TWO-LEVEL ANALYSIS OF AUTOMATIC MULTIGRID FOR NON-NORMAL AND INDEFINITE PROBLEMS

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SUMMARY

A two-level analysis method for several automatic multigrid methods (implemented with the red-black smoother) is presented. Unlike standard two-level analysis methods, based on Fourier analysis, it is based on spectral analysis; hence, it is applicable to non-normal problems. For indefinite equations, it provides a prediction of the optimal mesh size for the coarsest grid used. Furthermore, it motivates the definition of an improved version of BlackBox multigrid. Numerical examples confirming the analysis are presented.

1 INTRODUCTION

Some of the most powerful multigrid techniques for the solution of (possibly nonsymmetric and with variable coefficients) elliptic boundary value problems are based on matrix dependent construction. This means that the coarse grid operators and the transfer operators among grids are generated as functions of the coefficient matrix of the system. Once this is done in a set-up phase, these operators are used in all the multigrid cycles. Among these methods, called in the sequel automatic multigrid methods, are the method of [1], BlackBox multigrid [10] [11] [12] (denoted here by BBOX), the separable method of [12] (denoted by D2), the method of [29], AutoMUG [17] [18] [19] [20] and the semi-coarsening method of [16] [21] (denoted by SC). For problems with constant coefficients, the success of these methods may be explained by smoothing analysis. This is done for several kinds of Gauss-Seidel smoothers [4] [5] and the Incomplete LU (ILU) smoother [22]. For parallel implementations, multi-color Gauss-Seidel smoothing (e.g., red-black (RB) relaxation for matrices with property A) is desirable. It appears that for isotropic equations the RB smoother is efficient. For moderately anisotropic problems, it is shown in [26] that the Red-Black Successive Over Relaxation (RBSOR) smoother is preferable. For highly anisotropic problems, line relaxation or semi coarsening should be considered [4] [21]. A survey of smoothing properties for various point and line relaxation methods is presented in [25]. Note that the coarse grid matrices
in the automatic multigrid methods of [1], [10] and [29], lack property \( A \); hence, multi-color relaxation should be considered, rather than the usual red-black (RB) relaxation.

For standard multigrid algorithms, two-level analysis have also been developed for some model cases. A two-level analysis method for RB and "zebra" Gauss-Seidel smoothers for the Poisson equation and anisotropic diffusion equations with constant coefficients is presented in [23]. Another (equivalent) method is presented in [15]. These methods are based on Fourier analysis, hence are restricted to normal problems with constant coefficients. For automatic multigrid methods, no two-level analysis method have been yet developed.

In this work, a new two-level analysis method for RB and "zebra" Gauss-Seidel smoothers in automatic multigrid is introduced. In particular, it provides the spectrum of the iteration matrix of a two-level implementation in terms of that of the coefficient matrix alone. Since AutoMUG reduces to standard multigrid for the Poisson equation and anisotropic diffusion equations with constant coefficients, it covers the case of [23]. Furthermore, it applies also to non-normal problems and to certain problems with variable coefficients. For indefinite equations, where standard multigrid is inferior to automatic one [18] [19] [20], the method provides an improved version of BBOX and a prediction of an optimal mesh-size for the coarsest grid used. Numerical experiments confirm the analysis and show the validity of this prediction for the solution of highly indefinite Helmholtz equations.

In the sequel, matrix elements, vector components and constants are members of a field \( F \). The numerical computations in Sections 6.2 and 7 assume \( F = R \) and \( F = C \), respectively.

The contents of the paper are as follows. In Section 2 several automatic multigrid methods are described. In Section 3 modified versions of these methods, suitable for the present analysis, are presented. In Section 4 a simplified definition of these methods for separable problems is presented. In Section 5 some features of matrices with property-A, which are used later in the analysis, are discussed. In Section 6 the method of two-level analysis is introduced and applied to model problems. In Section 7 Numerical experiments are presented. In Section 8 concluding remarks are made.

## 2 AUTOMATIC MULTIGRID METHODS

### 2.1 A General Definition of a Two-Level Method

We begin with the definition of a general two-level (TL) method for the solution of the linear system of equations

\[
Ax = b,
\]

in which \( A \) is a square nonsingular matrix. Let \( \tilde{S} \) be some smoothing procedure and \( \nu_1 \) and \( \nu_2 \) small positive integers denoting, respectively, the number of presmoothings and number of postsmoothings used. The operators \( R \) (restriction), \( P \) (prolongation) and \( Q \) (coarse grid
coefficient matrix) will be defined later. The TL procedure is defined by

$$\text{TL}(X_{in}, A, b, x_{out}): \quad x_{out} = \tilde{S}^{\nu_2} \left( \tilde{S}^{\nu_1} X_{in} + PQ^{-1} R(b - A\tilde{S}^{\nu_1} X_{in}) \right).$$

(2)

An iterative application of TL is given by

$$\begin{align*}
x_0 &= 0, \quad k = 0 \\
\text{while} \quad &\|A x_k - b\|_2 \geq \text{threshold} \cdot \|A x_0 - b\|_2 \\
\text{TL}(x_k, A, b, x_{k+1}) \\
k &\leftarrow k + 1
\end{align*}$$

(3)

2.2 Definition of BBOX and D2

Assume that (1) is a discrete approximation of an elliptic PDE, e.g.,

$$-\nabla D \nabla u + \vec{c} \cdot \nabla u + \beta u = f$$

(4)

in $\Omega \subset \mathbb{R}^d$ with suitable boundary conditions, where $d$ is the dimension of the problem, $D$ is a $d \times d$ uniformly positive definite matrix and $\vec{c}$ is a $d$-dimensional vector. Assume that $A$ acts in $V(\bar{\Omega})$, where $\bar{\Omega} \subset \{\vec{j} = (j_1, \ldots, j_d)\} \subset \mathbb{Z}^d$ and, for any set $g \subset \mathbb{Z}^d$, $V(g)$ is the linear space of functions defined on $g$. In this case, $A$ may be considered a tensor $A = (a_{i,j})_{i,j \in \bar{\Omega}}$. In the following, we will treat $A$ both as a matrix and a tensor, assuming that the $i$th equation in the matrix $A$ corresponds to grid point $\vec{i}$. Assume that $A$ is of $3^d$-coefficient stencil, that is,

$$|\vec{i} - \vec{j}|_\infty > 1 \Rightarrow a_{i,j} = 0.$$

For any integer $m$, denote ”$m$ is even” by $2|m$ and ”$m$ is odd” by $2 \not{|} m$. Define the disjoint index sets $s^{(k)}$, satisfying $\bigcup_{k=0}^d s^{(k)} = \bar{\Omega}$, by

$$s^{(k)} = \left\{ \begin{array}{ll}
\{ \vec{j} \in \bar{\Omega} | \exists s \subset \{1, \ldots, d\}, |s| = k, 2|j_m \Rightarrow m \in s \} & \text{for BBOX} \\
\{ \vec{j} \in \bar{\Omega} | 2|j_1, \ldots, 2|j_k, 2 \not{|} j_{k+1} \} & \text{for D2} \end{array} \right.$$

Typically, $s^{(d)}$ serves as a coarse grid. In the sequel, we will also use the notation $c = s^{(d)}$ and $f = \bar{\Omega} \setminus c$. This induces a block form for $A$:

$$A = \left( \begin{array}{cc}
A_{ff} & A_{fc} \\
A_{cf} & A_{cc} \end{array} \right),$$

(5)

where $A_{ff}$ and $A_{cc}$ are of order $|f|$ and $|c|$, respectively. It is assumed that the order of variables in $f$ is according to blocks corresponding to $s^{(k)}$, $k = 0, \ldots, d-1$. The prolongation $P : c \rightarrow f$ will be defined later as a recursive procedure, determining the values on the subgrid $s^{(d-k)}$ at step $k$, $k = 1, \ldots, d$. 

