The Double Baseline is Rearrangeable

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Abstract

Feng and Wu [4] introduced the Baseline network and showed that it is isomorphic to the Butterfly. The Double Baseline results from linking two copies of the Baseline in tandem. We use the Layered Cross Product, introduced by Even and Litman [3], to prove that the Double Baseline is rearrangeable.

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1 Introduction

The Baseline network is known to be isomorphic to the Butterfly network. It is known that the Baseline network is not rearrangeable\(^3\). It is also known that the Beneš network, which is isomorphic to the Baseline linked to its mirror image, is rearrangeable [1], [6](see pages 451-456, Theorem 3.11), [7].

The question of whether the Double Butterfly (consists of two copies of the Butterfly network, in tandem) is rearrangeable has been open for many years\(^4\).

We use the fact that the Beneš network is rearrangeable and a variation of the Layered Cross Product, [3], to show that the Double Baseline is rearrangeable. In this variation, called the Ordered Layered Cross Product, OLCP, the order in which vertices appear in a layer is significant.

Since the Double Baseline is not isomorphic to the Double Butterfly, the question of whether the Double Butterfly is rearrangeable remains open.

2 Definitions

2.1 The Baseline Network

Let \( l \) be a positive integer. Let \( BL_1 \) denote the Baseline network with \( 2^l \) inputs and \( 2^l \) outputs, defined as follows:

1. \( BL_1 \) is defined as in Figure 1.

\[ \begin{array}{c}
\text{Figure 1}
\end{array} \]

2. \( BL_i = (V_0, V_1, \ldots, V_i, E) \), where

(a) \( V_0, V_1, \ldots, V_i \) are ordered disjoint sets of vertices. Each set contains \( n = 2^i \) vertices. (See Figure 2). In each row all vertices have the same label).

\(^3\)A network with \( n \) inputs and \( n \) outputs is called rearrangeable if for every permutation of the inputs to the outputs there is an appropriate routing. For a more detailed definition see the next section.

\(^4\)Leighton [6] reports that this problem has been solved (see page 778). However, the quoted source is inaccessible.
(b) The first \( l \) layers \( \{V_0, V_1, \ldots, V_{l-1}\} \) of \( BL_l \), with the set of edges between them, are composed of two copies of \( BL_{l-1} \).

(c) There is an edge between the \( i \)-th vertex \((0 \leq i \leq 2^l - 1)\) in layer \( l-1 \) and \((2 \cdot i \mod 2^l)\)-th vertex in layer \( l \).

(d) There is an edge between the \( i \)-th vertex \((0 \leq i \leq 2^l - 1)\) in layer \( l-1 \) and \((2 \cdot i + 1 \mod 2^l)\)-th vertex in layer \( l \).

Let the vertices of \( V_0 \) be the inputs of \( BL_l \) and the vertices of \( V_l \) be the outputs of \( BL_l \). Many variants of the Baseline appear in the literature under various names such as Butterfly, FFT Network, Omega network, etc; these directed networks are isomorphic to the Baseline network - see \([2, 5]\). However, for our purposes the order in which vertices are arranged in each layer is of significance. Thus our result relates only to the Baseline variant.

### 2.2 The Beneš Network

Let \( BL'_l \) denote the mirror image of \( BL_l \); i.e., the direction of the edges is reversed, the inputs (outputs) of \( BL'_l \) are the outputs (inputs) of \( BL_l \) (See Figure 3).

The Beneš network with \( 2^l \) inputs and \( 2^l \) outputs, \( BEN_l \), is obtained by the tandem of \( BL_l \) and \( BL'_l \): The outputs of \( BL_l \) are identified with the inputs of \( BL'_l \) (See Figure 4).
2.3 The Double Baseline Network

Let $BL_i^{(1)}$ and $BL_i^{(2)}$ be two copies of $BL_i$. The Double Baseline network with $2^i$ inputs and $2^i$ outputs, $DBL_i$, is obtained by the tandem of $BL_i^{(1)}$ and $BL_i^{(2)}$; i.e. the outputs of $BL_i^{(1)}$ are identified with the inputs of $BL_i^{(2)}$ (see Figure 5).

