


References


with, as well as other types of $R^2 \rightarrow R^2$ distortions.

The theory readily carries over to objects with curved boundaries. This topic and extensions to detect symmetries even under partial occlusion are currently under investigation.
while the range of the indicator values of all the other (false) vertex symmetry assumptions was

\[ 0.2 \leq 6 \]

4 Concluding Remarks

We have presented a general framework for skew-symmetry detection, based on the theory of invariant planar curve representation. The message of the general theory is that detection of skew symmetry is a process of detecting invariance under a concatenation of two transformations: one characterizing the symmetry sought after and the second, a viewing distortion. We have shown how this theory works on polygons distorted by affine and projective viewing transformations, however other types of symmetries can be similarly dealt
to produce low values for true symmetry axes, and significantly higher values for all the other possibilities.

Note that the line intersections, which are used in obtaining the affine and the projective signatures, may sometimes be extremely sensitive features, see e.g. [24], [23]. We therefore had to pay special attention to those places where the geometry of the (transformed) polygon indicated that small changes in the location of the vertices may cause large changes in the signature values. To do that, we calculated linear estimations of the error amplification from vertex position errors to errors in the signature values. We used the sensitivity analysis to normalize signature differences, thereby obtaining more stable symmetry indicators.

After the normalization described above the results where as follows: All the 12 polygons in Figures 8, 9 are projective skew symmetric. The range of projective symmetry indicator values for the true symmetry axis (over all the 12 images) was

\[ 2 \cdot 10^{-17} \quad \text{to} \quad 8 \cdot 10^{-6} \]

The range of projective symmetry indicator values over all the other (false) vertex symmetry assumptions, and all the 12 polygons was

\[ 2 \cdot 10^{-4} \quad \text{to} \quad 0.2 \]

We note that for each polygon considered individually, the results where always sharper than may appear from the above cumulative results. The ratio of the indicator value of the true axis to the smallest false indicator value was always less than \(10^{-3}\), while the range of false symmetry indicator values was never more than 1:100.

The range of affine symmetry indicator values for the true symmetry axis (over all the 6 affinely distorted polygons) was

\[ 6 \cdot 10^{-16} \quad \text{to} \quad 4 \cdot 10^{-14} \]
triangle as in Figure 6b. We want to show that we can find the symmetry inducing initial condition by sliding $Q_{i+2}$ on a line parallel to the line through $Q_{i-1}, Q_{i+1}$, see Figure 6b. Denote the position of $Q_{i+2}$ on the parallel line by $\lambda$. It is possible to show that for a symmetric position of $Q_{i-2}$, $\varphi^L_i(\lambda) = \varphi^R_i(\lambda)$ is a one to one function. Thus, for every given $\varphi^L_i = \varphi^R_i$ there is a unique $\lambda$ which causes $Q_{i-2}$ to be symmetric to $Q_{i+2}$, which provides a symmetric initialization for a fully symmetric polygon. Here also we can account for the six degrees of freedom in the translation, rotation, in axis scaling, and off axis scaling of the initial condition. The last degree of freedom is the ratio between the height of the isosceles triangle and the distance between $Q_{i+2}$ and the line through $Q_{i-1}, Q_{i+1}$.

### 3 Results

We have checked the proposed symmetry detection scheme for two symmetric polygons, shown in Figure 7. The polygons were distorted by applying several affine and projective transformations see Figures 8, 9 respectively. For each of the 12 polygons presented in the figures, we calculated two affine invariant signature sequences $\{\rho^L(i), \rho^R(i)\}$ and two projective invariant signature sequences $\{\varphi^L(i), \varphi^R(i)\}$.

The palindromic structure of all the sequences was analyzed as follows: A vertex symmetric sequence about the $i$th element obeys

\[
\rho^R(i-k) \approx \rho^L(i+k) \\
\rho^L(i-k) \approx \rho^R(i+k)
\]

The symmetry assumption was checked separately for each $i$, the vertex symmetry indicator $E(i)$, being the maximal absolute difference over all possible values of $k$,

\[
E(i) = \max_k \{ \rho^R(i-k) - \rho^L(i+k), \rho^L(i-k) - \rho^R(i+k) \}
\]

Similar indicators were used for edge symmetric sequences. The indicators were expected
once sufficient initial conditions are provided. Since the projective transformation group has eight degrees of freedom, at least eight numbers are necessary as initial conditions for the reconstruction. Indeed it is possible to reconstruct the polygon from $\varphi^L$, $\varphi^R$ and four consecutive point locations provided as initial conditions. Different positions of the initial conditions result in different projective transformations of the reconstructed polygon.