3
For any \( \tilde{i} \in f \), define the sets

\[
t_i = \begin{cases} 
\{ \tilde{m} \in \tilde{\Omega} \mid 2|\tilde{k}| \Rightarrow m_k = i_k, \ k = 1, \ldots, d \} & \text{for BBOX} \\
\{ \tilde{m} \in \tilde{\Omega} \mid \tilde{i} \in S^{(k)} \Rightarrow m_l = i_l, \ l = 1, \ldots, d, \ l \neq k + 1 \} & \text{for D2}
\end{cases}
\]

and, for each \( \tilde{j} \in t_i \), the sets

\[
s_{i,j} = \begin{cases} 
\{ \tilde{m} \in \tilde{\Omega} \mid 2|\tilde{f}| \Rightarrow m_k = j_k, \ k = 1, \ldots, d \} & \text{for BBOX} \\
\{ \tilde{m} \in \tilde{\Omega} \mid \tilde{i} \in S^{(k)} \Rightarrow m_{k+1} = j_{k+1} \} & \text{for D2}
\end{cases}
\]

In the prolongation, \( t_i \) is the set of grid points contributing to \( \tilde{i} \) and \( s_{i,j} \) is the set of grid points on which a stencil sum is made to compute this contribution. For \( 0 \leq k \leq d - 1 \), define the rectangular matrices

\[
A^{(k)} = (a_{i,j}^{(k)})_{(i,j) \in \tilde{\Omega}} \text{ by }
\]

\[
a_{i,j}^{(k)} = \begin{cases} 
\sum_{\tilde{m} \in s_{i,j}} a_{\tilde{m},\tilde{j}} \tilde{j} \in t_i \\
0 \text{ otherwise}
\end{cases}
\]

Let

\[
(U \ B) = \begin{pmatrix} A^{(0)} & \cdots & A^{(d-1)} \\
A^{(d-1)} & \cdots & A^{(0)} \end{pmatrix},
\]

where \( U \) and \( B \) are of size \(|f| \times |f|\) and \(|f| \times |c|\), respectively. Note that \( U \) is an upper triangular matrix, hence the action \( U^{-1} \) is easily performed by back substitution. Define

\[
P_{f,i} = -U^{-1}B \text{ and } P = \begin{pmatrix} P_{f,i} \\
I \end{pmatrix},
\]

where \( I \) denotes an identity matrix of a suitable order.

Let \( \tilde{U} \) and \( \tilde{B} \) be defined by the same procedure applied to \( A^* \), the hermitian conjugate of \( A \). Define

\[
R_{c,f} = -((\tilde{U}^{-1}\tilde{B})^* \text{ and } R = (R_{c,f} \ I).
\]

With the terminology of [11], \( P \) is based on \( A \) and \( R \) is the conjugate hermitian of a prolongation based on \( A^* \). As a matter of fact, the implementation recommended in [11] and [27] uses \( P \) based on \( (A + A^*)/2 \), the symmetric part of \( A \); this may be also considered by using \((a_{i,m} + a_{m,i})/2\) in the right hand side of (6).

This definition is suggested in [11] (see also Section 4.1 below). The definition of BBOX and D2 is completed by the Galerkin approximation \( Q = RAP \).

### 2.3 Definition of the Semi Coarsening Method SC

Consider the coarse grid

\[
e = \{ \tilde{j} \in \tilde{\Omega} \mid 2|\tilde{j}| \}
\]
and the complementary subgrid \( f = \tilde{\Omega} \setminus c \). With these definitions, \( f \) (respectively, \( c \)) consists of odd (respectively, even) numbered hyperplanes in \( \tilde{\Omega} \). This splitting induces a block form (5), with blocks corresponding to \( f \) and \( c \). Each block in (5) can be further written in a block form, where a block corresponds to a single hyperplane.

For any matrix \( M \), let \( p(M) \) be the matrix obtained by the "probing" procedure [9]; that is, \( p(M) \) is a \((2k+1)\)-diagonal matrix (for some nonnegative integer \( k \)) satisfying \( p(M)v_j = Mv_j \) for some vectors \( v_1, \ldots, v_{2k+1} \). Here we consider \( k = 0 \) only; \( v_1 \) is typically close to a nearly singular eigenvector of \( M \), e.g., the constant vector [21] or a vector with almost constant flux in the uncoarsened directions [16]. For any tensor \( T \), let \( \tilde{p}(T) \) be the tensor obtained by applying \( p \) to each hyperplane corresponding block in \( T \). Define

\[
P_{fc} = -\tilde{p}(A_{ff}^{-1}A_{fc}) \quad \text{and} \quad R_{cf} = -\tilde{p}(A_{cf}A_{ff}^{-1}).
\]

The definition of SC is completed by

\[
P = \begin{pmatrix} P_{fc} & I \\ I & \end{pmatrix}, \quad R = \begin{pmatrix} R_{cf} & I \\ I & \end{pmatrix} \quad \text{and} \quad Q = RAP.
\]

Note that one can also use

\[
P_{fc} = -\tilde{p}\left(\left[(A_{ff} + A_{cf}^*)/2\right]^{-1} (A_{fc} + A_{ff}^*)/2\right) \quad \text{and} \quad R_{cf} = -\tilde{p}(A_{ff}^{-1}A_{cf}^*)^*.
\]

this way, \( P \) is based on the symmetric part of \( A \) and \( R \) is the hermitian conjugate of a prolongation based on \( A^* \), as recommended in [11] [27].

### 2.4 Square Restriction and Prolongation Matrices

In [11], it is recommended to use the residual \( r \) in the prolongation; that is, in the prolongation step from \( s^{(k+1)} \) to \( s^{(k)} \), the amount \( r_{\gamma_k}/a_{\gamma_k}^\gamma \) is added to the prolonged vector at each \( \gamma \in s^{(k)} \). Here we consider a slightly different approach, using the amount \( r_{\gamma_k}/a_{\gamma_k}^{(k)} \) (for BBOX and D2) and \( r_{\gamma_k}/U_{\gamma_k}^\gamma \), where \( U = \tilde{p}(A_{ff}) \) (for SC). We show that this is equivalent to using square matrices for \( R \) and \( P \). Indeed, by letting \( J : V(\tilde{\Omega}) \to V(c) \) and \( \tilde{J} : V(\tilde{\Omega}) \to V(f) \) be injections, namely

\[
(Jv)_{\tilde{\gamma}} = v_{\tilde{\gamma}}, \quad v \in V(\tilde{\Omega}), \quad \tilde{\gamma} \in c \quad \text{and} \quad (\tilde{J}v)_{\tilde{\gamma}} = v_{\tilde{\gamma}}, \quad v \in V(\tilde{\Omega}), \quad \tilde{\gamma} \in f,
\]

and defining

\[
P = \begin{pmatrix} I & P_{fc} \\ 0 & I \end{pmatrix}, \quad R = \begin{pmatrix} I & 0 \\ R_{cf} & I \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} U & 0 \\ 0 & JRAPJ^t \end{pmatrix},
\]

one obtains

\[
PQ^{-1}R = JU^{-1}\tilde{J} + PJ(JRAPJ^t)^{-1}JR.
\]
Note also that, in this case, the coarse grid preconditioner
\[
(PQ^{-1}R)^{-1} = R^{-1}QP^{-1}
\]
\[
= \begin{pmatrix}
I & 0 \\
-R_{cf} & I
\end{pmatrix}
\begin{pmatrix}
U & 0 \\
JRAP & I
\end{pmatrix}
\begin{pmatrix}
I & -P_{fc} \\
0 & I
\end{pmatrix}
\]
is a perturbation of
\[
A = \begin{pmatrix}
I & A_{ff} & 0 \\
A_{cf} & A_{cc} & 0 \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
A_{ff} & 0 \\
0 & A_{cc} - A_{cf}A_{ff}^{-1}A_{fc} \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
I & -P_{fc} \\
0 & I
\end{pmatrix}
\]
in which \(A_{ff}, A_{ff}^{-1}A_{fc}\) and \(A_{cf}A_{ff}^{-1}\) are replaced, respectively, by \(U, -P_{fc}\) and \(-R_{cf}\). More precisely, we have

**Theorem 1** Let
\[
G = \begin{pmatrix}
U - A_{ff} & 0 \\
0 & 2R_{cf}A_{ff}P_{fc} + R_{cf}A_{fc} + A_{cf}P_{fc}
\end{pmatrix},
\]
\[
H = \begin{pmatrix}
0 & A_{fc} + A_{ff}P_{fc} \\
A_{cf} + R_{cf}A_{ff} & 0
\end{pmatrix}.
\]

There holds
\[
R^{-1}(Q - G)P^{-1} = A - H
\]

