

[7] M. Abramowitz, I.A. Stegun - Handbook of Mathematical Functions,

coefficients permits the reconstruction of the function precisely only at the *same collocation points*, from where the Fourier coefficients were red. Fig. 14 shows an interpolant function computed in the intermediate points $x_{j+\frac{1}{2}}$. The Gibbs phenomenon is clearly seen as well as in the Galerkin case.

![Graph showing the Gibbs phenomenon](image)

Figure 14: F-G interpolant $\tilde{f}_{N,m}^\lambda$ for the function (3.1) in the intermediate points $x_{j+\frac{1}{2}}$ using the collocation Fourier coefficients.

We conclude that the Fourier-Gegenbauer expansion with a large number of terms exhibits the Gibbs phenomenon both in the Galerkin and in collocation cases, inspite of the fact that the truncation and the regularization errors are exponentially small.

**References**


Figure 12: Maximum error for the collocation procedure using the "modified" Fourier coefficients with zero real components; \( N = 16, \lambda = 20 \).

Furthermore, we can obtain even better accuracy with smaller \( \lambda \) as shown in Fig.13.

Figure 13: Maximum error for the collocation procedure using the "modified" Fourier coefficients; \( N = 16, \lambda = 1 \).

Why did we obtain a Gibbs-free F-G interpolant using the approximate collocation Fourier coefficients, whereas the use of the exact Galerkin procedure leads to the Gibbs phenomenon for large \( m \). It turns out that the F-G algorithm with the collocation Fourier
computed. However, the irregularity in the computed solution has quite a specific form of the Gibbs oscillations. Besides, in the case of the continuous wave function, $w_k(x) = e^{ikx}$ with $k = 15$, the computation of the same order Gegenbauer terms, $T_i \approx 10^{10}$, gives the accuracy of $O(10^{-6})$, see Fig. 2. Thus, the Gibbs phenomenon manifests some intrinsic properties of the Gegenbauer expansion at large $m$ (this fact is not proven theoretically).

**Conclusion 4.** Inspite of the fact that the regularization error becomes exponentially small for large $m$, the F-G expansion exhibits the Gibbs phenomenon. Computations with different $N$ produce the same effect. The profiles of the F-G approximation are plotted in Fig. 11 for $N = 8$, 16, and 32 (in every particular case, $m$ is chosen such that $T_i < 10^{-40}$). The Gibbs phenomenon is seen distinctly in all these cases.

![Figure 11: Galerkin F-G approximation $f_{m,N}(x)$ for the function (3.1) for $N = 8$, $m = 40$ (rigid line) and $N = 32$, $m = 160$ (dashed line); $\lambda = 20$.](image)

### 4.3 Accuracy of the Collocation F-G Approximation for Large $m$

In the collocation case, the presence of non-zero real Fourier components $\hat{f}_k^R$, $k = 1, \ldots, N$, appearing due to the use of the Temperton’s DFFT routine, does not allow us to obtain an accurate F-G approximation at large $m$. Note that for small $m$ the presence of these false components did not affect the accuracy. An appropriate amendment of the DFFT algorithm will solve the problem. Meanwhile, let us see what happens if we pad the real components with zeros. The inverse DFFT, applied to the ”modified” vector with zero real components, results in the same function $f(x_j) = x_j$ but with zero boundary values at $x = \pm 1$.

The maximum error in this case is shown in Fig. 12. The lower curve corresponds to the interior part of the interval except for two boundary points. We see that the F-G approximation converges to the ”modified” distribution with zero values on the boundaries.
Figure 9: Maximum error for the Galerkin procedure; $N = 16$, $\lambda = 20$; lower curve corresponds to the interval without two boundary points $x = \pm 1$.

The F-G expansion $f_{m,N}^{\lambda}$ in the Galerkin case is plotted in Fig. 10 for $N = 16$, $\lambda = 20$, $m = 90$. We see that the $O(1)$ error for large $m$ in Fig. 9 corresponds to the Gibbs oscillations near the boundaries. Thus, the Gibbs phenomenon is "recovered" in the region of large $m$.

Figure 10: Galerkin F-G approximation $f_{m,N}^{\lambda}(x)$ for the function (3.1); $N = 16$, $\lambda = 20$, $m = 90$.

