to $N_i$. Clearly, $U_i$ is a unit of $F_S$ for any $i \in I$. Thus we get assertion (ii) for terminals.

Let $U$ be a stretched unit. It participates in all bunches that correspond to the edges of its projection. The projection is contained either in one of $H_i$, or in $H_i \cup [N_0, N_k]$ for some $i$, or in $H_i \cup [N_i, N_j] \cup H_j$. The rest of assertion (ii) follows, as well as assertion (iii). \(\square\)

7. **Incremental transformations of the connectivity carcass**

7.1. **Transformations of the components of the connectivity carcass.** In this section we describe the transformations of all the three components of the connectivity carcass caused by insertion of a new edge $(u_1, u_2)$ that preserves the value of $\lambda_S$. In what follows $U_i$ stands for the unit containing $u_i$, $Q_i$ for the cell containing $U_i$, and $P_i$ for the projection $\pi_S(U_i)$,
i = 1, 2. If $U_1 = U_2$, then the carcass does not change. So in what follows we assume that $U_1 \neq U_2$. If $P_1$ and $P_2$ are edge-disjoint, we denote by $L_1$ and $L_2$, respectively, the endpoints of the link $L = L(P_1, P_2)$. Otherwise, we set $L = \emptyset$ and denote by $[M, N]$ the intersection of the projections; furthermore, in this case we assume w.l.o.g. that if $P_1 = [M_1, N_1]$ and $P_2 = [M_2, N_2]$, then the paths $[M_1, M_2]$ and $[N_1, N_2]$ do not intersect and $M \in [M_1, M_2]$, $N \in [N_1, N_2]$. In both cases we denote by $T$ the minimal subtree of $H_S$ containing both $P_1$ and $P_2$. Recall that we put hat over any notation (e.g., $\hat{H}_S$, or $\hat{F}_S$) to denote the corresponding object after edge insertion.

The main concern of this Section is to distinguish, in terms of the connectivity carcass, the $S$-mincuts that do not separate $U_1$ from $U_2$, since these are exactly the $S$-mincuts that are preserved under the insertion.

The transformations of the skeleton under edge insertion are as follows.

**Theorem 7.1.** If $L$ contains at least one structural edge, then all the nodes and structural edges of $L$ are contracted to a single new node; otherwise the skeleton remains the same.

*Proof.* Let $\hat{H}_S$ be obtained from $H_S$ by contraction of $L$ into a new node $N$ (or $\hat{H}_S = H_S$ if $L$ does not contain any edges). We define $\hat{\varphi}_S$ as follows: $\hat{\varphi}_S = \varphi_S$ if $\varphi_S \notin L$ and $\hat{\varphi}_S = N$ otherwise.

Let $\varepsilon$ be an arbitrary structural edge in $H_S$.

If $\varepsilon \notin T$, then, by Theorem 6.7, both $U_1$ and $U_2$ lie in the same terminal of the strip $W(\varepsilon)$. Hence, in this case the new edge does not affect any cut of this strip.

If $\varepsilon \in T$, but $\varepsilon \notin L$, then at least one of the units $U_1$ and $U_2$ is a stretched unit in $W(\varepsilon)$. Hence, in this case at least one tight cut of $W(\varepsilon)$ is not affected by the new edge, and thus the corresponding 2-partition of $S$ is still an $S$-mincut.

Finally, if $\varepsilon \in L$, then, by Theorem 6.7, $U_1$ and $U_2$ lie in distinct terminals of $W(\varepsilon)$. Hence, in this case no cuts represented by $W(\varepsilon)$ survive, and thus the corresponding 2-partition of $S$ ceases to be an $S$-mincut.

Thus, by Theorem 5.6, $\hat{H}_S$ is indeed the skeleton of the graph $G \cup (u_1, u_2)$. □

The transformations of the flesh under edge insertion are as follows.

**Theorem 7.2.** The new flesh is obtained by contraction of a certain subset of units of $F_S$ into a single new unit $U^{\text{new}}$. This subset contains:

(i) $U_1$ and $U_2$;

(ii) all the units lying on coherent paths between $U_1$ and $U_2$;

(iii) all the units $U$ such that $\pi_S(U) \subseteq L$;

(iv) in the case when an endpoint $K$ of $P_i$ is the unique common point of $P_i$ and $P_j \cup L$, $i, j \in \{1, 2\}$, $i \neq j$, all the units $U$ such that $U \in R_K(U_1)$ and $\pi_S(U) \subseteq P_i \cup L$. 
Proof. First, let us prove the following statement concerning the initial graph $G$.

**Claim A.** If a unit $U$ satisfies at least one of the conditions of Theorem 7.2, then any $S$-mincut that separates $U_1$ from $U$ separates also $U_1$ from $U_2$.

Indeed, if $U = U_2$, then the claim is trivial.

Let $U$ belong to a coherent path between $U_1$ and $U_2$. Let $C$ be an $S$-mincut that separates $U_1$ from $U$ but fails to separate $U_1$ from $U_2$, then we get a coherent path with both endpoints on one side of $C$ and an intermediate unit $U$ on the other side, a contradiction.

Let $\pi_S(U) \subseteq \mathcal{L}$, and let $\mathcal{C}$ be an $S$-mincut separating $U_1$ from $U$. By Theorem 6.7, the structural edge $\varepsilon$ representing $\mathcal{C}$ in $H_S$ lies either in $\mathcal{L}$, or in $P_1$. If $\varepsilon \in \mathcal{L}$, then $U_1$ and $U_2$ are separated by any cut represented by $\varepsilon$ (see the proof of Theorem 7.1). If $\varepsilon \in P_1$, then $U_1$ is a stretched unit of $\mathcal{W}(\varepsilon)$, while both $U$ and $U_2$ lie in the same terminal of $\mathcal{W}(\varepsilon)$, and the claim follows.

Assume now that an endpoint $K$ of $P_1$ is the unique common point of $P_1$ and $P_2 \cup \mathcal{L}$, and besides, $U \in R_K(U_1)$ and $\pi_S(U) \subseteq P_1 \cup \mathcal{L}$. Let $\mathcal{C}$ and $\varepsilon$ be as in the previous case, then, by Theorem 6.7, $\varepsilon$ belongs to $P_1 \cup \mathcal{L}$. If $\varepsilon \in \mathcal{L}$, we proceed as above. If $\varepsilon \in P_1 \cap \pi_S(U)$, then both $U_1$ and $U$ are stretched units of $\mathcal{W}(\varepsilon)$. Moreover, $U_2$ lies in one of the terminals of $\mathcal{W}(\varepsilon)$, and, by Theorem 6.10, $U$ lies in the cone of $U_1$ containing this terminal. Thus, $\mathcal{C}$ separates $U_1$ from $U_2$. If $\varepsilon$ lies in $P_1$, but not in $\pi_S(U)$, then, by Theorem 6.10, both $U$ and $U_2$ lie in the same terminal of $\mathcal{W}(\varepsilon)$, and again $\mathcal{C}$ separates $U_1$ from $U_2$.

