


original PDE. It is based on a $V(1,1)$ multi-level cycle, in which one presmoothing and one postsmoothing with the ILU(1,1) method are performed on each level. For star-type schemes, it conserves the star structure in all grids. Our experiments suggest that MAMG is more robust than standard AMG; for instance, nonsymmetric terms in the equation do not significantly slow down the convergence, whether the characteristics are closed or open. Non-uniform grids and non-rectangular domains are handled with the same efficiency, and no special treatment of subdomains is needed. It is also applicable to singular perturbation problems and indefinite Helmholtz equations. It seems to perform almost equally well as the Black Box multigrid of [9] [10]; though his results are sometimes better than ours, they are obtained for problems of size smaller than that of our examples. In addition, MAMG does not restrict the coefficient matrix $A$ to be of 9-diagonal type, as does "black Box" multigrid.

Though performing slightly slower than the MBF method in some cases, MAMG has the advantage of using the coefficient matrix $A$ as a whole, without requiring the additional information concerning decomposing $A$ into a sum of spatial derivative discretizations. Moreover, it has a potential to be generalized to differential equations with mixed derivative terms, differential operators of order $> 2$, differential equations with more than one unknown function, etc.. In some cases it may be combined with MBF; when the PDE is written as a sum of differential terms, the coarse grid discretization of each of them is generated by MAMG (or, if possible, by MBF) and the overall summation is done by the MBF approach.

References


with \( \beta = 100 \), complex boundary conditions of the third kind

\[
\frac{\partial n}{\partial n} + 100i u = g \quad (x, y) \in (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})
\]

where \( \vec{n} \) is the outer normal vector and \( i = \sqrt{-1} \), and Dirichlet boundary conditions on the rest of \( \partial \Omega \). Like the previous problem, this problem is indefinite, hence the use of acceleration is crucial.

The advantage of \( MBF \) for the Helmholtz equation becomes more significant for \( \beta \)'s larger than that used in this example, even with Dirichlet boundary conditions alone.

In Table 1, results for the \( ILU \) smoother with acceleration are presented for all examples. In Table 2, results for the \( RB \) smoother without acceleration are presented, for examples for which neither \( ILU \) nor acceleration is important in achieving satisfactory rates of convergence.

Table 1: \( GMRES(10) \) with \( ILU \) smoother: number of iterations needed to reduce \( \|res\|_2 \) by \( 10^6 \).

<table>
<thead>
<tr>
<th>example</th>
<th>AMG</th>
<th>MAMG</th>
<th>MBF</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>5</td>
<td>5</td>
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<td>2</td>
<td>8</td>
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<td>3</td>
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<tr>
<td>6</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>&gt; 20</td>
<td>6</td>
<td>8</td>
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<tr>
<td>8</td>
<td>&gt; 30</td>
<td>9</td>
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<td>&gt; 30</td>
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<td>13</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>11</td>
<td>&gt; 80</td>
<td>17</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 2: \( RB \) smoother, no acceleration: number of iterations needed to reduce \( \|res\|_2 \) by \( 10^6 \).

<table>
<thead>
<tr>
<th>example</th>
<th>AMG</th>
<th>MAMG</th>
<th>MBF</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>8</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>

4 DISCUSSION

Modified \( AMG \) (\( MAMG \)) is a multi-level method which is automatic in the following sense: it depends only on the coefficient matrix of the linear system of equations and not on the
For this example the RB smoother was unsatisfactory for all methods.

8. The convection-diffusion equation with fan-like streamlines

\[ u_{xx} + u_{yy} + 256(u_x + yu_y) = f \]

whose characteristics are rays starting at the origin so that they all intersect a boundary. This kind of equations is hard to solve with multigrid approach, since the coarse grid equation does not supply an appropriate correction for error terms which are smooth in the convection direction and oscillates in the perpendicular direction [7]. For this example, it was found that the use of GMRES does not improve the convergence of MBF, but affects the convergence of MAMG: without acceleration, MAMG converged in more than 20 iterations. The use of ILU, on the other hand, affects the convergence of both methods: with 2 RB presmootherings and 2 RB postsmootherings, MAMG converged in 17 iterations and MBF in 10 iterations.

