An easy planarity test for unflippable modules

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Abstract

Given a set of VLSI modules, with labeled terminals on each, and such that each terminal appears in exactly two modules, we present a simple, linear-time test for determining whether this set of modules has a planar embedding, in which terminals with equal labels are connected without crossovers, and where it is not allowed to flip the modules. By using a property of oriented polyhedrons, our algorithm is very simple, and much simplifies previous algorithms for this problem.

1 Introduction

Wire-routing problems are extensively studied in the area of VLSI circuits. Though being NP-complete (see Kramer and van Leeuwen [8]), many special cases of it are efficiently solvable (e.g., Dolev et al [2], Leiserson and Pinter [9] and Pinter [10]).

Given a set of VLSI modules, with labeled terminals on each, and such that each terminal appears in exactly two modules, we consider the problem of determining whether this set of modules has a planar embedding, in which terminals with equal labels are connected without crossovers, and where it is not allowed to flip the modules. This planarity property stems from the requirement to have a one-layer embedding of the VLSI chip. As noted in [9], this problem is a two-dimensional extension of the river routing problem.
A linear-time algorithm for this problem was first given by Pinter [10] (for both the flippable and unflippable cases), based on the planarity testing algorithm of Hopcroft and Tarjan [7]. Another algorithm was later suggested by Amir [1], which does not use this planarity test, but rather exploits the special structures of the graphs resulting from the given problem.

We present a much simpler linear-time algorithm, that solves the embedding problem. For our solution, the graph containing the set of modules, together with the links connecting corresponding terminals, is shown to be an oriented polyhedron, to which we simply apply Edmonds’ permutation technique [3] to determine the genus of an orientable surface onto which an embedding is possible (and we thus obtain a planar embedding as a special case).

In Section 1 we present the problem. Basic observations are discussed in Section 3. The algorithm is discussed in Section 4, and its correctness is the subject of Section 4.

2 The problem

We are given a set of modules \( \mathcal{M} = \{m_1, m_2, ..., m_t\} \), each with a given ordering of terminals. If a module \( m_i \) contains the terminals \( A_{i_1}, A_{i_2}, ..., A_{i_s} \) clockwise, we denote this as \( m_i = < A_{i_1}, A_{i_2}, ..., A_{i_s} > \). \( A_{i_1} \) and \( A_{i_2} \), \( A_{i_2} \) and \( A_{i_3} \), ..., and \( A_{i_s} \) and \( A_{i_1} \), are called neighboring terminals. Each of the terminals appears in exactly two distinct modules (see 3rd remark at the end of the paper). If two modules contain a terminal \( A \), we term them neighboring modules. To simplify the exposition we denote the two occurrences of terminal \( A \) as \( A^1 \) and \( A^2 \), and call them twin terminals.

We have to embed the modules on the plane, and to connect all equally labelled terminals without crossovers, where no flips of the modules are allowed. If such an embedding exists, we term the set of modules planar.

Example 1: Let the set of modules be \( \mathcal{M}' = \{m_1, m_2, m_3, m_4\} \), where \( m_1 = < A^1, C^1, B^1 > \), \( m_2 = < I^1, J^1, C^2, D^2 > \), \( m_3 = < J^2, I^2, G^1, H^1, B^2 > \), and \( m_4 = < A^2, H^2, G^2, D^1 > \). A planar embedding of \( \mathcal{M}' \) is shown in Figure 1. □□□
Figure 1: A planar embedding of $\mathcal{M}'$

3 Preliminaries

Given a set of modules $\mathcal{M}$ as above, we define the graph $G(\mathcal{M}) = (V, E)$, where $V$ is the set of terminals that appear in $\mathcal{M}$, that is, $V = \bigcup_{i=1}^{n} \bigcup_{j=1}^{k} \{A_{ij}\}$, and $E$ contains the edges $(A, B)$, where $A$ and $B$ are neighboring terminals or twin terminals. We denote by $F$ the set of faces of $G(\mathcal{M})$. We get:

**Corollary:** A given set $\mathcal{M}$ of modules is planar if and only if the graph $G(\mathcal{M})$ has a planar embedding, that contains a face $x_1x_2...x_i$ for every module $m \subseteq x_1, x_2, ..., x_i \in \mathcal{M}$, and in the same orientation.

Such an embedding of $G(\mathcal{M})$ is termed *legal*.

A crucial property of legal embeddings (to be justified in Section 5) is that the faces are of two kinds:

- Faces corresponding to the modules.
- Faces that contain edges that connect neighboring terminals and edges that connect twin terminals, alternately.