It is quite clear that because $\varphi^L$ is not sufficient to determine the polygon up to a projective transformation, a palindromic structure of $\varphi^L(i)$ is not sufficient to conclude that the polygon is projective skew symmetric. Next we show, in a constructive manner, that a joint palindromic structure of $\varphi^L$ and $\varphi^R$ is sufficient to determine projective skew symmetry. We will do that by reconstructing a mirror-symmetric projective transform of the given polygon. Note that a projective transformation of a symmetric shape whose symmetry axis is aligned with the $y$ axis, loses its symmetry property only if either the skew or the $x$-tilt parameter $w_x$ are non zero. Specifying an initial condition for a symmetric shape will "cost" here two degrees of freedom.

If the sequence of invariants is edge symmetric about the space between elements $i$ and $i-1$, then a symmetric trapezoid initial condition as in Figure 6a will cause the reconstructed polygon to be symmetric. Changing the position of the initial configuration will translate the reconstructed polygon. Rotating the initial condition will rotate it. Scaling the configuration in the symmetry axis direction or perpendicular to it, will scale the reconstructed polygon accordingly. Changing the relative size of the trapezoid bases accounts for tilting the object plane in the symmetry axis direction. Altogether these variations account for the six remaining degrees of freedom.

Suppose the sequence of invariants is vertex symmetric about the $i$th element, and an initial condition is given by $Q_{i-1}$, $Q_i$, $Q_{i+1}$, $Q_{i+2}$, with $\Delta(Q_{i-1}, Q_i, Q_{i+1})$ an isosceles...
respectively. The definition for $\varphi^R$ is similar. In Figure 4c $\varphi^L(i)$ is the cross ratio of points denoted by square bullets, and $\varphi^R(i)$ the cross ratio of points denoted by circular bullets.

As in the Euclidean and affine cases it is easy to see that only one invariant associated with the vertices is not sufficient to determine the polygon up to a projective transformation. In order to do that, we need two invariant quantities associated to each vertex, and $[\varphi^L(i), \varphi^R(i)]$ are sufficient, as can be seen from the argument below.

Assume a single invariant $\varphi^L$, is given with the position of any number of consecutive vertices as initial condition, see Figure 5a. Denote as before, the first known vertex position by $Q_l$, and the last known vertex position by $Q_k$.

Considering $\varphi^L(l+1)$ we note that since the location of $Q_{l-1}$ is not yet determined, also the location of $A_{l+1}^-$ can not be defined. Remember now that the cross ratio is identical for all point quadruples created by the intersection of a pencil of four lines with another line. Note that $\varphi^L(l+1)$ is the cross ratio of the intersection points of the line through $A_{l+1}^-$, $Q_{l+3}$ with the pencil from $Q_l$ to $Q_{l+3}$, $Q_{l+2}$, $Q_{l+1}$, and $A_{l+1}^-$. Since the cross ratio is invariant to the line crossing the pencil, the location of $A_{l+1}^-$ and hence also $Q_{l-1}$ can be confined to the dotted line through $Q_l$ in Figure 5a. Similar arguments are valid for the reconstruction from the other side of the initial condition, the dotted line through $Q_{k-2}$ being the constraint induced by $\varphi^L$ on the location of $Q_{k+1}$, see Figure 5b.

It is clear that the pair of invariants $\varphi^L$ and $\varphi^R$ are sufficient to reconstruct a polygon.
2.2 Projective-Symmetry Detection for Planar Polygons

Suppose now that we are given a planar polygon \( \{Q_1, Q_2, Q_3 \ldots \} \) and we wish to determine whether it can be the projective image of a mirror-symmetric polygon. The projective group of transformations \( T_p : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is given by

\[
[x', y'] = \frac{1}{1 + w_xx + w_yy} [x, y]A + [t_x, t_y]
\]

with \( A \) an invertible matrix. The projective group of transformations has eight degrees of freedom: Two in the translation vector, four in the matrix \( A \), and two tilt parameters \( w_x = \frac{\partial H}{\partial x} \) and \( w_y = \frac{\partial H}{\partial y} \) indicating the tilt of the object plane \( H \) in the camera coordinate system. As in the affine case the four degrees of freedom in the invertible matrix \( A \) can be assigned to \( \phi \) the rotation, \( a_x \) and \( a_y \) the scales in the \( x \) and \( y \) directions, and \( s \) the skew parameter. It is well known that \( T_p \) preserves cross ratios. The cross ratio is defined for four collinear points \( P_1, P_2, P_3, P_4 \) ordered on a line.