The possible explanation for this effect is that the accuracy is corrupted by the round-off errors. Such errors are introduced when the large amplitude terms, $T_i$ with $l \approx \mu$, are
The amplitude of the Gegenbauer terms $T_{2q-1}$ as a function of $q$ is plotted in Fig. 8 both for the Galerkin and collocation methods (terms with even indexes $T_{2q}$ are equal to zero due to the symmetry of the function $f(x) = x$).

In accordance with our expectations, the Gegenbauer series does not behave monotonically: the terms grow for $q < q_* \approx 20$ and decay for $q > q_*$. The ”bump” of this distribution corresponds to the location $\mu = 2q_* - 1 \approx 40$ of the maximum term in the Gegenbauer series for the wave function $w_k(x)$ with $k = 16$, see Eq.(2.25).

### 4.2 Accuracy of the Galerkin F-G Approximation for Large $m$

The question about the accuracy of the F-G approximation at large $m$ seems trivial as we already know that the individual terms in the Gegenbauer expansion decay and eventually become very small for large enough $m$. Therefore, the resolution error is small by definition. In particular, for the F-G expansion with $m = 90$ terms in the case of Fig. 8, we expect the accuracy to be no less than $10^{-4} - 10^{-5}$.

The maximum error of the Galerkin F-G approximation in this case is plotted in Fig. 9 (the upper curve corresponds to the whole interval, the lower curve corresponds to the interior part of the interval without two extreme points $x = \pm 1$). For $l < 40$ the error increases with $l$ up to a very large number of order 20. Then for $l > 40$ the error decays. For $l > 85$ the error remains of order one.
The error is smaller if $N$ is larger for a given $m$. This is because we need a sufficiently large number of terms in the sum (2.6) to enable it to converge precisely and thus to have a small truncation error for the coefficients $\hat{g}^{\lambda}(l)$. On the other hand, we can reduce the error by taking a smaller $m$ as shown in Fig. 7.

![Figure 7: Accuracy of the Galerkin F-G approximation increases as $m$ becomes smaller; $N = 16$, $\lambda = 20.$](image)

4 Convergence of the Gegenbauer Series: The Case of Large $m$

We saw that the accuracy of the F-G approximation deteriorates as $m$ grows. Would it be worse as $m$ increases? What will happen if $m$ becomes very large?

4.1 Behavior of the Gegenbauer Series for Large $m$

The indication for the behavior of the Gegenbauer series at large $m$ can be found in our previous results concerning the convergence of the F-G series for the function $w_k(x) = \exp(ikx)$, see section 2.3. We can think about a small truncation error, incurred while Eq.(2.6) is computed, as a superposition of the wave functions $w_k(x)$ with the maximum wave number $k = N$ allowed for a given resolution $2N+1$. Since the amplitudes of the terms in the Gegenbauer expansion increase exponentially with $k$, Eq.(2.6), the behavior of the Gegenbauer series at large $m$ will be determined chiefly by the mode with the largest $k$. In particular, we may expect non-monotonic behavior of the Gegenbauer series not only for the function of Eq.(2.21) but for any function.
Conclusion 2. The truncation error, incurred while the Gegenbauer coefficients are computed, is severely amplified "along the Gegenbauer series" (as the number $l$ of the term $T_l$ increases) due to the rapid growth of the Gegenbauer polynomials $|C^l_\lambda(\pm 1)|$ with $l$. Therefore, if the function is smooth enough, i.e., a small number of terms in the Gegenbauer expansion is enough to get a small regularization error, the truncation error will also remain small.

From Conclusions 1 and 2 Conclusion 3 follows.

Conclusion 3. For a smooth and non-periodic function a highly accurate F-G approximation can be constructed with a small number of the Fourier modes $N$ and the Gegenbauer terms $m$.

The accuracy of the F-G approximation for the function (3.1) is shown in Table 5 for $N = 16$ and $m = 1, 2$ and 4.

<table>
<thead>
<tr>
<th>$m$</th>
<th>collocation</th>
<th>Galerkin</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$8.31335D - 13$</td>
<td>$1.21930D - 09$</td>
</tr>
<tr>
<td>2</td>
<td>$5.65097D - 10$</td>
<td>$1.21930D - 09$</td>
</tr>
<tr>
<td>4</td>
<td>$1.80426D - 08$</td>
<td>$7.71467D - 07$</td>
</tr>
</tbody>
</table>

Table 5: Maximum error of the $(m + 1)$-term Gegenbauer expansion, $N = 16$, $\lambda = 20$

We clearly see that the requirement for $N$ to be large is not necessary in order to obtain high accuracy of the F-G approximation for a non-periodic function. Quite a moderate number of spectral modes, $N = 16$, is enough to ensure a good convergence of the sum over $k$ in Eq.(2.6) (that is to say, to "resolve" the discontinuity at the boundaries).