**Example 2:** A legal embedding of the graph $G(\mathcal{M}')$ (Example 1) is drawn in Figure 2, with the faces of the first kind shaded. They are $f_1 = A^1C^1B^1$, $f_2 = I^1J^1C^2D^2$, $f_3 = J^2I^2G^1H^1B^2$ and $f_4 = A^2H^2G^2D^1$, corresponding to the modules $m_1, m_2, m_3$ and
The other faces are $f_5 = A^2 A^1 B^1 B^2 H^1 H^2$, $f_6 = B^2 B^1 C^1 C^2 J^1 J^2$, $f_7 = J^2 J^1 I^1 I^2$, $f_8 = H^2 H^1 G^1 G^2$, $f_9 = G^2 G^1 I^2 I^1 D^2 D^1$, and $f_{10} = A^1 A^2 D^1 D^2 C^2 C^1$.

Note that if we traverse the edges of each face in order, (e.g., the edges of $f_1 = A^1 C^1 B^1$ are traversed $A^1 \rightarrow C^1$, $C^1 \rightarrow B^1$, and $B^1 \rightarrow A^1$), then each of the edges of $G(\mathcal{M}')$ is traversed exactly once in each direction.

Note also that $|V| = 16$, $|E| = 24$, $|F| = 10$, and $|V| - |E| + |F| = 2$. □□□

![Figure 2: A legal embedding of $G(\mathcal{M}')$](image)

Clearly, not every set of modules has a planar embedding.

**Example 3:** Consider the set of modules $\mathcal{M}'' = \{m_1, m_2\}$, where $m_1 = \langle A^1, B^1, C^1 \rangle$ and $m_2 = \langle A^2, B^2, C^2 \rangle$ (see Figure 3). $\mathcal{M}''$ is clearly not planar. However, it is possible to embed $\mathcal{M}''$ on a torus. Note that in the graph $G(\mathcal{M}'')$ we have $|V| = 6$, $|E| = 9$, $|F| = 3$, and $|V| - |E| + |F| = 0$. Also, if it were allowed to flip the modules, then $\mathcal{M}''$ would have a planar embedding. □□□
4 The algorithm

We now present the algorithm to determine whether a given set \( \mathcal{M} \) of modules is planar, that is, whether \( G(\mathcal{M}) \) is an oriented polyhedron (\([12]\)), that can be embedded on a plane. We will first determine the sets \( \mathcal{V} \), \( \mathcal{E} \) and \( \mathcal{F} \) of vertices, edges and faces, correspondingly, of \( G(\mathcal{M}) \). We will then determine \( g \), such that \( G(\mathcal{M}) \) can be embedded on a surface of genus \( g \) (see \([5]\)).

The algorithm will always stop after exploring all faces (exploring each edge once in each direction). In case of a planar set of modules, it will determine that \( g = 0 \) (as in Example 1). If it will determine that \( g = 1 \), this will mean that the set of modules is not planar, but it can be embedded on a torus (as in Example 2), etc.

The faces are explored according to their types: first we traverse the faces that correspond to the modules, then we explore the faces that contain, alternately, twin terminals and neighboring terminals.

Formally, we are given a set of modules \( \mathcal{M} = \{m_1, m_2, \ldots, m_t\} \), such that for every \( i \) \( m_i = \langle A_{i_1}, A_{i_2}, \ldots, A_{i_{\nu_i}} \rangle \). We assume that the graph \( g(\mathcal{M}) = (\mathcal{M}, E) \), where \( (m, m') \in E \) iff \( m \) and \( m' \) are neighboring modules, is connected (otherwise, one has to find its connected components \( (\mathcal{M}_j, E_j) \) (see \([4]\)), and to apply the algorithm to each of them; \( \mathcal{M} \) is planar iff all the \( \mathcal{M}_j \)'s are).

We explore the faces of \( G(\mathcal{M}) \) as follows:

1. For each module \( m_i = \langle A_{i_1}, A_{i_2}, \ldots, A_{i_{\nu_i}} \rangle \), add the face \( A_{i_1} A_{i_2} \ldots A_{i_{\nu_i}} \) to \( \mathcal{F} \).

2. Let \( A \rightarrow B \) be an un-traversed edge \(^1\), where \( A \) and \( B \) are twin terminals. Extend it to a face by moving from \( B \) to its anti-clockwise neighboring terminal \( C \), and from there to \( C \)'s twin terminal, and apply this process (of going to an anti-clockwise neighboring terminal and then to a twin terminal) until you return to

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\(^{1}\)An edge \((A, B)\) is viewed as two directed edges \( A \rightarrow B \) and \( B \rightarrow A \).
A. Add the face to $\mathcal{F}$.

3. If all edges have been explored in both directions, then goto (4), else goto (2).

4. Let $g = 1 + \frac{1}{4} \sum_{i=1}^{n} s_i - \frac{1}{2} |\mathcal{F}|$.

5. If $g = 0$, then $\mathcal{M}$ is planar (otherwise, it is not planar, but it has an embedding with no cross-overs on a surface of genus $g$).