**Proof:** The theorem follows from the Schur complement representation
\[
\begin{pmatrix}
A_{ff} & -A_{ff}P_{fc} \\
-R_{cf}A_{ff} & A_{cc}
\end{pmatrix}
= \begin{pmatrix}
I & 0 \\
-R_{cf} & I
\end{pmatrix}
\begin{pmatrix}
A_{ff} & 0 \\
0 & A_{cc} - R_{cf}A_{ff}P_{fc}
\end{pmatrix}
\begin{pmatrix}
I & -P_{fc} \\
0 & I
\end{pmatrix}
= R^{-1}(Q - G)P^{-1}.
\]

Theorem 1 shows that, when \(A_{ff}, A_{ff}^{-1}A_{fc}\) and \(A_{cf}A_{ff}^{-1}\) are spectrally close to \(U, -P_{fc}\) and \(-R_{cf}\), respectively, \(G\) and \(H\) are negligible and the coarse grid preconditioner is spectrally close to \(A\). Furthermore, if \(A_{ff}, A_{ff}^{-1}A_{fc}\) and \(A_{cf}A_{ff}^{-1}\) have the same effect on nearly singular eigenvectors of \(A\) as do \(U, -P_{fc}\) and \(-R_{cf}\), respectively, then these error modes are efficiently handled by the coarse grid correction and the complete two-level method converges rapidly, provided that the remaining modes are efficiently handled by a suitable smoothing procedure. For BBOX and D2, this happens at least for isotropic problems, since the spectrum of \(A_{ff}\) is bounded away from zero and the prolongation is almost correct for vectors with a smooth flux. For SC, this is the case also for anisotropic problems, unless the hyperplanes are weakly connected in \(A\) (case for which the spectrum of \(A_{ff}\) is not bounded away from zero).

### 3 AUTOMATIC MULTIGRID FOR INDEFINITE PROBLEMS

#### 3.1 Modified BBOX and D2

The arguments following Theorem 1 are also relevant for indefinite problems, e.g., the Helmholtz equation
\[
-\triangle u - \beta u = f
\]
in $\Omega \subset \mathbb{R}^d$, with $\Re(\beta) > 0$ and suitable boundary conditions. Here, however, there are too many nearly singular error modes, that is, modes with frequency $\bar{\omega}$ satisfying

$$|\bar{\omega}|^2 = \sum_{i=1}^{d} \omega_i^2 \sim \beta.$$ 

It is possible, though, to handle at least some of them, e.g., those for which

$$\omega_i^2 \sim \alpha(i) \beta, \quad 1 \leq i \leq d,$$

where the $\alpha(i)$’s are nonnegative parameters satisfying $\sum_{i=1}^{d} \alpha(i) = d$. This is achieved by requiring that the prolongation and restriction will be almost accurate for these modes. To this end, write $A = G + H$, where $H$ corresponds to a discretization of the term $\beta u$ in (4) or (7). As in (6), define the rectangular matrices $G^{(k)} = (g_{i,j}^{(k)})$ and $H^{(k)} = (h_{i,j}^{(k)})$ based on $G$ and $H$, respectively. In order that the prolongation be almost accurate for the above modes, $h_{i,j}^{(k)}$ should be multiplied by some averaging of the $\alpha(m)$’s for coordinates $m$ along which prolongation is done, times the diffusion along these coordinates. This is done as follows.

For any tensor $M = (m_{i,j})$, define the off-diagonal sums

$$q_r(M) = \sum_{j \in \Omega, j \neq i} m_{i,j},$$

and the $\alpha(i)$-average

$$\bar{\alpha}_r = \frac{\sum_{1 \leq k \leq d, \exists j \in t_r, j_k \neq i_k} \alpha(k)}{\left|\{1 \leq k \leq d \mid \exists j \in t_r, j_k \neq i_k\}\right|}.$$

Modify $H^{(k)}$ to read

$$h_{i,j}^{(k)} \leftarrow h_{i,j}^{(k)} \bar{\alpha}_r \cdot q_{i+\vec{r}}(G^{(k)}) / q_{i+\vec{r}}(G),$$

where $\vec{c} = \vec{c}(i) \in \mathbb{Z}^d$ is a minimal (in some norm) vector for which $\vec{c} + \vec{c}$ is in the interior of $\Omega$ (it is used to avoid the effect of boundary conditions on the discrete approximation of the diffusion). $R$ and $P$ are then obtained by setting $A^{(k)} = G^{(k)} + H^{(k)}$ and proceeding as in Section 2.2. The resulting coarse grid operator

$$Q = RGP + RH P$$

is given automatically in a decomposed form, suitable for recursion. The resulting modification of BBOX and D2 is denoted, respectively, by Modified BlackBox (MBOX) and Modified D2 (MD).

### 3.2 Finite Difference Discretizations

In the remainder of this paper, we consider rectangular prolongation and restriction matrices only; i.e., residuals are not used in the prolongation. For this implementation, the reasoning
of the modified methods presented in Section 3.1 can be shown at least for finite difference discretization schemes.

Let $A$ represent a $(2d + 1)$-coefficient stencil of a second order elliptic PDE (e.g., (4)). Assume that

$$A = \sum_{i=1}^{d} X_i,$$

(8)

where $X_i$ represent a 3-point discretization of the derivatives in the $i$th spatial direction. It is assumed that the term $\beta u$ in (4) or (7) contributes to $\text{diag}(A)$ only, giving the amount

$$\alpha(i)\beta \cdot q_{j+e_i}(X_i)/q_{j+e_i}(A)$$

(9)

to $(X_i)^{j,\bar{j}}$, $1 \leq i \leq d$. In order to guarantee consistency, it is required that

$$\sum_{i=1}^{d} \alpha(i) \cdot q_{j+e_i}(X_i)/q_{j+e_i}(A) = 1,$$

for every $\bar{j} \in \bar{\Omega}$. Here we assume that a somewhat stronger condition holds, namely, that either $\alpha_i = 1$, $1 \leq i \leq d$ or the problem is isotropic, that is, $q_j(X_i)$ is independent of $i$. For MBOX, assume that either

- periodic boundary conditions are imposed and the number of grid points in each spatial direction is even, or
- nonperiodic boundary conditions are imposed, the number of grid points in each spatial direction is odd and the problem is of constant coefficients at least near boundaries, that is, in the cells including $\bar{i}$ and $\bar{i} + \bar{c}(\bar{i})$ (when $\bar{c}(\bar{i}) \neq 0$).

For MD, assume that periodic boundary conditions are imposed and the number of grid points in each spatial direction is even. For any set $s \subset \{1, \ldots, d\}$, let $\Gamma(s)$ denote the group of permutations on $s$ and $\Pi_s$ a product of elements indexed in $s$, with an increasing index order. Let $J_i : V(\bar{\Omega}) \rightarrow V(\{\bar{j} \in \bar{\Omega} \mid 2|\bar{j}_k\}, 1 \leq i \leq d$, be injections. For $k = 1, \ldots, d$, let

$$P^{(k)} : V(\cup_{i=d-k+1}^{d} s^{(i)}) \rightarrow V(\cup_{i=d-k}^{d} s^{(i)})$$

be the $k$th step in the prolongation $P$. Let $P_i = (2\text{diag}(X_i) - X_i)J_i^tJ_i$, $1 \leq i \leq d$. The following result will be useful in the uniform formulation of automatic multigrid for separable problems (Section 4.2).