Now we can give a new interpretation for the results concerning the influence of $N$ on the accuracy of the F-G approximation. Fig. 6 shows the error as a function of $x$ for $m = 16$ and $N = 16$, 32 and 64.

Figure 6: Accuracy of the Galerkin F-G approximation for different $N$, $m = 16$, $\lambda = 20$. 

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Table 3: First 16 Gegenbauer terms computed using the Galerkin or collocation procedure; $N = 32$, $\lambda = 20$

Although the Gegenbauer coefficients $\bar{g}_N(l)$ in Table 2 have the same order of accuracy for all $l$, the error in the Gegenbauer terms is amplified with $l$. This is because the Gegenbauer polynomials grow with $l$, see the last column in Table 2.

### 3.1 Accuracy of the F-G Approximation for Small $m$

As we can learn from Table 3, the main contribution to the error gives the highest Gegenbauer terms. If we retain less terms in the Gegenbauer expansion then we get better accuracy, see Table 4.

Table 4: Maximum error for the $(m + 1)$-term Gegenbauer expansion, $N = 32$

Influence of $\lambda$. The results in Table 4 show that the accuracy is better for $\lambda = 24$ than for $\lambda = 8$. This is in accordance with the fact that the convergence of the sum over $k$ in (2.6) is faster for $\lambda = 24$ than for any other $\lambda$, see Fig. 3. Hence, the Gegenbauer coefficients are most accurately computed for such a $\lambda$. 

<table>
<thead>
<tr>
<th>$l$</th>
<th>$T_l$ Galerkin</th>
<th>$T_l$ collocation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.000000 (+00)</td>
<td>-0.867362 (-17)</td>
</tr>
<tr>
<td>1</td>
<td>0.100000 (+01)</td>
<td>0.100000 (+01)</td>
</tr>
<tr>
<td>2</td>
<td>0.000000 (+00)</td>
<td>0.305419 (-15)</td>
</tr>
<tr>
<td>3</td>
<td>0.488211 (-12)</td>
<td>0.258349 (-14)</td>
</tr>
<tr>
<td>4</td>
<td>0.000000 (+00)</td>
<td>-0.402126 (-13)</td>
</tr>
<tr>
<td>5</td>
<td>0.370015 (-10)</td>
<td>-0.164293 (-12)</td>
</tr>
<tr>
<td>6</td>
<td>0.000000 (+00)</td>
<td>-0.510594 (-11)</td>
</tr>
<tr>
<td>8</td>
<td>0.766639 (-09)</td>
<td>-0.285619 (-11)</td>
</tr>
<tr>
<td>9</td>
<td>0.000000 (+00)</td>
<td>-0.291007 (-09)</td>
</tr>
<tr>
<td>10</td>
<td>-0.257099 (-07)</td>
<td>-0.291490 (-10)</td>
</tr>
<tr>
<td>12</td>
<td>0.000000 (+00)</td>
<td>-0.805064 (-08)</td>
</tr>
<tr>
<td>13</td>
<td>-0.171455 (-05)</td>
<td>0.170950 (-08)</td>
</tr>
<tr>
<td>14</td>
<td>0.000000 (+00)</td>
<td>-0.110591 (-06)</td>
</tr>
<tr>
<td>15</td>
<td>-0.401355 (-04)</td>
<td>0.658875 (-07)</td>
</tr>
</tbody>
</table>
(theoretical proof the fact that (3.4) is small and $\nu = O(1)$ independent of the function $f(x)$ is not completed yet).

**Conclusion 1.** The criteria (2.15) on $N$ and $\lambda$ for having an exponentially small truncation error seems to be too much restrictive. An appropriate small factor is the sum over $k$ in Eq.(3.4) and not the individual values of the Fourier coefficients $|\hat{f}(k)|$.

