Figure 10:
Figure 3:
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List of Figure Captions

Figure 1: The projection of an infinitesimal geodesic circle on the surface forms a tilted ellipse on the \((x, y)\)-plane.

Figure 2: Problems in curve evolution implementation: a. Development of singularities - Even a smooth initial curve can develop singularities when evolving with velocity \(\mathbf{n}\). The first singularity occurs at the center of curvature corresponding to the maximal curvature of the initial curve (marked point). b. Topological changes - An initial connected curve, evolving with velocity \(\mathbf{n}\), can become disconnected when evolving in time.

Figure 3: Embedding the curve in a higher dimensional function automatically solves a number of topological problems. The evolving curve is obtained as a level set of the evolving surface, which remains continuous (and connected) even when topological changes occur.

Figure 4: A simple “mountain” surface, where the source and destination are located at \((\frac{1}{4}, \frac{1}{4})\) and \((\frac{3}{4}, \frac{3}{4})\) in the planar coordinates. The surface is taken on \([0, 1] \times [0, 1]\). See text for more details.

Figure 5: An “egg-box” surface. See text for details.

Figure 6: Introducing obstacles in the “egg-box” surface. See text for details.

Figure 7: Finding the minimal path between two areas.

Figure 8: a. Equal distance contours are propagated from \(p\), until the contour first touches \(\alpha(u)\). b. The tangent point, observe that at \(q\): \(\tilde{t}^\beta|_{u=u_0} \parallel \tilde{t}^\gamma|_{v=v_0}\).

Figure 9: When considering only one minimal geodesic between two points, the path is the trace of a constant parameter along the evolution, see text.

Figure 10: a. A black thick curve segment passes through the middle of the geodesic circle on the surface. The dotted line is the normal to the 3D curve, on the surface. And \(l\) is the surfaces’ gradient direction. b. The projection of the geodesic circle on to the plane without tilting the ellipse (for simplicity). The \(l\) axis is scaled by \(\lambda\) due to the projection to the plane. c. The projection of the geodesic circle from surface on to the \((x, y)\)-plane. The axis is first scaled by \((1, \lambda)\) and then rotated by \(\psi\).


References


we can write the planar normal velocity component as the following simple equation which depends on the surface first derivatives

\[ V_N = \sqrt{a n_1^2 + b n_2^2 - c n_1 n_2}. \]

the result obtained in the previous section.
circle on the plane is a tilted ellipse with \( \lambda = \frac{1}{\sqrt{1+p^2+q^2}} \) and the tilted angle is given by 
\[
\tan(\psi) = -\frac{p}{q} \text{ or by }
\]
\[
s \equiv \sin(\psi) = -\frac{p}{\sqrt{p^2 + q^2}}
\]
\[
c \equiv \cos(\psi) = \frac{q}{\sqrt{p^2 + q^2}}.
\]
See Figure 10.c.

The planar normal \( \bar{n} \) can be written as a vector \( \bar{n} = \{n_1, n_2\} \), or in the tilted ellipse coordinate system
\[
\bar{n} = \{\tilde{n}_1, \tilde{n}_2\}
\]
\[
= \{n_1 c + n_2 s, n_2 c - n_1 s\}
\]
\[
= \frac{1}{\sqrt{p^2 + q^2}} \{qn_1 - pn_2, qn_2 + pn_1\}.
\]

By translating the above parameters to the ellipse coordinate system we get
\[
\tau = \tan(\theta) = \frac{\tan(\hat{\theta})}{\lambda} = -\frac{\tilde{n}_1}{\tilde{n}_2 \lambda}
\]

and
\[
\bar{w} = \{\bar{w}_k, \bar{w}_l\} = \frac{1}{1 + \tau^2} \{\tau, -\lambda\}
\]
\[
= -\frac{\lambda \tilde{n}_2}{\sqrt{\lambda^2 \tilde{n}_2^2 + \tilde{n}_1^2}} \{\frac{\tilde{n}_1}{\tilde{n}_2 \lambda}, \lambda\}.
\]

The planar normal propagation velocity \( V_N \) can now be computed
\[
V_N = \bar{w} \cdot \bar{n}
\]
\[
= -\sqrt{\tilde{n}_1^2 + \lambda^2 \tilde{n}_2^2}
\]
\[
= -\frac{1}{\sqrt{p^2 + q^2}} \sqrt{(pn_1 - pn_2)^2 + \lambda^2 (qn_2 + pn_1)^2}
\]
\[
= -\frac{1}{\sqrt{p^2 + q^2}} \sqrt{(p^2 + \lambda^2 p^2)n_1^2 + (p^2 + \lambda^2 q^2)n_2^2 - 2pq(1 - \lambda^2)n_1 n_2}
\]

Defining \( a, b \) and \( c \) to be
\[
a = \frac{1 + q^2}{1 + p^2 + q^2}
\]
\[
b = \frac{1 + p^2}{1 + p^2 + q^2}
\]
\[
c = \frac{2pq}{1 + p^2 + q^2},
\]
**Proof.** Define $\tau \equiv \tan(\theta)$ then

\[
\begin{align*}
\tan(\bar{\theta}) &= \lambda \tau \\
\tan(\bar{\xi}) &= -\frac{1}{\tan(\bar{\theta})} = -\frac{1}{\lambda} \\
\tan(\tilde{\xi}) &= \lambda \tan(\bar{\xi}) = -\frac{\lambda}{\tau} \\
\tan(\gamma) &= -\frac{1}{\tan(\bar{\theta})} = -\lambda \tau.
\end{align*}
\]

Then, the normal direction can be written as

\[\text{direction of}\{\bar{n}\} = \{-\lambda \tau, 1\}.\]

The ellipse formula can be written as

\[
\bar{k}^2 + \frac{\bar{l}^2}{\lambda^2} = 1
\]

and the lower part (negative $\bar{l}$ values) of the ellipse can be written as the function

\[
\bar{l} = -\lambda \sqrt{1 - \bar{k}^2}.
\]

The lower part of the ellipse may also be written as the vector $\bar{r}(\bar{k}) = \{\bar{k}, -\lambda \sqrt{1 - \bar{k}^2}\}$. Define the functional $F(\bar{k}) \equiv \bar{r}(\bar{k}) \cdot \bar{n} = -\lambda \tau \bar{k} - \lambda \sqrt{1 - \bar{k}^2}$. In order to find $\max_{\bar{k}} \{\bar{r}(\bar{k}) \cdot \bar{n}\}$, compute

\[
\frac{d}{d\bar{k}} F(\bar{k}) = -\lambda (\tau - \bar{k}(1 - \bar{k}^2)^{-\frac{1}{2}})
\]

