The 3-Edge-Components and a Structural Description of 3-Edge-Cuts in a Graph *

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Abstract

Let $G = (V, E)$ be an undirected graph. We denote by $V^k$ the partition of $V$ into classes of $k$-edge-connectivity: maximal vertex subsets indivisible by $k'$(edge)-cuts of the whole $G$, $k' \leq k - 1$. The factorgraph of $G$ corresponding to $V^3$ is known to give a clear representation of the system of cuts of $G$ with 1 or 2 edges, and of $V^2$ and $V^3$. Here, a graph invariant structural description of the system of 3-cuts and of $V^3$ for an arbitrary graph $G$ is suggested. It is based on a new concept of a 3-edge-connected component of a graph. The 3-cuts of $G$ are classified into bunches so that the bunches are naturally 1:1 correspondent to the cuts of 3-edge-connected components. The classes of 4-edge-connectivity of $G$ are exactly the classes of 4-edge-connectivity of the 3-edge-connected components of $G$. The description of 3-cuts and $V^4$ of a 3-edge-connected component is given by the known “cactus model”. The space complexity of the description suggested is $O(|V|)$ (though the total number of 3-cuts may be a cubic function in $|V|$). The time complexity of the construction of this description is

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*This research was done partly when the author was with IBM Israel, Science & Technology, Haifa and was supported partly by the Fund for the Promotion of Research at the Technion, Israel.

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Moreover, for a graph and every $l \geq 4$, a partial decomposition of the set of $l$-cuts and a full decomposition of $V^{l+1}$ to the sets of cuts and $(l+1)$-classes of the 3-edge-connected components are presented.

Key words. graph, data structure, edge connectivity, edge-cut, bottleneck, connectivity class.

AMS(MOS) subject classifications. 68R10, 68P05, 68Q20, 05C40

1 Introduction

We are interested here in methods of describing and finding all edge-cuts with a bounded number of edges - a kind of a “bottleneck” - and vertex subsets indivisible by them (“highly-connected”) in a graph. This may be useful for decomposition of various network problems. If the graph represents a communication network, then small cuts are sensitive sets of links in the network. Thus, their number and/or their location are essential for a reliability analysis of the network. A structural representation of the system of these cuts enables one to calculate globally, and hence efficiently, some of their characteristics (for example, the traffic through everyone of them). A description of the highly-connected vertex sets and of the bottlenecks separating them, for a network, may serve a network designer. This information may also be relevant to the layout of networks.

From the theoretical point of view, the development of a clear structural model for edge-cuts of a bounded cardinality and highly-connected vertex sets in a graph can serve as a basis for related studies. These include problems of dynamic maintenance of edge-connectivity structures [GI93], [D93], [W93]. Previous results on minimum edge-cuts, [DKL76] and [KT86], were employed essentially in [NGM90] for the development of its clear and fast schemes and algorithms for graph augmentation; also, [W93] contains applications in this field. Graph invariants obtained by work in the direction discussed here can serve for graph classification.

Consider an undirected unweighted connected graph $G = (V, E)$ with $n \geq 2$ vertices. Multiple edges are allowed. A minimal, or simple, edge-cut of $G$ is a set $C$ of its edges whose removal disconnects $G$ and no proper subset of $C$ disconnects $G$. We call it simply a cut, and an $l$-cut if $|C| = l$. A 1-cut is also called a bridge. For a subset $S$ of $V$, the subgraph induced by $S$ in
$G$ is denoted by $G(S)$. For a partition $(X, \overline{X})$ of $V$, denote by $E(X, \overline{X})$ the set of edges whose ends are in different parts. Clearly, $E(X, \overline{X}) = E(X, X)$. It is well known that the removal of a simple edge-cut $C$ results in dividing $G$ into exactly two disjoint connected subgraphs $G(C)$ and $G(\overline{C})$, and $C$ coincides with $E(X, \overline{X})$ (an equivalent definition of a simple edge-cut). Both partitions $(X, \overline{X})$ and $(\overline{X}, X)$ are called partitions of $V$ by $C$. In this paper, depending on the context, a cut $C$ will denote either set of edges or the corresponding vertex partition. In particular, we say that a subset $S$ of $V$ and the induced subgraph $G(S)$ are divided (broken) by a cut $C$ if both $S \cap X_C$ and $S \cap \overline{X_C}$ are non-empty. When a subset $S$ of $V$ is distinguished and $C$ does not break it, we assign to $C$ the induced partition $(X_C, \overline{X_C}) : S \subseteq X_C$. Then, $X_C$ is called the inner and $\overline{X_C}$ the outer part.

A graph is called $k$-edge-connected if it has no $k'$-cuts, $k' \leq k - 1$. The edge-connectivity $c(G)$ is defined as the maximum $k$ such that $G$ is $k$-edge-connected (equivalently: $c(G)$ is the minimum number of edges in a cut in $G$). In this paper, we usually omit the word “edge” when we refer to edge-cuts and edge-connectivity. (Similar notions of “vertex-cuts” and “vertex-connectivity”, when needed, are spelled out explicitly.) We say that a subset of vertices is $k$-connected if it is not divided by any $k'$-cut of the whole $G$, $k' \leq k - 1$. We define a class of $k$-edge-connectivity, or, for simplicity, a $k$-class, of $V$ to be an inclusion-maximal $k$-connected vertex subset in $G$.

The set of all $k$-classes forms a partition of $V$ because the relation on pairs $(u, v)$ of vertices “$u$ and $v$ cannot be divided by a $k'$-cut, $k' \leq k - 1$” is an equivalence. (The same equivalence relation is defined by the property: “existence of $k$ edge-disjoint paths between $u$ and $v$” [K56].) \footnote{In [GI93] these classes are called $k$-edge-connected components. We reserve the term “component” for a graph (together with [T66], [HT73]), not for a vertex set.} The partition of $G$ to $k$-classes is denoted by $\mathcal{V}^k = \mathcal{V}^k(G)$. Obviously, $\mathcal{V}^l$ is a subdivision of $\mathcal{V}^k$, provided $l > k$. For a $k$-class $S$, let $\mathcal{V}^l_S = \mathcal{V}^l \cap S$, $l > k$, denote the restriction of $\mathcal{V}^l$ to $S$. For any $l > k$; $\mathcal{V}^l = \bigcup_{S \in \mathcal{V}^k} \mathcal{V}^l_S$. The subgraph $G(S)$ induced by a $k$-class $S$ is called a $k$-piece of $G$.