**Influence of $\lambda$.** $\lambda$ must be sufficiently large for a fast convergence of the sum over $k$ in (2.6). Fig. 5 shows the logarithmic value of this sum as a function of $k$ for $l = 3$ and several $\lambda$ (the exact value of this sum in the present case is zero). The best convergence takes place for $\lambda = 24$ while for $\lambda = 8$ and $\lambda = 40$ it is slower. Therefore, the Chebyshev or Legendre polynomials, which are particular cases of the Gegenbauer polynomials at $\lambda = 0$ and $\lambda = \frac{1}{2}$ corresponding, are not a good choice for the present technique.

![Figure 5](image.png)

**Gegenbauer Terms**

Now we look at the terms in the Gegenbauer expansion. The amplitudes $T_i = \hat{g}_{\lambda}^N (l) \ C_i (1)$ of the first 16 terms are listed in Table 3 for the Galerkin and collocation cases.
Table 2: Gegenbauer coefficients $\hat{g}_N^l(i)$ in the Galerkin and collocation cases; $N = 32, \lambda = 20$. The last column is the Gegenbauer polynomials $C_l(1)$.

As a matter of fact, the sum over $k$ in expression (2.6) converges exactly to the same limit value for both $\hat{f}(k)$ or $\hat{f}_C(k)$ as shown in Fig. 4 (it is plotted for $N = 8$ to see clearly the difference between the Galerkin and collocation cases).

This observation illuminates an appropriate small parameter which determines the magnitude of the truncation error. It would not be

$$TE \propto A \left( \frac{2}{\pi N} \right)^{\lambda - 1}, \quad \left| \hat{f}(k) \right| \leq A$$

(see Eq.(2.15)) but rather

$$TE \propto \sum_{|k| \geq \nu} J_{l+\lambda}(\pi k) \left( \frac{2}{\pi k} \right)^{\lambda} \hat{f}(k)$$

---

```
Table: 0 0.000000 (+00) -0.867362 (-17) 0.100000 (+01)
1 0.250000 (-01) 0.250000 (-01) 0.400000 (+02)
2 0.000000 (+00) 0.372462 (-18) 0.820000 (+03)
3 0.425271 (-16) 0.225043 (-18) 0.114800 (+05)
4 0.000000 (+00) -0.325845 (-18) 0.123410 (+06)
5 0.346711 (-16) -0.151282 (-18) 0.108601 (+07)
6 0.000000 (+00) -0.626876 (-18) 0.814506 (+07)
7 0.143231 (-16) -0.533621 (-19) 0.535247 (+08)
```
Function

\[ f(x) = x, \quad -1 < x < 1 \] (3.1)

discretized on a uniform grid \( x_j, \ j = 1, \ldots, 2N + 1 \).

Fourier Coefficients

If the Fourier series approximates a real function then the Fourier coefficients (2.1) have the property:

\[
\hat{f}(k) = \hat{f}^{(R)}(k) + i \hat{f}^{(I)}(k),
\]

\[
\hat{f}(-k) = \hat{f}^{(R)}(k) - i \hat{f}^{(I)}(k).
\] (3.2)

The Fourier coefficients for the function (3.1) are listed in Table 1 both in spectral and in pseudospectral cases.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \hat{f}^{(R)}(k) )</th>
<th>( \hat{f}^{(I)}(k) )</th>
<th>( \hat{f}_{G}^{(R)}(k) )</th>
<th>( \hat{f}_{G}^{(I)}(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.000000</td>
<td>0.000000</td>
<td>-0.015625</td>
<td>0.000000</td>
</tr>
<tr>
<td>1</td>
<td>0.000000</td>
<td>0.318309</td>
<td>-0.015625</td>
<td>0.318054</td>
</tr>
<tr>
<td>2</td>
<td>0.000000</td>
<td>0.159154</td>
<td>-0.015625</td>
<td>0.158643</td>
</tr>
<tr>
<td>3</td>
<td>0.000000</td>
<td>0.106103</td>
<td>-0.015625</td>
<td>0.105335</td>
</tr>
<tr>
<td>4</td>
<td>0.000000</td>
<td>0.079577</td>
<td>-0.015625</td>
<td>0.078552</td>
</tr>
<tr>
<td>5</td>
<td>0.000000</td>
<td>0.063661</td>
<td>-0.015625</td>
<td>0.062378</td>
</tr>
<tr>
<td>6</td>
<td>0.000000</td>
<td>0.053051</td>
<td>-0.015625</td>
<td>0.051508</td>
</tr>
<tr>
<td>7</td>
<td>0.000000</td>
<td>0.045472</td>
<td>-0.015625</td>
<td>0.043668</td>
</tr>
</tbody>
</table>

Table 1: The first 8 Galerkin Fourier coefficients \( \hat{f}(k) \) and the collocation \((N = 32)\) Fourier coefficients \( \hat{f}_{C}(k) \) for the function (3.1).