Using $\frac{d}{d\bar{k}} F(\bar{k}) = 0$ and the vector definition $\bar{r}(\bar{k})$ we obtain

\[
\bar{w}_k = \frac{\tau}{\sqrt{1 + \tau^2}} \\
\bar{w}_l = -\lambda \frac{1}{\sqrt{1 + \tau^2}}
\]

that results in the angle of the tangent point

\[
\tan\{\text{Angle of maximal tangent}\} = \frac{\bar{w}_l}{\bar{w}_k} = -\frac{\lambda}{\tau} = \tan(\bar{\xi}).
\]

\[\text{Q.E.D.}\]

We proved that the maximal projection of the ellipse on the planar normal $\bar{n}$ is actually the projection of the projected 3D curve normal $\bar{w}$ on $\bar{n}$. Define the first order directional derivatives of the surface $z(x, y)$ as $p = \frac{dz}{dx}$ and $q = \frac{dz}{dy}$. The projection of the geodesic
We have noted earlier that the planar unit normal to any level set of a given continuous function \( \phi \) can be written as \( \tilde{n} = \frac{\nabla \phi}{\| \nabla \phi \|} \). Using this notation in equation (11) we obtain

\[
V_N = \frac{1}{\| \nabla \phi \|} \sqrt{a \phi_x^2 + b \phi_y^2 - c \phi_x \phi_y}.
\]

The multipliers \( a, b \) and \( c \) are approximated in our examples by a central finite difference approximation of the sampled surface \( z_{i,j} \equiv z(i \Delta x, j \Delta y) \). That is, by using

\[
p_{i,j} = \frac{z_{i+1,j} - z_{i-1,j}}{2 \Delta x}, \quad \text{and} \quad q_{i,j} = \frac{z_{i,j+1} - z_{i,j-1}}{2 \Delta y},
\]

and a one side finite difference approximation on the boundary. The multipliers \( a_{i,j} \equiv a(i \Delta x, j \Delta y), b_{i,j} \equiv b(i \Delta x, j \Delta y) \) and \( c_{i,j} \equiv c(i \Delta x, j \Delta y) \), are calculated once at the beginning of the process, and then used as constants in the numerical approximation.

## A.5 Geometric Interpretation of the Normal Velocity

In this part we obtain the normal component of the velocity by which the equal distance contour, \( C(t) \), propagates using the projection of a small geodesic circle around each point. Let us first observe an infinitesimal geodesic circle on the surface through which an infinitesimal segment of an equal distance contour passes (see Figure 1). In Figure 10.a a circle representing the geodesic circle is shown. The distance contour drawn as the thick black line segment passes through the middle of the circle. The normal to this contour segment is drawn with the dotted line. Observing the surface from above, the geodesic circle is projected to the \((x, y)\)-plane as a tilted ellipse as shown in Figure 10.c.

We search for the velocity's normal component to the projection of the equal distance contour to the \((x, y)\)-plane. The contour propagates on the surface with a velocity of 1, in the direction of the three dimensional contour's normal (the dotted line). The planar normal component of this unit velocity yields the propagation velocity of the projected equal distance contour.

**Lemma 5** The planar normals' velocity component is the maximal projection of \( \tilde{r}(\eta) \) on \( \tilde{n} \), where \( \tilde{r}(\eta) (\eta \in [0, 2\pi]) \) is a vector representation of the projected geodesic circle to the \((x, y)\)-plane (the ellipse), and \( \tilde{n} \) is the planar normal of \( C \) (the projection of \( \alpha \)). See Figure 10.b.
The projection of this curve evolution on the \( (x, y) \)-plane may by written using
\[ C_t = V_N \bar{n} \], or
\[
(z_t, y_t) = V_N \frac{(-y_u, x_u)}{\sqrt{x_u^2 + y_u^2}}
\]

We are looking for the velocity \( V_N \), which is the projection of \( (w_1, w_2) \) on the planar normal \( \bar{n} \). This may be written as
\[
V_N = \langle (w_1, w_2), \bar{n} \rangle = \frac{-w_1y_u + w_2x_u}{\sqrt{x_u^2 + y_u^2}}
\]
Let us first calculate the vector product
\[
(x_u, y_u, z_u) \times (-p, -q, 1) = (y_u + qz_u, -x_u - pz_u, -x_uq + py_u),
\]
recalling the denominator, we get
\[
(w_1, w_2) = \frac{(y_u + qz_u, -x_u - pz_u)}{\sqrt{1 + p^2 + q^2} \sqrt{x_u^2 + y_u^2 + z_u^2}}
\]

Now, we are ready to calculate \( V_N \),
\[
V_N = \frac{(y_u + qz_u, -x_u - pz_u)}{\sqrt{1 + p^2 + q^2} \sqrt{x_u^2 + y_u^2 + z_u^2}} \cdot \frac{(-y_u, x_u)}{\sqrt{x_u^2 + y_u^2}}
\]

Using the chain rule \( z_u = z_x x_u + z_y y_u = px_u + qy_u \), we get
\[
V_N = \frac{(x_u^2 + y_u^2) + (px_u + qy_u)^2}{\sqrt{x_u^2 + y_u^2} \sqrt{1 + p^2 + q^2} \sqrt{x_u^2 + y_u^2 + z_u^2}}
\]
\[
= \sqrt{\frac{x_u^2(1 + p^2) + y_u^2(1 + q^2) + 2pq x_u y_u}{(1 + p^2 + q^2)(x_u^2 + y_u^2)}}
\]

Writing the planar normal as its components \( \bar{n} = (n_1, n_2) = \frac{(-y_u, x_u)}{\sqrt{x_u^2 + y_u^2}} \), we conclude with
\[
V_N = \sqrt{n_1^2 \frac{1 + q^2}{1 + p^2 + q^2} + n_2^2 \frac{1 + p^2}{1 + p^2 + q^2} - n_1 n_2 \frac{2pq}{1 + p^2 + q^2}}
\]
or
\[
V_N = \sqrt{an_1^2 + bn_2^2 - cn_1 n_2},
\]
where
\[
a = \frac{1 + q^2}{1 + p^2 + q^2}
\]
Assume there is a parameterization point \( u_1 \neq u_0 \) through which the minimal path passes at \( t = a + \epsilon \) (\( \dot{t} = b - \epsilon \)). According to that assumption there is a minimal path \( P_1 \sim \alpha_s(u_1, a) \sim \alpha_s(u_1, a + \epsilon) \sim P_2 \) of length \( a + \epsilon + (b - \epsilon) \). However, part of this path, \( P_1 \sim \alpha_s(u_1, a) \), is not equal to the original sub-path of the minimal path \( \alpha(u_0, a) \), and this contradicts the assumption that the minimal path should pass through \( \alpha(u_0, a) \), and concludes the proof.

Q.E.D.

A.4 Planar Normal Velocity Calculation

In this section the normal component of the projection of the 3D velocity vector on the plane, is calculated. The equal distance contour evolution on a 3D surface may be given as the projection of the evolution on the \((x, y)\)-plane. This results in the following equation for planar curve evolution which corresponds to the 3D evolution,

\[
C_t = V_N \vec{n},
\]

where \( C \in \mathbb{R}^2 \) is the trace of the projection of the 3D equal distance contour on the \((x, y)\) plane, \( V_N \) is a scalar velocity representing the planar normal component of the velocity, and \( \vec{n} \) is the planar normal of \( C \). The way we achieve the results in this section is based on the the local theory of regular surfaces in \( \mathbb{R}^3 \).