**Theorem 2** Under the above assumptions, the prolongation matrix $P$ is given by

$$P = \begin{cases} 
(\sum_{\sigma \in \Gamma(1, \ldots, d)} \Pi_{1 \leq i \leq d} \left(\text{diag}[\sum_{i=d} \sigma(\bar{i}) X_{\sigma(\bar{i})}]^{-1} P_{\sigma(\bar{i})}\right) J^t & \text{for MBOX} \\
(\Pi_{1 \leq i \leq d} \text{diag}(X_i)^{-1} P_i) J^t & \text{for MD} \end{cases}$$

Furthermore, if the matrices $\text{diag}(X_i)^{-1}X_i$, $1 \leq i \leq d$, commute with each other, then the prolongation $P$ is given by

$$P = (\Pi_{1 \leq i \leq d} \text{diag}(X_i)^{-1} P_i) J^t$$

for MBOX also.
Proof: For MD, the theorem follows from the definition. For MBOX, proceed as follows. For any index set $s \subset \{1, \ldots, d\}$, define the set of planes

$$g(s) = \{ \vec{j} \in \hat{\Omega} | \exists j_m \Rightarrow m \in s \}.$$ 

Let $v \in V(c)$. Define $v^{(0)} = J^i v$ and, for $k = 1, \ldots, d$,

$$(v^{(k)})_{\vec{j}} = \left\{ \begin{array}{ll} (P(k)v^{(k-1)})_{\vec{j}} & \vec{j} \in \bigcup_{i=1}^d \mathcal{X}(s^{(i)}) \\ 0 & \text{otherwise} \end{array} \right.$$ 

By induction on $k = 1, \ldots, d$, one obtains that, for any set $s \subset \{1, \ldots, d\}$ of size $k$ and any $\vec{j} \in g(s)$,

$$\left( \left( \text{diag} \left[ \sum_{i \in s} X_i \right] \right)^{-1} \sum_{i \in s} P_i v^{(k-1)} \right)_{\vec{j}}$$

$$= \left( \left( \text{diag} \left[ \sum_{i \in s} X_i \right] \right)^{-1} \left[ \sum_{j \in s, 2|j_i} P_i v^{(k-1)} + \sum_{j \in s, 2|j_i} P_i v^{(k-1)} \right] \right)_{\vec{j}}$$

$$= \left( \left( \text{diag} \left[ \sum_{i \in s} X_i \right] \right)^{-1} \left[ \text{diag} \left[ \sum_{i \in s, 2|j_i} X_i \right] v^{(k-1)} + \text{diag} \left[ \sum_{i \in s, 2|j_i} X_i \right] v^{(k-1)} \right] \right)_{\vec{j}}$$

$$= (v^{(k)})_{\vec{j}}.$$ 

Note also that, for any set $s \subset \{1, \ldots, d\}$,

$$\sum_{\sigma \in \Gamma(s)} \Pi_{i \in s} \left( \text{diag} \left[ \sum_{i \leq m \leq d, m \in s} X_{\sigma(m)} \right] \right)^{-1} P_{\sigma(i)}$$

$$= \left( \text{diag} \left[ \sum_{m \in s} X_m \right] \right)^{-1} \sum_{i \in s} P_i \sum_{\sigma \in \Gamma(s) \setminus \{i\}} \Pi_{i \in s \setminus \{i\}} \left( \text{diag} \left[ \sum_{i \leq m \leq d, m \in s \setminus \{i\}} X_{\sigma(m)} \right] \right)^{-1} P_{\sigma(i)}.$$ 

Consequently, by induction on $k = 1, \ldots, d$, we have that, for any set $s \subset \{1, \ldots, d\}$ of size $k$ and $\vec{j} \in g(s)$,

$$(v^{(k)})_{\vec{j}} = \left( \sum_{\sigma \in \Gamma(s)} \Pi_{i \in s} \left( \text{diag} \left[ \sum_{i \leq m \leq d, m \in s} X_{\sigma(m)} \right] \right)^{-1} P_{\sigma(i)} v^{(0)} \right)_{\vec{j}}.$$ 

The first part of the Theorem follows by setting $k = d$. For the second part, note that, for any $1 \leq i, j \leq d$ and $\vec{m} \in \hat{\Omega}$,

$$(X_i J_i J_i X_j J_j^i, J_j)_{\vec{m}, \vec{n}} = \left\{ \begin{array}{ll} (X_i X_j)_{\vec{m}, \vec{n}} & \vec{n} \in g(\{1, \ldots, d\} \setminus \{i, j\}) \\ 0 & \text{otherwise} \end{array} \right.$$
Consequently, commutativity of the $\text{diag}(X_i)^{-1}X_i$’s implies that of the $\text{diag}(X_i)^{-1}P_i$’s. By induction on $|s|$ in (10), we have that

$$
\sum_{\sigma \in \Gamma(s)} \prod_{i \in s} \left( \text{diag} \left( \sum_{i \leq m \leq d, m \in s} X_{\sigma(m)} \right) \right)^{-1} P_{\sigma(i)}
= \left( \text{diag} \left( \sum_{m \in s} X_m \right) \right)^{-1} \prod_{i \in s} \text{diag}(X_i) \left( \text{diag}(X_i)^{-1} P_{\prod_{i \in s \setminus \{i\}} \text{diag}(X_i)^{-1} P_i} \right)
= \prod_{i \in s} \text{diag}(X_i)^{-1} P_i
$$

The second part of the theorem follows by setting $s = \{1, \ldots, d\}$. □

The straightforward representation of $P$ (and, similarly, for $R$) introduced in Theorem 2 for separable problems is the basis of the two-level analysis method introduced in Section 6.

3.3 Multiple Coarse Grid Correction

The definition of MBOX and MD does not specify the parameters $\alpha(i)$, $1 \leq i \leq d$. As discussed in Section 3.1, different choices of these parameters can handle different nearly singular error modes. For example, the choice $\alpha(i) = 1, 1 \leq i \leq d$, corresponds to the smoothest ones, for which $\omega^2 = \beta/d$, $1 \leq i \leq d$. This seems to be the optimal choice, and serves as a default in the remainder of this paper. However, multiple coarse grid corrections, corresponding to restriction, prolongation and coarse grid operators resulting from multiple choices of the $\alpha(i)$’s, are also possible. For example, in the 2-d case, the choices

$$
\alpha(i) = 2 \sin^2(\pi/4 + (2i - 3)j\pi/(2k)), \quad i = 1, 2, \quad |j| \leq |k/2|
$$

gives a uniform cover of the circle $\omega_1^2 + \omega_2^2 = \beta$ in the frequency space with resolution $2k/\pi$, where $k$ is a positive integer. In order to achieve high resolution, large $k$ should be used and $k + (k + 1 \mod 2)$ different constructions of $R$, $P$ and $Q$ should be performed. This approach can also be used for AutoMUG (Section 4.2), since multiple choices of the $\alpha(i)$’s induce (e.g., via (9)) multiple choices of the $X_i$’s decomposing $A$ in (8) and, consequently, multiple coarse grid problems. Each coarse grid problem may be solved by a $V$-cycle, using the relevant coarse grid operator $Q$ with the same choice of $\alpha(i)$’s to generate further restrictions, prolongations and coarse grid operators by the automatic multigrid method used. A similar idea for multiple coarse grid correction was introduced by Achi Brandt in an informal talk at the 8th Copper Mountain Conference on Multigrid Methods, April, 95, in the context of standard multigrid.

It seems, though, that the lower bound for the coarsest grid resolution derived in Section 6.2 below and in [17] [18] holds, since the nearly singular modes cannot be approximated on too coarse grids. This difficulty may be partially relaxed by using semi-coarsening, at least for the coarse grid correction of the highly oscillating nearly singular modes. For the 2-d case, for example, this means that semi-coarsening in direction $i$ is performed for $\alpha(i) < \alpha(3 - i)$,
i = 1, 2. With this approach, for any nearly singular mode correction, coarsening is done only in the direction of maximal smoothness; hence, the coarsest grid used might be coarser than that of the full coarsening approach.

4 AUTOMATIC MULTIGRID FOR SEPARABLE PROBLEMS

In this section we deal with separable problems. We use Theorem 2 to show that, for such problems, two-level analysis of automatic multigrid is available. For this we need some definitions.