The direction of studies suggested here is to find a structural description of the system of all $k$-cuts of $G$ (henceforth, the $k$-cuts structure of $G$). The “twin” $k$-classes description is derivable. If there exist suitable $k'$-cuts structures, for all $k' \leq k - 1$, then appears, also, an adequate characterization of the partition of $G$ to $k$-classes. Providing we had this for some $k$, we use
the terms “the $k$-cuts problem” and “the $k$-cuts/($k+1$)-classes problem” for the step to the next cardinality.

Given a partition of $V$, the corresponding factor-graph is defined to be the result of shrinking each part with all edges linking its vertices (i.e., the induced subgraph) to a single node. (We use the term “node” instead of “vertex” for all model graphs). We call the factor-graph $Q^k = Q^k(G)$ corresponding to the partition $V^k(G)$, together with the factoring transformation $f^k : G \rightarrow Q^k(G)$, the $k$-quotient of $G$ (for illustration, see Figs. 1, 2 and 3 below). Clearly, the set of all $k'$-cuts, $1 \leq k' \leq k-1$, in $Q^k(G)$ and the set of those in $G$ are naturally 1:1 correspondent using the factoring transformation $f^k$ (both edges of cuts and node/vertex divisions by cuts are correspondent). So a $k'$-cuts structure of $Q^k(G)$ serves as a $k'$-cuts structure of $G$, $k' \leq k - 1$.

Though $Q^k(G)$ can be less complex in comparison with the original graph $G$, the $(k-1)$-cuts problem for it is not simple in the general case. However, the cases $k = 2, 3$ are clear: the 2- and 3-quotients themselves serve, respectively, as the 1-cuts/2-classes and (1,2)-cuts/(2,3)-classes structures. The subgraph induced by a 2-class is 2-edge-connected and is called a 2-edge-connected component of $G$. For every $k \geq 2$, the $k$-cuts/($k+1$)-classes problem for $G$ is naturally decomposed to the same problems for its 2-edge-connected components.

This paper is devoted mainly to the 3-cuts/4-classes problem for a general graph $G$. The crucial break is its decomposition to the same problems for auxiliary 3-connected graphs corresponding to 3-classes of $G$. For a 3-class $S$ we construct the 3-edge-connected component $\overline{S}$ of $G$ as a natural extension (the closure) of the induced subgraph $G(S)$ [D92] (reinvented in [W93]). The closure is done by complementing $G(S)$ by some virtual edges, which refer to the “environment” of $G(S)$ in the 3-quotient of $G$. (These virtual edges are essentially the same (but not all) as those of the “split operations” when constructing Tutte’s decomposition of a graph into its 3-vertex-connected components [T66], [HT73]2.) The description of 3-cuts of $G$ is composed of the descriptions of 3-cuts for all 3-edge-connected components and of some simple rules of passage from them to cuts of $G$ using “virtual” edges. The partition of the whole $G$ into 4-classes coincides with the union of the partitions of 3-edge-connected components of $G$ into their 4-classes. A

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2An approach to 4-vertex-connectivity problems based on the theory of 3-vertex-connected components was employed in [KTBC91].
clear 3-cuts/4-classes structure for a 3-edge-connected component is given by means of the known cactus model [DKL76]. Such a two-level representation of 3-cuts of $G$ is very compact: linear in $n$, while the total number of 3-cuts may be cubic in $n$. We suggest an algorithm that constructs a representation of 1-, 2-, and 3-cuts and build all 2-, 3-, and 4-classes of $G$ with the time complexity $O(m + n \log(n))$, where $m$ is the number of vertex pairs connected by an edge in $G$. The time complexity of the explicit generation of all 1-, 2-, and 3-cuts of $G$ from this representation is linear in their number. Moreover, for a graph and every $l \geq 4$, a partial decomposition of the $l$-cuts problem and a full decomposition of the $(l+1)$-classes problem to the same problems for the 3-edge-connected components are suggested.

The paper is organized as follows. In Section 2 we describe properties of 2- and 3-quotients and 2-edge-connected components. The cactus model is introduced. A 3-piece is considered as a possible relevant object for connectivity structures. The difficulties and an idea for overcoming them are shown by a simple example. Section 3 presents our concepts of a 3-edge-connected component and of a 3-cuts/4-classes structure of a general graph. A compact representation of the set of 1-, 2- and 3-cuts of a graph is introduced. Some partial generalization of this to $l$-cuts and $(l+1)$-classes, $l \geq 4$, is presented. An example of the 3-cuts/4-classes structure is given for a concrete graph. Section 4 is devoted to the algorithm of construction and the space and time complexity analysis for the representation suggested. Section 5 concludes with a discussion of some possible extensions of these studies.

2 Basis

It is well known that the 2-quotient $Q^2(G)$ is a tree (see Fig. 1ab, where areas emphasized by dotted lines are 2-pieces and edges between them are bridges of $G$). All cuts of $Q^2(G)$ are 1-cuts; the pre-images of all nodes of $Q^2(G)$ form the partition of $V$ into 2-classes $Y^2(G)$. This is the ideal intrinsic property of a cuts/classes model. Furthermore, “gates-to-recursion” are open here. One can easily observe that every $k$-cut $C$, $k \geq 2$, of $G$ breaks exactly one 2-class (otherwise, $G(X_C)$ and $G(\overline{X_C})$ cannot both be connected). Moreover, such a cut coincides with a $k$-cut of a particular 2-piece of $G$ (as an edge set) and vice versa. Because of this, one can study $k$-cuts and $(k+1)$-classes, $k \geq 2$, of $G$ separately in all 2-pieces, as their $k$-cuts and $(k+1)$-classes.
2-pieces are 2-connected; they are called 2-edge-connected components (or bridge-components) of the graph \( G \) (henceforth, “2-components”).

The structure of the 3-quotient \( Q^3(G) \) for a 2-connected graph \( G \) is a tree-of-cycles. A tree-of-cycles is defined to be a union of simple cycles (possibly of length 2) connected by common nodes, each being an articulation node (plus the degenerate case of an isolated node). An equivalent characterization of a tree-of-cycles is that every edge belongs to precisely one simple cycle. This model is a particular case of the cactus model [DKL76] and has been invented independently in its explicit form in [GI93].

For arbitrary \( G \), its 3-quotient is the union of the 3-quotients of all 2-components of \( G \), which are connected by images of \( G \)-bridges (see Fig. 1ac, where shaded areas are 3-pieces). Its structure is a tree-of-edges-and-cycles (so called “cactus”) defined analogously with the above definition of a tree-of-cycles. (An equivalent characterization is that every edge belongs to at most one simple cycle). The cuts of a tree-of-edges-and-cycles are only: (i) all non-cycle edges (1-cuts) and (ii) all possible pairs of edges from the same cycle (2-cuts) (for illustration, see cuts indicated by dashed lines in Fig. 1c). So, for the 3-quotient \( f^3 : G \rightarrow Q^3(G) \), there is the natural 1:1 correspondence between the set of all cuts in \( Q^3(G) \) with the clear structure, and the set of cuts in \( G \).
1- and 2-cuts of the original graph.