Let the surface \( S \subset \mathbb{R}^3 \) be given as a graph surface, that can be written as \( S = \{ x, y, z(x, y) \} \), and let the equal distance contour be represented as the parametric curve \( \alpha(u) = (x(u), y(u), z(u)) \in S \). Define the parameterization of surface \( S \) as \( Z = (x, y, z) \).

Using these definitions the surface normal at every point may be written as

\[
N = \frac{Z_x \times Z_y}{|Z_x \times Z_y|} = \frac{(-p, -q, 1)}{\sqrt{1 + p^2 + q^2}}
\]

where \( p \equiv \frac{\partial z}{\partial x} \) and \( q \equiv \frac{\partial z}{\partial y} \).

Let \( T_{p_0}(S) \) be the tangent plane to surface \( S \) at point \( p_0 = \alpha(u_0) \). The tangent vector to the curve \( \alpha(u) \in S \) at the parameterization point \( u_0 \) is given by

\[
\vec{t} = \left. \frac{\alpha_u}{|\alpha_u|} \right|_{u = u_0} = \left. \frac{(x_u, y_u, z_u)}{\sqrt{x_u^2 + y_u^2 + z_u^2}} \right|_{u = u_0},
\]

where \( \vec{t} \in T_{p_0}(S) \).

The equal distance contour evolution may now be written as a differential equation involving the vector product of the surface normal and the tangent

\[
\alpha_t = N \times \vec{t}.
\]

The above equation may be written as

\[
(x_t, y_t, z_t) = \frac{(x_u, y_u, z_u)}{\sqrt{x_u^2 + y_u^2 + z_u^2}} \times \frac{(-p, -q, 1)}{\sqrt{1 + p^2 + q^2}} = (w_1, w_2, w_3).
\]
Considering the constant parameter traces along the evolving contours as the radial geodesics \( \beta(t) = \gamma(t) \), and the contours themselves as the geodesic circles \( \alpha(u) = \eta(u) \), we obtain that the equal distance contour evolution rule is given by the asserted equation, when starting from a point. To be more precise, we start from a given infinitesimal geodesic circle around the point.

Now we shall proceed and generalize the result to any given curve \( \alpha(u, 0) = \alpha(0) \) on the surface.

Let \( P \) be the set of points forming the equal distance contour of distance \( d \) from \( \alpha(u, 0) \), on the given surface. Propagate an equal distance contour \( \eta(v, t) \) starting from any point \( p \in P \). Stop the propagation when the equal distance contour first touches \( \alpha(u) \), let say at \( q = \alpha(u_0) = \eta(v_0, \tau) \). According to the construction \( \tau = d \), and therefore, \( q = \eta(v_0, d) \). See Figure 8.a.

At \( q, \eta(v, d) \) and \( \alpha(u) \) osculate, which means that \( \tilde{t}^\alpha|_{u=u_0} \) is parallel to \( \tilde{t}^\eta|_{v=v_0} \), see Figure 8.b.

We have shown earlier that the shortest path from \( p \) to \( q \) is given by the radial geodesic \( \beta(t) = \gamma(t) \), and where \( \beta(t) \) and \( \eta(t, v) \) are orthogonal along \( v = v_0 \) (Gauss lemma). Hence,

\[
\tilde{t}^\beta|_{t=d} - \tilde{t}^\eta|_{t=d, v=v_0} \]

and therefore,

\[
\tilde{t}^\beta|_{t=d} - \tilde{t}^\alpha|_{u=u_0}.
\]

We have just shown that the shortest paths from each point in the set \( P \) to \( \alpha \), is given by the geodesics starting from \( \alpha(u) \) and orthogonal to \( \tilde{t}^\alpha \). This geodesic is the one obtained via the asserted evolution rule; using the continuity of \( \alpha \), the equal distance contour from \( \alpha(u, 0) \), is obtained by the asserted evolution rule, \( \alpha_t = N \times \tilde{t}^\alpha \). Q.E.D.

A.3 Proof of Lemma 4

We shall prove that the trace of the curve formed by the tangent point of two equal distance contours, one propagating inwards and the other outward, from a source point and a destination point, respectively, is actually a geodesic. This is achieved by proving that the tangent point of \( \alpha_s(u, t) \) and \( \alpha_d(u, t) \) while \( t + t = g_m \), which generates the minimal path from point \( P_1 \) to point \( P_2 \), lies on a constant parameter \( u = u_0 \) \((\tilde{u} = \tilde{u}_0)\) of the propagating curve \( \alpha_s(u, t) \) \((\alpha_d(\tilde{u}, \tilde{t}))\).

Proof. The shortest path built by \( \frac{d}{dt} \alpha_s = N \times \tilde{t}^\alpha \) and tangent at \( t = a \) at \( u = u_0 \) to the path built by \( \frac{d}{dt} \alpha_d = N \times \tilde{t} \) at \( t = b \) at \( \tilde{u} = \tilde{u}_0 \), is of length \( a + b \) (see Lemma 3).

Assume that there is only one minimal path between \( P_1 \) and \( P_2 \), and assume that the trace of that minimal path passes through two different parameterization points \( u_0 \) and \( u_1 \). See Figure 9.

Let the length of the shortest distance from \( P_1 \) to \( \alpha_s(a, u_0) \) be \( a \), and the distance from \( P_2 \) to \( \alpha_d(\tilde{u}_0, b) \) be \( b \). The shortest path is of length \( a + b \), and it is given by the path \( P_1 \sim \alpha_s(u_0, a) \sim \alpha_d(\tilde{u}_0, b) \sim P_2 \).
Therefore, in order to show that \( \langle \alpha_t, \frac{\alpha_u}{|\alpha_u|} \rangle = 0 \), it is enough to show that \( \langle \alpha_t, \frac{\alpha_u}{|\alpha_u|} \rangle_t = 0 \).

Define the metric along the curve to be \( g \equiv |\alpha_u| \), and compute

\[
\frac{d}{dt} \frac{\alpha_u}{|\alpha_u|} = \frac{d}{dt} \frac{\alpha_u}{g} = \frac{\alpha_{ut} g + \alpha_u g_t}{g^2} = \frac{\alpha_{ut}}{g} + \frac{\alpha_u g_t}{g^2}.
\]

Using this result we proceed

\[
\langle \alpha_t, \frac{\alpha_u}{|\alpha_u|} \rangle_t = \langle \alpha_t, \frac{\alpha_{ut}}{g} + \frac{\alpha_u g_t}{g^2} \rangle = \langle \alpha_t, \frac{\alpha_{ut}}{g} \rangle + \langle \alpha_t, \frac{\alpha_u g_t}{g^2} \rangle = \frac{1}{g} \langle \alpha_t, \alpha_{tu} \rangle + \frac{g_t}{g^2} \langle N \times \vec{t}^{\alpha}, \vec{t}^{\alpha} \rangle = \frac{1}{g} \frac{1}{2} \frac{d}{du} \langle \alpha_t, \alpha_t \rangle + \frac{g_t}{g^2} 0 = \frac{1}{g} \frac{1}{2} \frac{d}{du} 1 = 0.
\]

Therefore, \( \langle \beta_t, \vec{t}^{\alpha} \rangle = 0 \), and this proves that \( \beta(t) \) is a geodesic.