9. The circulating flow equation

\[ \sin(\pi(y - 0.5))\cos(\pi(x - 0.5))u_x - \sin(\pi(x - 0.5))\cos(\pi(y - 0.5))u_y = f \]

The region is a square with a hole at the middle of it. For the algebraic system, this means that for the middle grid point we have a trivial identity equation instead of the difference equation. According to [7], an upwind scheme is inadequate for this problem. Following [8], we have used an isotropic artificial viscosity discretization. For this example, it was found that the use of ILU and GMRES is crucial in achieving satisfactory rates of convergence. With the use of ILU and GMRES, the results for MAMG are far better than those of the V-cycle in [8]. The coarse grid operators generated by MAMG may thus be used in conjunction with the defect correction approach of [8] to accelerate convergence.

10. The Helmholtz equation

\[ u_{xx} + u_{yy} + \beta u = f \]

with \( \beta = 64 \). Since this problem is indefinite, divergence occurs if acceleration is not used. Standard multigrid will not do, for coarse grid operators are ill-conditioned. This type of problems is discussed in [5]. The projection approach given there requires additional work and special treatment and is limited to small \( \beta \)'s.

11. The Helmholtz equation

\[ u_{xx} + u_{yy} + \beta u = f \]
4. The Poisson equation with a Tchebycheff-type grid; it is discretized via central differences on the 2-dimensional grid

\[ P(j,k) \equiv \left( \frac{1 - \cos \left( \frac{j\pi}{N+1} \right)}{2}, \frac{1 - \cos \left( \frac{k\pi}{N+1} \right)}{2} \right) \quad 1 \leq j,k \leq N \]

The matrix operator for this scheme may be used as a preconditioner for a Tchebycheff-collocation discretization of the Poisson equation (see [18] and the references therein). As in the previous 2 examples, the RB smoother was unsatisfactory. The reason for this is that the equation is locally anisotropic in most of the mesh cells.

5. The Poisson equation in a square with a slit. The actual shape is 127 x 127 grid minus a narrow strip of width 2 points and length 64 points. This problem is more difficult than the Poisson equation in a square, since the domain has a branch point in the middle of it. The smoothness of the solution does not make it easier, since the initial guess is random; similar rates of convergence were obtained when the solution has a jump over the slit.

6. The Poisson equation in a domain approximating the North Atlantic, from about 10° to 53° north. A definite Helmholtz problem stems from explicit time-stepping in the Quasigeostrophic version of the Shallow-Water equations. Here we solve the Poisson equation, which is more difficult than any definite Helmholtz equation. The domain is embedded in a Cartesian grid of 45 x 93 points. The bounding points are assumed to lie on that grid.

7. The Poisson equation with the Neumann-type boundary conditions

\[ \frac{\partial u}{\partial n} = g \quad (x,y) \in (\{1\} \times [0,1]) \cup ([0,1] \times \{1\}) \]

and Dirichlet boundary conditions of the rest of \( \partial \Omega \). For this problem only, \( N = 128 \) is used rather than \( N = 127 \), since the convergence of MAMG was unsatisfactory for \( N = 127 \). This is clear from the graphical interpretation of the coarse grid operator \( Q \) of MAMG: for \( N = 127 \), \( G(A(2I - D^{-1}A)) \) includes edges carrying flow to boundary nodes which are missing in \( G(Q) \); this is negligible for Dirichlet boundary conditions, where zero boundary conditions are acceptable for the coarse grid correction. But for boundaries assigned Neumann conditions, an appropriate coarse graph \( G(Q) \) must have no missing boundary sinks. This restriction also appears in the numerical experiments in [9] with pure Neumann boundary conditions: the odd number of fine grid points in each spatial direction used there guarantees that any coarse grid will have representative points on all boundaries. MBF, however, does not suffer this restriction and converges similarly for \( N = 127 \) and \( N = 128 \).
In our numerical experiments, one presmoothing and one post-smoothing are performed on each grid in a V-cycle.