\[
CR(P_1, P_2, P_3, P_4) = \frac{\| (P_1, P_3) \| \cdot \| (P_2, P_4) \|}{\| (P_1, P_4) \| \cdot \| (P_2, P_3) \|}
\]

\( T_p \) also preserves line intersections. If a closed polygonal curve \( P = \{P_1, P_2 \ldots P_N\} \) undergoes a projective transformation the vertices \( P_i \) will be mapped into the vertices \( Q_i \) of the resulting image \( Q = \{Q_1, Q_2 \ldots Q_N\} \) and clearly the ordered sequencing of the vertices acts as a projective invariant “arc-length” parameterization. As in the affine case, we ask ourselves what type of invariant signatures we could associate to each vertex of the polygonal curve. Recall that we need to produce a signature sequence associated with \( Q \) (by invariance the same as the one computed for \( P \) ) that will enable the unique reconstruction of the equivalence class of all planar polygonal curves that are projective-equivalent to \( P \). The signature sequence \( \{\varphi(1), \varphi(2) \ldots \varphi(N)\} \) associated to \( P \) or \( Q \) should be projective invariant, hence it will be based on cross ratios of collinear points anchored at either the vertices of the polygon or intersections of lines through the vertices.

For example we could define \( \varphi(i) \) to be the cross ratio of \( Q_{i-1}, C_i^L, C_i^R \), and \( Q_{i+1} \), where \( C_i^L \) and \( C_i^R \) are the intersections of the line through \( Q_{i-1}, Q_i \), with the lines through \( Q_i, Q_{i+2} \) and \( Q_i, Q_{i-2} \) respectively, see Figure 4a. Or, the cross ratio of \( D_i^-, Q_i, D_i, \) and \( D_i^+ \). Where \( D_i^- \) is the intersections of the line through \( Q_{i-2}, Q_{i-1} \) with the line through \( Q_{i+2}, Q_{i+1} \), and \( D_{i-}, D_{i+} \) are the intersections of the line through \( D_i^-, Q_i \), with the lines through \( Q_{i-1}, Q_{i+1} \) and \( Q_{i-2}, Q_{i+2} \) respectively, see Figure 4b.

In the projective case, again, we shall need two invariants at each vertex of the polygon. In order to maintain left right symmetry in the invariant pair, we use the projective invariants \( \varphi^L \) and \( \varphi^R \). \( \varphi^L \) is the cross ratio of \( A_i^-, A_i, A_i^+ \), and \( Q_{i+2} \). \( A_i^- \) is the intersections of the line through \( Q_{i-2}, Q_{i-1} \) with the line through \( Q_{i+1}, Q_i \), and \( A_i, A_i^+ \) are the intersections of the line through \( A_i^-, Q_{i+2} \), with the lines through \( Q_{i-1}, Q_i \) and \( Q_{i-1}, Q_{i+1} \).
parameter is non zero. Hence, specifying an initial condition for a symmetric shape will “cost” us only one degree of freedom (i.e will set the skew to zero).

We shall separate two generic cases of palindromic sequences:

- **Vertex symmetric** sequences are palindromic about, say the $i$th element, i.e. $\rho^R(i-k) = \rho^L(i+k)$ and $\rho^L(i-k) = \rho^R(i+k)$.

- **Edge symmetric** sequences are palindromic about the space between elements $i$ and $i-1$, i.e. $\rho^R(i-k-1) = \rho^L(i+k)$ and $\rho^L(i-k-1) = \rho^R(i+k)$.

If the sequence of invariants is vertex symmetric about its $i$th element, then an isosceles triangle initial condition $\|(Q_{i-1}Q_i)\| = \|Q_iQ_{i+1}\|$ will cause the reconstructed polygon to be mirror symmetric, see Figure 3a. Changing the position of the initial triangle will translate the reconstructed polygon. Rotating the initial condition around $Q_i$ will rotate it. Changing the height of the isosceles triangle or its width, will scale the polygon in directions parallel or perpendicular to the symmetry axis. Altogether these variations account for the five degrees of freedom remaining after the skew was set to zero.