4.1 Definitions

For any two matrices $S$ and $T$, $S = (s_{i,j})_{1 \leq i \leq K, 1 \leq j \leq L}$, define the tensor product $T \otimes S$ by

$$T \otimes S = \begin{bmatrix}
  s_{1,1}T & \cdots & s_{1,L}T \\
  \vdots & \ddots & \vdots \\
  s_{K,1}T & \cdots & s_{K,L}T 
\end{bmatrix}$$

Let $N$ be a fixed positive integer and $I$ the identity matrix of order $N$. Let $J$ be the injection operator defined by

$$J : F^N \to F^{[N/2]}, \quad J(v_1, v_2, \ldots, v_N) = (v_2, v_4, v_6, \ldots, v_{2\lfloor N/2 \rfloor}),$$

where $F$ denotes a field (see Section 1). For any matrix $S$, let

$$P(S) = (2I - diag(S)^{-1}S)J^t, \quad R(S) = J(2I - S diag(S)^{-1}) = P(S^*)^*,$$

and

$$T(S) = p\left(J(2I - diag(S)^{-1}S)\right),$$

where $p$ is the "probing" operator defined in Section 2.3, using a constant vector. Note that when $S$ is a tridiagonal matrix, $P(S)$ and $R(S)$ may be viewed as prolongation and restriction operators, respectively. With the terminology of [11], the prolongation $P(S) : F^{[N/2]} \to F^N$ is based on $S$ and the restriction $R(S) : F^N \to F^{[N/2]}$ is the transpose of a prolongation based on $S^*$. This is in the spirit of the method of [11]. In order to follow it exactly, one should use the modification $P(S) \leftarrow P((S + S^*)/2)$. With this modified definition the analysis presented in Section 6 is valid, provided the coefficient matrix is normal; here, however, we prefer considering the original definition, with which this assumption is not required.
4.2 The Separable 3-dimensional Case

Consider the separable PDE

$$\sum_{i=1}^{d} (a_i(x_i)u_{x_i})_{x_i} + \sum_{i=1}^{d} c_i(x_i)u_{x_i} + \beta u = g$$  \hspace{3cm} (11)$$

in $\Omega = (0,1)^d$ with suitable boundary conditions. Here we prefer discussing the 3-d case; the 2-d one is immediately obtained from it by setting $\eta = 0$ and omitting the third member in the triple tensor products below. Clearly, for such problems, the $X_i$'s in finite difference discretizations of the form (8) commute with each other. Following [24], this is equivalent to writing $A$ in the form

$$A = X \otimes I \otimes I + \xi I \otimes Y \otimes I + \eta I \otimes I \otimes Z,$$ \hspace{3cm} (12)$$

where $X$, $Y$ and $Z$ are tridiagonal matrices of order $N$. By the terminology 'tridiagonal' we also include periodically extended tridiagonal matrices, namely, matrices $M$ for which $M_{i,N} \neq 0$ or $M_{N,1} \neq 0$. In this case, however, we require that $N$ is even, so the matrix has property A.

In order to develop the analysis, we also assume that $diag(X) = diag(Y) = diag(Z) = I$, restricting the discussion to problems of the form (11) with constant $a_i$'s and second order central discretization schemes. For example, if

$$X = Y = Z = tridiag(-1,2,-\beta h^2/(1+\xi+\eta),-1)$$ \hspace{3cm} (13)$$

(where $h = 1/(N+1)$ is the cell-size) then $A$ represents a central discretization of an anisotropic Helmholtz equation in a box with Dirichlet boundary conditions.

For SC, assume that coarsening in the $x$ direction only is performed and that the ”probing” procedure (Section 2.3) uses the vectors $v_i = v_{y,i} \otimes v_{z,i}$, where $1 \leq i \leq N$ is even and $v_{y,i}, v_{z,i} \in F^N$ satisfy $Y v_{y,i} = \lambda_{y,i} v_{y,i}, Z v_{z,i} = \lambda_{z,i} v_{z,i}$ for some scalars $\lambda_{y,i}, \lambda_{z,i}$ (e.g., $Y$ and $Z$ are circulant Toeplitz matrices and the $v_{y,i}$'s and $v_{z,i}$'s are the constant vector). Typically, $v_{y,i}$ and $v_{z,i}$ have smooth flux in the $yz$ plane, hence the $\lambda_{y,i}$ and $\lambda_{z,i}$ are close to zero (see Section 2.3). Define

$$\Lambda = diag\left( (1 + \xi \lambda_{y,z} + \eta \lambda_{z,y})^{-1} \right)_{1 \leq i \leq \lfloor N/2 \rfloor}.$$ 

For MBOX and MD, assume that the assumptions of Section 3.1 hold. Using Theorem 2, the operators $R, P$ and $Q$ for the two-level method (2) are defined by

$$R = \begin{cases} R(X) \otimes R(Y) \otimes R(Z) & \text{for MBOX, MD and AutoMUG} \\ J \otimes I \otimes I + (\Lambda(R(X) - J)) \otimes I \otimes I & \text{for SC} \end{cases}$$

$$P = \begin{cases} P(X) \otimes P(Y) \otimes P(Z) & \text{for MBOX, MD and AutoMUG} \\ J^t \otimes I \otimes I + ((P(X) - J^t) \Lambda) \otimes I \otimes I & \text{for SC} \end{cases}$$
Note that, when the $a_i$ and $\beta$ in (11) are constant (and a uniform grid is used, so that their discrete values are also constant) and $N$ is odd, the definition of the coarse-grid operator $Q$ for AutoMUG simplifies to read

$$Q = \left\{ \begin{array}{l}
R(X)XJ^t \otimes T(Y) \otimes T(Z) \\
+ \xi \cdot T(X) \otimes R(Y)YJ^t \otimes T(Z) \\
+ \eta \cdot T(X) \otimes T(Y) \otimes R(Z)ZJ^t
\end{array} \right\} \text{ for AutoMUG}$$

Consequently, for the anisotropic diffusion equation obtained from (13) by setting $\beta = 0$, AutoMUG is equivalent to standard multigrid. In our analysis for AutoMUG we will assume that this simplification is valid.

5 SOME FEATURES OF MATRICES WITH PROPERTY-A

Our aim in this section is to present some features of matrices with property-A and non-vanishing diagonal elements. These matrices are characterized by the fact that their Jacobi iteration matrices (possibly after some permutation of rows and columns) are of the form

$$B = \begin{pmatrix} 0_{k_1} & T_2 \\ T_1 & 0_{k_2} \end{pmatrix},$$

where $k_1$ and $k_2$ are positive integers, $0_m$ is the null matrix of order $m$ and $T_1$ and $T_2$ are rectangular matrices.

For any vector $v = (v_1, v_2, \ldots, v_{k_1+k_2}) \in F^{k_1+k_2}$, let

$$J_1v = (v_1, v_2, \ldots, v_{k_1}) \quad \text{and} \quad J_2v = (v_{k_1+1}, v_{k_1+2}, \ldots, v_{k_1+k_2}).$$

Lemma 1 For $l = 1, 2$, let

$$U_l = \{ v \in F^{k_1+k_2} \mid Bv = 0, \ J_{3-l}v = 0 \}.$$

Then

(a) $\dim(U_l) = k_l - \text{rank}(T_l), \ l = 1, 2.$

(b) $\dim \{ v \in F^{k_1+k_2} \mid Bv = 0 \} = k_1 - \text{rank}(T_1) + k_2 - \text{rank}(T_2).$

Proof: Part (a) follows from

$$U_l = J_l^t \{ u \in F^{k_l} \mid T_l u = 0 \}, \ l = 1, 2.$$

Part (b) follows from the observation that, for any vector $v \in F^{k_1+k_2}$, there holds

$$Bv = 0 \Leftrightarrow T_l J_l v = 0, \ l = 1, 2 \Leftrightarrow v \in U_1 \oplus U_2.$$
which implies
\[ \dim \{ v \in F^{k_1+k_2} \mid Bv = 0 \} = \dim(U_1) + \dim(U_2). \]

The following theorem gives a necessary condition for diagonalizability of \( B \).

**Lemma 2** If \( B \) is diagonalizable, then \( \text{rank}(T_1) = \text{rank}(T_2) \).

**Proof:** From the assumption, it follows that \( B^2 = \text{diagblock}(T_2T_1, T_1T_2) \) is also diagonalizable. From part (a) of Lemma 1 we have
\[ \dim \{ v \mid T_{3-\ell}T_1v \neq 0 \} = \text{rank}(T_1), \quad \ell = 1, 2. \]

Let \( \gamma \neq 0 \) be an eigenvalue of \( T_2T_1 \) with the corresponding eigenvector \( u \). Then \( v = T_1u \) is an eigenvector of \( T_1T_2 \) with the same eigenvalue \( \gamma \), and \( \gamma^{-1}T_2v = u \). Hence there exists an invertible mapping of eigenvectors corresponding to nonvanishing eigenvalues of \( T_2T_1 \) onto those of \( T_1T_2 \). Consequently,
\[ \dim \{ v \mid T_2T_1v \neq 0 \} = \dim \{ v \mid T_1T_2v \neq 0 \}. \]

**Lemma 3** Assume that \( B \) is diagonalizable. Then there exist sets \( V_1 \) and \( V_2 \) of eigenvectors of \( B \) such that
\[ \#(V_1 \cap V_2) = \text{rank}(T_1) = \text{rank}(T_2), \quad J_{3-\ell}(V_{\ell} \setminus V_{3-\ell}) = \{0\}, \quad \ell = 1, 2 \]
and \( J_{\ell}V_{\ell} \) form a basis in \( F^{k_{\ell}}, \ell = 1, 2 \).