In conjunction with the AMG, MAMG or MBF iteration, we have used the acceleration method GMRES(10) of [20]. The initial guess is random. Double precision arithmetic is used.

The integers in following tables represent the number of AMG, MAMG or MBF iterations needed to reduce the $l_2$ norm of the residual by $10^6$. The maximum norm of the error was also computed; in most cases it was reduced by about $10^6$. Only for the Helmholtz equation it was reduced by about $10^4$.

In the following examples, unless stated otherwise, the use of acceleration and of the ILU smoother is not crucial; avoiding the use of the GMRES acceleration, and replacing ILU by the red-black Gauss-Seidel (RB) smoother, achieves quite similar rates of convergence. Nevertheless, for the anisotropic, nonsymmetric and indefinite cases, it is necessary to use acceleration and/or ILU, according to the next subsection. Moreover, note that the RB smoother is not purely automatic, since it requires a-priori knowledge of the structure of the region on which the PDE is defined and of the order of variables.

### 3.1 List of Examples

1. The Poisson equation.

2. The anisotropic equation

   \[ u_{xx} + 10^{-2} u_{yy} = 0 \]

   For this example, we have observed a drastic reduce in efficiency when the ILU smoother is replaced by an RB relaxation (see [3]). The ILU smoother seems to be an appropriate choice for this problem.

3. The anisotropic equation

   \[ a(x)u_{xx} + a(y)u_{yy} = 0 \]

   with the discontinuous coefficient

   \[ a(t) = \begin{cases} 
   0.01 & 0 < t < 0.495 \\
   1 & 0.495 < t < 1 
   \end{cases} \]

   As in the previous example, it was observed that the RB smoother is unsatisfactory. We would like to emphasize that for this problem to be solved it was crucial to choose $N = 127$, so that all grids are centered. For $N = 128$, where coarse grids are biased towards the boundary, none of the methods have converged (see [21]).
results for DAMG since they are almost identical to those of MAMG, as is expected from the explanation of the definition of MAMG in Section 2.2. The problems solved are of the type
\[ Lu(x, y) = f(x, y) \quad (x, y) \in \Omega \]
with Dirichlet boundary conditions (except Examples 7 and 11, where boundary conditions of the second and third kind, respectively, are given on part of \( \partial \Omega \)) and exact solution \( u = xy \). With the exception of Examples 5, 6 and 9, in all our examples \( \Omega \) is the unit square \([0,1] \times [0,1]\). Also, with the exception of Example 4, in all the examples we have used uniform grids. The equations are discretized via a second-order central difference scheme. The number of grid points in each space direction is \( N = 127 \).

For MAMG, the prolongation operators \( \tilde{P} \) and \( P \) of Section 2.2 are chosen to be the 2-dimensional injection operator \( J_2 \) and the bilinear interpolation operator \( W_2 \) of Section 2.3, respectively. The restriction operators \( \tilde{R} \) and \( R \) are chosen to be \( W_2^* \) and \( \frac{1}{4} W_2^* \), respectively. For AMG, \( \tilde{P} \) is \((4I - \frac{1}{2} \tilde{A})J_2 \), where \( \tilde{A} \) denotes the 2-dimensional Poisson star, \( P \) is \( W_2 \) (so that \( \tilde{P} \) and \( P \) have the same column-sums) and the restriction operators \( \tilde{R} \) and \( R \) are both equal to \( \frac{1}{4} W_2^* \). With this choice, the amount of arithmetic operations of AMG and MAMG is equal. For both AMG and MAMG we have also used the modification of the last part of Subsection 2.3. With this modification, for the Poisson equation in a square both AMG and MAMG are equivalent to classical multigrid. The stopping parameter \( q \) in Conditions (3) and (7) is equal to 1 for all examples but Example 11, where it is equal to 64. The reason for this exception is that for the 2-dimensional indefinite Helmholtz equation \( u_{xx} + u_{yy} + \beta u = f \), waves with \((k, l)\) wave-numbers satisfying \( \pi^2(k^2 + l^2) \leq \beta \) may appear in the solution even for smooth right-hand-sides \( f \). Hence an appropriate stencil for this equation must be fine enough for these \((k, l)\) waves to be visible. For Example 11, this amounts to grids with at least \( 8 \times 8 \) points. In matrix theory language, the \( 8 \times 8 \) grid is the last one for which the sign pattern of the operator is an \( M \)-matrix pattern: positive on the main diagonal and negative elsewhere. For an automatic code, Conditions (3) and (7) are to be replaced by a stopping criterion detecting the ill-conditionedness of a coarse grid operator (e.g. by observing negative elements on the main diagonal).

