


Figure 8: **Upper row:** A complicated surface and its shading image (under Lambertian shading rule). **Lower row:** The logical combination which is 'correct' result and the error when subtracting the reconstructed surface form the original one.
In the last example a more complicated surface, with 11 singular points of all possible types, is dealt with. Figure 8 (upper row on the left) presents the original surface and its shading image (the upper row on the right). The final reconstructed surface and the error are displayed in the lower row.

7 Concluding remarks

We have presented a new approach to solve the global shape from shading problem. The topological properties of simple smooth surfaces were used to construct an algorithm that uses a logical combination of the weighted distance transforms which are calculated form each of the singular points.

It is possible to implement the first step of the algorithm in parallel. The calculation efforts in computing the weighted distance transform of each of the singular points is determined by the maximal weighted distance from the singular point to the image boundary \( \partial \Omega \in \mathbb{R}^2 \). Define this maximal distance as:

\[
g = \sup_{i \in \{1, \ldots, N\}} \{z_i(x) : x \in \partial \Omega\}.
\]

The second step may also be performed in parallel. The computation complexity in this case is at the order of the number of merging steps that should be performed until the desired result is obtained. The number of iterations in the second step is therefore bounded by the number of saddle points, and according to the mountaineers theorem this number is given by \((N - 1)/2\).

The total calculation complexity for a parallel machine is of order \(O(g + (N - 1)/2)\). The memory involved in the solution should be at least of order \(O(\Omega N)\), and its upper bound is determined by the number of possible merging steps.

8 Acknowledgments

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References


Figure 7: A shading image (upper right) of an upside-down volcano model (upper left), is given as an input to the global algorithm. The reconstructed surface (lower left), is the weighted distance transform of the global minimum point. Observe that the saddle is of the first type in this distance map. The other maps on the lower row (middle and right) do not contribute to the global reconstruction process in this case.
The first example presents a simple surface containing two minimum (maximum) points and a saddle between them. The upper row in Figure 6 displays the surface and its shading image. The three weighted distance maps from each of the singular points are then computed and presented in the middle row of Figure 6. The logical combination, as previously defined, of the two minimal points, yields the surface in the lower row of Figure 6. This surface is the logical combination of the left and right surfaces in the middle row which share the same saddle point.

Figure 6: The minimum points are shown as maximum points. **Upper row:** The original surface and the shading image. **Middle row:** The weighted distance transform from each of the singular points (from the two minimal points and the saddle point). **Lower row:** The 'logical combination' of the distance maps from the two minimal points and the saddle as the merging point.

Figure 7 presents a volcano surface (upside down). In this example it is enough to start form the global minimum point (the ‘top’ of the upside-down volcano) in order to reconstruct the surface without any merging steps. When starting the propagation from the global minimum point, the saddle topology is of the first type, see Figure 3.a, and there is no need to perform any topological corrections in the reconstruction procedure. The weighted distance transform is the desired surface.
Figure 5: The weighted distance transforms from each of the singular points and the merging up to the first legal solution are presented as level contours, numbered 1 to 13, left to right top to bottom.
Figure 4: **Upper row:** The original surface and its shading image. The equal height contours of the original object are drawn on the shading image. **Lower row:** The result of the algorithm and the error surface obtained by subtracting the solution from the original object.
Having the weighted distance and the first saddle of the second type for each of the singular points the merging process begins. The procedure $\text{Merge}(k, l, N_T)$ checks for possible merging of the the map $k$ and the map $l$. If it is possible to merge the two maps, a new map is created and its second type saddle is located. The index variable $N_T$ is incremented by one and assigned as a label to the new map. In case the two maps are merged and a second type saddle is not detected, the merge of the two maps is a legal solution.