2.4 Rearrangeability

An ordered layered graph (or network), of $\lambda + 1$ layers, $G = (V_0, V_1, ..., V_\lambda, E)$, consist of:

1. $\lambda + 1$ layers of vertices; $V_i$ is the (nonempty) ordered set of vertices in layer $i$.
2. $E$ is a set of edges. Every edge $\langle u, v \rangle$ connects two vertices of adjacent layers; i.e. if $u \in V_i$ then $v \in V_{i+1}$. 

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Let a network \( N = (V_0, V_1, ..., V_\lambda, E) \) have an equal number of inputs and outputs; i.e. \( |V_0| = |V_\lambda| = n \).

1. A request for \( N \) is an ordered pair \((u, v)\) comprising an input \( u \) and an output \( v \).

2. A legitimate path from \( u \) to \( v \), is a path that contains \( \lambda \) edges, starts at \( u \), passes through one vertex in each layer and ends in \( v \).

3. \( N \) can satisfy the request \((u, v)\), if there is a legitimate path in \( N \) that connects \( u \) to \( v \).

4. Two requests are compatible if they have no input or output in common.

5. A permutation for \( N \), is a set of \( n \) mutually compatible requests for \( N \).

6. A routing \( R \), of \( N \), is a set of \( n \) vertex disjoint legitimate paths.

7. \( N \) can satisfy a permutation \( \Pi \) if there is a routing \( R \) such that for each request in \( \Pi \) there is a path in \( R \) through which \( N \) satisfies the request.

8. \( N \) is rearrangeable if \( N \) can satisfy all \( n! \) different permutations.

It is well known that a single Baseline is not rearrangeable; for example, for \( l > 1 \), the requests \((0, 0)\) and \((1, 1)\) cannot be satisfied simultaneously. It follows that none of the Baseline variants is rearrangeable. (The standard proof, that none of the Baseline variants is rearrangeable, uses a counting argument.)
2.5 The Ordered Layered Cross Product

Let \( A = \{a_1, a_2, ..., a_m\} \) and \( B = \{b_1, b_2, ..., b_n\} \) be two ordered sets. \( A \times B \) consist of the ordered set \( \{(a_i, b_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\} \) where \((a_p, b_q)\) precedes \((a_r, b_s)\) if \( p < r \) or, \( p = r \) and \( q < s \). Let \( G^{(1)}, G^{(2)} \) be ordered layered graphs, each of \( \lambda + 1 \) layers; i.e., for \( j \in \{1, 2\} \), \( G^{(j)} = (V^{(j)}_0, V^{(j)}_1, ..., V^{(j)}_\lambda, E^{(j)}) \). Their Ordered Layered Cross Product, \( \text{OLCP} \), \( G^{(1)} \times G^{(2)} \) is an ordered layered graph, \( G' = (V^{(1)}_0, V^{(1)}_1, ..., V^{(1)}_\lambda, E') \), where:

1. For every \( 0 \leq i \leq \lambda \), \( V'_i = V^{(1)}_i \times V^{(2)}_i \).
2. There is an edge \( \langle u', v' \rangle \) in \( G' \), connecting vertices \( u' = (u^{(1)}, u^{(2)}) \) and \( v' = (v^{(1)}, v^{(2)}) \), if and only if \( \langle u^{(1)}, v^{(1)} \rangle \) and \( \langle u^{(2)}, v^{(2)} \rangle \) are edges in \( G^{(1)} \) and \( G^{(2)} \), respectively.

3 The Rearrangeability of The Double Baseline

**Theorem 3.1** The Baseline network is the \( \text{OLCP} \) of two binary trees, one with its root on the right and one with its root on the left (as drawn in Figure 6).

**Proof:** By induction. Suppose it is true for \( l - 1 \) (\( BL_{l-1} \)). It is easy to see that in the graph \( G \), resulting from the \( \text{OLCP} \) of the two trees, there are \( l + 1 \) layers, each one contains \( 2^l \) vertices. In the first \( l \) layers we have \( \text{OLCP} \) of two full binary trees of height \( l - 1 \) with one full binary tree of height \( l - 1 \). By the inductive hypothesis, we get two \( BL_{l-1} \), i.e. the first \( l \) layers \( \{V_0, V_1, ..., V_{l-1}\} \) of \( G \), with the set of edges between them, are composed of two disjoint copies of \( BL_{l-1} \). Let us look now at layers \( V_{l-1} \) and \( V_l \). In the upper tree, let us label the 2 vertices of \( V_{l-1} \) by 0,1 and the vertex of \( V_l \) by 0. In the lower tree, let us label the vertices of \( V_{l-1} \) by 0,1,...,\( 2^{l-1} - 1 \) and the vertices of \( V_l \) by 0,1,...,\( 2^{l-1} - 1 \). In the lower tree vertex \( i \) in layer \( V_{l-1} \) is connected to vertices \( 2i \) and \( 2i + 1 \) in layer \( V_l \).