Finally, if an endpoint $K$ of $P_2$ is the unique common point of $P_2$ and $P_1 \cup \mathcal{L}$, and besides, $U \in R_K(U_2)$ and $\pi_S(U) \subseteq P_2 \cup \mathcal{L}$, one has to interchange the roles of $U_1$ and $U_2$ in the above reasoning.

Therefore, Claim A is proved, and as an immediate corollary we get that all the units listed in Theorem 7.2 are indeed contracted into a single new unit.

Let us now prove another statement concerning the initial graph $G$.

**Claim B.** If a unit $U$ violates all the conditions of Theorem 7.2 and $U' \neq U$ is an arbitrary unit, then there exists an $S$-mincut that separates $U$ from $U'$ and does not separate $U_1$ from $U_2$.

Indeed, let $P = \pi_S(U)$, $Q$ be the cell containing $U$, and assume first that $U$ is a stretched unit.

Suppose that there exists a structural edge $\varepsilon \in P$ that does not belong to $\mathcal{T}$. Then $U$ is a stretched unit in $\mathcal{W}(\varepsilon)$, while both $U_1$ and $U_2$ lie in the same terminal of $\mathcal{W}(\varepsilon)$ (by Theorem 6.7). Therefore, any $S$-mincut that separates $U$ from $U'$ in $\mathcal{W}(\varepsilon)$ satisfies the assertion of the claim.

From now on we may assume that $P$ lies entirely in $\mathcal{T}$. By the assumptions of the Theorem, $P$ does not lie entirely in $\mathcal{L}$, hence, there exists an edge
\(\varepsilon \in P\) that belongs (w.l.o.g.) to \(P_1\). If \(\varepsilon\) belongs also to \(P_2\), then all the three units \(U, U_1,\) and \(U_2\) are stretched in \(W(\varepsilon)\). If \(U_1\) and \(U_2\) lie in the opposite cones of \(U\) in \(W(\varepsilon)\), then \(U\) lies on a coherent path from \(U_1\) to \(U_2\), which is prohibited by the assumptions of the claim. So, there exists a cone \(\mathcal{R}\) of \(U\) in \(W(\varepsilon)\) that does not contain neither \(U_1\), nor \(U_2\). If \(\mathcal{R}\) does not contain \(U\) as well, then the cut of \(W(\varepsilon)\) defined by \(\mathcal{R}\) satisfies the assertion of the claim. Otherwise, we use instead of \(\mathcal{R}\) the cone \(\mathcal{R}'\) of \(U'\) in \(W(\varepsilon)\) that is contained in \(\mathcal{R}\).

If \(\varepsilon \not\in P_2\), then in \(W(\varepsilon)\) both \(U\) and \(U_1\) are stretched, while \(U_2\) lies in a terminal. Let \(\mathcal{R}\) be the cone of \(U\) in \(W(\varepsilon)\) that contains the other terminal. If \(U_1\) does not belong to \(\mathcal{R}\), then the cut of \(W(\varepsilon)\) defined by \(\mathcal{R}\) (or by \(\mathcal{R}'\) as above, if \(U' \in \mathcal{R}\)) satisfies the assertion of the claim.

Let now \(U_1 \in \mathcal{R}\); this means, in particular, that \(U\) belongs to some cone of \(U_1\) in \(W(\varepsilon)\), and thus in \(\mathcal{F}_S\); say, \(U \in \mathcal{R}_K(U_1)\). By Theorem 6.10, this means that \(Q\) is \(\pi\)-reachable from \(Q_1\) in direction \(K\). Therefore, if \(\varepsilon_1\) is the edge of \(P_1\) incident to \(K\), then \(\varepsilon_1 \in P\). If \(\varepsilon_1 \in P_2\), we proceed in the same way as above; so, we may assume that \(\varepsilon_1 \not\in P_1\). Let us consider the strip \(W(\varepsilon_1)\). Both \(U\) and \(U_1\) are stretched in \(W(\varepsilon_1)\), moreover, \(U\) lies in a cone \(\mathcal{R}_1\) of \(U_1\), while \(U_2\) lies in a terminal. If this terminal does not belong to \(\mathcal{R}_1\), we consider the cone \(\mathcal{R}\) of \(U\) that lies inside \(\mathcal{R}_1\). It is easy to see that the cut of \(W(\varepsilon_1)\) defined by \(\mathcal{R}\) (or by \(\mathcal{R}'\) as above, if \(U' \in \mathcal{R}\)) satisfies the assertion of the claim.

Finally, let the terminal containing \(U_2\) in \(W(\varepsilon)\) lie in \(\mathcal{R}_1\). It follows immediately from Theorem 6.7 that in this case \(K\) is the unique common point of \(P_1\) and \(P_2 \cup L\). By the assumptions of the claim, \(P\) does not lie entirely in \(P_1 \cup L\), hence, there exists and edge \(\varepsilon_2\) of \(P\) such that \(\varepsilon_2 \in P_2\). Evidently, in \(W(\varepsilon_2)\) both \(U\) and \(U_2\) are stretched, while \(U_1\) lies in a terminal. Let \(U\) does not lie in the cone of \(U_2\) in \(W(\varepsilon_2)\) that contains this terminal, and let \(\mathcal{R}\) be the cone of \(U\) in \(W(\varepsilon_2)\) that contains the other terminal; then the cut of \(W(\varepsilon_2)\) defined by \(\mathcal{R}\) (or by \(\mathcal{R}'\) as above, if \(U' \in \mathcal{R}\)) satisfies the assertion of the claim. Otherwise, we use the same reasoning as above to see that one of the endpoints of \(P_2\) is the unique common point of \(P_2\) and \(P_1 \cup L\), and hence \(T\) is just a path. Therefore, \(Q_1\) and \(Q_2\) are \(\pi\)-reachable from \(Q\) in opposite directions. Thus, the coherent path from \(U_1\) to \(U\) in \(W(\varepsilon_1)\) can be concatenated with a coherent path from \(U\) to \(U_2\) in \(W(\varepsilon_2)\), and we get that \(U\) lies on a coherent path from \(U_1\) to \(U_2\), in a contradiction with the assumptions of the claim.

Assume now that \(U\) is a terminal unit, and let \(P' = \pi_S(U')\). Observe that \(P\) from now on is a node of \(H_S\).

If \(P \cap P' = \emptyset\), then, by the assumptions of the claim, there exists a structural edge \(\varepsilon\) that belongs to the link of \(P\) and \(P'\) and does not lie in \(L\). Evidently, the units \(U\) and \(U'\) lie in the opposite terminals of the strip \(W(\varepsilon)\). On the other hand, the units \(U_1\) and \(U_2\) cannot lie in the opposite terminals in such a strip. Therefore, one of the tight cuts in direction \(\varepsilon\) satisfies the
assertion of the claim.