The preconditioner \( Z \) in all calls to AMG and MAMG is the one provided by the \( ILU(1,1) \) decomposition of \([16] \ [17] \ [24] \). This amounts to a purely automatic smoother, independent of the domain on which the PDE is defined. An \( ILU(1,1) \) iteration is about twice as expensive as a Gauss-Seidel relaxation, if \( \nu \) smoothings are used, \( \nu \) \( ILU(1,1) \) iterations are about \((\nu + 1)/\nu \) times as expensive as \( \nu \) Gauss-Seidel relaxations, provided the trick of [12] is used. For the solution of a given problem, the \( ILU \) decompositions of all operators are generated once and for all in the first AMG (or MAMG, or MBF) iteration.
in the upper-left part of the template of \(A(2I - D^{-1}A)\) by

\[
\begin{bmatrix}
NW/4 & 0 & NW/4 \\
0 & 0 & 0 \\
NW/4 & 0 & NW/4
\end{bmatrix}
\]

which is equivalent to the addition of

\[
\begin{bmatrix}
NW/4 & 0 & NW/4 \\
0 & -NW & 0 \\
NW/4 & 0 & NW/4
\end{bmatrix}
\]

Similar additions appear on the other corners of the template. The effect of this inexactness is cancelled by the addition of the skew-Laplacian

\[
\begin{bmatrix}
-NW/4 & 0 & 0 & 0 & -NE/4 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & (NW + NE + SW + SE)/4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-SW/4 & 0 & 0 & 0 & SE/4
\end{bmatrix}
\]

to the coarse grid operator \(Q\), which amounts to throwing the corner elements of the template of \(Q\) onto the main diagonal.

From the graphical point of view, the above modification compensates for the large distances in the coarse grid graph: since fine edges of \(G((1 - D^{-1}A)^2)\), i.e. edges which do not appear in \(G(Q)\), are interpolated when multiplied by \(\bar{R}\) to twice longer edges of \(G(Q)\), only half the flow on these edges is to be interpolated, and the other half is to remain at the origin, or, in matrix language, on the main diagonal. This amounts to the template

\[
\bar{R} = \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/4 & 1 & 3/4 \\ 1/4 & 3/4 & 1/2 \end{bmatrix}
\]

(or some circulation of it, depending on the location of variable) rather than the usual \(W_2^*\), which is equivalent to the application of \(W_2^*\) followed by throwing corner template elements to the main diagonal.

For the Poisson equation in a square, the \(MAMG\) with the above modification is equivalent to classical multigrid. It is used in the numerical experiments of the next section, and is to be used also for general sparse linear systems.

3 Numerical Experiments

In this section, the automatic methods \(AMG\) and \(MAMG\) and the pseudo-automatic method \(MBF\) of [21] [22] [23] are compared for various kinds of equations. We do not show the
Then $R$ and $\bar{R}$ will be defined as $\frac{1}{2}W_1'$ and $W_1'$, respectively. The reason to the additional factor $1/2$ in $R$ is that it serves as a restriction operator in the algorithm, hence its row-sum has to be equal to 1, as opposed to $\bar{R}$, which only helps to create the coarse grid operator $Q$ and does not really act on fine grid vectors. With this choice of transfer operators $MAMG$ is independent of any of the features of the original PDE. Nevertheless, more efficiency may be achieved if the choice of transfer operators depends on the structure of the domain of the PDE; for problems which arise from PDE’s defined on $d$-dimensional regions, for example, it is natural to use $\bar{P} = J_d$, $P = W_d$, $R = \frac{1}{2}W_d^*$ and $\bar{R} = W_d^*$, where $J_d$ and $W_d$ are the $d$-time tensor product of $J_1$ and $W_1$, respectively:

\[J_d \equiv J_1 \otimes J_1 \cdots \otimes J_1 \quad (d \text{ times})\]
\[W_d \equiv W_1 \otimes W_1 \cdots \otimes W_1 \quad (d \text{ times})\]

It is also possible to take for $P$ the operator suggested in [9] and [14], where a coarse grid vector is continued to a fine grid point such that the equation corresponding to this point is satisfied. Then $R$ will be defined as $\frac{1}{2}P^*$.