![Figure 3: Initial conditions causing a symmetric reconstruction from affine invariants.](image)

If the sequence of invariants is edge symmetric about the space between elements $i$ and $i-1$, and an initial condition is given by $Q_{i-1}, Q_i, Q_{i+1}$: We have that changing the position of the initial triangle and rotating it, will translate the reconstructed polygon and rotate it accordingly. Scaling the initial condition parallel (perpendicular) to the direction $Q_{i-1}Q_i$, will scale the polygon perpendicular (parallel) to the symmetry axis. The only degree of freedom left is the position of $Q_{i+1}$ on a line parallel to $Q_{i-1}Q_i$. Since $\rho^R(i-1) = \rho^L(i) = \lambda$ it easily shown that $Q_{i-2}$ is on the same line parallel to $Q_{i-1}Q_i$ as $Q_{i+1}$, and its horizontal distance from $Q_{i-1}$ is $\frac{d}{x} - z$, where $d$ is the length of the edge $Q_{i-1}Q_i$ and $z$ is the horizontal distance between $Q_i$ and $Q_{i+1}$, see Figure 3. Clearly, selecting the last degree of freedom to be $z = \frac{d(1-2\lambda)}{2\lambda}$ causes $Q_{i-2}$ to be symmetric to $Q_{i+1}$, and the rest of the reconstruction is naturally symmetric as well. Selecting another $z$, would cause the reconstruction to be skewed.
invariant quantities associated to each vertex, and \([\rho^L(i), \rho^R(i)]\) from the example above are indeed sufficient, as can be seen from the argument below.

Assume a single scalar invariant \(\rho^L\) is given, with the position of a finite number of consecutive vertices as an initialization for the reconstruction process, see Figure 2. Denote the first known vertex position by \(Q_1\), and the last known vertex position by \(Q_k\). From \(\rho^L(i+1)\) and the known length of the segment \(Q_iQ_{i+2}\), we can determine the direction in which \(Q_{i-1}\) is located. This limits its location to the dotted line, see Figure 2a. In order to fully specify the location of \(Q_{i-1}\) further information is needed. If for example, another invariant quantity, say \(\rho^R\) is given for each vertex, then \(\rho^R(i)\) would uniquely determine the location of \(Q_{i-1}\) on the line. Similar arguments are valid for reconstruction starting at the other side of the initial condition, the dotted line in Figure 2b being equivalent to the dotted line in Figure 2a.

![Figure 2: A single affine invariant \(\rho^L\) is not sufficient for reconstruction.](image)

We have shown that the pair of values \(\rho^L\) and \(\rho^R\) associated to each vertex, are sufficient to reconstruct a polygon, once sufficiently many initial conditions are provided. Since the affine transformation group has six degrees of freedom, at least six values must be set by the initial conditions to enable reconstruction of the polyline. Indeed, it is clearly possible to reconstruct the polygon from \(\rho^L\), \(\rho^R\) and three consecutive vertex locations given as an initial condition. Different positions of the initial conditions result in various affine transformations of the reconstructed polygon.

It is quite clear that because \(\rho^L(i)\) is not sufficient to determine the polygon up to an affine transformation, a palindromic structure of \(\rho^L(i)\) is not sufficient to conclude that the polygon is skew symmetric. Next we shall show, in a constructive manner, that a joint palindromic structure of \(\rho^L\) and \(\rho^R\) is indeed sufficient to determine affine skew symmetry. We will do that by reconstructing a mirror-symmetric affine transformation of the given polygon. Note that an affine transformation of a symmetric shape whose symmetry axis is aligned with either the \(x\) axis or the \(y\) axis, looses its symmetry property only if the skew
class of all planar polygonal curves that are affine-equivalent to \( P \). The signature sequence \( \{ \rho(1), \rho(2), \ldots, \rho(N) \} \) associated to \( P \) or \( Q \) should be affine invariant and hence the \( \rho(i) \)'s can be (functions of) area ratios of various shapes “anchored” at each of the vertices \( \{ P_i \} \) or \( \{ Q_i \} \).