Let now \( U' = U_1 \) and \( P \) be a node of \( P_1 \). If \( P \) is not an endpoint of \( P_1 \), then at least one of the two \( P \)-tight cuts in directions of the edges of \( P_1 \) incident to \( P \) does not contain \( U_2 \) (otherwise \( P \) would belong to \( \mathcal{L} \)). If \( P \) is an endpoint of \( P_1 \), then the \( P \)-tight cut in direction of the edge of \( P_1 \) incident to \( P \) does not contain \( U_2 \) (for the same reason as above).

This completes the proof of Claim B, since if \( U' \) is stretched and satisfies the assumptions of the claim, then we can use the same reasoning as above interchanging the roles of \( U \) and \( U' \), whereas if \( U' \) violates the assumptions of the claim, then any cut that separates \( U \) from \( U_1 \) separates also \( U \) from \( U' \), by Claim A.

As an immediate corollary of Claim B we get that all the units not listed in Theorem 7.2 remain uncontracted, and thus the Theorem is proved. \( \square \)

The transformations of the projections under edge insertion are as follows.

**Theorem 7.3.**

(i) The projection of \( U' \) is either the new node obtained by the contraction of \( \mathcal{L} \) (if \( \mathcal{L} \) contains at least one structural edge), or \( P_1 \cap P_2 \), otherwise.

(ii) The intersection with \( \mathcal{L} \) is deleted from the projection of any noncontracted unit.

(iii) For each unit \( U \in \mathcal{R}_{\mathcal{N}_1}(U_1) \), the common part of \( \pi_S(U) \) and the path between \( M_1 \) and \( M \) (or \( I_1 \)) is deleted from \( \pi_S(U) \). Similar transformations are applied to \( \mathcal{R}_{\mathcal{M}_1}(U_1), \mathcal{R}_{\mathcal{N}_1}(U_2), \mathcal{R}_{\mathcal{M}_2}(U_2) \).

**Proof.** Assume first that \( U \) is an arbitrary stretched unit of the initial graph, and let \( \varepsilon \) be an arbitrary structural edge of \( \pi_S(U) \).

If \( \varepsilon \notin \mathcal{T} \), then both \( U_1 \) and \( U_2 \) lie in the same terminal of \( \mathcal{W}(\varepsilon) \), and thus the new edge does not affect any cut of \( \mathcal{W}(\varepsilon) \). Hence, \( \varepsilon \) is preserved in the new projection of \( U \).

If \( \varepsilon \in \mathcal{L} \), then \( U_1 \) and \( U_2 \) lie in the opposite terminals of \( \mathcal{W}(\varepsilon) \), and thus all the cuts of \( \mathcal{W}(\varepsilon) \) do not survive. Hence, \( \varepsilon \) is deleted from the new projection of \( U \).

If \( \varepsilon \in P_1 \cap P_2 \), then both \( U_1 \) and \( U_2 \) are stretched units of \( \mathcal{W}(\varepsilon) \). Therefore, both \( \mathcal{W}(\varepsilon) \) remain unaffected by the new edge, and hence \( \varepsilon \) is preserved in the new projection of \( U \).

The only remaining case is when \( \varepsilon \) belongs to exactly one of the \( P_1 \) and \( P_2 \); assume w.l.o.g. that \( \varepsilon \in [M_1, K] \), where \( K = M \) if \( P_1 \cap P_2 \neq \emptyset \), and \( K = I_1 \) otherwise. Then \( U_1 \) is a stretched unit of \( \mathcal{W}(\varepsilon) \), while \( U_2 \) lies in the terminal of \( \mathcal{W}(\varepsilon) \) that belongs to \( \mathcal{R}_{\mathcal{N}_1}(U_1) \) (or, more exactly, to the cone of \( U_1 \) in \( \mathcal{W}(\varepsilon) \) that lies in \( \mathcal{R}_{\mathcal{N}_1}(U_1) \)). Therefore, if \( U \notin \mathcal{R}_{\mathcal{N}_1}(U_1) \), then \( \varepsilon \) is preserved in the new projection of \( U \), while otherwise it is deleted from the new projection.

Assume now that \( U \) is a terminal of \( \mathcal{F}_S \); evidently, the new edge cannot convert it to a stretched unit, and thus all we have to do is to establish that the new projection of \( U \) cannot shift.
By Theorem 6.5(ii), $U$ lies in the intersection of all the $K$-tight cuts, where $K = \pi_S(U)$. It follows easily from Theorem 7.1 that if $\varepsilon$ is incident to $K$ and $\varepsilon \notin L$, then the new $K$-tight cut in direction $\varepsilon$ dominates the old one. Therefore, if $K \notin L$, or $K = L$, then $K$ remains to be the projection of $U$ in the new skeleton.

Finally, let $K \in L$ and $K \neq L$; we denote by $\varepsilon_i$ and $\varepsilon'_i$ the edges of the path $[K, L_i]$ incident to $K$ and $L_i$, respectively $(i = 1, 2)$. By Lemma 6.1, any $L_i$-tight cut, except for the $L_i$-tight cut in direction $\varepsilon'_i$, dominates the $K$-tight cut in direction $\varepsilon_i$. By Theorem 7.1, all the edges incident to $L_i$ except for $\varepsilon'_i$ are preserved in the new skeleton, and are incident to its new node. Evidently, the tight cut of the new node in any of these directions dominates the $L_i$-tight cut in the same direction. Thus, the $U^{\text{new}}$ lies in the intersection of all the tight cuts of the new node. \(\square\)

Remark. It follows from Theorems 7.2 and 7.3 that if a unit must be contracted, then its new projection obtained formally via Theorem 7.3 coincides with that of $U^{\text{new}}$.

Finally, let us describe the transformation of the inherent 2-partition under edge insertion.

**Theorem 7.4.**

(i) The inherent 2-partitions at noncontracted units remain the same.

(ii) If $P_1 \cap P_2$ does not contain any edges, then the inherent 2-partition at $U^{\text{new}}$ is trivial. Otherwise, the inherent 2-partition for $U^{\text{new}}$ is glued from those for contracted units. Namely, the part of the star of $U_1$ labeled by $N_1$ (resp. $M_1$) is glued with that for $U_2$ labeled by $N_2$ (resp. $M_2$). For other units, two parts are glued together if the corresponding reachability cones contain the same unit out of $U_1$ and $U_2$.

Proof. (i) Follows immediately from Theorem 7.3, since the 2-partition at a unit $U$ of the flesh is either trivial, or inherited from the 2-partition at the same unit of any strip $W(\varepsilon), \varepsilon \in \pi_S(U)$.

(ii) The first part follows immediately from Theorem 7.3.