For linear systems which arise from discretization of a vector PDE involving $k > 1$ unknown functions, the diagonal matrix $D$ in Definition (8) may be replaced with $D \equiv \text{diagblock}(A)$, where $\text{diagblock}(A)$ is a nonsingular matrix consisting of blocks of order $k$ representing the connection between different unknown functions on a given grid point. This approach is also used for Black Box multigrid in [11]. For staggered grids, a $k$-order block may connect different unknown functions defined on adjacent grid points. Neglecting the overhead of solving $k \times k$ systems, this implementation actually reduces the complexity of the original vector PDE to that of a scalar PDE, since the $k$ unknown functions are treated as they were one function. The only information from the PDE needed for this implementation is the integer parameter $k$ and the order of variables.

For star discretization of second order PDE’s, we introduce another modification of $AMG$: after obtaining the coarse grid matrix $Q$, enforce upon it the star structure of $A$ by throwing off-diagonal elements to the main diagonal. In other words, the elements in the corners of the 9-point template (in the 2-dimensional case) representing $Q$ are added to the central element, then set to zero. From the differential point of view, this modification may be considered as a compensation to the skew-Laplacian added to some elements of $A(2I - D^{-1}A)$ when multiplied by $\bar{R}$. In the 2-dimensional case, for example, multiplying by $\bar{R} = W_2^*$ amounts to replacing the element

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & NW & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
stop;
endif
smooth(x_in, A, b, x_in)
\[ D \equiv \text{diag}(A) \]
\[ Q \equiv \bar{R}A(2I - D^{-1}A)\bar{P} \]
\[ DAMG(0, Q, R(Ax_in - b), \bar{e}) \]
\[ x_{in} = x_{in} - (2I - D^{-1}A)P\bar{e} \]
\[ \text{smooth}(x_{in}, A, b, x_{out}) \]

for some small positive integer \( q \). As in AMG, the computation of the \( Q \) operators is accomplished in the first \( DAMG \) iteration solely. Note that the row-sums of \( A \) are often equal to 0. Since \( P\bar{e} \) in Equation (9) is a smooth vector, it is natural to replace \( 2I - D^{-1}A \) in that equation by \( \text{rowsum}(2I - D^{-1}A) = 2I \), to get what we call the Modified AMG (MAMG) method. This method is also obtained from AMG by setting

\[ \bar{P} = (I - \frac{1}{2}D^{-1}A)\bar{P} \]

in Definition (4), for some prolongation operator \( \bar{P} \). For sparse coefficient matrix \( A \), MAMG is practically not more expensive than AMG.

For star-type schemes for second order PDE’s, the MAMG defined above is not much different from the Black Box multigrid of [9] [10]. Nevertheless, it is defined in terms of the coefficient matrix \( A \) solely, independent of its structure. Hence it may be applied to general sparse linear systems.

## 2.3 Implementation of MAMG

For the successful implementation of MAMG it is necessary that no element on the main diagonal of \( A \) is equal to zero. For nonsingular coefficient matrices, a suitable order of variables and equations exists such that this condition is satisfied. The most recommended order is that for which the coefficient matrix is as diagonally dominant as possible.