For example, we could define \( \rho(i) \) to be the triangle area ratio for the two triangles \( \Delta(Q_{i-2}, Q_{i-1}, Q_i) \) and \( \Delta(Q_{i-1}, Q_i, Q_{i+2}) \), see Figure 1a, or the area ratios of the triangles \( \Delta(Q_{i-2}, Q_{i-1}, Q_i) \) and \( \Delta(Q_i, Q_{i+1}, Q_{i+2}) \) and the triangle \( \Delta(Q_{i-1}, Q_i, Q_{i+1}) \). Or we could even use the (invariantly defined) segment intersections \( q_i^L, q_i^R \) (see Figure 1b) to define as \( \rho(i) \) the vector \([\rho^L(i) \rho^R(i)]\) where

\[
\rho^L(i) = \frac{\| \Delta(q_i^L, Q_{i-1}, Q_i) \|}{\| \Delta(Q_{i-1}, Q_i, Q_{i+1}) \|}
\]
\[
\rho^R(i) = \frac{\| \Delta(Q_i, Q_{i+1}, q_i^R) \|}{\| \Delta(Q_{i-1}, Q_i, Q_{i+1}) \|}
\]

Here we have clearly

\[
\rho^L(i) = \frac{\| (q_i^L, Q_{i-1}) \|}{\| (Q_{i-1}, Q_{i+1}) \|}
\]
\[
\rho^R(i) = \frac{\| (Q_{i+1}, q_i^R) \|}{\| (Q_{i-1}, Q_{i+1}) \|}
\]

In the above \( \| \Delta(A, B, C) \| \) denotes the area of the triangle \( \Delta(A, B, C) \), and \( \| (A, B) \| \) the length of the line segment from \( A \) to \( B \).

Similar types of invariants were proposed for affine invariant recognition of polygons under occlusion by [22].

Suppose we have a single invariant quantity associated with every vertex of the polygon. It is easy to see that the scalar invariant series \( \{ \rho(i) \} \), in conjunction with any initialization involving a finite number of vertices is not sufficient to reconstruct the entire polygon up to an arbitrary affine transformation. (just like the sequence of edge-length is a valid Euclidean-invariant signature but is clearly insufficient for reconstruction of the shape - although it might be good enough for model-based recognition!). We need two independent
signature should have the property that from it the boundary is uniquely determined up to a transformation $T$, (i.e., the equivalence class of all shapes that are $T$-distortions of each other is uniquely characterized by $\rho(\tau)$).

As we shall see in the sequel, for planar polygons, such signatures are readily found for the groups of affine and projective transformations.

In case we deal with continuous curves, the classical theory of differential invariants yields signatures with the desired property, however, those are unfortunately based on using high derivatives of the parameterized curve representations [12], [13], [15], [17]. Several approaches have been developed to circumvent the need for high derivatives, through the use of global point matches, [18], [19], via local frames [21], or by the local use of global invariants in conjunction with finite differences w.r.t the $T$-invariant arclength [20].

We shall next discuss the easier case of planar polygons and develop methods for reflexive (mirror) symmetry detection under affine and projective transformations.

### 2.1 Affine-Symmetry Detection for Planar Polygons

Suppose we are given a planar polygon $Q = \{Q_1, Q_2, Q_3, \ldots\}$ and we wish to determine whether it can be the affine image of a mirror-symmetric polygon. The affine group of transformations $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$[x', y'] = [x, y]A + [t_x, t_y]$$

where $A$ is an invertible matrix. The affine group of transformations has six degrees of freedom: Two in the translation vector and four in the matrix $A$. Any $2 \times 2$ invertible matrix $A$ can be multiplicatively decomposed as follows

$$A = R \cdot D \cdot S = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \cdot \begin{bmatrix} a_x & 0 \\ 0 & a_y \end{bmatrix} \cdot \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