Example 4:

1. We apply the algorithm to the set $\mathcal{M}'$ of Example 2. The faces $f_1 - f_4$ are explored in Step (1), and the faces $f_5 - f_{10}$ in Step (2), all in the order of vertices as specified in this example. Then $|\mathcal{F}| = 10$, $\sum_{i=1}^{4} s_i = 3+4+5+4 = 16$, and $g = 1 + \frac{16}{4} - \frac{10}{2} = 0$, thus determining that $\mathcal{M}'$ is planar.

2. We apply the algorithm to the set $\mathcal{M}''$ of Example 3. In Step (1) we explore the faces $f_1 = A^1B^1C^1$ and $f_2 = A^2B^2C^2$, and in Step (2) the face $f_3 = A^1A^2C^2C^1B^1B^2A^2A^1C^1C^2B^3B^1$. Then $|\mathcal{F}| = 3$, $\sum_{i=1}^{2} s_i = 3 + 3 = 6$, and $g = 1 + \frac{6}{4} - \frac{3}{2} = 1$, thus determining that $\mathcal{M}''$ is not planar, but it can be embedded on a torus.

\[\square\square\square\]

5 Proof of correctness

1. Termination

We view the traversal procedure as done on the directed graph $\bar{G}(\mathcal{M}) = (\mathcal{V}, \mathcal{E})$, constructed from $G(\mathcal{M})$ such that each edge $(a, b)$ in $G(\mathcal{M})$ exists once in each direction.

Since in $\bar{G}(\mathcal{M})$ every vertex has an in-degree and an out degree of 2, and since $\bar{G}(\mathcal{M})$ is strongly connected, then an easy induction shows that each time Step (2) is executed, it must terminate at the vertex $A$ in which it started. This clearly implies termination of our algorithm (this is similar to exploring Eulerian tours in Eulerian graphs ([3]), or to exploring closed walks in graphs with rotation ([11])).
2. Correctness

In this way we have constructed polygons. Since each edge was traversed exactly twice, and the graph is connected, then these polygons constitute an orientable polyhedron (see [11], pp. 39-40). According to a result of Edmonds [2] there exists a unique embedding of the graph into an orientable surface, with these sets of vertices, edges and faces. This also justifies (see [11], pp. 179-180) our claim that the faces of an embedding of a given set of modules are of two kinds; namely, the faces corresponding to the modules, and the faces that contain edges that connect neighboring terminals and edges that connect twin terminals, alternately.

The genus $g$ of this surface is then determined using Euler’s formula, (see [5], or ch. 5 in [12] for a more detailed discussion)

$$|V| - |E| + |F| = 2 - 2g,$$

with the special case $g = 0$ implying planarity of $M$.

We next use the following lemma (whose proof follows immediately from the definition of $G(M)$):

**Lemma:** $|V| = \sum s_i$, and $|E| = \frac{1}{2} \sum s_i$.

By Euler’s formula and the Lemma, it follows that $g$, as calculated in Step (4) of the algorithms, contains the required genus.

3. Complexity

Since each edge was traversed exactly once in each direction, the complexity of the algorithm is clearly linear (the data structures needed for this linear implementations are left to the reader).

We end with few remarks.

1. Extending the result to the case where there might exist more than two twin terminals will have to be done using planarity testing algorithms, as is the case in [10] and [1]. Our approach cannot be extended to this case.

2. Extending the result to the case where the modules can be flipped (see remark at the end of Example 3) requires different techniques (see [10]), since there is no known analogue to Edmonds’ Theorem for this case.

3. In the definition of the problem we assumed that the two modules on which twin terminals reside are distinct. This assumption is not necessary, since if in $M$ we have terminals $A^1$ and $A^2$ on a module $m$, we can change in $m A^2$ to $A^1 - A$, being

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2Called Edmonds’ permutation technique in [12]; also observed by Heffter [6], and stated in terms of graphs with rotation [11].
a new terminal - and add to $\mathcal{M}$ a module $m' = \langle A_1^2, A_2^2 \rangle$, thus getting a new set of modules, that is planar iff $\mathcal{M}$ is.

4. Our discussion could be presented slightly differently, using graphs with rotation. In such graphs a cyclic (clockwise or anti-clockwise) ordering of the edges around each vertex is given. For a given set of modules $\mathcal{M}$ we thus could define a graph $G'(V', E')$, such that $V' = \mathcal{M}$ and $E'$ is the set of terminals (twin terminals are considered here as identical), and such that the edges around a vertex $m_i$, where $m_i = \langle A_{i_1}, A_{i_2}, ..., A_{i_s} \rangle$, correspond to a rotation $A_{i_1}, A_{i_2}, ..., A_{i_s}$ around it.

5. In our algorithm it would suffice to only count the number $|\mathcal{F}|$ of faces in $\mathcal{F}$, instead of listing them.

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References