**Proof:** By Lemma 2, there exists an integer \( r = \text{rank}(T_1) = \text{rank}(T_2) \). By part (b) of Lemma 1, \( 2r = \dim \{ v \mid Bv \neq 0 \} \). By Theorem 5.2.2(a) of [28], there exist sets \( \Lambda \) consisting of \( r \) (not necessarily distinct) eigenvalues of \( B \) satisfying \( \lambda \in \Lambda \Rightarrow -\lambda \not\in \Lambda \) and \( V(\Lambda) \) consisting of the independent eigenvectors of \( B \) corresponding to the eigenvalues of \( \Lambda \). From Section 7.1 of [28], it follows that \( J_{\ell}V(\Lambda) \) consists of independent eigenvectors of \( T_{3-\ell}T_1 \). Let \( \bar{U}_i \) be a basis of the subspace \( U_i \) defined in Lemma 1. The assertion follows from the definition \( V_{\ell} \equiv V(\Lambda) \cup \bar{U}_i \).

## 6 THE TWO LEVEL ANALYSIS METHOD

### 6.1 The Two-Level Analysis Method

The two-level analysis of MBOX and AutoMUG introduced in this section follows from the representation of their iteration matrix in the basis constructed in Lemma 3. The analysis of MD is identical to that of MBOX. Analysis for SC can be derived in the same way, and is left to the reader.

For any positive integer \( m \), let \( I_m \) denote the identity matrix of order \( m \). It is obvious that the Jacobi iteration matrix of a tridiagonal matrix assumes the structure of \( B \) of Section 5.
after a suitable odd-even reordering (see, for example, [18]). With this ordering and the notation of Section 5, \( k_1 = k_2 \) if \( N \) is even and \( k_1 = k_2 + 1 \) if \( N \) is odd. Define the operators \( J_e \) and \( J_o \) by

\[
J_e : F^N \to F^{[N/2]}, \quad J_e(v_1, v_2, \ldots, v_N) = (v_2, v_4, v_6, \ldots, v_{2[N/2]})
\]

\[
J_o : F^N \to F^{[N/2]}, \quad J_o(v_1, v_2, \ldots, v_N) = (v_1, v_3, v_5, \ldots, v_{2[N/2] - 1})
\]

("e" for even, "o" for odd). Assume that the matrices \( X, Y \) and \( Z \) of Section 4.2 are diagonalizable; then Lemma 3 is applicable to each of them (after a suitable odd-even reordering). For simplicity, assume that they are also irreducible; then, with odd-even ordering and the notation of Section 5, we have, for their Jacobi iteration matrices,

\[
r = \text{rank}(T_1) = \text{rank}(T_2) = \min(k_1, k_2) = \lfloor N/2 \rfloor = \#V_2.
\]

Consequently, they have, respectively, sets of eigenvectors \( \{\bar{x}_i\}_{1 \leq i \leq \lfloor N/2 \rfloor}, \{\bar{y}_i\}_{1 \leq i \leq \lfloor N/2 \rfloor}, \{\bar{z}_i\}_{1 \leq i \leq \lfloor N/2 \rfloor} \), with corresponding eigenvalues \( x_i, y_i, z_i \), such that, for any \( v \in F^N \), it is possible to write

\[
J_o v = \sum_{i=1}^{\lfloor N/2 \rfloor} a_i J_o \bar{x}_i, \quad J_e v = \sum_{i=1}^{\lfloor N/2 \rfloor} c_i J_e \bar{x}_i
\]

and similarly for \( y_i \) and \( z_i \). The \((a_i, c_i)\) are independent variables denoting the coefficients of \( v \) when represented in the basis \( V_x \equiv V_{x,o} \cup V_{x,e} \), where

\[
V_{x,o} \equiv \{ J_o^t J_o \bar{x}_i \}_{1 \leq i \leq \lfloor N/2 \rfloor}, \quad V_{x,e} \equiv \{ J_e^t J_e \bar{x}_i \}_{1 \leq i \leq \lfloor N/2 \rfloor}.
\]

Moreover, from Section 7.1 of [28] it follows that, for normal matrices, the transformation from the standard basis to the coefficient basis is orthogonal.

Let \( \dot{J} = (0, 1) \) and, for any \( c \in F_e \),

\[
M(c) = \begin{pmatrix}
1 & c - 1 \\
c - 1 & 1
\end{pmatrix}.
\]

For a fixed \( 1 \leq i \leq \lfloor N/2 \rfloor \) the set \( \{ J_o^t J_o x_i, J_e^t J_e x_i \} \subset V_x \) is invariant under the operations \( X, R(X) \) and \( P(X) \); these operations are represented, in the corresponding \( i \)th coefficient couple \((a_i, c_i)\), by

\[
\dot{X} = M(x_i), \quad \dot{R}(X) = \dot{J} M(2 - x_i) \quad \text{and} \quad \dot{P}(X) = M(2 - x_i) \dot{J}^t.
\]

For the 3-d case (12), we consider the tensor product basis \( V_x \otimes V_y \otimes V_z \). The set

\[
\{ J_p^t J_p \bar{x}_i \otimes J_q^t J_q \bar{y}_j \otimes J_r^t J_r \bar{z}_k \}_{p, q, r \in \{o, e\}} \subset V_x \otimes V_y \otimes V_z
\]

is invariant under the operations \( A, R \) and \( P \) of (12) and (14); these operations are represented, in the corresponding \( ijk \)th coefficient set, by

\[
\dot{A} = M(x_i) \otimes I_2 \otimes I_2 + \xi I_2 \otimes M(y_j) \otimes I_2 + \eta I_2 \otimes I_2 \otimes M(z_k)
\]

\[
\dot{R} = \dot{J} M(2 - x_i) \otimes \dot{J} M(2 - y_j) \otimes \dot{J} M(2 - z_k)
\]

\[
\dot{P} = M(2 - x_i) \dot{J}^t \otimes M(2 - y_j) \dot{J}^t \otimes M(2 - z_k) \dot{J}^t.
\]
Moreover, this tuple is also invariant under relaxation over "black" points, the matrix representation of which is given by

\[ \hat{S}_b \equiv \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{\omega(1-x_i)}{1+\xi+\eta} & 1 - \omega & 0 & 0 & 0 & 0 \\
\frac{\omega(1-y_j)}{1+\xi+\eta} & 0 & 1 - \omega & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\frac{\omega(1-z_k)}{1+\xi+\eta} & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}. \]

Let

\[ I_{rb} \equiv \begin{pmatrix}
0 \\
I_4 \\
I_4 \\
0 \\
I_4 \\
0 \\
\end{pmatrix} \]

be a permutation matrix exchanging "red" and "black" coefficients. A relaxation over red points is then represented by \( \hat{S} = I_{rb} \hat{S}_b I_{rb} \). Consequently, the representation of the RBSOR iteration is given by \( \hat{S} = \hat{S}_b \hat{S}_r \). Finally, the coarse grid operator \( Q \) of (14) is acting in the linear space spanned by \( V_v \otimes V_v \); its representation in this basis is diagonal and its operation on the basis-vector \( J_v J_x x_i \otimes J_v J_y y_j \otimes J_v J_z z_k \) is just a multiplication by the scalar

\[ \hat{Q} = \hat{R} \hat{A} \hat{P} = x_i(2-x_i)[1 + (1 - y_j)\frac{2}{\beta h^2}] [1 + (1 - z_k)\frac{2}{\beta h^2}] + \xi [1 + (1 - x_i)\frac{2}{\beta h^2}] [1 + (1 - y_j)\frac{2}{\beta h^2}] + \eta [1 + (1 - x_i)\frac{2}{\beta h^2}] [1 + (1 - y_j)\frac{2}{\beta h^2}] [1 + (2 - z_k)\frac{2}{\beta h^2}] \]

for MBOX and

\[ \hat{Q} = [1 + (1 - x_i)\frac{2}{\beta h^2}] [1 + (1 - y_j)\frac{2}{\beta h^2}] [1 + (1 - z_k)\frac{2}{\beta h^2}] \]

for AutoMUG. This provides the following theorem:

**Theorem 3** Let \( A \) be as in (12) with \( X, Y \) and \( Z \) tridiagonal diagonalizable irreducible matrices satisfying \( \text{diag}(X) = \text{diag}(Y) = \text{diag}(Z) = 1 \). Let \( M_\omega \) be the iteration matrix of the TL method (2) implemented with the \( R, P \) and \( Q \) defined in (14) and the smoothing procedure RBSOR(\( \omega \)). Then there exists a nonsingular matrix \( K \) such that

\[ KM_\omega K^{-1} = \text{diagblock} \{ J_{i,j,k} \hat{S}^{\omega} (I_{2d} - \hat{P} \hat{Q}^{-1} \hat{R} \hat{A}) \hat{S}^{\omega} J_{i,j,k} \}_{1 \leq i, j, k \leq \lceil N/2 \rceil}, \]

where \( J_{i,j,k} = J_i \otimes J_j \otimes J_k \) and, for \( 1 \leq m \leq \lceil N/2 \rceil \), \( J_m = (1, 0) \) if \( m = (N + 1)/2 \) and \( J_m = I_2 \) otherwise. Furthermore, if \( X, Y \) and \( Z \) are normal, then \( K \) is orthogonal.