Let $N$ be the number of singular points, the following simple procedure reconstructs the surface:

$$N_T = N$$

$$\text{for } k = 2 \text{ to } N$$

$$\text{for } l = 1 \text{ to } k - 1$$

$$\text{Merge}(k, l, N_T)$$

$$\text{for } k = N + 1 \text{ to } N_T$$

$$\text{for } l = 1 \text{ to } N_T$$

$$\text{if (}k \text{ and } l \text{ were not Merged before)} \text{ Merge}(k, l, N_T)$$

Note that the end-of-loop variable, $N_T$, in the second $k$-loop may change due to merging.

In order to better understand the merging algorithm consider the following simple example. Let the original object be composed of a maxima surrounded by four minimas, see Figure 4.

After solving the local shape from shading problem for each of the 9 singular (maxima minima and saddle) points, the merging process went as follows, see Figure 5: 4 and 8 merged into 10 via the point 7. 7 and 9 merged into 11 via the point 8, 1 and 3 merged into 12 via the point 2, and 11 and 12 merged into 13 via the point 4 resulting in the first legal solution, where there are no points of the second saddle type.

It is possible that merging that do not lead to any solution are performed (e.g., 4 and 8 resulting 10), however, the final result of the algorithm is always a legal solution that satisfies the topology restrictions and the irradiance equation. We can let the merging process go on after the first legal solution is obtained and thereby detect all possible solutions (in cases there are more than one).

6 Examples

In the following examples the shape from shading via level set algorithm [10] was used as the basic procedure that calculates the weighted distance transform from each singular point. Any other local shape form shading algorithm or weighted distance transform may be used for the same purpose at the first step of the algorithm. The shading images are given as $64 \times 64$ gray-level pixel arrays. The surfaces are shown upside down in the first and last examples, making it easier to understand the reconstruction process and the surfaces structure.
Step 2 An iterative search for the ‘proper solution’ that satisfies the smoothness demands is performed. A ‘merge’ of two distance transforms is performed if the following conditions are satisfied:

- The two distance maps have the same singular point as the minimal distance second type saddle \( (P_i = P_j) \).
- The equal height contours, that correspond to the height defined by the second type saddle \( (D_i(x) = H_i \) and \( D_j(x) = H_j) \), osculate only at the singular point, and do not cut or touch each other elsewhere, i.e.

\[
\{ x : D_i^{-1}(H_i) = D_j^{-1}(H_j) \} = P_i(= P_j).
\]

When those two conditions are satisfied, a merge or ‘logical combination’ of the two distance maps is performed as follows:

\[
D_{ij}(x) = \min\{ D_i(x), D_j(x) + H_j - H_i \}.
\]

The map \( D_{ij} \) and its minimal distance second type saddle if encountered \( P_{ij} \) (of height \( H_{ij} \)) are added to the array of distance maps. The merging steps are repeated until a map, for which there are no second type saddle points, is obtained. This map will be a ‘legal’ solution to the reconstruction problem. However, it is possible to continue the merging process until there are no more connections (second type saddle points) between any of the entries in the array. All the legal maps corresponding to the data will be found this way.

The second step of the algorithm finds all possible solutions to the given shape from shading problem, solutions that obey the Morse conditions at singular points.

5 Discrete approximation

The algorithm was implemented on a grid of pixels. The weighted distance map from each of the singular points was calculated using the equal height contour evolution via level sets implemented on the grid. In [10], the numerical properties of the level set algorithm, like accuracy, stability, noise effects, efficiency and its performance on real images were explored. The singular points were determined by considering the grid point of highest intensity (the brightest point) within a connected area of pixels, these White areas being isolated by thresholding the shading image.

Given a weighted distance transform of a singular point, the first saddle point of the second type should be located. Consider a small simple closed curve around each of the singular points. The number of zero crossings along that curve, where the zero is the height of that singular point, classify that point. For a saddle of the first type - four zero crossings will occur, for a maximum point there are no zero crossings, whereas for a saddle of the second type there are two zero crossings. On the grid, the zero crossings were searched along a cyclic chain code representation of a square boundary around each singular point.
point. The first one is when the saddle is reached from two opposite sides, see Figure 3.a. In this case the propagation may proceed with no difficulties. After encountering such a saddle point the contour splits into two separate curves and the internal curve may encounter a maximum point while propagating inwards and vanish at that point without violating any of the topology conditions. We shall refer to these two events as the saddle of first type and the maximum point event, respectively.

