A Note on the Determination of the Zeros of Certain Polynomials

by

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1 Introduction

Let the polynomial \( P(\lambda) \) be given by

\[
P(\lambda) = \begin{vmatrix}
1 & \lambda & \lambda^2 & \ldots & \lambda^k \\
a_{10} & a_{11} & a_{12} & \ldots & a_{1k} \\
a_{20} & a_{21} & a_{22} & \ldots & a_{2k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{k0} & a_{k1} & a_{k2} & \ldots & a_{kk}
\end{vmatrix},
\]

(1.1)

where \( a_{ij} \) are known constants that may be complex in general. We would like to determine the zeros of \( P(\lambda) \) when the only thing known about \( P(\lambda) \) is the set of the \( a_{ij} \).

Polynomials of this kind arise in different problems. For example, the denominator of the \((m/k)\) Padé approximant associated with the formal power series \( \sum_{i=0}^{\infty} c_i z^i \) is known to be, see [Ba],

\[
Q(z) = \begin{vmatrix}
z^k & z^{k-1} & \ldots & z^0 \\
c_m & c_{m+1} & \ldots & c_{m+k} \\
c_{m+1} & c_{m+2} & \ldots & c_{m+k+1} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m+k-1} & c_{m+k} & \ldots & c_{m+2k-1}
\end{vmatrix},
\]

(1.2)

which, upon making the substitution \( z = \lambda^{-1} \), becomes \( \lambda^{-k} P(\lambda) \), where \( P(\lambda) \) is of the form (1.1) with \( a_{ij} = c_{m+i+j-1} \).

Provided \( P(\lambda) \neq 0 \), we can express \( P(\lambda) \) as

\[
P(\lambda) = \alpha \sum_{j=0}^{k} c_j \lambda^j, \quad \text{some } \alpha \neq 0,
\]

(1.3)

where the \( c_j \) are the solution of the homogeneous linear system

\[
\sum_{j=0}^{k} a_{ij} c_j = 0, \quad 1 \leq i \leq k.
\]

(1.4)

Provided the cofactor of \( \lambda^k \) in (1.1) is nonzero, we have \( c_k \neq 0 \). Thus we can pick \( c_k = 1 \) and solve the \( k \) equations in (1.4) for \( c_0, c_1, \ldots, c_{k-1} \) in a unique fashion. For this see, e.g., [SBr].

In view of (1.3) and (1.4) the most direct way of determining the zeros of \( P(\lambda) \) is by computing the coefficients \( c_j \) from (1.4) and then by solving the polynomial equation \( \sum_{j=0}^{k} c_j \lambda^j = 0 \). The success of this approach depends on the accuracy of the computed \( c_j \) and on the conditioning of
the polynomial \( \sum_{j=0}^{k} c_j \lambda^j \). Because of this one may conclude that this way of determining the zeros of \( P(\lambda) \) may lead to serious errors in some cases, although it may produce satisfactory results in others.

It is the purpose of this note to propose an alternative and possibly more convenient approach to the problem of determination of the zeros of \( P(\lambda) \).

## 2 Generalized Eigenproblem Formulation

**Theorem 2.1:** Let \( P(\lambda) \) be given as in (1.1), and define the matrices \( A \) and \( B \) by

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1k} \\
a_{21} & a_{22} & \cdots & a_{2k} \\
\vdots & \vdots & & \vdots \\
a_{k1} & a_{k2} & \cdots & a_{kk}
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
a_{10} & a_{11} & \cdots & a_{1,k-1} \\
a_{20} & a_{21} & \cdots & a_{2,k-1} \\
\vdots & \vdots & & \vdots \\
a_{k0} & a_{k1} & \cdots & a_{k,k-1}
\end{bmatrix}.
\tag{2.1}
\]

Then

\[
P(\lambda) = \det(A - \lambda B),
\tag{2.2}
\]

which implies that the zeros of \( P(\lambda) \) are the eigenvalues of the matrix pencil \((A, B)\), i.e., the roots of \( \det(A - \lambda B) = 0 \).