For 3-cuts of a general graph, we might not get a structure with analogous properties by means of only its 4-quotient. For example, for any cubic graph $G'$ (i.e., a graph with all vertex degrees equal to three): $Q^4(G') = G'$, so the 4-quotient does not help describe non-trivial 3-cuts of $G'$ (see, for example, the cut $C'$ in Fig. 2b). However, if a graph is 3-connected, another technique is used. The paper [DKL76] deals with $c(G)$-cuts (i.e., minimum cuts) and presents the model $\mathcal{H}(G)$ of the system of vertex partitions by these cuts. The graph $\mathcal{H}(G)$ is a cactus with $O(n)$ vertices and edges. It is provided with a mapping $\varphi(G)$ from $V$ to the set of its nodes. Thus, pre-images of nodes of $\mathcal{H}(G)$ are disjoint subsets of $V$ whose union is $V$ (some of them can be empty). All the non-empty subsets from these are exactly all $(c(G) + 1)$-classes of $G$. The partitions of nodes given by cuts of the graph $\mathcal{H}(G)$ are in an almost 1:1 correspondence, via $\varphi^{-1}(G)$, with the vertex partitions given by minimum cuts of $G$. The cactus model of a graph with the connectivity one (resp. two) is essentially the same as its 2- (resp. 3)-quotient. When the connectivity of $G$ has an arbitrary odd value, $\mathcal{H}(G)$ has the form of a tree (i.e., its cuts are its single edges) and the aforementioned cut correspondence is a bijection. For example, see Fig. 2ac, where $c(G) = 3$ and small white circles denote “empty” nodes.

A naive idea for solving the 3-cuts/4-classes problem for an arbitrary graph $G$ is to decompose $G$ into 3-pieces and, provided they are 3-connected, to proceed with each of them separately using the aforementioned tree-model $\mathcal{H}(G)$. It does not, however, come out directly. In particular, a 3-piece might
not be 3-connected. In Fig. 3abc there is a simple example. The graph $G$ is the union of three disjoint paths between vertices $s$ and $t$. The 3-cuts of such a graph are all triples of edges, one by one from each path. Their number is a cubic function of $n$ (here $3 \cdot 4 \cdot 5 = 60$; three cuts are shown explicitly by dashed lines). The 3-classes of this graph are the pair $\{s,t\}$ and singletons $\{u\} : u \in V \setminus \{s,t\}$. Notice that the 3-piece $G(\{s,t\})$ is not connected; it has the two vertices $s$ and $t$ and no edges. Consider an analogous graph with four, not three, paths between $s$ and $t$. There are no 3-cuts in this graph, but $\{s,t\}$ is a 3-class of it too, and the 3-piece $G(\{s,t\})$ is identical. So $G(\{s,t\})$ itself is not adequate to describe the 3-cuts.

We present a kind of “closure” of 3-pieces. In this example (according to the definition given in the next section), the 3-edge-connected component based on the vertex set $\{s,t\}$ is the subgraph $G(\{s,t\})$ with three additional “virtual” edges between $s$ and $t$. Each virtual edge has a reference to a corresponding cycle in the 3-quotient, which itself consists of the edges of one of the three $s,t$-paths in $G$ (see Fig. 3bd). This component is 3-edge-connected. Its 4-classes $\{s\}$ and $\{t\}$ are 4-classes of $G$. From its single 3-cut, we can formally generate all 3-cuts of the original graph: each of the virtual edges is replaced independently by any one of the edges of the corresponding cycle of $Q^3(G)$.

### Figure 3: An example with a disconnected 3-piece

3 Edge-Cuts and 3-Edge-Components of a Graph

In the sequel we shall logically identify cuts, edges, and nodes of a factor-graph with their pre-images in $G$: edges with edges and nodes with subgraphs
of $G$ (or with subsets of $V$). A pair of distinct partitions of a set $V$ into two non-empty subsets is called crossing if, together, they induce a partition of $V$ into four non-empty subsets and parallel otherwise (in this case, there are three non-empty subsets). Cut pairs are called analogously, depending on the relation between the vertex set partitions defined by them. The square of a pair of crossing cuts will denote the 4-node factor-graph of their mutual partition of $V$. The pre-image in $V$ of any of its nodes is called a corner part.

**Lemma 1**

(i) If $C_2$ and $C_3$ are two crossing cuts with 2 and 3 edges, respectively, then their square is a cycle with one edge in $C_3$ doubled (see Fig. 4a).

(ii) Each cut of the square separating a single node from the three other nodes induces, via the inverse of the factor-mapping, a cut of $G$ (a corner cut).

**Proof:** In principle, a square can have (possibly multiple) side and diagonal edges. All four side edges must exist in the square of any pair of crossing cuts. In fact, if, for example, the left-side edge is absent, then the left-part subgraph of the vertical cut might not be connected. Therefore, since $C_2$ already contains two side edges, there cannot be any diagonal edges (see Fig. 4b). So, the only possibility is that one side edge belonging to $C_3$ has the multiplicity two.

Both parts of the partition of $V$ corresponding to $C_3$ consist of two corner parts. The subgraph of $G$ induced by any one of them is connected (by the definition of a cut) and has a side edge of the square as its bridge. Clearly, the

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3In fact, a tree structure of $\mathcal{H}$-model, $c(G)$ odd, of an arbitrary graph (see Sect. 2) is a direct consequence of the mutual parallelism in all pairs of $c(G)$-cuts, in the case $c(G)$ odd.
elimination of a bridge from a connected graph breaks it into two connected subgraphs. Hence, each corner part is connected. Its complement in $G$ is also connected, as the union of two connected graphs with at least one edge between them. Thus, each corner partition implies a cut. □

Lemma 2
(i) An $l$-cut, $l \geq 3$, divides at least one 3-class.
(ii) A 3-cut divides exactly one 3-class.

Proof: First, if an $l$-cut, $l \geq 3$, does not divide any 3-class, it induces an $l$-cut of $Q^3(G)$. However, we know that in the 3-quotient there are only 1- or 2-cuts. A contradiction. Suppose now that some 3-cut $C3 \text{ divides two distinct 3-classes } S1$ and $S2$. Then, there is a 2-cut $C2$ which separates the whole of $S1$ from the whole of $S2$ (by the definition of 3-classes). According to our assumptions, the cuts $C3$ and $C2$ cross, and so, by Lemma 1, they define a (1-1-1-2)-type square. Each of the four corner cuts $C1, C2, C3, C4$ divides one of $S1$ and $S2$ (see Fig. 4a). Two of these cuts are 2-cuts. But, by definition, a 2-cut cannot divide a 3-class. A contradiction. □

Definition 1 Let $C$ be a set of cuts of a graph $G$. It is partitioned relatively to the set of 3-classes of $G$: $C = C_{3,0} \cup C_{3,1} \cup C_{3,2}$. A cut $C \in C$ belongs to $C_{3,0}$ if it does not break any 3-class, belongs to $C_{3,1}$ if it breaks exactly one 3-class, and belongs to $C_{3,2}$ if it breaks at least two 3-classes. The $S$-family $C_{3,S} \subseteq C_{3,1}$ consist of cuts that break the same 3-class $S$, and only this 3-class. A cut from the $S$-family is called an $S$-cut.