A similar theorem holds when the irreducibility assumption is not presented. If, for example, \( X \) is reducible, then the eigenvectors \( \tilde{x}_i \) are taken from the sets \( V_v \) and \( V_e \) (corresponding to the sets \( V_1 \) and \( V_2 \) of Lemma 3) rather than from the set \( \{ \tilde{x}_i \}_{1 \leq i \leq \lceil N/2 \rceil} \); the operators \( J_{i,j,k} \)
of Theorem 3 are then replaced by \( J_{x,y,z} = J_x \otimes J_y \otimes J_z \), where \( J_x = (1, 0) \) if \( x \in V \setminus V_c \), 
\( J_x = (0, 1) \) if \( x \in V_c \setminus V \) and \( J_x = I_2 \) if \( x \in V \cap V_c \).

The above analysis method applies also to line and plane relaxation. For example, for
line relaxations in the \( x \) direction and red-black ordering in the \( yz \) planes ("zebra" pattern) the above \( \hat{S}_k \) should be replaced by

\[
\hat{S}_k = \begin{pmatrix}
I_2 & 0 & 0 & 0 \\
\omega(x(1-y_j)L(x_i) & (1-\omega)I_2 & 0 & 0 \\
\omega(y(1-z_k)L(x_i) & 0 & (1-\omega)I_2 & 0 \\
0 & 0 & \omega(z(1-y_j)L(x_i) & 0 \\
\end{pmatrix},
\]

where \( L(x) \equiv [(\xi + \eta)I_2 + M(x)]^{-1} \).

### 6.2 Numerical Computation of the Spectrum of MBOX and AutoMUG

Here we employ Theorem 3 (with \( \omega = 1 \)) to compute the spectrum of the iteration matrices of MBOX and AutoMUG, implemented with two levels and the RB smoother and applied to 2-d problems of the form

\[-u_{xx} - u_{yy} + \epsilon u_x + \epsilon u_y - \beta u = g\]

in the unit square (where \( \epsilon, \beta \in \mathbb{R} \)) with Dirichlet boundary conditions. The discretization is done via a second-order central difference scheme over an \( N \times N \) uniform grid with cell-size \( h = 1/(N + 1) \). It is assumed that \( |\epsilon|, |\beta| < 2/h \); the coefficient matrix \( A \) is then similar to the symmetric matrix \( D^{-1}AD \), where

\[D = diag \{(1 - \epsilon h^2/2)^{1/2})_{i \leq N} \otimes diag \{(1 + \epsilon h^2/2)^{1/2})_{i \leq N} \}.
\]

Assume that \( A \) is scaled as in (12); the spectrum of \( D^{-1}AD \), as well as that of \( A \), is then given by \( x_i(\epsilon) + y_j(\epsilon) \), \( 1 \leq i, j \leq N \) where, for any \( t \) satisfying \( |t| < 2/h \) and \( 1 \leq k \leq N \),

\[x_k(t) = 1 - \frac{2\sqrt{(1-th/2)(1+th/2)}}{2-\beta h^2/2} \cos(\pi k h).
\]

We start with the Poisson equation, for which \( \epsilon = \beta = 0 \). In Table 1 we display the spectral radius \( \rho \) of the iteration matrix of MBOX and AutoMUG for various \( \nu = \nu_1 + \nu_2 \) (only the sum matters, as discussed in [15]). Since, for odd \( N \), AutoMUG is equivalent to standard multigrid, it is expected that the results for AutoMUG will be very close to the bound presented in [15]. Furthermore, since MBOX employs a Galerkin approach, its spectral radius is expected to be smaller than that of AutoMUG. Table 1 confirms both expectations.

Next, we examine the (non-normal) convection diffusion equation

\[-u_{xx} - u_{yy} + \frac{15}{8h}(u_x + \gamma u_y) = g.
\]
Table 1: The spectral radius of the MBOX and AutoMUG iteration matrices for the Poisson equation with Dirichlet boundary conditions, implemented with two levels and the RB smoother. The bound for the spectral radius presented by Kuo and Levy is displayed in the last column.

<table>
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<th>N</th>
<th>(\nu)</th>
<th>MBOX</th>
<th>AutoMUG</th>
<th>MBOX</th>
<th>AutoMUG</th>
<th>[15]</th>
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<td>0.0407</td>
<td>0.0212</td>
<td>0.0408</td>
<td>0.0410</td>
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</tbody>
</table>

Table 2: The spectral radius of the MBOX and AutoMUG iteration matrices for the convection diffusion equation \(\nabla u = g\) with Dirichlet boundary conditions, implemented with two levels, the RB smoother and \(\nu_1 = \nu_2 = 1\).

<table>
<thead>
<tr>
<th>N</th>
<th>(\gamma = 1)</th>
<th>(\gamma = 0.8)</th>
<th>(\gamma = 1)</th>
<th>(\gamma = 0.8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>MBOX 0.0008</td>
<td>AutoMUG 0.0060</td>
<td>MBOX 0.011</td>
<td>AutoMUG 0.023</td>
</tr>
<tr>
<td>31</td>
<td>MBOX 0.0369</td>
<td>AutoMUG 0.0741</td>
<td>MBOX 0.012</td>
<td>AutoMUG 0.024</td>
</tr>
<tr>
<td>63</td>
<td>MBOX 0.0554</td>
<td>AutoMUG 0.0740</td>
<td>MBOX 0.012</td>
<td>AutoMUG 0.024</td>
</tr>
</tbody>
</table>

\(\nu_1 = \nu_2 = 1\) is used in the analysis. The results for the cases \(\gamma = 1\) and \(\gamma = 0.8\) are summarized in Table 2.

Finally we have considered the highly indefinite Helmholtz equation

\[-u_{xx} - u_{yy} - 790u = g.\]

The (scaled) coefficient matrix \(A\) for this problem, with (respectively) \(N = 15, 31\) and 63, has (respectively) 38, 32 and 30 distinct negative eigenvalues, the ten smallest (in magnitude) of which are displayed in the 1st, 4th and 7th column of Table 3. This information shows that the problem is nearly singular in the sense of [6]. In the other columns of Table 3, the ten largest (in magnitude) eigenvalues of the iteration matrix of MBOX and AutoMUG are displayed.

The five largest (in magnitude) eigenvalues of the AutoMUG iteration matrix for the \(N = 31\) example (6th column) were compared to those computed directly by a Lanczos-type method. The relative error was not greater than 2%.

Although the basic iteration (3) is divergent, it is seen that for \(N = 31\) or \(N = 63\) there exist only few isolated eigenvalues of magnitude larger than or close to one; consequently, a Lanczos-type acceleration method applied to (3) will result in a satisfactory convergence, as is confirmed in the numerical experiments in Section 7. Moreover, the analysis in [3] and the computations in [17] [18] show that the number of levels may be arbitrarily enlarged, provided an appropriate mesh-size for the coarsest grid is used; a \(15 \times 15\) or a \(31 \times 31\) grid is thus suitable to serve as a coarsest grid in a multi-level implementation.
Table 3: Part of the spectrum of $A$ (scaled) and the MBOX and AutoMUG iteration matrices for the Helmholtz equation $-u_{xx} - u_{yy} - 790u = g$ with Dirichlet boundary conditions, implemented with two levels, the RB smoother and $\nu_1 = \nu_2 = 1$.