where $R$ is a unitary (rotation) matrix, $D$ a diagonal matrix, and $S$ an upper triangular “skew” matrix. In this decomposition $\phi$ is the rotation parameter, $a_x$ and $a_y$ the scale parameters in the $x$ and $y$ directions respectively, and $s$ the skew parameter. It is well known that $T_\alpha$ preserves area-ratios, since under $T_\alpha$ areas of shapes are uniformly scaled by $|\det A| = a_xa_y$. If a closed polygonal curve $P = \{P_1, P_2, \ldots, P_N\}$ undergoes an affine transformation the vertices $P_i$ will be mapped into the vertices $Q_i$ of the resulting $T_\alpha$ transformed image $Q = \{Q_1, Q_2, \ldots, Q_N\}$. Clearly the ordered sequencing of the vertices readily acts as an affine invariant “arclength” parameterization. We now ask ourselves what type of invariant signatures we could associate to each vertex of the polygonal curve. Recall that we need to produce a signature sequence associated with $Q$ (that by invariance will be the same as the one computed for $P$) enabling the unique reconstruction of the equivalence
The next section will discuss at length the problem of detecting reflexive symmetry for planar shapes distorted by affine and projective viewing transformations, however the reader should realize that the approach is quite general and can be readily carried over to other types of symmetries and different $R^2 \rightarrow R^2$ distorting maps.

2 General Approach to Skew Symmetry Detection

A planar object with reflexive or mirror symmetry has a boundary curve $C(s)$ with a (periodic) curvature versus arclength description $K(s)$ that clearly displays the symmetry. If the length of the boundary is $L$, we must have a starting point $P_o$ on the curve such that if $C(0) \neq P_o$ then

$$K(s) = K(L - s) \text{ for } s \in [0, L/2]$$

Hence mirror symmetry induces a palindromic structure on the $K(s)$ representation. Conversely, if $K(s)$ has this structure the curve we can reconstruct from it (uniquely, up to a Euclidean transformation!) will necessarily be mirror symmetric. Thus the $K(s)$ representation elegantly solves the problem of mirror symmetry detection for shapes whose instances are “distorted” by Euclidean and even similarity transformations in the plane.

Suppose however that we pose the following general question: Given a shape $S$ and a continuous group of plane transformations $T_{\theta} : R^2 \rightarrow R^2$ (parameterized by $\theta$) that distort the shape, how could we detect whether the “original” undistorted shape is mirror-symmetric given a distorted instance of $S$, i.e., $T_{\theta_0}[S]$.

The answer to this question is the following: Suppose that we can find a $T$-invariant metric on the planar curve (like the Euclidean-invariant arclength metric) and furthermore assume we can find a signature function (like the Euclidean curvature) that is also $T$-invariant. Such invariant metrics and signatures will be based on some local, differential or geometric properties of the curve. Denote the signature versus the $T$-invariant arclength function by $\rho(\tau)$. All $T_{\theta}$-transformed versions of $S$ will then have the same $\rho(\tau)$ modulo some initialization for the (clockwise) traversal of the boundary curve. (The initialization clearly induces a shift in $\tau : \tau \rightarrow \tau - \pi_0$).

Since the identity transformation always belongs to the group $T_{\theta}$, the original, truly symmetric, instance of $S$ will have the same $\rho(\tau)$. But the reparameterization and $\rho(\tau)$ are computed based on some local, geometric properties of the boundaries (see [18], [19] [20], [21]), therefore the $\rho(\tau)$ description of a mirror symmetric $S$ will exhibit the same type of palindromic symmetry like $K(s)$ for the Euclidean-invariant case.

Therefore, the problem of symmetry detection under the distorting $T$-transformation is quite elegantly solved if an invariant signature versus $T$-invariant arclength is found. This
we, humans, are exhibiting remarkable capabilities of detecting symmetries in spite of such viewing distortions. It is considered important to have the capability of analyzing and recognizing shapes and their symmetries, even when those underwent quite severe distortions due to the, generally nonlinear, projection involved in the image acquisition process. Hence, the problem of detecting skew symmetries received a lot of attention in the machine vision literature, see for example the survey of the “state-of-the-art” in a recent paper by Gross and Boult [6] and the references therein. Most of the work in this context, including [6] (and also [7], [8], [9]), dealt with skew symmetries as defined by T. Kanade [10], [11], assuming distortions due to orthographic projections and devising various, most often global and sometimes local and feature-based, ad-hoc procedures to determine the slanted symmetry axis of distorted planar objects, with reflexive symmetry. Some of the work in this area was quite clever and elegant. However, it seems to us that a general approach never emerged.

In this paper we propose a general framework for symmetry analysis of planar shapes based on the use of invariance theory. Indeed, differential, semi-differential and various types of local invariants were recently proposed for the descriptions of planar shapes that would enable their recognition even in distorted and partially occluded instances. The idea is to use an invariant “signature”-based boundary curve description, generalizing the commonly used curvature versus arclength representation that is invariant only under Euclidean distortions. Through the recent work of several teams of researchers in computer vision, work based on the classical theory of differential and geometric invariants for the affine and projective groups of transformations [12] [13], a wealth of invariant signatures, of “generalized curvatures” versus “invariant arclength” representations are now available. See the papers [14]-[21].