In the second case, see Figure 3b, the saddle point is touched is from one side only. In this case a topology-based procedure is needed to correct the contour propagation to cause the interior to obey the mountaineers theorem. Propagating the contour beyond that point will cause that point to become a shoulder (inflection), a pathological phenomenon not permitted in Morse functions. However, note that propagating beyond that point causes false results only in a well defined area which is a subset of the region swept by the propagating equal height contour. The rest of the contour is not affected and can still be used in the reconstruction process. This fact helps us in producing the global algorithm while keeping the calculation efforts low.

Starting from minimum points as sources of propagating equal distance contours, the merging points of the propagating contours should be saddles of the second type. A rule for the merging of the contours should be devised. In the following section a rule that leads to a global shape from shading algorithm is presented.

4 The global shape from shading algorithm

As mentioned before, the main problems are to classify the singular points, propagate from the minimum points and use the second type saddle points as merging points. The solution to this ‘three phase’ problem seems impossible when one tries to break it into small sequential parts, as was traditionally done. Considering the global problem in hand we must use a global approach and the smoothness conditions (the surface being a Morse function) as the key to the solution. The algorithm has in two main steps.

Step 1 For each singular point, \( m_i \) \((i \in \{1,.., N\})\), where \( N \) is the number of singular points), calculate the ‘weighted distance transform’

\[
D_i(x) = \inf_{l(s) \in L_i} \left\{ \int_0^l f(l(s)) ds : \left| \frac{\partial l}{\partial s} \right| = 1, \ l(0) = x, \ l(l) = m_i \right\},
\]

in all of the image domain \( \Omega \in \mathbb{R}^2 \). This transform is the correct solution in the case of a single singular point known to be a minimum (the maximum is the dual case).

The first singular point: \( m_k (\neq m_i) \), that is identified as a saddle of the second type (Figure 3.b) according to the minimal distance in \( D_i \), is labeled \( P_i = m_k \), its corresponding distance being \( H_i \):

\[
H_i = \{D_i(m_k) \mid m_k \text{ is a second type saddle in } D_i \},
\]
Figure 3:  

(a). The equal height contour reaches the saddle from two opposite sides, and the mountaineers theorem remains valid when the propagation continues. 

(b). The contour reaches the saddle from only one side and continuing the propagation will cause that saddle to appear as a higher order local extrema that looks like a shoulder, which is forbidden according to our smoothness assumptions.
where \( d \) is some constant. This local behavior in the neighborhood of singular points yields three local types of surfaces as presented in Figure 2. Being able to classify the singular points greatly helps solving the global shape from shading problem. In fact, given the height of all minimum points \( \{ z(m) \mid m \text{ is a minimum} \} \), the global solution is given by

\[
z(x) = \inf \{ z(m) + \int_0^{l|l|} f(l(s))ds : \quad l(s) \in L_m, \\
\frac{\partial l}{\partial s} = 1, \quad l(0) = x, \quad l(|l|) = m, \quad m \text{ is a minimum} \},
\]

where \( L_m \) is the set of all planar curves starting at \( x \) and ending at \( m \), [17, 20].

Propagating from an assumed local minimum singular point, the evolving contour meets several kinds of singular points. There are only three types of such singularities that are of interest to us. According to the mountaineers theorem [16, 6], the number of extrema located within a closed equal height contour of a smooth surface exceeds by one the number of saddle points within that contour. Therefore, when tracking an equal height contour that starts as a small circle around a minimum, the first singular point that the propagating contour meets must be a saddle point. There are two possible ways for the contour to reach a saddle point.