Let now $P_1 \cap P_2$ contain at least one edge $\varepsilon$. By Lemma 5.3, it is enough to prove the statement for the 2-partition at $U$ in the strip $W(\varepsilon)$. It follows from Theorems 7.1-7.3 that to get $\hat{W}(\varepsilon)$ from $W(\varepsilon)$ one has just to insert the edge $(U_1, U_2)$ into $W(\varepsilon)$ and to transform it accordingly. This transformation was already considered in Sect. 4.1, so the statement follows. \(\square\)

### 7.2. Local reachability cones.

The transformations of the connectivity carcass described in Theorems 7.2 and 7.3 are formulated in terms of reachability cones. Therefore, to maintain the carcass, we need to maintain the cones as well. However, reachability cones in an acyclic locally orientable graph behave in a more complicated way than those in a dag. In particular, insertion of an edge may cause a reduction of a reachability cone. Indeed, if a stretched unit belonging to a cone is contracted into a terminal (see
Theorem 7.2, then any coherent path passing through this unit cannot be traced behind it in a modified flesh. To avoid these difficulties, we introduce a simpler structure, called a local reachability cone, which preserves all the useful properties of reachability cones, but is easier to handle.

Let \( \mathcal{R} \) be a reachability cone of an arbitrary unit \( U \). We define the local reachability cone \( \mathcal{R}^{\text{loc}} \) as the set of units \( U' \in \mathcal{R} \) such that \( \pi_S(U) \cap \pi_S(U') \) contains at least one edge (in particular, the local cone of a terminal is empty). One can check easily that local reachability cones of \( \mathcal{R} \) and deleting the terminals. Observe that reachability cones in the assumptions of Theorems 7.2 and 7.3 can be replaced by the corresponding local reachability cones.

Indeed, for any unit \( U \) whose projection is changed according to Theorem 7.3(iii), the intersection \( \pi_S(U) \cap P_1 \) contains at least one edge, and hence such a unit belongs to \( \mathcal{R}^{\text{loc}}(U_1) \) (or, similarly, to \( \mathcal{R}^{\text{loc}}(U_1), \mathcal{R}^{\text{loc}}(U_2) \), or \( \mathcal{R}^{\text{loc}}(U_2) \)).

The units \( U \) distinguished by condition (iv) of Theorem 7.2 satisfy \( U \in \mathcal{R}(U_1) \) and \( \pi_S(U) \subseteq P_1 \cup \mathcal{L} \). However, if \( \pi_S(U) \cap P_1 \) contains any edges, the latter inclusion implies \( \pi_S(U) \subseteq \mathcal{L} \), and hence \( U \) is already distinguished by condition (iii). Otherwise \( \pi_S(U) \cap P_1 \) contains at least one edge, and this together with the former inclusion implies \( U \in \mathcal{R}^{\text{loc}}(U_1) \).

Finally, let us consider the units distinguished by condition (ii) of Theorem 7.2. Evidently, these units form the intersection of the two reachability cones \( \mathcal{R}(U_1) \) and \( \mathcal{R}(U_2) \) (for the sake of simplicity, we do not specify the labels of the cones). Let us prove that it suffices to consider only the units lying in the intersection \( \mathcal{R}^{\text{loc}}(U_1) \cap \mathcal{R}^{\text{loc}}(U_2) \). Indeed, if \( \mathcal{R}(U_1) \cap \mathcal{R}(U_2) = \emptyset \), then \( \mathcal{R}^{\text{loc}}(U_1) \cap \mathcal{R}^{\text{loc}}(U_2) = \emptyset \), and the assertion is trivial. Otherwise, by Theorem 6.10, the minimum tree containing \( P_1 \) and \( P_2 \) is a path, and the projection of any unit \( U \in \mathcal{R}(U_1) \cap \mathcal{R}(U_2) \) belongs to this path. If \( \pi_S(U) \) intersects both \( P_1 \) and \( P_2 \) at least by an edge, then \( U \in \mathcal{R}^{\text{loc}}(U_1) \cap \mathcal{R}^{\text{loc}}(U_2) \). If \( \pi_S(U) \) intersects both \( P_1 \) and \( P_2 \) at most by a node, then \( \pi_S(U) \subseteq \mathcal{L} \), and \( U \) is distinguished by condition (iii) of Theorem 7.2. Finally, if \( \pi_S(U) \) intersects at least by an edge exactly one of \( P_1 \) and \( P_2 \), then the assumptions of Theorem 7.2(iv) are satisfied, and \( U \) is distinguished by condition (iv).

8. Construction and incremental maintenance of the connectivity carcass

8.1. Construction of the connectivity carcass. We build the connectivity carcass by a recursive algorithm based straightforwardly on Lemma 6.14; in what follows we use the notation of this lemma.

First of all, we choose an arbitrary vertex \( s \in S \) and find maximal flows from \( s \) to all the other vertices \( t \in S \); this allows to define \( \lambda_S \) as the minimal value of these flows. All the vertices \( t \in S \) such that the value of a maximal flow from \( s \) to \( t \) exceeds \( \lambda_S \) are contracted together with \( s \); in what follows
s denotes this new vertex. We then choose an arbitrary of the remaining vertices of S (denoted by t) and execute the following procedure.

We build the strip $F = W_s t$, as explained in Sect. 3.2. To build the skeleton $H = H_S(G)$ we find a topological order of the units in $F$. Let $U_0 \ni s, U_1, \ldots, U_k \ni t$ be the heavy units in this order; we define $H$ as a $k$-edge path $(N_0, N_1, \ldots, N_k), k \geq 1$.

To find the corresponding projections $\pi = \pi_S$, we execute DFS in $F$ in direction $U_k$ first from $U_{k-1}$, then from $U_{k-2}$, and so on up to $U_0$. In any such execution we assign $N_i$ as a coordinate to the units found from $U_i$ and backtrack each time when we discover a unit already visited in the previous search. To get the other coordinate we repeat the same process in direction $U_0$ first from $U_1$, then from $U_2$, and so on up to $U_k$.

Next, for each $i$, $0 \leq i \leq k$, we turn the unit $U_i$ into a terminal. If $U_i$ contains vertices of $S$ distinct from $s$ and $t$, we do the following. We build the set $S'$ by taking all the vertices of $S$ that do not belong to $U_i$, together with $s$ and $t$, if they are not yet included in $S'$. We contract all the vertices of $S'$ into a new vertex $s'$, thus obtaining the new graph $G'$ and the new set $S'$ in it. We then scan the vertices $t'$ in $S' \setminus s'$ and find a maximum flow from $s'$ to $t'$. If its value exceeds $\lambda_S$, then we contract $t'$ to $s'$. Otherwise, we stop scanning and build the connectivity carcass $(H', F', \pi')$ of $S'$ in $G'$ by a recursive execution of the same procedure, with $s = s', t = t'$. This carcass is then merged with the current triple $(H, F, \pi)$ as follows.