Q.E.D.

A.2 Proof of Lemma 2

Let \( \alpha(u, t) \) be a 3D curve propagating on the surface \( S \subset \mathbb{R}^2 \), where \( u \) is the parameter and \( t \) is the propagation time. We have to prove that the equal distance contour evolution is given by

\[ \alpha_t = N \times \vec{t}^{\alpha} \quad \text{given} \quad \alpha(0), \]

where \( N \) is the surface normal and \( \vec{t}^{\alpha} \) is the tangent to the contour.

Proof. As a first step we shall use Gauss Lemma to show that the asserted evolution rule formulate geodesic polar coordinates when starting from a point. The geodesic circles in this coordinates system are the equal distance contours. This result is then generalize to any given initial curve.

Define the tangent plane to a point \( p \) on the surface \( S \) as \( T_p(S) \). Let \( \vec{w} \in T_p(S) \), \(|\vec{w}| = 1\) be a unit vector indicating a direction from \( p \) in the tangent plane. Use a polar coordinate system to define \( \vec{w} \) on \( T_p(S) \) as \( \vec{w}(u) = \sin(u) \hat{z} + \cos(u) \hat{y} \). Define \( \gamma^u(t) \) to be the geodesic which starts from \( p \), with \( \gamma^u(0) = p \), and \( \gamma^u_t(0) = \vec{w}(u) \), where \( t \) stands for the arclength.

According to Gauss Lemma, see e.g. [17] pp. 287, the radial geodesics \( \gamma^u(t) \) together with the geodesic circles \( \eta(u, t) = \gamma^u(t), t = \text{const.} \subset S \), form geodesic polar coordinates, in which the geodesic circles are orthogonal to the radial geodesic.
A  Appendix

In the appendix we first provide the proofs of the lemmas given in the paper. Then, we calculate the planar normal component of the projected velocity. In the last section we present a geometric interpretation of the velocity, calculated using the projection of small geodesic circle on the plane.

A.1 Proof of Lemma 1

We prove that given $\beta(t) = \alpha(u = u_0, t)$. $\beta(t)$ is a geodesic.

*Proof.* The trace $\beta(t)$ is determined by the evolution of $\alpha_t = N \times \bar{t}^\alpha$, hence $\beta_t = N \times \bar{t}^\alpha$. Since $|\beta_t| = |N \times \bar{t}^\alpha| = 1$, the $t$ parameter is the arclength of $\beta$.

In order to show that $\beta$ is a geodesic we recall the definition of geodesics and prove that

$$\frac{d^2}{dt^2} \beta = \lambda N,$$

where $\lambda(t)$ is a scalar function. A geometric interpretation of this formula is that the second derivative of the curve (its normal direction $k \tilde{n}^\beta$) is in the surface normal direction $N$. To prove the result it is sufficient to verify that

$$\langle \beta_{tt}, \beta_t \rangle = 0$$
$$\langle \beta_{tt}, \bar{t}^\alpha \rangle = 0,$$

or

$$\left\langle \frac{d}{dt} [N \times \bar{t}^\alpha], N \times \bar{t}^\alpha \right\rangle = 0$$
$$\left\langle \frac{d}{dt} [N \times \bar{t}^\alpha], \bar{t}^\alpha \right\rangle = 0$$

which clearly force $\beta_{tt} = \lambda N$. Let us first define $k, \tilde{n}^\beta$ and $\bar{t}^\beta$ to be the curvature, normal and tangent of $\beta(t)$, respectively. Using the Frenet formulas we first have

$$\left\langle \frac{d}{dt} [N \times \bar{t}^\alpha], N \times \bar{t}^\alpha \right\rangle = \left\langle \beta_{tt}, \beta_t \right\rangle = \left\langle \beta_t, \beta_t \right\rangle = 0.$$

The second expression may be written as follows

$$\left\langle \frac{d}{dt} [N \times \bar{t}^\alpha], \bar{t}^\alpha \right\rangle = \left\langle \beta_{tt}, \bar{t}^\alpha \right\rangle = \left\langle \alpha_{tt}, \frac{\alpha_u}{|\alpha_u|} \right\rangle.$$

Therefore, we should prove that $\left\langle \alpha_{tt}, \frac{\alpha_u}{|\alpha_u|} \right\rangle = 0$. First note that

$$\frac{d}{dt} \left( \frac{\alpha_u}{|\alpha_u|} \right) = \frac{d}{dt} \left( \frac{N \times \bar{t}^\alpha}{|N \times \bar{t}^\alpha|} \right) = \frac{d}{dt} 0 = 0.$$

Using inner product rules we have

$$\frac{d}{dt} \left( \frac{\alpha_u}{|\alpha_u|} \right) = \left( \alpha_{tt}, \frac{\alpha_u}{|\alpha_u|} \right) + \left( \alpha_t, \frac{\alpha_u}{|\alpha_u|} \right)_t.$$
effort is of order $O(\frac{L}{\log m \cdot n})$, where $L$ is the length of the shortest geodesic path and $m \cdot n$ is the number of grid points.

It was shown that wavefront propagation methods in fluid dynamics also provide a nice approach to the problem of finding the minimal geodesics.

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7 Examples and Results

Let us demonstrate the performance of the algorithm by applying it to a number of synthetic surfaces and finding the paths of minimal length. The synthetic surfaces are given on a mesh of $256 \times 256$ points.

Two simple surfaces are considered. In the first three cases the source and destination areas are points located at $(x, y) = (0.25, 0.25)$ and $(x, y) = (0.75, 0.75)$. In the first example, Figure 4.a, the source and destination are on opposite sides of the “mountain”. Figures 4.b and 4.c show the distance map obtained from the source and destination points, and their sum is presented in Figure 4.d. We search for the minimal level set of that sum, shown as black smooth area in Figure 4.e. Here the gray lines represent the equal distance contour from each point, and the pixel chain line in the middle of the minimal level set is the desired path obtained by a simple thinning algorithm. The thinning algorithm make use of the distance values of the pixels in the interior of the minimal level set. In Figure 4.f the two minimal geodesics are drawn as black lines on the surface.

Another simple example is the “egg-box” surface presented in Figure 5.a. The sum of the distance maps from the source and destination is presented in Figure 5.b. The threshold minimal level set as smooth black lines on the map of equal distance contours from the two points is presented in Figure 5.c. The pixel chain path in the middle of the zero level set is the desired path. Figure 5.e is the black minimal geodesic connecting source and destination on the egg-box.