Figure 2: The three types of singular points that characterize the Morse function are the minimum, the saddle and the maximum points. The lower frames display the equal height contours painted on a gray-level image that represents the heights.
case, the problem may be solved by the level sets propagation algorithm. The reconstructed surface is the weighted distance transform of the minimum point. The height ('weighted distance') at each point $x \in \mathbb{R}^2$ is determined by

$$z(x) = \inf_{l(s) \in L} \left\{ \int_0^{|l|} f(l(s)) ds : \frac{\partial l}{\partial s} = 1, \ l(0) = x, \ l(|l|) = m \right\},$$

where $L$ is the set of all curves connecting point $x$ to the given minimum point $m$.

Other non direct (iterative) methods that solve this local shape from shading problem were recently presented, see e.g. [20, 5, 1], and may replace our core level-set based procedure that solves the local problem. The basis for our global algorithm is the knowledge of 'how to combine' the local solutions together.

### 3 Topological properties of smooth surfaces

Our basic assumption is that the object we try to reconstruct has a simple and smooth surface, described by a Morse function formally defined in [2]. Such a surface is assumed to have no pathological singularities like the *monkey-saddle*, see Figure 1. All singular points of

![Figure 1: A pathological case of a singular point known as the monkey saddle. The surface on the left, and its equal height contours painted on a gray-level map which represents the heights on the right.](image)

a Morse function may be locally described, up to a mere change of coordinates in the plane [2], by

$$d \pm x^2 \pm y^2.$$
the local shape from shading problem is similar to the weighted distance transform. Starting from all the singular points we calculate via level set propagation the weighted distance transform for each of those points. Section 3 briefly reviews the topological properties of simple smooth surfaces. Those properties are used in Section 4 to formulate the global shape from shading algorithm. The algorithm implements a 'logical combination' of the weighted distance transforms of all the singular points. Some examples that clarify the algorithm are presented in Sections 5 and 6, followed by some remarks on the complexity of the algorithm in Section 7.

2 Local shape from shading

We have chosen to use the simple case where the light source and the viewer are located at the same place. Consider a smooth surface \( z(x, y) \). According to the Lambertian shading rule, the brightness map (shading image) \( E(x, y) \) is equal to the inner product of the light source direction \( \hat{l} = (0, 0, 1) \) and the normal \( \hat{N}(x, y) \) to the surface. This relation is known as the irradiance equation and may be written as

\[
E(x, y) = R(p, q) = \hat{l} \cdot \hat{N} = \frac{1}{\sqrt{1 + p^2 + q^2}},
\]

where \( p = \partial z/\partial x \) and \( q = \partial z/\partial y \) are the partial derivatives of the surface. The extension to the 'oblique' light source becomes simple considering light source coordinates as was done in [12], see also [14, 5].

Starting from a small circle around a singular point one may extend the solution by using the equal height contour evolution as suggested by Bruckstein in [3]. Bruckstein observed that given an equal height contour \( C(s) : S^1 \to \mathbb{R}^2 \), it may serve as the initial condition, \( C(s, 0) = C(s) \), for the curve evolution equation

\[
\frac{\partial C(s, t)}{\partial t} = \frac{E}{\sqrt{1 - E^2}} \vec{n},
\]

where \( \vec{n} \) is the planar normal of the evolving curve. The evolving curve tracks the equal height contours of the surface, and thereby reconstructs the surface. The 'time', \( t \), in the evolution equation represents the height. This direct shape from shading method was recently reformulated via level sets propagation and numerically implemented by the Osher–Sethian algorithm [19, 21], in [10, 12]. In [10] the equal height contour propagation via level sets was introduced and compared to other shape from shading algorithms. The algorithm was applied to real images and its performance under different types of noise was demonstrated.