There are many possible choices for the transfer operators \( P, \bar{P}, R \) and \( \bar{R} \) used in MAMG. A sufficient choice for \( \bar{P} \) seems to be the 1-dimensional injection operator \( J_1 \), whose symbol is

\[ [0 \ 1 \ 0]^t, \]

and for \( P \) the 1-dimensional linear interpolation operator \( W_1 \), whose symbol is

\[ \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}^t. \]
2.2 Modified AMG Methods

In this subsection we will define the Double AMG (DAMG) method. To accomplish this we will define the trivial two-level direct solver

\[
\text{solve}(x_{\text{in}}, A, b, x_{\text{out}}): \\
D \equiv \text{diag}(A) \\
Q \equiv A(2I - D^{-1}A) \\
\bar{e} = Q^{-1}(Ax_{\text{in}} - b) \\
x_{\text{out}} = x_{\text{in}} - (2I - D^{-1}A)\bar{e}. \\
\]

(6)

Of course, this solver is not practical, since Action (6) may be more expensive than the original problem. Note that if \( B \) is some nonnegative matrix, one may define the directed graph \( G(B) \) by

\[
G(B) \equiv \{(i, j) \mid B_{i,j} \neq 0\}
\]

Then one has

\[
G(B^2) = \{(i, j) \mid (B^2)_{i,j} \neq 0\} = \{(i, j) \mid \exists k, B_{i,k} \neq 0, B_{k,j} \neq 0\}
\]

that is, \((B^2)_{i,j} \neq 0\) if and only if there exists a path in the graph \( G(B) \) decomposed of 2 edges leading from \( i \) to \( j \). Hence \( B^2 \) represents the two-step graph generated by \( B \), and since

\[
D^{-1}Q = I - (I - D^{-1}A)^2
\]

\( I - D^{-1}Q \) represents the two-step graph generated by \( I - D^{-1}A \), and \( Q \) is suited to serve as a coarse grid approximation to \( A \). Moreover, the following lemma indicates that if \( A \) approximates an elliptic operator, so does \( Q \):

**Lemma 1** If \( A \) is an \( M \)-matrix, so is \( Q \).

**Proof:** Since \( A \) is an \( M \)-matrix, \( I - D^{-1}A \geq 0 \). From Theorem 3.10 in [25] it follows that \( \rho(I - D^{-1}A) < 1 \), where \( \rho \) denotes the spectral radius. Hence \((I - D^{-1}A)^2 \geq 0\) and \( \rho((I - D^{-1}A)^2) < 1 \), and the lemma follows from Theorem 3.8 in [25]. \( \square \)

This leads to the following algorithm:

\[
\text{DAMG}(x_{\text{in}}, A, b, x_{\text{out}}): \\
\text{if } A \text{ is of order } \leq q \\
x_{\text{out}} = A^{-1}b
\]

(7)
The modified AMG algorithms are described in Section 2. In Section 3 numerical results are presented. In Section 4 the algorithms and the numerical results are discussed.

2 DESCRIPTION OF ALGORITHMS

2.1 The AMG Method

Let $A$ be an $n \times n$ matrix. Let $x$ and $b$ be $n$-dimensional vectors of unknowns and data, respectively. Consider the problem

$$Ax = b. \quad (1)$$

An iteration of the Algebraic Multigrid (AMG) method is defined by

$$\text{AMG}(x_{in}, A, b, x_{out}):$$

if $A$ is of order $\leq q$  

\[ x_{out} = A^{-1}b \]  

stop;

endif

$$\text{smooth}(x_{in}, A, b, x_{in})$$

$$Q \equiv RAP$$

$$\text{AMG}(0, Q, R(Ax_{in} - b), \varepsilon)$$

\[ x_{in} = x_{in} - P\varepsilon \]

$$\text{smooth}(x_{in}, A, b, x_{out})$$

where $R$ and $R$ are some restriction operators, $P$ and $P$ are prolongation operators, $q$ is some positive integer and the subroutine smooth is defined by

$$\text{smooth}(x_{in}, A, b, x_{out}):$$

\[ x_{out} \leftarrow x_{in} - Z(Ax_{in} - b) \]

for some preconditioner $Z$ to be defined later. An iterative application of AMG is given by

\[ x_0 = 0, \; i = 0 \]

while $\|\text{residual}\| \geq \varepsilon$

\[ x_i = \text{AMG}(x_i, A, b, x_{i+1}) \]

$\; i \leftarrow i + 1$

endwhile

The operators $Q$ in the AMG calls may be generated once and for all in the first AMG iteration, then reused in all other iterations.
to handle discontinuous coefficients are given in [1] and [14], while the non-symmetric case is discussed in [7] [8]. Slightly indefinite problems are discussed in [5]. These approaches involve special treatment of problems according to the original PDE, and the need for a uniform approach is not yet fulfilled.

