Lemma 3 For any \( n \geq 0 \) and \( m \geq 1 \), there holds

\[
f^{(m)} - f_n^{(m)} = (-1)^n f^{(m)} \prod_{k=1}^n \frac{f^{(m+k)}}{1 + f^{(m+k)}_{n-k}}.
\]

Proof: The lemma follows from part (a) of Theorem 1 by taking \( l \to \infty \) in Lemma 2. □

Lemma 4 For any \( l > n \geq 0 \) and \( m \geq 1 \), there holds

\[
|f_i^{(m)} - f_n^{(m)}| \leq |h_i^{(m)} - h_n^{(m)}|.
\]

Proof: By applying Lemma 2 to \( f_n^{(m)} \) and \( h_n^{(m)} \) and using Lemma 1, one obtains

\[
|f_i^{(m)} - f_n^{(m)}| = |f_i^{(m)} \prod_{k=1}^n \frac{f^{(m+k)}_{i-k}}{1 + f^{(m+k)}_{n-k}}| \leq | - h_i^{(m)} \prod_{k=1}^n \frac{-h^{(m+k)}_{i-k}}{1 + h^{(m+k)}_{n-k}}| = |h_i^{(m)} - h_n^{(m)}|. □
\]

Lemma 5 For any \( n \geq 0 \) and \( m \geq 1 \), there holds

\[
|f^{(m)} - f_n^{(m)}| \leq |h^{(m)} - h_n^{(m)}|.
\]

Proof: The lemma follows from part (a) of Theorem 1 by taking \( l \to \infty \) in Lemma 4. □

Proof of Theorem 2: By setting once \( g_{l+1} = 0 \) and once \( g_{n+1} = 0 \) in part (b) of Theorem 1, one obtains \( h_1^{(1)} = 1/S_1 - 1 \) and \( h_n^{(1)} = 1/S_n - 1 \). The first assertion of the theorem follows from Lemma 4; the second one follows from part (a) of Theorem 1 by taking \( l \to \infty \). □

References


Proof: The assertion follows from Corollary 1 and Theorem 2, respectively, for the cases \( g < 1/2 \) and \( g = 1/2 \). □

This result is better than that of [2], p. 118; furthermore, as opposed to the result of [2], it applies also to \( g = 1/2 \), giving an optimal version of the theorem of Worpitzky ([1], p. 513, Problem 2).

In the next section we introduce some lemmas which provide a proof for Theorem 2.

2 Proof of Theorem 2

Lemma 1 For all \( m \geq 1 \) and \( n \geq 0 \), \( |f_n^{(m)}| \leq -h_n^{(m)} \).

Proof: From part (c) of Theorem 1 we have that \( h_n^{(1)} \leq 0 \) for all \( n \geq 0 \); consequently, \( (1-g)h_n^{(1)} \leq 0 \) and hence \( h_n^{(m)} \leq 0 \) for all \( m \geq 1 \).

The proof follows from a mathematical induction on \( n \geq 0 \) for all \( m \geq 1 \). By definition, the assertion holds for \( n = 0 \) for all \( m \geq 1 \). Suppose it holds for some \( n \geq 0 \) for all \( m \geq 1 \). Then, for each \( m \geq 1 \),

\[
|f_{n+1}^{(m)}| = \left| \frac{a_m}{1 + f_{n+1}^{(m+1)}} \right| \leq \frac{-b_m}{1 + h_n^{(m+1)}} = -h_n^{(m+1)}. \quad \Box
\]

Lemma 2 For any \( l > n \geq 0 \) and \( m \geq 1 \), there holds

\[
f_{i-1}^{(m)} - f_{n+1}^{(m)} = (-1)^n f_i^{(m)} \prod_{k=1}^n \frac{f_{n-k}^{(m+k)}}{1 + f_{n-k}^{(m+k)}}.
\]

Proof: The proof follows from a mathematical induction on \( n \geq 0 \) for all \( l > n \) and \( m \geq 1 \). The assertion holds trivially for \( n = 0 \). Suppose it holds for some \( n \geq 0 \). Then, for all \( l > n \) and \( m \geq 1 \),

\[
f_{i-1}^{(m)} - f_{n+1}^{(m)} = \frac{a_m}{1 + f_{i}^{(m+1)}} - \frac{a_m}{1 + f_{n+1}^{(m+1)}} = \frac{f_{i-1}^{(m)} - f_{n+1}^{(m+1)}(f_{i}^{(m+1)} - f_{n+1}^{(m+1)})}{1 + f_{i}^{(m+1)}}
\]

\[
= \frac{f_{i-1}^{(m)}}{1 + f_{i}^{(m+1)}}(-1)^{n+1} f_{i}^{(m+1)} \prod_{k=1}^n \frac{f_{n-k}^{(m+k)}}{1 + f_{n-k}^{(m+k)}} = (-1)^{n+1} f_{i+1}^{(m)} \prod_{k=1}^{n+1} \frac{f_{n+1-k}^{(m+k)}}{1 + f_{n+1-k}^{(m+k)}}
\]

\[
= (-1)^{n+1} f_{i+1}^{(m)} \prod_{k=1}^{n+1} \frac{f_{n+1-k}^{(m+k)}}{1 + f_{n+1-k}^{(m+k)}} \quad \Box
\]
Assume \( \sum_d = 0 \) and \( P_d = 1 \). For any \( n \geq 0 \), let

\[
S_n = 1 + \sum_{p=1}^{n} \prod_{k=1}^{p} \frac{g_k}{1 - g_k}.
\]

Let \( S = \lim_{n \to \infty} S_n \) (possibly \( \infty \)). The fundamental result of Wall ([3], Theorem 11.1, pp. 45-46) states that

**Theorem 1**

(a) \( K(a_p/1) \) converges uniformly for \( |x_p| \leq 1 \), \( p = 2, 3, 4, \ldots \).