Collocation Fourier coefficients in Table 1 are obtained using the Temperton’s DFFT code. Since it implies that \( f(x_{2N+1}) = f(x_1) \), we have the constant \( 0.015625000 = 1/2N \) instead of 0. Such an inconvenience will cause some troubles which are discussed below.

Gegenbauer Coefficients

The difference between Galerkin and collocation Fourier coefficients for \( N = 32 \) is quite perceivable. For larger \( N \) it is less, but still about several percents for a first few coefficients. Inspite of this, the Gegenbauer coefficients obtained by using \( \hat{f}(k) \) or \( \hat{f}_{C}(k) \) have practically the same values. This can be seen from Table 2.
An additional "burst" of growing and decaying terms in the vicinity of \( l \approx 40 \) is related to the high-frequency error modes generated by the Discrete Fast Fourier Transform (DFFT), that is employed in the pseudospectral method. The highly oscillating components with large \( k \approx N \) produce the Gegenbauer terms which grow rapidly with their numbers according to Eq.(2.6). Therefore, the presence of high-frequency error modes, even with very small amplitude, may affect the accuracy considerably if the computation involves the Gegenbauer terms with the large numbers \( l \). Note that the maximum amplitude at \( l \approx 40 \) corresponds to the wave number \( k = 16 \) according to Eq.(2.26): \( 0.83 \times \pi \times 16 \approx 41.7 \).

3 Convergence of the Gegenbauer Series:

The Case of Small \( m \)

The meaning of "small" and "large" \( m \) will be explained below.

For the simplicity of consideration, the analysis is performed for the linear function. However, the results are applied for any analytic non-periodic function.
The amplitude of the largest term grows exponentially with $k$:

$$T_\mu \propto e^{\alpha \pi k}, \quad 0.58 < \alpha < 1$$

(2.26)

where $\alpha \approx 1$ for $\pi k \ll \lambda$, and $\alpha \approx 0.58$ for $\pi k \sim \lambda$. Note that this result does not contradict the fact that the Gegenbauer series converges exponentially with $m$, if $m$ is defined in Eq. (2.22), since the ranges for $\beta$ in Eqs. (2.25) and (2.22) do not overlap.

The accuracy of the F-G series is affected by the round-off errors incurred as the large amplitude terms $T_l$ with $l \approx \mu$ are computed. The logarithmic pointwise error is plotted in Fig. 2 for the periodic (continuous) wave function $w_k$ with $k = 1, 7$ and 15. The number of terms $m$ in the Gegenbauer expansion is retained such that $T_l < 10^{-40}$ for $l > m$. In spite of the diminutive value of the regularization error (2.12), the accuracy near the boundary points $x = \pm 1$ deteriorates as $k$ increases. This corresponds to the above results: the largest term grows exponentially with $k$, and the computation of large terms produces the round-off errors. Note that this is a new result, that did not appear in [5]. However, it is placed here for the consistency of representation.

![Figure 2](image-url)  
Figure 2: Accuracy of the F-G approximation for the periodic wave function

If the Fourier coefficients are computed using the pseudospectral procedure, the Gegenbauer series exhibits some additional features. Fig. 3 shows the amplitudes of the terms $T_l$ in this case for $k = 1$, $N = 16$, $\lambda = 24$. In the region $1 < l < 20$, these terms behave as in Fig. 1 for $k = 1$. However, unlike the spectral case, for $l > 20$ the terms start to grow, reach the maximum at $l = \mu \approx 40$ and, ultimately, decay in the region $l > \mu$. 

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2.3 Behavior of the Gegenbauer Series for the Wave Function

In this section we summarize some results obtained in [5] concerning the behavior of the Gegenbauer series for the wave function (2.21).