The skeleton $H$ is merged with $H'$ by identifying the node $N_i \in H$ with the node $\pi'(s') \in H'$.

The new terminal unit corresponding to $N_i$ is built according to Lemma 6.14(ii).

To merge the partitions of $V$ into units of $F$ and $F'$ we assume that each vertex $v \in V$ knows its units $U(v)$ and $U'(v)$, and each unit knows its cardinality. Now, for each $v \in V$ we check the units $U(v)$ and $U'(v)$. If their cardinalities are equal, we concatenate their projections into the new projection of $U(v)$ and cancel $U'(v)$. Otherwise, we set the new $U(v)$ to be the smallest of the two units involved, preserve its projection, and cancel the other unit.

The above described algorithm implies the following result (the proof is omitted).

**Theorem 8.1.** The connectivity carcass of an arbitrary vertex subset $S$ can be constructed in time dominated by the complexity of $2\sigma - 2$ max-flow computations in $G$.

**8.2 Incremental maintenance of the connectivity carcass.** Let $n$ be the number of vertices in the initial graph $G$, $\bar{\sigma} \leq \sigma$ be the number of $(\lambda_S + 1)$-connectivity classes in $S$, $\bar{n} \leq n$ be the initial number of flesh units, $\bar{m} \leq m$, be the initial number of flesh edges.
We maintain the connectivity carcass under an arbitrary sequence of updates (edge insertions) not changing the connectivity of $S$ and queries “Are vertices $u, v \in G$ separated by an $S$-mincut?”, denoted by $\text{Sep}(u, v)$, “Show an $S$-mincut separating $u, v \in G^*$”, denoted by $\text{Cut}(u, v)$, and “Construct the strip $W_{S_1, S_2}$ for $S_1, S_2 \subseteq S$”, denoted by $\text{Strip}(u, v)$.

In order to maintain the connectivity carcass efficiently, we propose to keep track of the projections (in fact, the coordinates) and of certain parts of reachability cones specified below. It follows from the discussion in Sect. 7.2 that instead of a reachability cone $\mathcal{R}$ it suffices to maintain any partial subcone $\mathcal{R}^\text{part}$ of it that contains the corresponding local reachability cone $\mathcal{R}^\text{loc}$. By Lemma 3.4, a reachability cone $\mathcal{R}$ of an arbitrary unit $W$, and hence $\mathcal{R}^\text{part}$, is globally orientable. Since it is acyclic, it can be treated in the same way as a dag. In what follows we assume that $W$ is the source in this dag. Following [I], we represent $\mathcal{R}^\text{part}$ by a directed spanning tree rooted at $W$ and make use of the reachability vector of length $\bar{n}$. Each entry of the vector takes one of the three values to distinguish the following situations: the unit is currently contained in $\mathcal{R}^\text{part}$; the unit was never contained in $\mathcal{R}^\text{part}$; the unit was previously deleted from $\mathcal{R}^\text{part}$. At the initial moment we set $\mathcal{R}^\text{part} = \mathcal{R}$, which takes $O(\bar{n}m)$ time for all the cones together.

We represent the skeleton as a rooted tree with an arbitrary fixed root and make use of the data structure proposed in [W92]. This data structure allows to perform an arbitrary sequence of $w$ Nearest Common Ancestor (NCA) queries and edge contractions in $O(w + t \log^2 t)$ time, where $t$ is the size of the initial tree. In our case this means that we can perform NCA queries in $O(1)$ amortized time with an overhead of $O(\bar{\sigma} \log^2 \bar{\sigma})$, since by Theorem 5.6 the size of the skeleton is $O(\bar{\sigma})$.

The set of units is represented with the help of the standard union-find technique (see [AHU]) that allows to perform an arbitrary sequence of $w$ operations in $O(w + t \log t)$, where $t$ is the size of the initial set. In our case this means that we can perform find operations in $O(1)$ worst-case time with an overhead of $O(\bar{n} \log \bar{n})$. Each side of a unit $W$ is represented as a linked list of edges, and the whole list is labeled by the corresponding coordinate of $W$.

When a new edge is inserted, it takes $O(1)$ worst-case time to locate the units $U_1$ and $U_2$ containing its endpoints. The nontrivial case, when $U_1$ and $U_2$ are distinct, can occur at most $\bar{n}$ times (since this causes the contraction of these units). For each nontrivial case we do the following.

On the first stage we analyze the relative position of the projections $\pi_S(U_1)$ and $\pi_S(U_2)$. If they are edge-disjoint, we find their link and, according to Theorem 7.1, contract the edges of the link. Otherwise, we find their intersection and establish the correspondence between the endpoints of the intersection and the endpoints of the projections (see the beginning of Sect. 7.1 for details). It is easy to check that in order to find the link or the intersection of two projections, and to establish the latter correspon-
dence, it suffices to apply a constant number of NCA queries. Since we spend $O(1)$ amortized time for each of $O(n)$ NCA queries with an overhead of $O(\sigma \log^2 \sigma)$ (covering all contractions), the total time for the first stage is $O(\sigma + n + \sigma \log^2 \sigma) = O(n + \sigma \log^2 \sigma)$.

On the second stage we change the projections. Evidently, the total cost of projection changes prescribed by Theorem 7.3(i) is $O(n)$. Further, since projections are represented by coordinates, contractions in the skeleton do not imply explicit changes in this representation. Thus, projection changes prescribed by Theorem 7.3(ii) does not require any time.

Finally, to find units whose coordinates must be changed according to Theorem 7.3(iii), we execute DFS in the spanning tree of each of the four partial cones involved and backtrack each time when we reach a unit whose projection no more intersects a certain path in the skeleton, see Theorem 7.3(iii) for details (e.g., for the case of $R_{N_1}(U_1)$ this path is either $[M_1, M]$ or $[M_1, L_1]$). The validity of such a backtracking is justified by Theorem 6.10. It is easy to see that all the operations performed on the second stage, other than the analysis of the relative position of two paths in the skeleton, require $O(1)$ amortized time per unit. The latter analysis can be performed in $O(1)$ amortized time per unit as well by means of NCA queries as above.

To estimate the total amount of time required by projection changes in this case, let us assign a weight to each edge in the flesh. The weight of an edge is equal to the sum of the lengths of the projections of its endpoints (the length of a projection is just the number of structural edges in it). Observe that each time when an edge is scanned, the projection of its tail is truncated. Hence, the length of the projection of the tail strictly decreases, and the same occurs to the weight of the edge considered. Since the initial weight of each edge is $O(\sigma)$, we see that the overall number of projection changes is $O(\sigma n)$, and that of edge scans is $O(\sigma m)$. Thus, the total amount of work on the second stage is $O(n + \sigma n + \sigma m + \sigma \log^2 \sigma) = O(\sigma m)$.