(b) \( f^{(1)}, f_n^{(1)} \in \{ z \mid |z| \leq 1 - \frac{1}{g} \}, \ n = 0, 1, 2, \ldots \) and \( h^{(1)} = \frac{1}{S} - 1 \).

(c) \( f^{(1)}, f_n^{(1)} \in \{ z \mid |z - \frac{1}{2 - g} - \frac{1}{g} \} \leq \frac{1 - g}{2 - g}, \ n = 0, 1, 2, \ldots \)

The aim of this work is to derive an optimal upper bound for the error \( |f^{(1)} - f_n^{(1)}| \). The main result is

**Theorem 2** For any \( l > n \geq 0 \),

\[
|f^{(1)}_l - f^{(1)}_n| \leq \frac{1}{S_n} - \frac{1}{S_l}.
\]

Furthermore, for any \( n \geq 0 \),

\[
|f^{(1)} - f^{(1)}_n| \leq \frac{1}{S_n} - \frac{1}{S}.
\]

These results are optimal in the sense that equality is achieved in them by the continued fraction \( K(b_p/1) \).

As a consequence, we have the following corollaries.

**Corollary 1** Assume that, for some integer \( N_0 \) and scalar \( 0 < g < 1/2 \), there holds \( g_p \leq g \) for every \( p > N_0 \). Then, for any \( n \geq N_0 \),

\[
|f^{(1)} - f_n^{(1)}| \leq \frac{1 - g}{S_n S (1 - 2g)} \left( \frac{g}{1 - g} \right)^{n+1}.
\]

This result is optimal in the sense that equality is achieved in it (when \( g_p = g \) for every \( p > N_0 \)) by the continued fraction \( K(b_p/1) \).

**Proof:** Clearly, \( S < \infty \). The assertion follows from Theorem 2 and

\[
\frac{1}{S_n} - \frac{1}{S} = \frac{S - S_n}{S_n S} \leq \frac{1}{S_n S} \sum_{p=n+1}^{\infty} \left( \frac{g}{1 - g} \right)^p. \quad \square
\]

**Corollary 2** Assume that all the \( g_p \) are equal to the same constant \( 0 < g \leq 1/2 \). Then, for any \( n \geq 0 \),

\[
|f^{(1)} - f_n^{(1)}| \leq \frac{1}{S_n} \left( \frac{g}{1 - g} \right)^{n+1}.
\]

This result is optimal in the sense that equality is achieved in it by the continued fraction \( K(b_p/1) \).
Optimal Upper Bounds for the Error in the Approximants of Continued Fractions

Yair Shapira, Avram Sidi and Moshe Israeli *

October 18, 1994

Abstract

Consider the continued fraction \( K(a_p/1) \), where \( a_1 = -g_1, a_p = (1-g_{p-1})g_p x_p \) with \( |x_p| \leq 1 \), \( p = 2, 3, 4 \ldots \) and the constants \( g_p \) satisfy \( 0 \leq g_p < 1, p = 1, 2, 3 \ldots \). It is known that \( K(a_p/1) \) converges. Let \( f \) and \( f_n \) be the limit and approximants, respectively, of \( K(a_p/1) \). In this work it is shown that 

\[
|f - f_n| \leq 1/S_n - 1/S, \quad S_n = 1 + \sum_{p=1}^{n} \prod_{k=1}^{p} \frac{g_k}{1-g_k} -_{n \to \infty} S \quad \text{(possibly \infty)}
\]

and that this result cannot be improved.

1 Introduction

Consider the continued fractions \( K(a_p/1) \) and \( K(b_p/1) \), where \( a_1 = b_1 = -g_1, a_p = (1-g_{p-1})g_p x_p \) with \( |x_p| \leq 1 \) and \( b_p = -(1-g_{p-1})g_p, p = 2, 3, 4 \ldots \) and the constants \( g_p \) satisfy

\[
0 \leq g_p < 1, \quad p = 1, 2, 3 \ldots
\]

For any integer \( m \geq 1 \), let

\[
\begin{align*}
f^{(m)}_0 &= 0 \\
h^{(m)}_0 &= 0 \\
f^{(m)}_n &= \frac{a_m}{1+} \frac{a_{m+1}}{1+} \cdots \frac{a_{m+n-1}}{1}, \quad n = 1, 2, 3 \ldots \\
h^{(m)}_n &= \frac{b_m}{1+} \frac{b_{m+1}}{1+} \cdots \frac{b_{m+n-1}}{1}, \quad n = 1, 2, 3 \ldots \\
f^{(m)} &= \lim_{n \to \infty} f^{(m)}_n \\
h^{(m)} &= \lim_{n \to \infty} h^{(m)}_n.
\end{align*}
\]

*Computer Science Department, Technion - Israel Institute of Technology, Haifa 32000, Israel.