In principle, the multigrid procedure is problem-dependent, and cannot serve as a "black box" that solves every problem. Special attention has to be given to the neighborhood of the boundary and to lines of discontinuity. In [4] [19] the algebraic multigrid method AMG for symmetric problems is developed. Though this method is automatic in the sense that it depends on the linear system of equations solely, it suffers the disadvantage of the coefficient matrices for coarse grids having 9-diagonals, even when the original matrix has only 5-diagonals [15]. This significantly enlarges the amount of arithmetic operations required to generate coarse grid operators, and also enlarges by 25% (for 2-dimensional problems) or 40% (for 3-dimensional problems) the cost of a multigrid V-cycle. The Black-Box Multigrid of [9] is a modification of AMG which is applicable to problems with discontinuous coefficients and to nonsymmetric problems [10]. None of these methods, however, handle indefinite problems. In [21] [22] a multi-level approach called Multi Block Factorization (MBF) is presented. For star discretizations of second order PDE's this is an automatic solver, provided the coefficient matrix is given in several data structures, each of which contains elements corresponding to one of the spatial derivatives only. For example, for 2-dimensional problems, a tridiagonal matrix containing the coefficients of the discretized $u_{xx}$ is to be accessible independently of that representing $u_{yy}$. For discretizations more accurate than the star scheme, the application of MBF requires an additional finite difference discretization of the PDE, organized in the above data structure [23]. Though performing well even for indefinite problems, MBF is not as automatic as AMG. Moreover, it cannot handle differential equations with mixed derivative terms or differential operators of order $> 2$.

The algorithm presented in this work gives a uniform approach that enables one to handle the above difficulties. It relies on the algebraic system of equations solely, and not on the original PDE. As opposed to Black Box multigrid, it is defined as a sparse linear system solver, hence is not restricted to 9-template operators and has a potential to be applied to more general classes of linear systems (e.g. $M$-matrix systems). It may be viewed as a modification of AMG, hence is called Modified AMG (MAMG). The operators for coarse grids are automatically defined when the coefficient matrix is given. It seems to be more robust than the classical multigrid method, as it solves non-symmetric problems (even singular perturbation problems with closed characteristics) in a small number of iterations. Moreover, it is applicable to non-uniform grids and non-rectangular domains as well, and does not require any special treatment of sub-domains.
A MODIFIED ALGEBRAIC MULTIGRID METHOD WITH APPLICATION TO NONSYMMETRIC AND INDEFINITE PARTIAL DIFFERENTIAL EQUATIONS

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SUMMARY

A modified algebraic multigrid method for the solution of sparse linear systems of equations is presented. It is automatic in the sense that it depends on the linear system of equations solely, unlike classical multigrid which requires information about the differential equation. It is well-defined regardless of any special structure of the coefficient-matrix. It is applicable to non-uniform grids and non-rectangular domains almost as efficiently as for uniform grids on a rectangle. When supplemented with an acceleration method, it converges for non-symmetric, singular perturbation and indefinite equations in a small number of iterations.

1 INTRODUCTION

The multigrid method is a powerful tool for the solution of linear systems which arise from elliptic PDE’s [3] [13]. This is an iterative method, in which the equation is first relaxed on the original fine grid in order to smooth the error; then the residual equation is sent to a coarser grid, to be solved there and to supply a correction term. Recursion is used to solve the coarser grid problem in the same way the original equation was handled. In order to apply this procedure, the differential operator has to be discretized on all grids, and restriction and prolongation operators are to be defined in order to send information from fine to coarse grid and vice versa. The multigrid method works well for the Poisson equation in the square, but difficulties arise with non-symmetric problems, indefinite problems and problems with discontinuous coefficients, complicated domains or non-uniform grids. In those cases, it is not easy to discretize the differential operator on coarse grids. Some suggestions about how