The contractions in the flesh are performed on the third stage. Evidently, the total cost of contractions described in Theorem 7.2(i) is $O(n \log \tilde{n})$. The set of units that must be contracted according to Theorem 7.2(ii) is the intersection of two opposite local reachability cones (see the discussion in Sect. 7.2). To find these units we scan the reachability vectors of the corresponding partial cones, and for each unit that belongs to the intersection of the cones, compare its projection with $\pi_S(U_1)$ and $\pi_S(U_2)$. Since each contraction itself takes $O(\log \tilde{n})$ amortized time, the total amount of time for contractions in this case is $O(n^2 + n \log \tilde{n} + n \log^2 \sigma) = O(n^3)$.

To find the units that must be contracted according to Theorem 7.2(iii) and (iv), we scan all the units and check whether their new coordinates coincide with that of the contracted unit (see the remark after Theorem 7.3). This takes $O(1)$ amortized time per unit (via techniques of [W92]), and thus $O(n)$ time for the whole scan. Each contraction itself takes $O(\log \tilde{n})$
amortized time. Since each occurrence of this case leads to a contraction in
the skeleton, the total number of such occurrences is $O(\tilde{\sigma})$. Hence, the total
amount of work in this case is $O(\tilde{\sigma} \tilde{n} + \tilde{n} \log \tilde{n} + \tilde{\sigma} \log^2 \tilde{\sigma}) = O(\tilde{\sigma} \tilde{n} + \tilde{n} \log \tilde{n})$.

To find the 2-partition at the new unit, provided it is stretched, we es-
tablish the correspondence between the labels of the sides involved and con-
catenate the corresponding linked lists. This can be done in $O(\tilde{n} + \tilde{\sigma} \log^2 \tilde{\sigma})$ time with the same technique as above.

On the fourth stage we change partial reachability cones. Let $R^{\text{part}}$ be
a partial cone of some unit $W$. As it was described above, $R^{\text{part}}$ is rep-
resented by a spanning tree rooted at $W$ and the three-valued reachability
vector. Since the set of the contracted units is already known, we scan the
Corresponding entries of the reachability vector and find the list $L(R^{\text{part}})$ of
the units in $R^{\text{part}}$ that should be contracted. Since each unit is contracted
at most once, the total amount of work for these scans is $O(\tilde{n}^2)$. If the list
$L(R^{\text{part}})$ is empty, the cone $R^{\text{part}}$ is not changed. Otherwise we proceed as
follows (see the proof of Theorem 8.2 for support).

If $U^{\text{new}}$ is a terminal, we just delete all the units belonging to $L(R^{\text{part}})$,
together with their subtrees, from the tree representing $R^{\text{part}}$, and change
the reachability vector accordingly. Since each unit is deleted at most once
from each tree, and the total number of trees is $O(\tilde{n})$, the total amount of
work in this case is $O(\tilde{n}^2)$.

If $U^{\text{new}}$ is not a terminal, we do the following. For each unit $U$ belonging
to the list $L(R^{\text{part}})$ and distinct from $W$ we check whether its parent belongs
to the list; if it does, we contract the edge from $U$ to its parent, and if not,
we mark this edge. If there are marked edges, we identify all their tails
into a new unit $W'$ and delete all marked edges except for an arbitrary
one. Finally, if exactly one of $U_1$ and $U_2$ (say, $U_1$) never belonged to $R^{\text{part}}$,
the other one (in our case, $U_2$) belongs to $R^{\text{part}}$, and $\pi_S(W) \cap (\pi_S(U_1) \cap
\pi_S(U_2))$ contains at least one edge, we add a certain subtree to $R^{\text{part}}$ by
identifying its root with $W'$, if defined, or with $W$ otherwise. To obtain this
subtree we execute DFS in the spanning tree of the corresponding partial
cone of $U_1$ and backtrack each time we get to a unit that belongs to
$R^{\text{part}}$ or was previously deleted from $R^{\text{part}}$. Clearly, the total amount of
work in this case is proportional to the number of edge operations, that is,
contractions, marking and deletions of edges currently in $R^{\text{part}}$ and scans of
edges currently not in $R^{\text{part}}$. Each edge in $R^{\text{part}}$ is contracted or marked and
deleted at most once. Besides, each time when $R^{\text{part}}$ is updated, at most one
edge is marked and not deleted. Finally, each scan turns an edge currently
not in the entire cone $R$ to an edge in $R$. Moreover, such an edge never
belonged to $R$ before, hence each edge is scanned at most once. Therefore,
the number of edge operations is $O(\tilde{n} \tilde{m})$.

To find whether two vertices of $G$ are separated by an $S$-mincut, it suff-
fices to find the corresponding units (in $O(1)$ worst-case time) and to check
whether they do not coincide. If this is the case, the partition of $S$ corre-
sponding to such a cut can be obtained via Lemma 6.13 using Theorem 6.7. Finding a cut of the skeleton separating two given nodes and finding an edge of a projection takes $O(1)$ amortized time (with the help of NCA queries). To find a tight cut as in Lemma 6.13(ii), we check for all the units the inclusion of Theorem 6.7 in $O(1)$ amortized time per unit.

To obtain the strip $W_{S_1,S_2}$ of Theorem 6.4 we execute $O(|S_1|+|S_2|)$ NCA queries to check whether $\mathcal{T}(S_1) \cap \mathcal{T}(S_2) = \emptyset$, and if this is the case, to find the link $\mathcal{L}(S_1,S_2)$. Next, for each unit $U$ we check whether its projection intersects the path $\mathcal{L}(S_1,S_2)$ by an edge. If this is not the case, a few NCA queries give us the endpoint $N \in \mathcal{L}(S_1,S_2)$ of the link of $\mathcal{L}(S_1,S_2)$ and $\pi_S(U)$. The unit $U$ belongs to the contracted subset corresponding to $N$. To obtain the orientation, it suffices to execute DFS by coherent paths from any terminal unit of the strip.

The complexity of the above described algorithm is given by the following statement.

**Theorem 8.2.** The connectivity carass of an arbitrary vertex subset $S$ can be maintained in $O(\bar{n}m + u + q_{sep} + q_{cut} n + q_{strip} \bar{m})$ time for an arbitrary sequence of $u$ edge insertions preserving the value of $\lambda_S$, $q_{sep}$ queries Sep$(u,v)$, $q_{cut}$ queries Cut$(u,v)$, and $q_{strip}$ queries Strip$(u,v)$. Moreover, each query Sep$(u,v)$ can be answered in $O(1)$ worst-case time.

**Proof.** It follows from the discussion above that the only thing one has to check is that partial reachability cones are maintained properly, that is, that at any moment they remain intermediate between local reachability cones and ordinary ones. Since for a terminal both the local cone and its partial cone maintained by the algorithm are trivial, we assume from now on that $W$ is a stretched unit and remains stretched in the modified flesh.