In the next example two cuts are made in the “egg-box”, see Figure 6.a. The infinite walls in the sum of the distance maps in Figure 6.b represent the impenetrable walls created by the cuts. The equal distance contours from only one point are shown in Figure 6.c, followed by 6.d, where there is a unique path between the two points. Figures 6.e, 6.f and 6.g are views from three angles on the surface and the shortest path.

The last example presents the possibility to find the minimal paths between two initial areas. In Figure 7, a circle and a square areas are located in opposite sides of the mountain. The algorithm finds the minimal paths, two in this symmetric case, which start from different locations on the boundary of the initial areas.

Ways of achieving more accurate results are by increasing the grid resolution, and by decreasing the time step ($\Delta t = 0.21$ in our examples).

8 Concluding Remarks

We have described a numerical method for calculating a distance map from a given area on a graph surface, so that topological problems in the propagated equal distance contours are inherently avoided. An algorithm for finding the geodesics of minimal Euclidean length between two areas on the surface based on the distance mapping was constructed. The algorithm works on a grid, therefore it is easy to implement the algorithm in parallel using each mesh point as a small calculating device which communicates with its four close neighbors. In each iteration we need to calculate the values of $\phi(x, y, t)$ in those grid points close to the current contour and the rest of the grid points serve as sign holders. This can be exploited to reduce calculation effort. When not considering any possible redundancy, the calculation
When more then two points are involved things become more complicated, and similar procedures as those that are used in the 2D case should be followed. A possible procedure that finds the projection of the diagram is the following

\[
\text{Voronoi Projection} = \{(x, y) | \forall i \forall j, i \neq j : \mathcal{M}_i - \mathcal{M}_j = 0 \cap \bigcap_{k \neq i,j} \mathcal{M}_i < \mathcal{M}_k\},
\]

where \(i, j, k \in \{1, 2, \ldots, N\}\) are indexes of the given points on the surface.

Using this procedure the Voronoi diagram between patches on the surface may also be computed.

6 Finding the Path of Minimal Energy

Consider the problem where we look for the planar path from source to destination, so that the accumulation of an energy function along that path is the minimal. This minimization problem may be defined as follows. Let \(s, d \in \mathbb{R}^2\) be the given source and destination points, and \(f(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}\) be a given energy function. Then, find the planar curve \(l^{\text{opt}}\) connecting \(s\) to \(d\) such that

\[
|l^{\text{opt}}| = \min_{l \in \mathcal{L}} \{ \int_s^d f(x, y) dl \},
\]

where \(\mathcal{L}\) is the set of all planar curves \(l \in \mathbb{R}^2\) connecting \(s\) to \(d\).

This problem is similar in many ways to Shape from Shading. In the recent years various solution to this problem were proposed [26, 27, 28, 29, 30, 31]. In [32, 10] we have described a solution to the shape from shading problem via level sets. Motivated by this solution and by the ideas presented in this paper, we shall point out a way to address the above problem.

As a first step propagate an equal energy contour [32] according to the planar curve evolution rule

\[
\frac{\partial C}{\partial t} = \frac{1}{f(x, y)} \hat{n},
\]

where \(\hat{n}\) is the unit normal to the curve. Formulating this evolution in the Eulerian representation yields the level set evolution version

\[
\phi_t = \frac{1}{f(x, y)} \|\nabla \phi\|,
\]

see [10] for the numerical implementation.

Next, we proceed following the same idea as for the minimal length problem, and search for the minimal level of \(h_s(x, y) + h_d(x, y)\), where \(h_s(x, y)\) is the equal accumulated energy contour when starting from \(s\). This may be represented in a more formal way by

\[
l^{\text{opt}} = \{(x, y) | h_s(x, y) + h_d(x, y) = \min_{(x, y)} \{h_s(x, y) + h_d(x, y)\}\}.
\]

The paths of minimal energy obtained in this way may serve as a simple and direct approach for many energy minimization problems.
The desired minimal geodesics are achieved by applying the contour finder (see previous section) on $\mathcal{M}_S + \mathcal{M}_D$ to find the level set $g_m + \epsilon$, for some very small $\epsilon$, and then applying a simple thinning algorithm which operates on the interior of the minimal level set.

### 3.6 Introducing Obstacles

It is possible to set obstacles on the surface by simply setting the multipliers $a_{i,j}, b_{i,j}$ and $c_{i,j}$ to zero at obstacle areas. The equal distance contours will not propagate through such areas that will act as regions surrounded by impenetrable walls.

### 4 Solving the Three Point Steiner Problem

The Steiner problem is defined as follows: given three points (or areas) on a surface, find the graph of minimal length path connecting the three points. On the plane the solution is quite simple and looks like a "Y", connecting the three points with one junction in the "middle". Given three points on a surface, it is easy to show that the solution is of the same type, but now there might be more than one solution, e.g. three points equally placed around a mountain, then there may be three different solutions.

In order to find the solution we first calculate the distance map from each of the given points on the surface, we then find the minimum on the sum of the three maps. There might be more than one point achieving this minimum if there is more than one solution. The point/s will be the junction points of the solution. The last stage involves computing the distance map from the junction, and determining the shortest path from the junction to each of the original regions using the technique described in the previous sections (that is finding the minimal level set of the sum of the distance map from the junction and the distance map from the point). Combining the three paths to the junction/s results in the desired solution. The length of the graph is the minimum of the sum of the three distance maps calculated on the first step.

This is only a simple example of a wide variety of possibilities for using the calculated distance map on the surface.

### 5 Voronoi diagram on surfaces

Another example of the possible use of the distance maps from several points is the following level set based procedure that finds the Voronoi diagram on surfaces. Observe that the surface curve that separates two points on the surface is given as the zero level set of the distance map from the first point, $\mathcal{M}_1$, subtracted from the the distance map form the second point, $\mathcal{M}_2$. The planar projection of that curve is given by $l = \{(x, y) | \mathcal{M}_1(x, y) - \mathcal{M}_2(x, y) = 0\}$. 
3.3 Initialization

The function \( \phi(x, y, 0) \) is actually an implicit representation of \( \partial \tilde{S} \) - the projection of the boundary of the source area \( \partial S \) to the \((x, y)\)-plane. The first demand for \( \phi(x, y, 0) \) is to follow

\[
X(0) \equiv \{(x, y)|\phi(x, y, 0) = 0\} = \{(x, y)|(x, y, z(x, y)) \in \partial S\} = \partial \tilde{S}.
\]

Furthermore \( \phi(x, y, 0) \) should admit smoothness, continuity and be negative in the interior of \( \partial \tilde{S} \), and positive in the exterior of \( \partial \tilde{S} \).