As a corollary, for an arbitrary set of cuts $C$, its subset $C_{3,1}$ is the disjoint union of all 3-class-families. Thus, the following statement is equivalent to Lemma 2(ii).

Theorem 1 The set of 3-cuts of a graph $G$ is partitioned by its 3-class-families.

Two cuts breaking a 3-class $S$ are called $S$-equivalent if they partition $S$ in the same way.

Lemma 3
(i) For every $l$-cut $C$, $l \geq 3$, breaking a 3-class $S$, there exists an $S$-cut $C'$,
\(|C'| \leq 1\), equivalent to it; moreover, if \(C\) breaks more than one 3-class, then there exists such a cut \(C'\): \(|C'| < 1\).

(ii) For every 3-class \(S\) and every \(l \geq 4\), the mutual partition of \(S\) by all \(S\)-cuts \(C'\) of \(G\), \(|C'| \leq l - 1\), coincides with the partition \(\mathcal{V}_S^l\) of \(S\) to \(l\)-classes of \(G\).

**Proof:** Let us consider a cut \(C'\) that is \(S\)-equivalent to \(C\) and has the minimum cardinality among such cuts (clearly, \(|C'| = l' \leq l\)). Assume that a 3-class \(S_1, S_1 \neq S\), is broken by \(C'\). Let \(C_2\) be a 2-cut separating \(S_1\) from \(S\). Consider the square of the crossing pair \((C', C_2)\) (see Fig. 4c). Analogously to the proof of Lemma 1, there are no diagonal edges, the two side edges belonging to \(C_2\) have the multiplicity one, and all corners of the square induce cuts of \(G\). Let the edge of the square from the side of \(S\) (resp. \(S_1\)) have multiplicity \(l_S\) (resp. \(l_{S_1}\)), \(l_S + l_{S_1} = l'\). We have \(l_S, l_{S_1} \geq 2\), because otherwise there would be corner cuts of the cardinality two dividing a 3-class (a contradiction). Any corner cut \(C''\) of the square from the side of \(S\) has the cardinality \(l_S + 1 < l'\). That contradicts the assumption of the minimality property of \(C'\), so \(C'\) is an \(S\)-cut. In the case of \(C\) breaking at least two 3-classes, the same construction applied to \(C\) itself provides an \(S\)-equivalent cut of strictly less cardinality. Thus \(|C'|\), as the minimum possible cardinality, is also strictly less than \(|C|\).

Now we advance to (ii). For \(l = 4\), only 3-cuts should be taken into account. According to Lemma 2, every 3-cut that divides \(S\) is an \(S\)-3-cut. So, cuts from \(C_{3,S}^3\) form the set \(\mathcal{V}_S^4(G)\) of 4-classes of \(G\) contained in \(S\). In the general case, accordingly to (i), every division of \(S\) by a \(p\)-cut, \(p \leq l - 1\), coincides with the division of \(S\) by some \(S\)-cut of cardinality \(l' \leq p \leq l - 1\). Therefore, when constructing \(l\)-classes of \(V\) contained in \(S\), all non-\(S\)-cuts can be omitted. \(\Box\)

Let us, for a fixed 3-class \(S\), build an object which will help to study \(S\)-cuts. Recall that no 1- or 2-cut of \(G\) breaks any 3-class. Such a cut \(C\) is called **neighboring to** \(S\) if the inner part \(X_C\) of the corresponding partition contains \(S\) and is inclusion-minimal among such cuts. From the structure of \(Q^3(G)\), it is evident that a neighboring cut \(C\) is either a bridge incident to \(S\) or a pair of edges incident to \(S\) of the same neighboring to \(S\) cycle \(L(C)\) of \(Q^3(G)\). Clearly, for any two distinct neighboring cuts \(C, C'\), their outer parts \(\overline{X_C}, \overline{X_{C'}}\) do not intersect.
**Definition 2** The 3-edge-connected component (henceforth, “3-component”) \( \overline{S} \) of a graph \( G \), corresponding to a 3-class \( S \), is the following closure of the 3-piece \( G(S) \).

(i) For a neighboring 1- or 2-cut \( C \) whose edges are incident to \( S \) in a single attachment vertex \( v \) (type 1) the subgraph \( G(X_C \cup \{v\}) \) is shrunk to \( v \).

(ii) For a neighboring 2-cut \( C_2 \) incident to \( S \) in two distinct attachment vertices \( v_1 \) and \( v_2 \) (type 2), the subgraph \( G(X_{C_2} \cup \{v_1, v_2\}) \) is shrunk to a new virtual edge \( e(S, C_2) = (v_1, v_2) \). The list of edges of \( L(C_2) \) is associated with \( e(S, C_2) \). (For illustration, see Fig. 5, where 3-pieces are shaded.)

According to the definition of a 3-class, the intersection of inner parts of all cuts neighboring to \( S \) coincides with \( S \). Since all the vertices of outer parts disappear when constructing the closure, the set of vertices of \( \overline{S} \) is \( S \). The set of its edges consists of edges of the induced subgraph \( G(S) \) and of new virtual edges. Our current aim is to show that the relation “to partition \( S \) in the same way” provides a bijection between \( S \)-equivalence classes of \( S \)-cuts of \( G \) and cuts of \( \overline{S} \) which preserves cut cardinalities (as a corollary, any two \( S \)-equivalent \( S \)-cuts have the same cardinality).

For further studies we need some formalization. Let us give a definition to a meaning of the expression “the cut \( C \) of \( G \) divides the cycle \( L \) of \( Q^3(G) \)”. Let us call vertices from \( X_C \) white and vertices from \( \overline{X_C} \) black. Recall that \( L \)
is logically identified with the corresponding sequence of edges and 3-pieces of $G$. Let $W = W(L)$ be the cyclic sequence of endpoints of edges of $L$ in $G$, according to their order in $L$. We say that $C$ divides $L$ if there are vertices of both colors in $W(L)$ (see Fig. 6).

Lemma 4 If a cycle $L$ is divided by a cut $C$, then vertices of each color form a non-empty continuous subsequence of $W(L)$.