Let \( T_l \) be the amplitude of the \( l \)-term in the Gegenbauer expansion:

\[
T_l = \max_{-1 \leq x \leq 1} |\hat{f}^\lambda(l) C_l^\lambda(x)|
= |f^\lambda(l)| \cdot |C_l^\lambda(1)|. \tag{2.23}
\]

For the wave function, the amplitudes \( T_l \) have the following properties:

- The dependence of \( T_l \) on their index \( l \) is not monotonic; first, the quantities \( T_l \) grow with \( l \) until a maximum is reached at \( l = \mu \) where \( \mu \) satisfies

\[
T_\mu = \max_{0 < l < \infty} T_l.
\tag{2.24}
\]

Then, for \( l > \mu \), these terms decay.

The behavior of \( T_l \) versus \( l \) for the wave function (2.21) is shown in Fig. 1.

![Figure 1: Amplitudes of the Gegenbauer terms \( T_l \) for the wave function with \( k = 1, 6, \) and 12, computed by the spectral method; \( \lambda = 24 \).](image)

- If \( \gamma = \lambda / \mu \) and \( \mu \gg 1 \), then

\[
\mu \approx \beta(\gamma) \pi k, \quad 0.83 < \beta(\gamma) < 1 \tag{2.25}
\]

where \( \beta(\gamma) \to 1 \) for \( \gamma \ll 1 \) or \( \gamma \gg 1 \), and \( \beta \approx 0.83 \) for \( \gamma \sim 1 \).

Thus, the index \( \mu \) of the largest term in the Gegenbauer expansion increases in proportion to \( k \).
At the boundary $x = 1$, the regularization error satisfies the estimate
\[
RE(1, \lambda, m, N) \leq A \frac{C(\rho) \Gamma(\lambda) \Gamma(m + 2\lambda + 1)}{m (2\rho)^{m+1} \Gamma(2\lambda) \Gamma(m + \lambda)},
\]  
(2.17)
For a fixed $\lambda$ we can rewrite this expression as follows:
\[
RE(1, \lambda, m, N) \leq \tilde{A}(\lambda, \rho) \ m^q q^m, \quad q = \frac{1}{2\rho},
\]  
(2.18)
where $\tilde{A}$ includes all terms independent of $m$. The error decays exponentially with $m$ since $q < 1$.

- If $\lambda = \gamma m$, then
\[
TE(1, \lambda, m, N) \leq A q^m, \quad q = \frac{(1 + 2\gamma)^{1+2\gamma}}{\rho^{2^{1+2\gamma} \gamma \gamma} (1 + \gamma)^{1+\gamma}}.
\]  
(2.19)
It is easy to verify that $q$ is an increasing function of $\gamma$ and $q < \frac{1}{\rho} \leq 1$ for all $\gamma > 0$. Thus, the convergence is faster when $\gamma$ is smaller.

- The resolution properties of the F-G method are characterized by the ratio
\[
r = \frac{m}{k}
\]  
(2.20)
where $m$ is the number of terms in the Gegenbauer expansion required to obtain an exponentially accurate approximation to the wave function
\[
w_k(x) = e^{i\pi kx}, \quad k \in \mathbb{R}^+
\]  
(2.21)
(here the wave number $k > 0$ is not necessarily an integer).

If $r$ and $\lambda$ are constants, the estimate for the resolution error is similar to that in Eq.(2.18) where $q(\pi/r)$ depends on $r$. For $r > \pi$, the factor $q < 1$ and thus the exponential convergence takes place.

- If $\gamma = \lambda/m$ is constant, the minimum number of Gegenbauer terms per wave required to resolve the wave function with spectral accuracy grows with $\gamma$. Thus, $r = \pi$ for $\gamma < 1$, $r = 7.03$ for $\gamma = 1$, and $r = 8.1$ for $\gamma = 5$. In a slightly different form, the necessary relation between $m$ and $k$ writes
\[
m = \beta(\gamma) \pi k, \quad 1 < \beta(\gamma) < 2.6.
\]  
(2.22)
- In a particular case of the wave function Eq.(2.21) and for $r$, $\gamma$ and $\beta$ being constant, the truncation error decreases exponentially with $N$ if the number of Fourier harmonics per wave, $R = \frac{2N}{k}$, is large enough. It is $R_{\min} = 22.2$ at $\gamma = 1$ and more for $\gamma < 1$ or $\gamma > 1$. 