<table>
<thead>
<tr>
<th>$N = 15$</th>
<th>$N = 15$</th>
<th>$N = 15$</th>
<th>$N = 31$</th>
<th>$N = 31$</th>
<th>$N = 31$</th>
<th>$N = 63$</th>
<th>$N = 63$</th>
<th>$N = 63$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>MBOX</td>
<td>AMUG</td>
<td>$A$</td>
<td>MBOX</td>
<td>AMUG</td>
<td>$A$</td>
<td>MBOX</td>
<td>AMUG</td>
</tr>
<tr>
<td>-0.241</td>
<td>174.13</td>
<td>-886.52</td>
<td>-1.14-10^-3</td>
<td>-6.710</td>
<td>-17.56</td>
<td>-1.16-10^-3</td>
<td>-3.409</td>
<td>1.478</td>
</tr>
<tr>
<td>-0.368</td>
<td>119.77</td>
<td>-727.10</td>
<td>-1.90-10^-2</td>
<td>1.680</td>
<td>2.91</td>
<td>-8.42-10^-3</td>
<td>0.751</td>
<td>-0.928</td>
</tr>
<tr>
<td>-0.431</td>
<td>61.56</td>
<td>-594.85</td>
<td>-2.07-10^-2</td>
<td>1.028</td>
<td>-1.75</td>
<td>-9.97-10^-3</td>
<td>0.329</td>
<td>0.451</td>
</tr>
<tr>
<td>-0.528</td>
<td>-42.44</td>
<td>-511.42</td>
<td>-5.04-10^-2</td>
<td>-0.700</td>
<td>0.981</td>
<td>-1.08-10^-2</td>
<td>0.152</td>
<td>-0.183</td>
</tr>
<tr>
<td>-0.617</td>
<td>-33.37</td>
<td>-416.42</td>
<td>-6.03-10^-2</td>
<td>0.554</td>
<td>0.786</td>
<td>-1.63-10^-2</td>
<td>-0.136</td>
<td>-0.136</td>
</tr>
<tr>
<td>-0.785</td>
<td>-21.84</td>
<td>-294.54</td>
<td>-6.17-10^-2</td>
<td>-0.533</td>
<td>-0.673</td>
<td>-1.97-10^-2</td>
<td>0.130</td>
<td>0.126</td>
</tr>
<tr>
<td>-1.094</td>
<td>21.70</td>
<td>-288.96</td>
<td>-9.12-10^-2</td>
<td>-0.493</td>
<td>0.542</td>
<td>-2.01-10^-2</td>
<td>0.108</td>
<td>0.098</td>
</tr>
<tr>
<td>-1.189</td>
<td>-20.85</td>
<td>-237.74</td>
<td>-0.102</td>
<td>0.420</td>
<td>-0.530</td>
<td>-2.46-10^-2</td>
<td>-0.097</td>
<td>-0.096</td>
</tr>
<tr>
<td>-1.285</td>
<td>-15.76</td>
<td>-162.36</td>
<td>-0.109</td>
<td>0.351</td>
<td>-0.508</td>
<td>-2.85-10^-2</td>
<td>0.092</td>
<td>0.089</td>
</tr>
<tr>
<td>-1.349</td>
<td>-11.81</td>
<td>-122.89</td>
<td>-0.123</td>
<td>0.279</td>
<td>-0.423</td>
<td>-3.48-10^-2</td>
<td>-0.087</td>
<td>0.084</td>
</tr>
</tbody>
</table>

7 NUMERICAL RESULTS

We apply BBOX, MBOX and AutoMUG to the problem

$$-u_{xx} - u_{yy} - 790u = f, \quad (x, y) \in \Omega = (0, 1) \times (0, 1),$$

with Dirichlet boundary conditions on $\Gamma_D \subset \partial \Omega$, Neumann boundary conditions on $\Gamma_N \subset \partial \Omega$ and complex boundary conditions of the third kind

$$\frac{\partial u}{\partial n} + 10i u = g, \quad (x, y) \in \partial \Omega \setminus \Gamma_D \setminus \Gamma_N$$

(where $n$ is the outer normal vector). We consider the following cases:

(a) $\Gamma_D = \partial \Omega$ \hspace{1cm} $\Gamma_N = \emptyset$

(b) $\Gamma_D = \emptyset$ \hspace{0.5cm} $\Gamma_N = x = 0$, $x = 1$ or $y = 0$.

The equation is discretized via a second-order five-point difference scheme on a uniform $N \times N$ grid. For case (a), this is just (13); for case (b), the finite volume scheme [1] is used. The initial error is random in $(0,1)$. The multi level methods are implemented with the red-black Gauss-Seidel (RB) smoother in a V(1,1)-cycle. The second level equation is solved to 6 orders of magnitude accuracy.

In order to handle iteration matrix eigenvalues of magnitude larger than one, we apply to the basic two-level iteration (3) the Transpose Free Quasi Minimal Residual (TFQMR) acceleration method (Algorithm 5.2 of [14]). This technique was chosen because it avoids the computation of the transpose of the coefficient matrix and preconditioner (the latter is only implicitly given in (2), so its transpose is not available). TFQMR may be considered a modification of the Conjugate Gradients Squared (CGS) method of [22]. The costs of these acceleration techniques are comparable to that of the Conjugate Gradients method, that is,
about 1–1.5 work units per iteration. We found that the performance of CGS and TFQMR is similar; we preferred the latter, however, because of its smooth convergence curve.

We define the following measures of efficiency: the asymptotic convergence factor

$$\text{cf} = \frac{\|Ax_{last} - b\|_2}{\|Ax_{last-1} - b\|_2}$$

and the averaged convergence factor

$$\text{avcf} = \left(\frac{\|Ax_{last} - b\|_2}{\|Ax_0 - b\|_2}\right)^{1/last},$$

where $last$ is large enough to realize the process behavior (when the process converges, $last$ is the iteration number for which the $l_2$ norm of the residual is reduced by about $10^6$). When acceleration is used, the convergence factor often oscillates; hence, only avcf is reported. Note that when no acceleration is used, MBOX and AutoMUG diverges in a rate which is equal to the spectral radius of the iteration matrix computed by the analysis in Table 3.

Table 4: Convergence factors for a two-level implementation of AutoMUG and MBOX (with the RB smoother in a V(1,1) cycle) for the equation $-u_{xx} - u_{yy} - 790u = f$. When acceleration is used, avcf is reported. When acceleration is not used, cf is reported.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\partial \Omega$</th>
<th>acceleration</th>
<th>BlackBox</th>
<th>MBOX</th>
<th>AutoMUG</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>(a)</td>
<td>no</td>
<td>10.613</td>
<td>6.710</td>
<td>17.563</td>
</tr>
<tr>
<td>63</td>
<td>(a)</td>
<td>no</td>
<td>2.007</td>
<td>3.409</td>
<td>1.478</td>
</tr>
<tr>
<td>31</td>
<td>(a)</td>
<td>yes</td>
<td>&gt; .9</td>
<td>.669</td>
<td>.757</td>
</tr>
<tr>
<td>63</td>
<td>(a)</td>
<td>yes</td>
<td>.503</td>
<td>.317</td>
<td>.275</td>
</tr>
<tr>
<td>63</td>
<td>(b)</td>
<td>yes</td>
<td>.525</td>
<td>.411</td>
<td>.503</td>
</tr>
</tbody>
</table>

8 CONCLUSIONS

In this work, a new method for two-level analysis of automatic multigrid methods (implemented with the red-black smoother) is introduced. It gives the spectrum of the iteration matrix in terms of that of the coefficient matrix alone. Unlike existing Fourier analyses, it is applicable also to non-normal problems (e.g., entering flow problems) and to certain problems with variable coefficients.

For indefinite Helmholtz equations, where standard multigrid is inferior to automatic one [18] [19] [20], the analysis provides an optimal choice for the coarse grid resolution. Furthermore, it induces an improved version of Black Box multigrid. The numerical experiments show that, when no acceleration is used, the iteration diverges exactly in the rate predicted by the analysis method. When acceleration is used, they show the advantage of the new variant of Black Box multigrid.

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References