**Proof:** Let us multiply the \( k \)th column of the determinant in (1.1) by \( \lambda \) and subtract from the \((k+1)\)st column. The new \((k+1)\)st column is

\[
(0, a_{1,k} - \lambda a_{1,k-1}, a_{2,k} - \lambda a_{2,k-1}, \ldots, a_{k,k} - \lambda a_{k,k-1})^T,
\tag{2.3}
\]

and the rest of the columns are as before. Continuing, we multiply the \((j-1)\)st column by \( \lambda \) and subtract from the \( j \)th column, in the order \( j = k, k-1, \ldots, 2 \). As a result of this sequence of elementary column transformations we have

\[
P(\lambda) \approx \begin{vmatrix}
1 & 0 \cdots 0 \\
\vdots & \ddots & \ddots & \ddots \\
a_{10} & a_{20} & \ddots & a_{k0} \\
a_{11} & a_{21} & \cdots & a_{k1} \\
a_{12} & a_{22} & \cdots & a_{k2} \\
\vdots & \vdots & & \vdots \\
a_{1,k-1} & a_{2,k-1} & \cdots & a_{k,k-1}
\end{vmatrix}.
\tag{2.4}
\]
On expanding the determinant in (2.4) with respect to its first row, we obtain (2.2).

From (2.2) we see that, provided we have a good procedure for solving the generalized eigenvalue problem, the approach given in Theorem 2.1 may be more stable numerically in some cases. It also requires less involvement on the part of the user than the polynomial equation approach.

Note that when \( a_{ij} = c_{m+i+j-1} \), cf. second paragraph of Section 1, the matrices \( A \) and \( B \) are given by

\[
A = H^{(m+1)}_k \quad \text{and} \quad B = H^{(m)}_k,
\]

where \( H^{(r)}_s \) stands for the Hankel determinant

\[
H^{(r)}_s = \begin{vmatrix}
  c_r & c_{r+1} & \cdots & c_{r+s-1} \\
  c_{r+1} & c_{r+2} & \cdots & c_{r+s} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{r+s-1} & c_{r+s} & \cdots & c_{r+2s-2}
\end{vmatrix}.
\]

In this case both \( A \) and \( B \) are symmetric. This actually follows from Nuttal's compact formula for Padé approximants, see, e.g., [Ba]. It was this observation that provided the motivation for this note to some extent.

\section{A Further Development}

Let us now assume that the \( a_{ij} \) in (1.1) are given as

\[
a_{ij} = (u_{i-1}, u_j), \quad i \geq 1, \quad j \geq 0,
\]

where \( u_j \) are vectors in an inner product space and \( (\cdot, \cdot) \) is the inner product on this space. The homogeneity property of this inner product is such that \( (\alpha x, \beta y) = \overline{\alpha} \beta (x, y) \) for \( x \) and \( y \) vectors and \( \alpha \) and \( \beta \) scalars.

Note that when the vectors \( u_0, u_1, \ldots, u_{k-1} \) form a linearly dependent set, some rows of \( P(\lambda) \) in (1.1) are linear combinations of others, which causes \( P(\lambda) \equiv 0 \). We shall, therefore, assume throughout that \( u_0, u_1, \ldots, u_{k-1} \) form a linearly independent set. In addition, when \( \{u_0, u_1, \ldots, u_{k-1}\} \) is linearly independent, \( P(\lambda) \) is of degree at most \( k - 1 \) if \( \{u_1, u_2, \ldots, u_k\} \) is linearly dependent.

For simplicity of presentation we shall assume that the inner product space is \( \mathbb{C}^N \) and the inner product is arbitrary. Recall that any inner product in \( \mathbb{C}^N \) is ultimately of the form

\[
(x, y) = x^* M y, \quad M \text{ hermitian positive definite}.
\]
Theorem 3.1: Define the $N \times k$ matrices $U_j^{(n)}$ by
\[
U_j^{(n)} = [u_n | u_{n+1} | \cdots | u_{n+j}].
\] (3.3)

Then
\[
P(\lambda) = \det(U_{k-1}^{(0)*}MU_{k-1}^{(1)} - \lambda U_{k-1}^{(0)*}MU_{k-1}^{(0)}),
\] (3.4)

where $M$ is the matrix associated with the inner product through (3.2).

Proof: The truth of (3.4) can be shown by verifying that the matrices $A$ and $B$ of Theorem 2.1 are now $U_{k-1}^{(0)*}MU_{k-1}^{(1)}$ and $U_{k-1}^{(0)*}MU_{k-1}^{(0)}$, respectively. $\Box$

We would now like to employ Theorem 3.1 to devise a new method for determining the zeros of $P(\lambda)$ under the assumption that only the vectors $u_0, u_1, \ldots, u_k$ are known.