By the definition of the \( \text{OLCP} \) we get that in \( G \) vertex \( (0, i) \) (\( 0 \leq i \leq 2^{l-1} - 1 \)) in layer \( V_{l-1} \) is connected to vertices \( (0, 2i) \) and \( (0, 2i + 1) \) in layer \( V_l \). Vertex \( (1, i) \) (\( 0 \leq i \leq 2^{l-1} - 1 \)) in layer \( V_{l-1} \) is connected to vertices \( (0, 2i) \) and \( (0, 2i + 1) \) in layer \( V_l \). Thus, \( G \) is \( BL_l \).

**Theorem 3.2** The Double Baseline is rearrangeable.

**Proof:** Let \( BL_l \) denotes a Baseline network with \( l + 1 \) layers. Let \( BEN_l \) denotes a Beneš network with \( 2l + 1 \) layers. According to the definition of \( BEN_l \) (see section 2.2), \( BEN_l \) is the tandem of \( BL_l \) and \( BL'_{l} \).
Let $DBL_l$ denotes a Double Baseline network with $2l + 1$ layers. Let $II$ be a permutation to be satisfied by $DBL_l$. Clearly $BEN_l$ can satisfy $II$. Let $R$ denote a routing in $BEN_l$ that satisfies $II$. Notice that $R$ consists of two routings - $R_1$ in $BL_l$ and $R_2$ in $BL'_l$.

According to Theorem 3.1, $BEN_l$ is the OLCP of two structures $A$ and $B$. $A$ is a full binary tree of height $l$, which shares its root with its mirror image. $B$ is a full binary tree of height $l$, which shares its leaves with its mirror image (see Figure 7 for $l = 3$).

For every path $r$ in $R$, there are two projections on its multiplicands $A$ and $B$. Each of these projections is a path in the corresponding multiplicant. Also, $r$ is uniquely defined by its projections. Now we disconnect the two trees in each structure (Figure 8) and exchange the trees on the right hand side of $A$ and $B$ (Figure 9). This is done while the paths in each of the four binary trees are maintained; i.e., the pair of projected paths, for each path in $R_1$ and $R_2$, is recorded. We use the OLCP on the four trees to produce two copies of $BL_l - M_1$ on the left hand side and $M_2$ on the right hand side. Notice that both are drawn in the same direction (Figure 10). Also, the routing $R_1$ in $BL_l$ has not changed. Let $S_2$ denote the routing in $M_2$ which
results from the pairs of projected paths in the two trees on the right hand side of A and B. Suppose that a path \( r_2 \) in \( R_2 \) starts at input vertex \((0, j)\) and ends in output vertex \((k, 0)\). After exchanging the trees on the right hand side of A and B, and performing OLCP we have a path \( s_2 \) in \( S_2 \) that starts at input vertex \((j, 0)\) and ends in output vertex \((0, k)\). We connect the first \( BL_i \) to the second \( BL_i \) and we get \( DBL_i \).

Let \( r \) be a path in \( R \) which starts at input vertex \((i, 0)\) in \( BL_i \), passes through some output vertex \((0, j)\) in \( BL_i \), which is also an input vertex of \( BL_i' \), and ends in some output vertex \((k, 0)\) in \( BL_i' \). There is a path \( s \) in \( S \) which starts at input vertex \((i, 0)\) in \( M_1 \), passes through output vertex \((0, j)\) in \( M_1 \), which is also input vertex \((j, 0)\) of \( M_2 \), and ends in output vertex \((0, k)\) in \( M_2 \). Thus, we get a routing \( S \) which satisfies \( \Pi \). \( \square \)

![Figure 7]
References


Figure 9

Figure 10