Given \( \partial \tilde{S} \) it is possible to initialize \( \phi \) as follows

\[
\phi(x, y) = \begin{cases} 
+ d((x, y), \partial \tilde{S}) & (x, y) \in \text{exterior of } \partial \tilde{S} \\
- d((x, y), \partial \tilde{S}) & (x, y) \in \text{interior of } \partial \tilde{S} \\
0 & (x, y) \in \partial \tilde{S}
\end{cases}
\]

where \( d(\cdot, \cdot) \) denotes the (minimal) planar Euclidean distance of the point from the \((2D)\) planar zero distance contour \( \partial \tilde{S} \) (the projection of \( \partial S \) on the plane).

There are many ways to initialize \( \phi(x, y, 0) \), for examples see [12, 10, 25]. It is possible, for example, to truncate the values of \( \phi(x, y, 0) \) using the observation that we are interested in the function behavior only near the relevant contour, (the zero level set). Note that every \( \phi \) function which obeys the demands described earlier is sufficient. The Euclidean distance is only an example of such initialization.

3.4 Distance Assignment and Contour Finder

After the initialization is completed, the \( \phi \) function is propagated according to equation (9). While propagating the function, our goal is to find the distance of each grid point. A simple way of achieving (first order) accurate results is by interpolating the zero crossings. At every iteration step, for each grid point, check

If \( (\phi_{i,j}^n \cdot \phi_{i,j}^{n-1} < 0) \) then \( \text{Distance}_{i,j} = \Delta t(n - \frac{\phi_{i,j}^n}{\phi_{i,j}^n - \phi_{i,j}^{n-1}}) \).

Using the above procedure each grid point gets its distance at the “time” when the \( \phi \) functions’-zero level passes through it.

If the equal distance contours of the distance map are needed, a simple contour finder for \( X(t) \) can be generated following [11] in the following manner: Use the cell definition for each grid point \((i, j)\) as \( N_{ij} = \{\phi_{i,j}, \phi_{i+1,j}, \phi_{i,j+1}, \phi_{i,j+1}\} \). Now, if \( \max[N_{ij}] < 0 \) or \( \min[N_{ij}] > 0 \) then the contour \( X(t) \) does not pass through the cell. Otherwise find the entrance and exit points of \( \phi(t) = 0 \) by a linear interpolation; this provides a line segment of \( X(t) \) belonging to the contour. The line segments need neither be ordered nor directed in the same direction in order to display the desired contour.

Calculating \( M_D \) is similar to the above procedures, \( i.e. \) using \( \partial \tilde{D} \) instead of \( \partial \tilde{S} \) in the initialization procedure.

3.5 Finding Minimal Geodesics

Having \( M_S \) and \( M_D \) on the grid, the minimal geodesic may be found in a simple way. Recalling that \( g_m = \min (M_S + M_D) \), the projection of the minimal geodesic, \( G \), on to the
Using this relation in conjunction with the condition equation (2) we obtain

$$\phi_t = -V_N \| \nabla \phi \|, \quad (7)$$

where the curve $C(t)$ is obtained as the zero level set of $\phi$. This procedure is known as the Eulerian formulation [18].

This formulation of planar curve evolution processes frees us from the need to take care of the possible topological changes in the propagating curve, see Figure 3. The numerical implementation of (7) is based on monotone and conservative numerical algorithms, derived from hyperbolic conservation laws and the Hamilton–Jacobi “type” equations of the derivatives, (for details see [6]). For some normal velocities these numerical schemes automatically enforce the entropy condition, a condition equivalent to Huygens principle, see [20].

Using the normal component of the velocity, derived in Appendix A.4 equation (13), in equation (7), we get in our case

$$\phi_t = \sqrt{a(x,y) \phi_x^2 + b(x,y) \phi_y^2 - c(x,y) \phi_x \phi_y} \quad (8)$$

where $a(x,y)$, $b(x,y)$ and $c(x,y)$ are defined in Appendix A.4, equation (12). This equation describes the propagation rule for the surface $\phi$.

### 3.2 Finite Difference Approximation

In our implementation, which is motivated by the relation to the Hamilton–Jacobi type equations, we use the following finite difference approximation (see [6, 13, 23]).

Define the minmod finite derivative as

$$\text{minmod}\{a, b\} = \begin{cases} 
\text{sign}(a) \min(|a|, |b|) & \text{if } ab > 0 \\
0 & \text{otherwise}
\end{cases}$$

We use this definition to approximate $\phi_x \phi_y$ by

$$\phi_x \phi_y \bigg|_{x=i\Delta x, y=j\Delta y} \approx \text{minmod}(D_x^+ \phi_{i,j}, D_x^- \phi_{i,j}) \text{minmod}(D_y^+ \phi_{i,j}, D_y^- \phi_{i,j}).$$

where $D_x^+ \phi_{i,j} \equiv \phi_{i+1,j} - \phi_{i,j}$, $D_x^- \phi_{i,j} \equiv \phi_{i,j} - \phi_{i-1,j}$, $D_y^+ \phi_{i,j} \equiv \phi_{i,j+1} - \phi_{i,j}$ and $D_y^- \phi_{i,j} \equiv \phi_{i,j} - \phi_{i,j-1}$, for $\phi_{i,j} \equiv \phi(i\Delta x, j\Delta y, t)$ and $\Delta x = \Delta y = 1$.

A different approximation, that is also motivated by the hyperbolic conservation laws, is used for the squared partial derivatives, see [13], and is defined as

$$\phi_x^2 \bigg|_{x=i\Delta x, y=j\Delta y} \approx (\max(D_x^+ \phi_{i,j}, -D_x^- \phi_{i,j}, 0))^2$$
$$\phi_y^2 \bigg|_{x=i\Delta x, y=j\Delta y} \approx (\max(D_y^+ \phi_{i,j}, -D_y^- \phi_{i,j}, 0))^2$$

These finite difference approximations yield a first order numerical scheme for the equal distance contours evolution. Using a forward difference approximation in time gives the following numerical scheme for the propagation of the function $\phi_{i,j}^n = \phi(i\Delta x, j\Delta y, n\Delta t)$ on the $(x, y)$ rectangular grid

$$\phi_{i,j}^{n+1} = \phi_{i,j}^n + \Delta t a_{i,j} (\max(D_x^+ \phi_{i,j}, -D_x^- \phi_{i,j}, 0))^2 + b_{i,j} (\max(D_y^+ \phi_{i,j}, -D_y^- \phi_{i,j}, 0))^2$$
$$- c_{i,j} \text{minmod}(D_x^+ \phi_{i,j}, D_x^- \phi_{i,j}) \text{minmod}(D_y^+ \phi_{i,j}, D_y^- \phi_{i,j})$$
$$\quad (9)$$

which is the finite difference approximation of equation (8). This numerical scheme is stable and inherently overcomes topological changes in the evolving contour. For higher order accuracy numerical schemes that deal with such Hamilton Jacobi type of equations see [24].

7
3 The Numerical Approximation

When implementing curve evolution equations such as (2) on a digital computer, a number of problems must be solved.