Lemma 3.2: There exists a unique $N \times (j + 1)$ matrix $Q_j$ and a unique $(j + 1) \times (j + 1)$ upper triangular matrix $R_j$ given by
\[
Q_j = [q_0 | q_1 | \cdots | q_j] \text{ such that } (q_i, q_j) = q_i^* M q_j = \delta_{ij}
\] (3.5)

and
\[
R_j = \begin{bmatrix}
    r_{00} & r_{01} & \cdots & r_{0j} \\
    & r_{11} & \cdots & r_{1j} \\
    & & \ddots & \vdots \\
    & & & r_{jj}
\end{bmatrix},
\] (3.6)

such that
\[
U_j^{(0)} = Q_j R_j.
\] (3.7)

Proof: The developments in (3.5)-(3.6) are nothing but a weighted QR factorization of $U_j^{(0)}$ or, equivalently, the Gram-Schmidt orthogonalization with respect to the inner product in (3.2). We leave the details to the reader. $\Box$

Theorem 3.3: Define the $j \times j$ upper Hessenberg matrix $\tilde{R}_j$ by
\[
\tilde{R}_{j-1} = \begin{bmatrix}
    r_{01} & \cdots & r_{0,j-1} & r_{0j} \\
    r_{11} & \cdots & r_{1,j-1} & r_{1j} \\
    \vdots & \vdots & \vdots & \vdots \\
    r_{j-1,j-1} & \cdots & r_{j-1,j} & r_{jj}
\end{bmatrix}.
\] (3.8)
Then

\[ P(\lambda) = \det[R_{k-1}^* (\tilde{R}_{k-1} - \lambda R_{k-1})], \quad (3.9) \]

from which there follows

\[ P(\lambda) = 0 \iff \det(\tilde{R}_{k-1} - \lambda R_{k-1}) = 0. \quad (3.10) \]

**Proof:** We start by observing that

\[ U_j^{(1)} = Q_{j-1} \tilde{R}_{j-1} + r_j q_j e_j^T, \quad (3.11) \]

where \( e_j^T = (0, 0, \ldots, 0, 1) \) is the \( j \)th fundamental basis vector in \( \mathbb{R}^j \). Then, by the fact that \( Q_{j-1}^* M Q_{j-1} = I_{j \times j} \), the \( j \times j \) identity matrix, and by \( Q_{j-1}^* M q_j = 0 \), we have

\[ U_{k-1}^{(0)*} M U_{k-1}^{(1)} = R_{k-1}^* \tilde{R}_{k-1}. \quad (3.12) \]

Similarly,

\[ U_{k-1}^{(0)*} U_{k-1}^{(0)} = R_{k-1}^* R_{k-1}. \quad (3.13) \]

From (3.12) and (3.13) and Theorem 3.1, (3.9) follows. When \( U_{k-1}^{(0)} \) has full rank, \( \det R_{k-1}^* \neq 0 \) by Lemma 3.2. We can now factor out \( \det R_{k-1}^* \) from (3.9), and this leads to (3.10).

The approach implied by Theorem 3.3 then is as follows: Determine the \( QR \) factorization of the matrix \( U_k^{(0)} \). Form the upper triangular matrix \( R_{k-1} \) and the upper Hessenberg matrix \( \tilde{R}_{k-1} \), and solve the generalized eigenvalue problem in (3.10). Obviously, the success of this approach will, first of all, depend on how accurately we have determined \( R_k \). Therefore, the \( QR \) factorization must be performed with great care. In the common case of \( M = I \), i.e., the standard Euclidean inner product \( (x, y) = x^* y \), a very appropriate way of performing the \( QR \) factorization may be by Householder transformations, see, e.g., [GV].

If the coefficients of \( P(\lambda) \) are needed, then they can be obtained by solving the upper triangular system

\[ R_{k-1} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{bmatrix} = \begin{bmatrix} r_{0k} \\ r_{1k} \\ \vdots \\ r_{k-1,k} \end{bmatrix}, \quad (3.14) \]

which is equivalent to solving the linear least squares problem

\[ \| \sum_{j=0}^{k-1} e_j u_j + u_k \| = \text{minimum}, \quad (3.15) \]
for $c_0, c_1, ..., c_{k-1}$, where we have set $c_k = 1$. Here $|| \cdot ||$ is the vector norm induced by the inner product of (3.2). For these developments see [S].

References