Proof: Let $v'$ be a white vertex in $W$. Let $w'$ be the black vertex closest to $v'$ in some direction along $W$, and let $w''$ be the black vertex closest to $v'$ in the other direction (possibly $w' = w''$). Let $W_2$ be the continuous subsequence of $W$ between $w'$ and $w''$ not containing $v'$. Assume that there is some white vertex $v''$ in $W_2$. The connectedness of the subgraph $G(X_C)$ implies existence of a path in $G$ between $v'$ and $v''$ containing only white vertices. However, according to the structure of the 3-quotient, every path from $v'$ to $v''$ contains either the black vertex $w'$ or the black vertex $w''$ (because $\{w', w''\}$ is a 2-vertex-cut of $G$). A contradiction. So $W_2$ is all black while its complement in $W$ is all white. □

Lemma 4 implies that if $C$ divides $L$, then there are exactly two pairs of differently colored neighboring vertices in $W(L)$. If such a pair $(v_1, v_2)$ consists of the end-points of the same edge $e$, we say that $C$ divides $L$ at $e = (v_1, v_2)$. Otherwise, these two vertices belong to the same 3-piece $G(S)$. Therefore, $C$ divides $G(S)$ and we say that $C$ divides $L$ at $G(S)$.
Let us analyze the structure of an $S$-cut. It can include inner edges, which connect two vertices of $S$ (i.e., edges of $G(S)$), and other - outer - edges. Consider an $S$-cut $C$ including an outer edge $e$. By the definition of a 3-class-family, the edge $e$ cannot be inner to any other 3-piece. Evidently, $e$ is not a bridge. So $e$ belongs to some cycle $L$ of $Q^3(G)$, with $r \geq 2$ edges, and $C$ divides $L$ at $e$. The cut $C$ cannot divide $L$ at any other edge $e'$, in addition (otherwise, such a pair $\{e,e'\}$ would form a 2-cut not breaking $S$, contradicting the fact that $C$ breaks $S$ and the minimality of $C$). So $C$ divides $L$ at some 3-piece. This 3-piece must be precisely $G(S)$, with two attachment vertices $v_1$ and $v_2$ to $L$, separated by $C$ (see Fig. 5a). We see that the two edges $e_1,e_2$ of $L$ incident to $G(S)$ form a neighboring 2-cut $C2(S,L)$ of type 2 and derive the following statement.

**Lemma 5** Any outer edge of an $S$-cut belongs to a neighboring $S$ cycle of type 2.

The replacement of the outer edge $e$ by any other edge $e' \in L$, in the cut $C$, results in an $S$-cut $C' \neq C$ of the same cardinality. This cut $C'$ partitions $X_{C2(S,L)}$ and, in particular, $S$ in the same way as $C$ partitions them. We call $C$ and $C'$ $(S,L)$-equivalent cuts. Every $(S,L)$-equivalence class consists either of $r$ cuts, if $v_1$ and $v_2$ are separated by cuts of this class, or of a single cut, otherwise. In the latter case, it has no edges in common with $L$ and, moreover, with $G(X_{C2(S,L)})$.

Let us reduce the subgraph $G(X_{C2(S,L)})$ together with the two edges of $C2(S,L)$ to a new virtual edge $(S,L) = (v_1,v_2)$. This $(S,L)$-closure transformation $G \rightarrow G(S,L)$ (see Fig. 5ab) involves the following $S$-cuts correspondence. Any $(S,L)$-equivalence class of cuts separating $v_1$ and $v_2$ is represented by a single cut of $G(S,L)$ of the same cardinality. This cut almost coincides with each cut $C$ from this class (as an edge set), except that it does not contain the edge $e = C \cap L$, but contains the virtual edge $(S,L)$ in its place. The divisions of $X_{C2(S,L)}$ (and, in particular, of $S$) by corresponding cuts of $G$ and $G(S,L)$ coincide. An $S$-cut of $G$ not separating $v_1$ and $v_2$ is represented by the same (as an edge set) cut of $G(S,L)$. A sequence of $(S,L)$-transformations for all neighboring $S$ cycles of type 2 is equivalent to the shrinkings according to Defn. 2(ii) and leads to the cut equivalence and the cut correspondence we are looking for. Obviously, shrinkings according to Defn. 2(i) have no influence on $S$-cuts. This approach implies the following equivalent alternative definition of a 3-component.
Definition 3 The 3-edge-connected component $\overline{S}$ of a graph $G$ corresponding to a 3-class $S$ is the following closure of a 3-piece $G(S)$: $G(S)$ itself with additional virtual edges $e(S, L)$, for all cycles $L$ of $Q^3(G)$ incident to $S$ in two vertices; the virtual edge $e(S, L)$ is associated with the edge list of $L$.

Let us call a cut of a 3-component $\overline{S}$ $S$-local. For a cut $C$ of $G$ breaking $S$ let us define its $S$-induced cut as the $S$-local cut $C_\overline{S} = E(X_C \cap S, \overline{X_C} \cap S)$. The previous analysis can be summarized as the following statement.

Proposition 1
(i) For every $S$-cut $C$, its $S$-induced cut $C_\overline{S}$ has the same cardinality and is the following modification of $C$: if $C$ contains an edge $e$ belonging to a cycle $L$ of $Q^3(G)$, then $e$ is replaced by $e(S, L)$.
(ii) Inversely, every $S$-local cut $\overline{C}$ is $S$-induced by at least one $S$-cut of $G$ (an extended cut of $\overline{C}$).
(iii) The whole set of extended cuts of a fixed $S$-local cut $\overline{C}$ is non-empty, forms an $S$-equivalent class of $G$ whose members are $S$-cuts of the same cardinality $|\overline{C}|$, and is the result of all possible independent substitutions of every virtual edge $e(S, L)$ in $\overline{C}$ by an edge from the cycle $L$.

For an $S$-equivalence class of cuts, we will use the mnemonic term “bunch”. For example, the bunch of the cut $C$ shown in Fig. 5a consists of $5 \cdot 3 = 15$ cuts.

Theorem 2 Every 3-component $\overline{S}$ of a graph $G$ is 3-connected.

Proof: In fact, an $l$-cut of $\overline{S}$ generates a non-empty bunch of $l$-cuts of $G$ dividing $S$. By the definition of a 3-class, $l \geq 3$. □

The following statements are consequences from Prop. 1. Let $C'$ denote the set of $l$-cuts of $G$, $l \geq 3$. It is partially decomposed in the following way.