It is convenient to use the same notation as in Sect. 7, that is, $P_1 = [M_1, N_1]$ and $P_2 = [M_2, N_2]$ are the projections of $U_1$ and $U_2$, respectively, $[M,N]$ is their intersection (it contains at least one edge by our assumption concerning $W$), the paths $[M_1, M_2]$ and $[N_1, N_2]$ are disjoint, and $M \in [M_1, M_2]$, $N \in [N_1, N_2]$. Besides, $\hat{\pi}_S(U)$ stands for the modified projection of $U$, provided $U$ exists in the modified flesh, or for the projection of $U^{new}$ otherwise. In the latter case $U$ is one of the units constituting $U^{new}$, and hence is implicitly contained in it as a set of vertices, which justifies our notation. In the same sense, we say that a cone is increased upon edge insertion if the set of vertices that constitute its units increases.

Let us prove first the following statement.

**Lemma 8.3.** If an edge insertion takes a unit $U$ out of a local cone $\mathcal{R}^{loc}$ of $W$, then $U$ will never belong to $\mathcal{R}^{loc}$ again.

**Proof.** We prove, moreover, that if $U \in \mathcal{R}^{loc}$ and $U \notin \hat{\mathcal{R}}^{loc}$, then already $\hat{\pi}_S(W)$ and $\hat{\pi}_S(U)$ intersect at most by a node; since the projections can only shrink upon edge insertions, this would mean that $U$ never belongs to...
Remark. Observe that the monotonicity property of Lemma 8.3 does not hold for the cone $\mathcal{R}$ itself.

It follows immediately from Lemma 8.3 that when maintaining $\mathcal{R}^{\text{part}}$, one does not need to include in it anew the units that were previously deleted from it.

Observe that exactly the same reasoning proves that if $U_{\text{new}}$ is a terminal, then the whole subtree rooted at any unit in $(\mathcal{R}^{\text{part}})$ does not belong to the corresponding $\hat{\mathcal{R}}^{\text{loc}}$. Thus, the case when the new unit is a terminal is completed.

Assume now that $U_{\text{new}}$ is stretched; clearly, in this case $\mathcal{R}$ is not decreased. First of all, observe that if $U', U'' \in L(\mathcal{R}^{\text{part}})$ and there exists a coherent path from $W'$ to $U''$ passing through $U'$, then all the units on this path lying between $U'$ and $U''$ belong to $L(\mathcal{R}^{\text{part}})$ as well. Indeed, let $U$ be such a unit, and consider an arbitrary $S$-mincut separating $U$, say, from $U'$. It follows immediately from Theorem 5.5(ii) that the same $S$-mincut separates also $U'$ from $U''$, and hence it no more exists after the edge insertion. Therefore, $U$ is contracted into $U_{\text{new}}$ as well. This means that the units in $L(\mathcal{R}^{\text{part}})$ form a set of subtrees in the tree representing $\mathcal{R}^{\text{part}}$ in such a way that no root of a subtree is a descendant of another root. It follows immediately that the contraction-marking-deletion-identifying procedure described in the algorithm is well defined, and that the tree thus obtained represents the part of $\hat{\mathcal{R}}^{\text{part}}$ containing all the vertices constituting $\mathcal{R}^{\text{part}}$, and thus $\hat{\mathcal{R}}^{\text{loc}}$. Clearly, if $\hat{\mathcal{R}}^{\text{loc}}$ does not increase, we are done.

Let us consider the case when $\mathcal{R}^{\text{loc}}$ is increased. Evidently, this happens if and only if the new unit belongs to $\hat{\mathcal{R}}^{\text{loc}}$ and at least one of $U_1$ and $U_2$ (say, $U_1$) does not belong to $\mathcal{R}^{\text{loc}}$. Let us prove the following statement.

Lemma 8.4. The conditions $U_{\text{new}} \in \hat{\mathcal{R}}^{\text{loc}}$ and $U_1 \notin \mathcal{R}^{\text{loc}}$ are equivalent to the following three: $U_1 \notin \mathcal{R}^{\text{part}}$, $U_2 \in \mathcal{R}^{\text{part}}$, and $\pi_S(W) \cap (P_1 \cap P_2)$ contains at least one edge.

Proof. Indeed, $U_{\text{new}} \in \hat{\mathcal{R}}^{\text{loc}}$ implies that $\pi_S(W) \cap (P_1 \cap P_2)$ contains at least one edge, therefore, the same is true for $\pi_S(W) \cap (P_1 \cap P_2)$ (since projections can only shrink). Next, if $\pi_S(W) \cap (P_1 \cap P_2)$ contains at least one edge and $U_1 \notin \mathcal{R}^{\text{loc}}$, then $U_1 \notin \mathcal{R}$; indeed, $U_1 \in \mathcal{R}$ would imply together with $U_1 \notin \mathcal{R}^{\text{loc}}$ that $\pi_S(W)$ intersects $P_1$ at most by a node, a contradiction. Clearly, $U_1 \notin \mathcal{R}$ implies $U_1 \notin \mathcal{R}^{\text{part}}$. The same reasoning with $U_1$ replaced by $U_2$ shows that if $U_2 \notin \mathcal{R}^{\text{part}}$, then $U_2 \notin \mathcal{R}$. To get finally the condition
\( U_2 \in \mathcal{R}^{\text{part}} \), it remains to rule out the case \( U_1, U_2 \notin \mathcal{R} \). It follows easily from Theorem 7.2 that the units constituting \( U^{\text{new}} \) are, apart from \( U_1 \) and \( U_2 \), exactly those lying on coherent paths between \( U_1 \) and \( U_2 \). Therefore, \( U_1, U_2 \notin \mathcal{R} \) together with \( U^{\text{new}} \in \hat{\mathcal{R}}^{\text{loc}} \) would imply the existence of a unit \( U \in \mathcal{R} \) lying on such a path; hence, extending a coherent path from \( W \) to \( U \) behind \( U \) we can get either to \( U_1 \), or to \( U_2 \), a contradiction.

In the other direction, \( U_1 \notin \mathcal{R}^{\text{part}} \) implies \( U_1 \notin \mathcal{R}^{\text{loc}} \) and \( U_2 \in \mathcal{R}^{\text{part}} \) implies \( U^{\text{new}} \in \hat{\mathcal{R}} \). By Theorem 7.3, any edge contained in \( \pi_S(W) \cap (P_1 \cap P_2) \) is contained also in \( \hat{\pi}_S(W) \cap (P_1 \cap P_2) \), hence, the existence of an edge in the former intersection together with \( U^{\text{new}} \in \hat{\mathcal{R}} \) implies \( U^{\text{new}} \in \hat{\mathcal{R}}^{\text{loc}} \).