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can be split into two components as follows:

\[ |f(x) - f_{m,N}^\lambda(x)| = |f(x) - f_m(x) + f_m(x) - f_{m,N}^\lambda(x)| \]
\[ \leq |f(x) - f_m(x)| + |f_m(x) - f_{m,N}^\lambda(x)|. \]  

The second component is the truncation error (2.9). The first component

\[ RE(x, \lambda, m, N) = \sum_{l=0}^{\infty} \hat{f}^\lambda(l) C(l)(x) - \sum_{l=0}^{m} \hat{f}^\lambda(l) C(l)(x) \]  

arises due to the truncation of the Gegenbauer series. It is called the regularization error.

The F-G method can be extended in order to evaluate the derivatives (integrals) of the function \( f(x) \) when knowing the Fourier coefficients \( \hat{f}(k) \). The algorithm is similar to the one described above but in the capacity of \( \hat{f}(k) \) in (2.6), one should take \( ik \hat{f}(k) \) for the first derivative, \(-k^2 \hat{f}(k)\) for the second derivative, and so on. The analysis of the approximation error in these cases is given in [6].

2.2 Asymptotic Convergence and Resolution Properties of the F-G Method

In this section we describe the basic results on convergence of the F-G method derived in [1], [2]. These results are valid for any analytic function \( f(x), \ x \in [-1, 1] \) whose derivatives satisfy the following estimate:

\[ \max_{-1 \leq x \leq 1} \left| \frac{d^k f(x)}{dx^k} \right| \leq C(\rho) \frac{k!}{\rho^k} \]  

where \( \rho \geq 1 \) and \( C(\rho) \) are constants independent of \( k \).

- Both truncation and regularization errors, defined correspondingly in Eqs. (2.9) and 2.12, achieve their maximum at the boundaries \( x = \pm 1 \).

- At the boundary \( x = 1 \), the truncation error satisfies the estimate

\[ TE(1, \lambda, m, N) \leq A \frac{(m + \lambda)\Gamma(m + 2\lambda)\Gamma(\lambda)}{(m - 1)! \Gamma(2\lambda)} \left( \frac{2}{\pi N} \right)^{\lambda-1}. \]  

- If \( \lambda = \gamma m \) and \( m = \beta N \) where \( \gamma, \beta \) are positive constants, then the truncation error decays exponentially with \( N \) as follows:

\[ TE(1, \lambda, m, N) \leq AN^2 q^N, \quad q = \left( \frac{\beta^\gamma(1 + 2\gamma)^{1+2\gamma}}{2\pi e^{\gamma}} \right)^{\beta}. \]  

The convergence takes place for \( \beta < 0.64 \).
where $C^\lambda_l(x)$ is a two-parametric family of the Gegenbauer polynomials ($l$ is the order of the polynomial, $\lambda$ is a parameter; the numerical algorithm for computation of the polynomials $C^\lambda_l(x)$ can be found in [7], page 782).

The Gegenbauer coefficients are defined by

$$\hat{f}^\lambda(l) = \frac{1}{h^\lambda_l} \int_{-1}^{1} (1 - x^2)^{\lambda - \frac{1}{2}} f(x) C^\lambda_l(x) \, dx$$

(2.4)

where

$$h^\lambda_l = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda)(l + \lambda)}.$$

Since we do not know the function $f(x)$, but rather its truncated Fourier series (2.2), we have only an approximation to $\hat{f}^\lambda(l)$ which we denote by $\hat{g}^\lambda_N$ given by

$$\hat{g}^\lambda_N(l) = \frac{1}{h^\lambda_l} \int_{-1}^{1} (1 - x^2)^{\lambda - \frac{1}{2}} f_N(x) C^\lambda_l(x) \, dx.$$  

(2.5)

It is a remarkable fact that the approximate Gegenbauer coefficients $\hat{g}^\lambda_N(l)$ can be explicitly expressed in terms of the Fourier coefficients $\hat{f}(k)$ as follows:

$$\hat{g}^\lambda_N(l) = \delta_{0l} \hat{f}(0) + \Gamma(\lambda) i^l(l + \lambda) \sum_{0 < |j| \leq N} J_{l+\lambda}(\pi k) \left( \frac{2}{\pi k} \right)^\lambda \hat{f}(k)$$

(2.6)

where $\Gamma(\lambda)$ and $J_\nu(x)$ are the Gamma and the Bessel functions. The corresponding Gegenbauer expansion, based on the approximate coefficients $\hat{g}^\lambda_N(l)$, will then be:

$$f^\lambda_{m,N}(x) = \sum_{l=0}^{m} \hat{g}^\lambda_N(l) C^\lambda_l(x).$$

(2.7)

We shall refer to (2.6), (2.7) as the *Fourier-Gegenbauer (F-G)* approximation of $f(x)$.