Theorem 3 For any $l \geq 3$:
(i) $C_{3,0}^l = \emptyset$;
(ii) $C_{3,1}^l = \bigcup_{S \in \mathcal{S}} C_{3,S}^l$ (the union is disjoint);
(iii) for a 3-class $S$, $C_{3,S}^l$ is partitioned by bunches of cuts bijectively corresponding to all $l$-cuts of $\overline{S}$ (by the relation “to break $S$ in the same way”). All these bunches are non-empty.
The decomposition of Theorem 3 is full for 3-cuts (see Th. 1).

**Corollary 1** The set of 3-cuts of an arbitrary graph $G$ is represented in a natural way as the disjoint union of non-empty bunches corresponding to all 3-cuts of 3-components of $G$.

Thus, the 3-cuts/4-classes problem is reduced to known results. Indeed, a 3-component $S$ is either 4-connected, and so it has no 3-cuts and the whole $S$ is a 4-class, or $S$ has the connectivity three. In the latter case we can use the tree-like cactus model $\mathcal{H}(\overline{S})$ for its 3-cuts/4-classes description (see Sect. 2).

Summarizing, we have the following compact graph invariant representation of all cuts of $G$ with at most 3 edges.

**Representation 1**

1. The 1-cuts are members of the list $\mathcal{L}_1$ of bridges of $G$.

2. There is a set $\mathcal{L}_2$ of edges-lists of all cycles of $Q^3(G)$. The 2-cuts are pairs of members of the same edge-list from $\mathcal{L}_2$.

3. (a) The 3-cuts are grouped into 3-class-families.
   (b) The 3-cuts from an $S$-family are grouped into bunches corresponding to 3-cuts of the 3-component $\overline{S}$.
   (c) There is the list $\mathcal{L}_3(\overline{S})$ of 3-cuts for every 3-component $\overline{S}$; such a cut is an edge-triple consisting of edges of $G$ and virtual edges; the latter are linked each to an edge-list in $\mathcal{L}_2$.
   (d) Every bunch can be generated from the corresponding edge-triple $\tau$ by making all possible substitutions of every virtual edge belonging to $\tau$ by any member of its associated edge-list in $\mathcal{L}_2$.

We supplement Theorem 3 with a property of a pair: a cut $C$ and a 3-class $S$ broken by it, where $C$ is not an $S$-cut, and state that the $l$-classes problem for $G$ is decomposed into $l$-classes problems for its 3-components.

**Theorem 4**

(i) For every cut $C \in \mathcal{C}_{3,2}$ breaking a 3-class $S$, there is a non-empty set of $S$-cuts $S$-equivalent to it; the cardinality of all its members is the same number
which is strictly less than \(|C|\). Namely, this set is the \(S\)-bunch corresponding to the \(S\)-induced cut \(C_{\overline{S}}\), and the cardinality of its members is \(|C_{\overline{S}}| < |C|\).

(ii) For every \(3\)-class \(S\) and every \(l \geq 4\): \(\mathcal{V}^l_S = \mathcal{V}^l(\overline{S})\), so \(\mathcal{V}^l = \bigcup_{S \in \mathcal{V}^3} \mathcal{V}^l(\overline{S})\).

**Proof:** According to Lemma 3(i), there is at least one \(S\)-cut \(S\)-equivalent to \(C\) of cardinality less than \(|C|\). By Prop. 1, all cuts \(S\)-equivalent to \(C\) form the bunch of extended cuts of \(C_{\overline{S}}\) and so have the same cardinality \(|C_{\overline{S}}|\). By this (or by Lemma 3(ii)), it is sufficient to take only \(S\)-cuts to get the partition \(\mathcal{V}^l_S\). By Th. 3(iii), all divisions of \(S\) by \(S\)-cuts are its divisions by cuts of \(\overline{S}\) with the same cardinalities, and vice versa. According to the definitions of \(l\)-classes of \(G\) and of \(\overline{S}\), \(\mathcal{V}^l_S = \mathcal{V}^l(\overline{S})\). \(\square\)

Following is an example which illustrates the \(3\)-cuts/4-classes structure. In Fig. 7a, a graph \(G\) with 3-pieces \(A, B, C\) and \(D\) (emphasized by dotted lines) is presented. Three of its \(3\)-cuts are shown explicitly by dashed lines. The \(3\)-quotient \(Q^3(G)\) is shown in Fig. 7b. The four \(3\)-components \(\overline{A}, \overline{B}, \overline{C}\) and \(\overline{D}\) of \(G\), and the sequences of edges of the two cycles of its \(3\)-quotient, \((e_1, e_2, e_3)\) and \((e_4, e_5)\), are presented in Fig. 7c; the arrows point from virtual edges to the corresponding edge sequences. All four \(3\)-cuts of \(3\)-components are shown by dashed lines; the 4-classes of the \(3\)-components (and so of \(G\)) are emphasized by dotted lines.
There are three non-empty 3-cut families of $G$:

- **$A$-family.** The two 3-cuts of $\overline{A}$ generate each a bunch consisting of three 3-cuts of $G$; the virtual edge is replaced by any of $e_1$, $e_2$, or $e_3$.

- **$B$-family.** The single 3-cut of $\overline{B}$ consists only of real edges, and so is a 3-cut of $G$.

- **$C$-family.** The single 3-cut of $\overline{C}$ generates a bunch consisting of $3 \cdot 2 = 6$ 3-cuts of $G$. The upper virtual edge is replaced by any of $e_1$, $e_2$, or $e_3$ and the lower one by either $e_4$ or $e_5$.

All the thirteen 3-cuts of $G$ are listed above.

### 4 Algorithm and Complexity

Let us present the tools we use and their complexity. For checking, if the connectivity of a $k$-connected graph is exactly $k$, we use the algorithm [G91a], whose complexity, for a graph $G'$ with $n'$ vertices and $m'$ edges, is $O(m' + k^2 \cdot n' \cdot \log(n'/k))$. For construction of the cactus model of $G'$, we use the algorithm [G91b], whose complexity is $O(m' + c(G')^2 \cdot n' \cdot \log(n'/c(G')))).$

Observe that, for our cases $k, c(G') = 1, 2, 3$, all this is $O(m' + n' \log(n'))$.

Space complexity for both algorithms is $O(m')$.