The main thesis of this paper is that symmetries, if present, will always manifest themselves as special structures in the projection-invariant signature functions, thereby reducing the problem of symmetry detection and analysis to that of analyzing a (periodic) 1D function.

In fact detection of symmetries of planar shapes affected by the viewing projection that generated their images can be accomplished by encoding the boundaries of the objects in the image in ways that are invariant under those distorting transformations. This invariance implies that the original, undistorted and hence symmetric object (in the real “Euclidean” sense) will have the same description as the entire class of its possible (distorted) images.

Therefore: If symmetry in the Euclidean plane implies invariance under some $R^2 \rightarrow R^2$ transformations, like reflections about some axis passing through the centroid of the shape or rotations by some angles, symmetry of shapes viewed through some distorting, say projective of affine transformation implies invariance under the “conceptual” concatenation of the projection and symmetry transformation maps.
1 Introduction: Symmetries and Invariance

Patterns and symmetries are the source of endless enjoyment for all of us: we seek to detect symmetry and patterns whenever we are presented with visual or auditory stimuli and we try to design and produce things with interesting symmetries and structures. A. Weil in his remarkable book *Symmetry*, [1], shows that "symmetry in its several forms, bilateral, translatory, rotational, ornamental", etc. is a geometric concept closely related to the notion of "invariance of a configuration of elements under a group of automorphic transformations".

Computer Vision and Computational Geometry researchers interested in shape analysis have devoted much work to developing symmetry detection methods, see e.g., [2], [3], [4], [5]. The algorithms that were developed for symmetry detection exploit in clever ways the invariance of shapes or configurations of geometric elements implied by various types of symmetries.

If we have a bounded configuration of geometric elements (points, lines, basic simple shapes) in 2D, the groups of transformation whose invariances generate symmetries are rotations about the centroid and reflections about lines passing through it. If, furthermore, the number of feature points in the bounded geometric configuration is finite then the size of the group of transformations that could induce symmetry invariances is finite. Hence it is usually not very complicated to detect symmetries in such cases: We must perform a finite number of tests for reflection or for rotation symmetries.

Atallah [4] and Eades [5] have shown that symmetries of planar point configurations consisting of points and segments can be detected with efficient algorithms. For example, detecting reflection symmetry for polygons can be done in the following way: encode the closed polygonal line defined by the (cyclic) sequence of points \( P_1 P_2 \cdots P_N \) via an associated cyclic sequence of triplets, each associated to a break point \( P_i \),

\[
(d_N \alpha_1 d_1)(d_1 \alpha_2 d_2)(d_2 \alpha_3 d_3) \cdots (d_{N-1} \alpha_N d_N)
\]

where \( d_i \) is the length of the segment \( P_i P_{i+1} \) and \( \alpha_i \) is the angle in the range \([-\pi, \pi]\) at the \( i \)th vertex. This (obviously redundant) representation clearly enables us to reconstruct the polyline up to a Euclidean transformation, and the various symmetries of the object, if they exist, will become apparent in it. In particular, the reflection symmetry appears as a palindromic structure of the cyclic sequence \( \{ \cdots d_i \alpha_{i+1} d_{i+1} \cdots \} \) about some center point. This property is easily detected with a linear algorithm in \( N \), see [4] and [5].

A related problem of much interest to the Computer Vision community is the detection of "skewed" symmetries. To give an illustrative example, assume that a symmetric planar shape is projected into an image by a pin-hole camera. The image clearly loses its symmetry in the process, whenever the image plane is not parallel to the object plane. However
Skew-Symmetry Detection via Invariant Signatures

A.M. Bruckstein and D. Shaked
Center for Intelligent Systems
Technion, IIT, 32000 Haifa, Israel

Abstract

We propose a new approach to skew-symmetry detection, based on the theory of invariant signatures for planar objects. Invariant signatures associated to object boundaries are generalizations of the curvature versus arclength description of curves, invariant under geometric transformations more complex than the Euclidean ones. We show that symmetries of objects, and hence of closed boundaries, translate into simple structures in the invariant signature functions and are therefore readily detectable.