- Topological changes - Topological changes may occur while the curve evolves, i.e. the curve may change its topology from one connected curve to two separate evolving curves, or, two curves may merge into one, see Figure 2.b.

- Stability and accuracy - In [18, 6] some numerical problems which characterize a direct formulation of (2) are described. The problems are caused due to a time varying coordinate system \((u, t)\) of the direct representation (where \(u\) is the parameterization, and \(t\) - the time).

- Singularities - Even an initial smooth curve can develop curvature singularities (see Figure 2.a). The question is how to continue the evolution after singularities appear. The natural way is to choose the solution which agrees with the Huygens principle [20, 21]. Viewing the curve as the front of a burning flame, this solution states that once a particle is burnt, it cannot be re-ignited [18]. It can also be proved that from all the weak solutions of (2) part the singularities, the one derived from the Huygens principle is unique, and can be obtained by a constraint denoted as the entropy condition [6].

Sethian and Osher [18, 6] proposed an algorithm for curve and surface evolution that elegantly solves these problems. As a first step in constructing the algorithm, the curve is embedded in a higher dimensional function. Then, evolution equations for the implicit representation of the curve are solved using numerical techniques derived from hyperbolic conservation laws [22].

3.1 The Eulerian Formulation

Let the curve \(C(t)\) be represented by the zero level set of a smooth Lipschitz continuous function \(\phi : \mathbb{R}^2 \times [0, T) \rightarrow \mathbb{R}\), so that \(\phi\) is negative in the interior and positive in the exterior of the zero level set \(\phi = 0\). Consider the zero level set defined by

\[
\{ X(t) \in \mathbb{R}^2 : \phi(X, t) = 0 \} \tag{4}
\]

We have to find the evolution rule of \(\phi\), so that the evolving curve \(C(t)\) can be represented by the evolving zero level set \(X(t)\), i.e.,

\[
C(t) \equiv X(t). \tag{5}
\]

Using the chain rule on \(\phi(X(t), t) = 0\) we get

\[
\nabla \phi(X, t) \cdot X_t + \phi_t(X, t) = 0.
\]

Note that for any level set the planar normal can be written as

\[
\mathbf{n} = \frac{\nabla \phi}{\|\nabla \phi\|}. \tag{6}
\]
2.2 Finding The Minimal Path

The procedure that calculates the equal distance contours allows us to build a Euclidean distance map on the surface, from a given area. Assuming we have reliable distance map procedure in hand, we can construct a simple procedure that finds the minimal path from a source area $S$ to a destination area $D$ (where $S, D \in Z$).

Define $\mathcal{M}_A$ as the distance map of area $A$ as

$$
\mathcal{M}_A(x, y) = d_z((x, y, z(x, y)), A).
$$

we readily have the following result

**Lemma 3** All minimal paths between $S$ and $D$ on $Z$ are given by the set $G \subset Z$,

$$
G = \{(x, y, z(x, y))|\mathcal{M}_S(x, y) + \mathcal{M}_D(x, y) = g_m\}
$$

where $g_m = \min_{x, y}(\mathcal{M}_S + \mathcal{M}_D)$ is the global minimum of the sum of the source and destination distance maps.

**Proof.**

- **[p_a \in G \Rightarrow p_a \in set of minimal paths]** If the point $p_a$ is in $G$ then $d_z(p_a, S) + d_z(p_a, D) = g_m$. Therefore, there exist a path from $S$ to $p_a$ and from $p_a$ to $D$ which together form a minimal length path that passes through $p_a$.

- **[p_a \notin G \Rightarrow p_a \notin any minimal path]** If $p_a \notin G$ then $d_z(p_a, S) + d_z(p_a, D) > g_m$. Recalling the $d_z$ definition, all possible paths from $S$ to $D$ which pass through the point $p_a$ are longer than $g_m$ and, therefore, not minimal.

Q.E.D.

Now, we can prove the following result connecting geodesics to the trace of tangential points of the two equal distance contours $\alpha_s$ and $\alpha_d$, propagating from the source and destination.

**Lemma 4** The tangential points of $\alpha_s(u, t)$ and $\alpha_d(\bar{u}, \bar{t})$ for $\bar{t} + t = g_m$, generate the minimal paths from point $P_1$ to point $P_2$; i.e lies on a constant parameter $u = u_0$ ($\bar{u} = \bar{u}_0$) of the propagating curve $\alpha_s(u, t)$ ($\alpha_d(\bar{u}, \bar{t})$).

The proof of the above, appears in Appendix A.3. We also have the following result.

**Corollary 1** All minimal paths between $S$ and $D$ which defined by $G$ (equation (3)), are minimal geodesics.

In the next section a numerical scheme based on the level set representation of the evolving planar curve is presented. Note that the shortest paths are minimal value level sets of the function $\mathcal{M}_S + \mathcal{M}_D$. This observation will later be used on to find the minimal paths.
where $\tilde{t}^\alpha$ is the tangent unit vector to $\alpha$, and $N$ is the surface normal.

The traces of constant parameter along the curve evolving according to (1) are geodesics, and these geodesics are locally shortest paths. We have the following results:

**Lemma 1** Define the curve $\beta(t) = \alpha(u, t)|_{u = u_0}$. Then, for any $u_0$, the curve $\beta(t)$ is a geodesic.

See Appendix A.1 for the proof.

Let $\alpha(u, t)$ be a 3D curve propagating on the surface $Z \subset \mathbb{R}^3$, where $u$ is the parameter and $t$ is the propagation time. Then

**Lemma 2** The equal distance contour evolution is given by

$$\alpha_t = N \times \tilde{t}^\alpha \quad \text{given} \quad \alpha(0),$$

where $N$ is the surface normal and $\tilde{t}^\alpha$ is the tangent to the contour $\alpha$.

This lemma, proven in Appendix A.2, provides the evolution equation of the equal distance contour. Starting from the boundary of the source area

$$\alpha(0) = \{(x, y, z(x, y))|(x, y, z(x, y)) \in \partial S\},$$

it is possible to find the equal distance contour for any desired distance $d$, by using the evolution equation to calculate $\alpha(u, t)|_{u = d}$. This propagation may be used to build the distance map for each point on the surface.

Implementing the three dimensional curve evolution is quite a complicated task. We are therefore interested in considering the projection of the 3D curve on the $(x, y)$-plane,

$$C(t) = \{(x, y)|(x, y, z(x, y)) \in \alpha(t)\}.$$

Recall that the trace of the propagating planar curve may be determined only by its normal velocity. Let us consider the projection of the above evolution on the $(x, y)$-plane, see Figure 1. The knowledge of how this projected contour behaves allows us to construct a simple, accurate and stable numerical algorithm that can be used to produce these equal distance contours.