Given the cactus representation $\mathcal{H} = \mathcal{H}(G')$ with the mapping $\varphi = \varphi(G)$, one can enrich it by explicit lists of edges for all minimum cuts of $G'$, within the same complexity bound. We show this for the case $\mathcal{H}$ is a tree, where minimum cuts of $G'$ are represented by single edges of $\mathcal{H}$. Let us look over each edge $e = (u, v)$ of $G'$ and check if $u$ and $v$ are mapped by $\varphi$ into the same node of $\mathcal{H}$. If this is the case, then $e$ does not belong to any minimum cut of $G'$. If this is not the case, then $e$ belongs to minimum cuts corresponding to all edges of the path $P(u, v)$ between $\varphi(u)$ and $\varphi(v)$ in $\mathcal{H}$, and only to these cuts. To find $P(u, v)$, let us assume that $\mathcal{H}$ is rooted arbitrarily. We trace the paths from $\varphi(u)$ and $\varphi(v)$ to the root alternating steps along these two paths and mark traced vertices by “$u$” and “$v$”, respectively. When a node $N$ gets both marks, we are sure that $P(u, v)$ is the union of the traced paths from $\varphi(u)$ to $N$ and from $\varphi(v)$ to $N$ (the latter taken in the inverse direction). Clearly, the total number of tracing steps is at most twice the length of $P(u, v)$. Now we traverse $P(u, v)$ and add $e$ to the lists corresponding to all
its edges. The complexity of the initial check of all edges of $G'$ is $O(m')$, since data structures used in [G91b] provide a single value of $\varphi$ in $O(1)$ time. The total complexity of finding paths and updating lists is linear in the total length of the lists. There are $O(n')$ lists each of the length $c(G')$, so we have $O(m' + c(G') \cdot n')$ complexity.

Now, for an arbitrary graph $G$, we describe an algorithm of the construction of Representation 1 and finding all 2-, 3-, and 4-classes. In what follows items are items of Representation 1.

To item 1. We check if $G$ is 2-connected. If this is the case, there are no bridges, $G$ coincides with its single 2-component, and $V$ is the single 2-class. If this is not the case, the cactus model of $G$ brings us the list of bridges $L^1$ and all 2-classes and 2-components of $G$.

To item 2. We check the connectivity of every 2-component. Those of connectivity greater than two are themselves 3-components; the corresponding 2-classes are themselves 3-classes. For those with connectivity two we build their cactus models (of the type “tree-of-cycles”) coinciding with their 3-quotients. Trivial DFS searches for all of them bring us the set $L^2$ of edge-lists of all their cycles.

To item 3. Non empty pre-images in $V$ of nodes of the cacti of 2-components give us all 3-classes. For each 3-class $S$, the incidence relation of the corresponding node with cycles in the cactus is given. Hence, the construction of the 3-component $\overline{S}$ is straightforward. For every 3-component $\overline{S}$, we check its connectivity. If it is greater than three, then the corresponding 3-class is itself a 4-class and there are no 3-cuts in $\overline{S}$. If it is equal to three, we build the cactus model of $\overline{S}$. Thus, we find all its 3-cuts, i.e., the list $L^3(\overline{S})$, and all its 4-classes, i.e., the 4-classes of $G$ contained in $S$. The 3-cuts of $G$ corresponding to a cut of $\overline{S}$ can be found by a trivial Cartesian product generation.

Remark. It is well known that the bridge-tree and 2-components of a graph are provided by DFS search from any vertex of this graph. We are sure that one can find a simpler algorithm for the construction of 3-quotients, too. Our present choice was motivated by uniformity and by the fact that simplifying these steps cannot decrease the total complexity.

Observe, before we pass to the complexity analysis, that the linear size of the cactus model has been stated in previous works, but there is no published proof known to the author. So, in order to calculate the space and time complexities of the suggested representation, we prove some auxiliary
Claim 1  The total number of nodes of $Q^3(G)$ is bounded by $n$, and the number of its edges is at most $2n - 2$.

Proof: The first part of the statement follows directly from the fact that nodes of $Q^3(G)$ correspond bijectively to 3-classes: non-empty and non-intersecting subsets of $V$, $|V| = n$. It is sufficient to prove the second part in the form: every tree-of-edges-and-cycles $Q$ with $n'$ nodes has at most $2n' - 2$ edges. The statement is trivial when $Q$ is a node, an edge or a simple cycle. Assume, by induction, that our bound is true for every tree-of-edges-and-cycles with $n'' < n'$ nodes. Suppose that there is a terminal edge in $Q$ and eliminate it. The resulting graph has $n' - 1$ nodes and, by induction, $\leq 2n' - 4$ edges. Therefore, $Q$ has at most $2n' - 3$ edges. Suppose that $Q$ has a terminal cycle of length $r \geq 2$ and eliminate it. The resulting graph has $n' - r + 1$ nodes and, by induction, $\leq 2n' - 2r$ edges. The starting number of edges was more by $r$, i.e., at most $2n' - r \leq 2n' - 2$. □

Claim 2  The number of edges of $H(G)$ for an arbitrary graph $G$, $c(G)$ odd, (i.e., the number of $c(G)$-cuts of $G$) is at most $2n - 3$.

Proof: The graph $G$, $n \geq 2$, has at least one cut, and so at least one minimum cut. Thus, a model tree has at least two nodes and one edge. Let us call a node “poor” if it has degree 1 or 2, and “rich” otherwise. By the construction of the cactus model, it has no poor “empty” nodes [DKL76]. So it has at most $n$ poor nodes. We prove the statement in the form: every tree $T$ with $p$ poor nodes has at most $2p - 3$ edges. When $T$ has no rich nodes, i.e., is a path of the length $p - 1 \geq 1$, it has $p - 1 \leq (p - 1) + (p - 2) = 2p - 3$ edges. Suppose, by induction, that, for some $r \geq 1$, our bound is true for every tree with $r' < r$ rich nodes and assume that $T$ has $r$ rich nodes. Let us split $T$ in a rich node $v$ of degree $q \geq 3$ (see Fig. 8). Every one of the resulting trees has at most $r - 1$ rich nodes and gets a new poor terminal node implied by $v$. Let us denote $p_1, p_2, \ldots, p_q$ the numbers of poor nodes in these trees: $p = (p_1 - 1) + (p_2 - 1) + \cdots + (p_q - 1)$. The total number of edges has not been changed by splitting and, by induction, is bounded by $\sum_{i=1}^q (2p_i - 3) = \sum_{i=1}^q 2(p_i - 1) - q = 2p - q \leq 2p - 3$. □

In particular, $2n - 3$ is an upper bound for the number of 3-cuts of a 3-connected graph.
Claim 3
(i) The sum of the numbers of vertices of all 2-components or of all 3-components is \( n \).
(ii) The total number of 3-cuts of 3-components is at most \( 2n - 3 \).

Proof: The sets of vertices of 2- (resp. 3)-components are 2- (resp. 3)-classes, which form a partition of \( V \). Therefore, the sum of their cardinalities is \( n \). Let us denote by \( n_1, n_2, \ldots, n_q \) the numbers of vertices in 3-components of \( G \): \( n = n_1 + n_2 + \cdots + n_q \). According to Claim 2, the total number of their 3-cuts is at most \( \sum_{i=1}^{q} (2n_i - 3) = 2n - 3q \leq 2n - 3 \). \( \square \)

Claim 4 The sum of the numbers of edges of all 2-components or of all 3-components does not exceed \( |E| \).