In fact, in the algorithm, instead of checking condition \( U_1 \notin \mathcal{R}^{\text{part}} \), we check whether \( U_1 \) has never belonged to \( \mathcal{R}^{\text{part}} \). Indeed, if \( U_1 \) currently does not belong to \( \mathcal{R}^{\text{part}} \) but has belonged to it previously, then by Lemma 8.3 we do not need to include \( U_1 \) into \( \hat{\mathcal{R}}^{\text{part}} \).

![Fig. 8.1. To the dynamics of \( \mathcal{R}^{\text{part}} \)](image)

**Fig. 8.1. To the dynamics of \( \mathcal{R}^{\text{part}} \)**

Let us describe now the additional units acquired by \( \hat{\mathcal{R}}^{\text{loc}} \). Assume w.l.o.g. that the cones of \( W \) under consideration are in direction \( N_2 \) (by Theorem 6.10 this makes sense, since \( U_2 \notin \mathcal{R}^{\text{part}} \)). Let us prove that all the additional units belong to \( \mathcal{R}^{\text{loc}}_{N_1}(U_1) \) (see Fig. 8.1). Indeed, all such units evidently belong to \( \mathcal{R}_{N_1}(U_1) \). If \( U \in \mathcal{R}_{N_1}(U_1) \) but \( U \notin \mathcal{R}^{\text{loc}}_{N_1}(U_1) \), then \( \pi_S(U) \) does not have edges in common with \( \pi_S(U_1) \) and lies behind \( N_1 \) w.r.t. \( M_1 \). However, by Theorem 7.3(iii), the part of \( \pi_S(W) \) lying behind \( N_1 \) w.r.t. \( M_1 \) is deleted in the modified carcass. Hence, \( \hat{\pi}_S(W) \) intersects \( \hat{\pi}_S(U) \) at most by a node, and thus \( U \notin \hat{\mathcal{R}}^{\text{loc}} \). The backtracking rule in the cone \( \mathcal{R}^{\text{loc}}_{N_1}(U_1) \) is justified by Lemma 8.3. □

**8.3. Maintenance of the cell structure.** As one can see easily from the description of the algorithm in Sect. 8.2, the most time-consuming problem is to maintain the flesh, namely, the distribution of the vertices of the initial graph among units and reachability between units. Below we propose a less detailed, cell oriented approach. We for sure keep track of the distribution of vertices of \( G \) among cells, but do not guarantee the proper distribution of vertices among units. At any moment the partition of \( V \) into units represented by our data structure (henceforth, *preunits*) is, in a
sense, intermediate: it is a refinement of the true partition, while the initial partition is a refinement of this partition. More exactly, upon inserting a new edge we execute all the contractions as prescribed only if the resulting unit is a terminal; otherwise, we contract only the two preunits containing the endpoints of the inserted edge, as described in Theorem 7.2(i). Thus, the preflush can be not acyclic, but is a coherent locally orientable graph, and the true units are just its strongly connected components. Besides, we maintain only the 2-partitions at the preunits, but not reachability cones of any kind. The set of the preunits, the 2-partitions at each one of them, and the skeleton are represented in the same way as in the previous algorithm.

The preliminary stage of the new algorithm (distinguishing between trivial and nontrivial edge insertions) almost coincides with that of the previous one and has the same complexity $O(w)$. The only difference is that we locate not the units, but the preunits $U^p_1$ and $U^p_2$ that contain the endpoints of the new edge.

The first stage of the new algorithm (update of the skeleton) coincides literally with that of the previous one and has the same complexity $O(n + \sigma \log^2 \tilde{\sigma})$.

The only difference on the second stage (update of projections) is related to the case of units whose coordinates must be changed according to Theorem 7.3(iii). Instead of scanning the cones, we scan the two reachability subgraphs of $U^p_1$ and $U^p_2$. It is done by executing DFS four times (in both directions for both preunits) with backtracking each time when we reach a preunit whose projection no more intersects a certain path in the skeleton. It is easy to see that all the reasoning involving weights of edges remains valid in this case, hence the total amount of work on the second stage is $O(\sigma \tilde{m})$.

The third stage of the new algorithm (update of the preflush) is somewhat different from the corresponding stage of the previous one. If the new unit (and thus the new preunit) is not a terminal, we contract only the preunits $U^p_1$ and $U^p_2$. The 2-partition at the new preunit is maintained exactly as in the previous algorithm with the help of labels, and the total amount of time for this case is $O(\sigma \tilde{n} \log \tilde{n} + \sigma \log^2 \tilde{\sigma})$.

If the new unit (and thus the new preunit) is a terminal, we just contract all the preunits whose new projection coincides with the projection of the new terminal (see the remark after Theorem 7.3). Thus, the total amount of time in this case is $O(\sigma \tilde{n} + \tilde{n} \log \tilde{n} + \sigma \log^2 \tilde{\sigma}) = O(\sigma \tilde{n} + \tilde{n} \log \tilde{n})$.

To find whether two vertices of $G$ are separated by an $S$-mincut, we find the corresponding preunits (in $O(1)$ worst-case time). If they coincide, then such a cut does not exist. Otherwise, we check their projections. If the projections differ, then such a cut exists and is a cell cut. To find it we analyze the projections of all the preunits and verify the inclusion of Theorem 6.7. This can be done in $O(\tilde{n})$ amortized time by means of NCA queries as in the previous algorithm.
If the projections coincide, then the two preunits belong to the same non-tertiary cell. In this case we do not know at once whether a cut in question exists or not; thus we answer both queries simultaneously. According to Lemma 6.13, the work to be done is split into two parts. One of them consists in finding a certain cell cut and is done exactly in the same way as in the previous case and within the same time. The other part is to find a certain reachability cone of one of our preunits. It is done by scanning the reachability subgraph of this preunit. Since, unlike the previous algorithm, we have no special data structure supporting reachability cones, this is done similarly to the scanning on the second stage. Thus, the time for both queries in this case is $O(m)$.

Finally, to obtain the strip $W_{S_1,S_2}$ of Theorem 6.4 we proceed exactly as in the previous algorithm and get this strip up to contractions of strongly connected components. To get the true strip, we replace the ordinary DFS used in the previous algorithm for obtaining orientations by the extended DFS that also finds and contracts strongly connected components. The total time remains the same.

We thus get the following result.

**Theorem 8.5.** The cell structure of the connectivity carcass of an arbitrary vertex subset $S$ can be maintained in $O(\sigma m + m \log n + u + q m)$ time for an arbitrary sequence of $u$ edge insertions preserving the value of $\lambda_S$ and $q$ queries $\text{Sep}(u, v)$, $\text{Cut}(u, v)$, $\text{Strip}(u, v)$. Moreover, if at least one of the vertices $u,v$ is in $S$, or $u$ and $v$ belong to distinct cells, then the query $\text{Sep}(u, v)$ can be answered in $O(1)$ worst-case time, and the query $\text{Cut}(u, v)$ in $O(n)$ amortized time.

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