In the appendix we calculate the planar normal component of the projected velocity of the evolving equal distance contour, $V_N$. Using this velocity we construct a differential equation describing the projected equal distance contour evolution of the form

$$\frac{\partial}{\partial t} C = V_N \bar{n},$$

given

$$C(0) = \{(x, y)|(x, y, z(x, y)) \in \partial S\} = \partial \tilde{S},$$

where $\bar{n}$ is the planar normal direction, and $V_N$ depends on the surface gradient ($p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$) and $\bar{n}$.

It is possible to construct a direct numerical scheme for the above curve evolution equation, but implementing this direct formulation involves some difficulties which are discussed in detail in section 3.
geodesic circle map, or the map of equal distance contours on the surface.

In the next Section an analytic model for the equal distance contour evolution is discussed. In Section 3, a numerical implementation of the analytic propagation is presented. It is based on ideas of Osher and Sethian \([6, 18]\). Sections 4 and 6 present the solutions of two related minimization problems. The results of the numerical algorithm are demonstrated in several examples, see Section 7. We conclude with a discussion of some possible extensions of the algorithm and comment on its complexity in Section 8.

2 The Analytic Model

In this section we determine a differential equation describing the propagation of equal distance contours on a smooth surface, from a point or a source region on the surface. This equation describes the evolution of a planar curve, a curve that is the projection of the three dimensional equal distance contour (a 3D curve) on the \((x, y)\)-plane.

We first present a result from the general theory of planar curve evolution that will be used in deriving the equal distance contour evolution equation. Let \(C(u, t): S^1 \times [0, T) \to \mathbb{R}^2\) be a family of embedded closed planar curves, where \(u\) parameterizes each curve, and \(t\) denotes time. In our model time reflects distance, the propagation speed being constant. Assume that the curve evolution is described by

\[
\frac{\partial}{\partial t} C = V_T \tilde{t} + V_N \tilde{n},
\]

given the initial curve \(C(u, 0)\)

where \(\tilde{t}\) is the unit tangent, \(\tilde{n}\) is the unit normal vector, \(V_T\) and \(V_N\) are the tangent and normal components of the velocity \(\tilde{V}\) respectively. If the velocity \(\tilde{V}\) is smooth along the curve, it can be proved that the tangential component \(V_T\) does not affect the shape of the evolving curve \([19]\) since only the curve parameterization is affected by the tangential component. Being only interested in the shape of the curve, we shall consider the equations of the type

\[
\frac{\partial}{\partial t} C = V_N \tilde{n},
\]

given \(C(u, 0)\),

where \(V_N = \tilde{V} \cdot \tilde{n}\) is the projection of the velocity \(\tilde{V}\) on \(\tilde{n}\).

2.1 Equal Distance Contours

Let us first define the equal distance contour. Given a source area \(S \in \mathbb{R}^3\) (\(S\) is not necessarily connected), on a graph surface \(Z \in \mathbb{R}^3\) (a graph surface is described by a function \(z(x, y) : \mathbb{R}^2 \to \mathbb{R}\)), let the 3D equal distance contour of distance \(t\) from \(S\) be defined as

\[
\{p \in Z | d_z(p, S) = t\} = \alpha(\ast, t),
\]

where \(d_z(p, S)\) is the minimal Euclidean distance determined by the the shortest paths from a point \(p\) to an area \(S\) on surface \(Z\).

We shall prove that the 3D parametric representations of \(\alpha(\ast, t)\), on \(Z\), can be obtained by the equal distance contour propagation

\[
\alpha_t = N \times \tilde{t}^5, \quad \text{given} \quad \alpha(u, 0) = \alpha(u),
\]

(1)
1 Introduction

Finding paths of minimal length between two areas on a three dimensional surface is of great importance in many fields such as computer aided neuroanatomy, robotic motion planning (autonomous vehicle navigation), geophysics, terrain navigation, etc. Paths of minimal Euclidean distance between two points on a graph-surface will be referred to as geodesics\(^1\) in this paper.

A variational approach to finding a geodesic path on a given surface was presented by Beck et al. in [1]. In other cases it is natural to approximate a surface with planar polygonal patches. Some algorithms that solve this discrete geodesic problem were presented recently [2, 3]. Schwartz et al. used such an algorithm as a preliminary step in solving the mapmaker problem, which is the problem of the gradient-descent surface flattening of a polyhedral surface [4]. Kiryati and Székely [5] considered voxel representations of three-dimensional surfaces and used 3D-length estimators to derive an efficient algorithm for the minimal distance geodesic problem.

In this paper we introduce a different way of dealing with the problem of finding the minimal distance paths. The surface is given as height samples on a rectangular grid. The technique we developed operates only on graph surfaces \(\{x, y, z(x, y)\}\), however, the height resolution is not limited to the traditional \(z\)-voxel resolution. This fact makes it applicable in all sorts of navigation problems. As a first step, a distance map from the source area is calculated. The distance map is computed via equal distance curve propagation on the surface. Equal distance curves are calculated as the zero sets of a bivariate function evolving in time. This formulation of curve evolution processes is due to Osher and Sethian, [6]. It overcomes some topological and numerical problems encountered in direct implementations of curve evolutions using parametric representations. The implicit representation of the evolving curve produces a stable and accurate numerical scheme for tracing shock waves in fluid dynamics. This formulation has subsequently found applications in doing planar shape analysis in computer vision [7, 8, 9], in solving the shape from shading problem in computer vision [10], in simulating crystal growth [11], offsetting shapes in computer aided design [12], in implementing continuous-scale morphology operations on a digitized picture [13], and more [14, 15].

We propose a numerical scheme that is consistent with the continuous propagation rule. The consistency condition guarantees that the solution converges to the true one as the grid is refined and the time step in the numerical scheme is kept in the right proportion to the grid size. This is known not to be the case in general graph search algorithms that suffer from digitization bias due to the metrification error when implemented on a grid, see [16].

The relation between minimal paths, geodesics and equal distance contours may be found in elementary differential geometry textbooks, e.g. [17]. Geodesics are locally shortest paths in the sense that any perturbation of a geodesic curve will increase its length. The minimal length paths between two points are geodesics connecting those points. A simple way of determining minimal geodesics is by constructing a so-called geodesic polar coordinate system on the surface around the source area. Using such a coordinate system readily provides the

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\(^1\)By definition a geodesic line is not necessarily the shortest path, but the shortest path is always a geodesic line.
Finding Shortest Paths on Surfaces

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Abstract

This paper presents a new algorithm for determining minimal length paths between two points or regions on a three dimensional surface. The numerical implementation is based on finding equal distance contours from a given point or area. These contours are calculated as zero sets of a bivariate function designed to evolve so as to track the equal distance curves on the given surface. The algorithm produces all minimal length paths between the source and the destination areas on the surface given as height values on a rectangular grid. Complexity and accuracy are governed by the grid resolution and the distance step size in the iterative scheme. Using the distance maps from three areas we also solve the Steiners' problem (for three points) and Voronoi diagram on a surface.

Key Words: Curve evolution, Equal distance contours, Geodesic path, Partial differential equations, Digital implementation, Numerical algorithms.

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