Proof: The statement is evident for the 2-components, because their edges are edges of \( G \), without repetitions. According to the definition of a 3-component, at most one virtual edge representing a cycle \( L \) of \( Q^3(G) \), is added to any 3-piece incident to it. Assume that cycle \( L \) has \( r \) edges and, thus, has \( r \) incident 3-pieces. Then, it generates at most \( r \) virtual edges. This is compensated by the fact that the \( r \) edges of \( G \) belonging to \( L \) do not pass to the edge sets of 3-components. \( \square \)

These statements imply the following bounds. There are at most \( n - 1 \) bridges and at most \( (2n - 2) \) edges in all cycles of the 3-quotient of \( G \) (the total length of the edge-lists of \( L^2 \)). There are at most \( 3 \cdot (2n - 3) = 6n - 9 \) edges and references to \( L^2 \) in the records of all 3-cuts of 3-components.

Let us check that the worst case time of the algorithm suggested is \( O(m + n \log(n)) \), where \( m \) is the number of vertex pairs connected by an edge in \( G \) (\( m \leq n(n - 1)/2 \)). First, let us reduce to four the multiplicity of every edge in \( G \) provided it is greater than four; so, the number of edges
is reduced to at most \(4m\). Obviously, this operation has no influence to \(k\)-cuts and \((k + 1)\)-classes, \(k \leq 3\). Clearly, the complexity of the part related to item 1 is of the desired order. The checking and building related to a 2-component with \(n'\) vertices and \(m'\) edges costs \(O(m' + n' \log(n'))\) time. By Claims 3 and 4, the sums of the numbers of their vertices and edges are at most \(n\) and \(4m\), respectively. The convexity of the function implies the time bound \(O(m + n \log(n))\) for all 2-components. DFS search in a 2-component demands linear time, so time for all of them is linear in \(m\) (by Claim 1). Given 3-quotients of 2-components and, thus, all 3-pieces, time of the construction of all 3-components is linear in their total number of edges. By Claim 4, this is \(O(m)\). Time related to analysis of all 3-components is \(O(m + n \log(n))\), by the same reasons as for 2-components. The generation of a non-empty Cartesian product is, trivially, linear in the output. Thus we have the following statement.

**Theorem 5** For an arbitrary multigraph \(G\) with \(n\) vertices and \(m\) pairs of adjacent vertices:

(i) The Representation 1 of 1-, 2- and 3-cuts of \(G\) has the space complexity \(O(n)\);

(ii) There exists an algorithm that constructs Representation 1 and generates all 2-, 3-, and 4-classes of \(G\) with the time complexity \(O(m + n \log(n))\) and the space complexity \(O(m)\).

(iii) The time complexity of the generation of all 1-, 2-, and 3-cuts of \(G\) from Representation 1 is linear in their number.

5 Further Studies

There are a number of directions of further studies.

1. To develop an algorithm for on-line maintenance of the suggested structural description of 3-cuts and 4-classes of a graph.

2. To give a description of \(l\)-cuts, \(l \geq 4\), dividing more than one 3-component (i.e., cuts from the set \(C_3\)), as combined from \(l'\)-cuts, \(l' \leq l\), of 3-components.
3. To develop a definition of a \( k \)-component, \( k \geq 4 \), adequate to the \( k \)-cuts/\((k + 1)\)-classes investigation (firstly, for \( k = c(G) + 1 \)). Using this definition:

- to give a description \( l \)-cuts from a fixed \( k \)-class-family, \( l \geq k \);  
- to give a description \( l \)-cuts from \( C_{k,2}^l \) relative to the \( k \)-components system structure, \( l \geq k \), in particular, as combined from \( k \)-component-cuts of less or equal cardinalities.

4. To develop an edge \( l \)-cuts description related to the structure of the system of \( k \)-vertex-connected components of a graph, \( l \geq k \), \( k \geq 3 \).

5. To describe the system of minimum cuts of a graph \( G \), \( c(G) \geq 3 \), as edge-sets (at present, we have only the indirect description of these cuts, as corresponding to vertex partitions given by the cactus model).

It is desirable that descriptions developed be graph invariants. Following is an information concerning the current state of studies.

To item 1. An incremental maintenance algorithm for the case of inserting \( k \)-cuts is presented in [D93], [W93] (the same one, invented in parallel).

To item 2. Seems to be done, in general. A cut from \( C_{k,2}^l \) is represented by the set of all its \( 3 \)-class-induced cuts \( C_S^l \) of less cardinality, for all \( 3 \)-classes \( S \) broken by \( C \) (see Th. 4). The analysis of the system of these cuts implies that the relevant nodes and cycles of the \( 3 \)-quotient appear to define a subtree-of-cycles with either one, or two articulation nodes incident to each its cycle.

To item 3. There is a natural way of defining a \( k \)-component, \( k \geq 4 \), \( k = c(G) + 1 \), analogous to the definition of a \( 3 \)-component (see Sect. 3). However, virtual vertices of degree \( k - 1 \) appear in a \( k \)-component. This prevents straightforward generalizations of the results presented in this paper. For example, let \( G \) be a graph with all vertices of degree three, except for one vertex \( v_0 \) of greater degree, and with no non-trivial \( 3 \)-cuts (for example, graphs obtained from the dodecahedron graph or the Petersen graph by contraction of an edge to \( v_0 \)). The \( 4 \)-component of the \( 4 \)-class \( \{v_0\} \) coincides with \( G \) itself, while \( G \) can be complex for the analysis.

The connectivity carcass [DV93] for a \( k \)-class \( S \) seems to be a relevant data structure for a general \( k \). Using it, a representation of \( k \)-cuts of \( S \) is obtained, so a description of \((k + 1)\)-classes follows. An efficient algorithm
for incremental maintenance of this structure is suggested. No general $l$-cuts
description, $l \geq k + 1$, is known.

To item 4. The system of $k$-vertex-connected classes of a graph is, in
general, thinner than one of $k$-edge-connected classes. But we show in Fig. 9
that there can be 3-edge-cuts of 3-vertex-connected components which do
not generate any 3-edge-cuts of $G$ in any way. Therefore, we cannot achieve
an immediate connection between $k$-vertex-connected components, $k \geq 3$,
and edge-cuts (such as we have for 3-edge-connected components).

To item 5. A full classification is found for the case, where $k = 3$ and the
cactus model is a path.

6 Acknowledgments

The author is very grateful to G.Kant, A.Hartman, and Z.Nutov for com-
ments improving the presentation of this paper, and to G.Even, Y.Sagi, and
Sh.Katz for useful technical advises.
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