The difference between the Gegenbauer partial sum with $m$ terms of the function $f(x)$

$$f^\lambda_m(x) = \sum_{l=0}^{m} \hat{f}^\lambda(l) C^\lambda_l(x)$$

(2.8)

and that of the truncated Fourier series $f_N(x)$, is called the *truncation error*:

$$TE(x, \lambda, m, N) = |f^\lambda_m(x) - f^\lambda_{m,N}(x)|$$

$$= \left| \sum_{l=0}^{m} (\hat{f}^\lambda(l) - \hat{g}^\lambda_N(l)) C^\lambda_l(x) \right|.$$ 

(2.9)

It measures the error in the finite Gegenbauer expansion due to truncating the Fourier series.

The total error of the F-G approximation

$$E(x, \lambda, m, N) = |f(x) - f^\lambda_{m,N}(x)|$$

(2.10)
1 Introduction

The Fourier-Gegenbauer (F-G) method was introduced recently by D. Gottlieb et al. in [1] and analyzed extensively by D. Gottlieb and C.-W. Shu in [2],[3] and [4]. This method is aimed at removing the Gibbs phenomenon, in other words, recovering the point values of a non-periodic function from its Fourier coefficients. It is achieved by the reexpansion of the Fourier partial sums into the rapidly convergent Gegenbauer series.

In [5, 6] the F-G method was extended for evaluating the derivatives and the integrals of a piecewise analytic function and for the solution of differential equations in non-periodic domains.

In the current paper we discuss some new features of the F-G method observed numerically. In section 2, we provide a brief survey of the basic F-G technique (section 2.1) and the asymptotic convergence of the F-G expansion (section 2.2) based on the results in papers [1]-[3]. Here we also summarize results concerning the behavior of the F-G series in the representative case of the wave function obtained in [5] (section 2.3). The original results concerning the convergence of the F-G expansion are reported in section 3 (the case of short-term expansions) and section 4 (the case of large-term expansions).

2 Preliminaries

In this section we describe briefly the Fourier-Gegenbauer method developed in [1]-[4].

2.1 Basic Technique

Consider an analytic and non-periodic function \( f(x) \) defined in \([-1,1]\). Such a function has discontinuity at the boundary \( x = \pm 1 \) if it is extended periodically with period 2. The Fourier coefficients of \( f(x) \) are defined by

\[
\hat{f}(k) = \frac{1}{2} \int_{-1}^{1} f(x)e^{-ik\pi x} \, dx. \tag{2.1}
\]

Assume that the first \( 2N + 1 \) Fourier coefficients \( \hat{f}(k) \) are given. Our objective is to recover the function \( f(x) \) on \( x \in [-1,1] \) with exponential accuracy in the maximum norm.

The truncated Fourier series for a discontinuous function \( f(x) \)

\[
f_N(x) = \sum_{k=-N}^{N} \hat{f}(k)e^{ik\pi x}, \tag{2.2}
\]

converges slowly, like \( O(\frac{1}{N}) \), inside the interval and exhibits \( O(1) \) spurious oscillations near the boundaries \( x = \pm 1 \). This is known as the Gibbs phenomenon. Thus, there is no convergence in the maximum norm.

The basic approach of [1] consists of reexpansion of (2.2) into a rapidly convergent Gegenbauer series

\[
f(x) = \sum_{\lambda=0}^{\infty} \hat{f}^{\lambda}=0 C_{\lambda}^{\lambda}(x) \tag{2.3}
\]
Remarks on the Fourier-Gegenbauer Method

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Abstract

In this paper we investigate some numerical aspects of the Fourier-Gegenbauer method introduced in [1]. The asymptotic behavior of the Gegenbauer series is analyzed as well as the behavior of this series with small and moderate numbers of terms.

A new criteria for obtaining an exponentially small truncation error is found. It is shown that the exponential convergence takes place only for the short-term Gegenbauer expansion. The computed Fourier-Gegenbauer expansion with a large number of terms exhibits the Gibbs phenomenon in spite of the fact that both the resolution and the truncation errors are exponentially small.