Recently it was also shown that the local shape from shading problem is equivalent to the problem of computing the weighted distance transform [11, 13], see also [17, 20, 15]. Equation (1), describes the propagation of an equal cost contour where the cost function, defined over the image domain \( \Omega \in \mathbb{R}^2 \), is given as a function of the brightness by

\[
f(x, y) = \sqrt{1/E^2(x, y) - 1}.
\]

Having only one singular point that is known to be a minimum \( m \in \mathbb{R}^2 \), the surface may be reconstructed by calculating the weighted distance map from that singular point. In this
1 Introduction

The problem of reconstructing the three dimensional shape of an object from its shading (gray level) image was and still is of great interest to the computer vision community. Horn [7], formulated the problem as a first order non linear partial differential equation, and attempted to solve it by using the classical characteristic-strip expansion method. This direct approach suffers from numerical difficulties of inaccuracy and instability due to its local nature. The search for better approaches led to the development of a series of algorithms that solve regularized versions of the problem (see [9, 8], which represent only few of the many published works concerning such approaches.) Direct iterative procedures were recently presented in [17, 20], followed by [4, 1], the problem being reformulated to yield iterative numerical schemes that could be proven to converge to the ‘correct’ solution. However, these algorithms were aimed at solving the local problem, and some external information, like the nature of singular points, was necessary in order to patch the local solutions together.

The shaded image induces a relation between surface normals and the light source direction. According to the so called Lambertian shading rule, the image, i.e. the two dimensional array of gray levels at each pixel in the image, maps the cosine of the angle between the light source direction and the surface normal. The shape from shading problem is the inverse problem of reconstructing the 3D surface from this data. In order to resolve ambiguities, smoothness assumptions as well as other ‘clues’ in the image were explored, such as apparent contours, grazing light edges, self shadow edges [15], and (last but most important for our discussion) the nature of singular points. Singular points are those points in the image where there is no ambiguity in the normal direction. The normal of the surface at a singular point is known to be in the direction of the light source. These brightest points are the local minima, local maxima and saddle points of the surface. Horn [7] suggested to use small circles around singular points as initial conditions for the characteristic strip expansion method. Some attempts to use those points as a key to the global reconstruction were made in [18, 5], where the local behavior of characteristics around a singular point is used to determine the nature of that point. The local solution to the shape from shading problem is extended from maxima to minima and so forth, based on that classification procedure. This method requires choosing a singular point that is known not to be a saddle, based on local information. Extending the solution involves the inspection and classification of singular points on the boundary of the ‘area of attraction’ of a singular point, also a local operation. These local operations make no use of global properties of the surface, and may therefore lead to mistakes.

In this paper we address the problem of global shape from shading, where no boundary nor initial conditions are assumed besides the shading image itself, based on the topological properties of simple smooth Morse functions. We rely on the theory of differential topology to classify the singular points. The ‘correct solution’ is selected from all possible solutions as the one that satisfies the global topological properties and agrees with the local shading equation. There is no need to classify any of the singular points, and all those points take part in the process of merging areas until all possible solutions are found.

The structure of this paper is as follows: In Section 2, a local shape from shading via level set algorithm based on [3, 10, 11, 13] is shortly referred to. It was shown in [20, 12, 13] that
Global Shape from Shading

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Abstract

A new approach for the reconstruction of a smooth three dimensional object from its two dimensional gray level image is presented. An algorithm based on topological properties of simple smooth surfaces is provided to solve the problem of global reconstruction. Classifying singular points in the shading image as maxima minima and two kinds of saddle points, serves as the key to the solution of the problem. This classification is performed globally with no assumptions on the local behavior of characteristics near singular points. The global reconstruction procedure, being deterministic and using topological properties of the surface performs better than other approaches proposed so far, based on classification of singular points according to the local behavior of characteristics in their neighborhood. The proposed algorithm is simple, easy to implement and works remarkably fast on a parallel machine.

Keywords: Differential topology, Morse functions, Lambertian reflectance map, singularity theory.

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