2-Trees Optimal T-Join and Integral Packing of T-Cuts

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Abstract

Let G be an undirected graph, T an even subset of vertices and F an optimal T-join, which is a forest of 2 trees. The main theorem of this paper characterizes the cases, where \((G,T)\) has an optimal packing of T-cuts which is integral. This theorem unifies and generalizes a theorem of P. Seymour on packing of T-cuts and a theorem of A. Frank on planar edge disjoint paths. It also solves positively a conjecture by A. Frank. The proof of the main theorem implies a polynomial algorithm for optimal integral packing of T-cuts for the case where the optimal T-join consists of 2 trees. This algorithm is in fact a simple post optimality method, that can be applied to existing algorithms for \(\frac{1}{2}\) integral packing of T-cuts and also solves polynomially a certain planar integral multicommodity flow problem.

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1. Introduction.

Let $G = (V,E)$ be an undirected graph, and let $T \subseteq V$ such that $|T|$ is even. A $T$-cut is a cut $\delta(S)$, $S \subseteq V$, such that $|T \cap S|$ is odd. $F \subseteq E$ is a $T$-join if $F$ is minimal such that $|F \cap \delta(v)|$ is odd if and only if $v \in T$. Let $A$ be the matrix of all $T$-cuts in $(G,T)$ and let $y$ be a real vector indexed on the collection of $T$-cut in $G$ (the rows of the matrix $A$). Let $w$ be a non-negative real vector indexed on $E(G)$ and let $1$ be an appropriate vector of all ones. A minimum $T$-join is a solution to the following integer linear programming problem (ILP):

$$\min \{wx : Ax \geq 1, x \geq 0, x \text{ integral} \}$$

Edmonds and Johnson [EJ73] proved that the basic solutions of the following linear programming (LP) relaxation of (1.1) are minimum $T$-joins.

$$\min \{wx : Ax \geq 1, x \geq 0 \}$$

A packing of $T$-cuts is a feasible solution to the following dual (LP) of (1.2):

$$\max \{y^T A \geq w, y \geq 0 \}$$

An optimal solution $y$ is called an optimal packing of $T$-cuts and if $y$ is also integral, it is called an optimal packing of $T$-cuts which is integral. Lovasz [L73] proved that (1.3) has always optimal $y$ which is $\frac{1}{2}$-integral (a result that is also implicit in [EJ70]). Seymour [S81b] strengthened this result by proving that for every integral $w$ such that the weight of every circuit is even, (this includes unweighted bipartite graphs,) there is an optimal packing of $T$-cuts which is integral. Middendorf and Pfieffer [MP89] proved NP-hardness of the integral packing of $T$-cuts problem, which can be stated as the following ILP restriction of (1.3):

$$\max \{y^T A \leq w, y \geq 0, y \text{ integral} \}$$

Therefore, it is probably very hard to find an optimum integral packing of $T$-cuts, and to find a good characterization of instances $(G,T,w)$ for which there exists an optimum packing of $T$-cuts
which is integral. Hence, much effort has been devoted to finding solutions to special cases: Seymour [577] characterized pairs \((G,T)\) for which there is an optimal packing of \(T\)-cuts which is integral for every integral \(w\) and algorithmic proof of this case is given in [K82]. Sebo [Se88] gave a polynomial algorithm for optimum packing of \(T\)-cuts for instances where the cardinality of the optimum \(T\)-join is bounded by a constant. In [KP85] a tight bound on the gap between an optimum solution to (1.3) and an optimum one to (1.4) is given, together with a polynomial heuristic algorithm that achieves this bound. Seymour [S81b] has shown that integral packing of \(T\)-cuts has an important application to a planar integral multicommodity flows problem. Other results can be found in the literature, see for example: [B87], [FST84], [K82], [Se90] and [S81a].

In this paper we consider the following case. Assume that \((G,T)\) has an optimal \(T\)-join \(F\), which is a forest of 2 trees. The main theorem of this paper characterizes the cases, where \((G,T)\) has an optimal packing of \(T\)-cuts which is integral. This theorem unifies and generalizes a theorem of P. Seymour on packing of \(T\)-cuts and a theorem of A. Frank on planar edge disjoint paths given below. It also solves positively a conjecture by A. Frank. The proof of the main theorem implies a polynomial algorithm for optimal integral packing of \(T\)-cuts for the case where the optimal \(T\)-join consists of 2 trees. This algorithm is in fact a simple post optimality method, that can be applied to any existing algorithm for \(\frac{1}{2}\) integral packing of \(T\)-cuts (e.g. [B87], [K82]). It also solves polynomially a certain planar integral multicommodity flow problem.

2. Definitions.

In this section we introduce some definitions that will be of use later. Standard graph-theoretical terminology and notation follows [BM76]. Let \(F\) be a \(T\)-join in \((G,T)\). Then: the \textit{length of a path} \(P\) relative to \(F\) is \(|P\setminus F| - |P \cap F|\) and the \textit{length of a circuit} \(C\) relative to \(F\) is \(|C\setminus F| - |C \cap F|\). A \textit{zero-circuit} is a circuit \(C\) with length = zero.
For $S \subseteq V$ such that $\delta(S) \cap F = \{f\}$ and $e \in \delta(S) \setminus F$ we say that the distance between $e$ and $f$ through $S$ is zero if for every simple path $P = (e, e_1, e_2, \ldots, e_k, f)$ such that the ends of every $e_i$, $1 \leq i \leq k$ are in $S$, its length is non-negative and there is at least one such path with length zero. A set $S$ is a level set relative to $F$ if $\delta(S) \cap F = \{f\}$ and the distance between $f$ and any $e \in \delta(S) \setminus F$ through $S$ is zero.

Let $(s_1, t_1)(s_2, t_2)\ldots (s_k, t_k)$ be $k$ pairs of nodes (called terminals) in $G$. The edge disjoint path problem is to find $k$ pairwise edge-disjoint paths $P_1, \ldots, P_k$ such that $P_i$ connects $s_i$ and $t_i$. If we connect each pair of terminals $(s_i, t_i)$ by an edge $f_i$ called a demand edge and $F$ is the set of all the demand edges then $H = (V, F)$ is called the demand graph.

Let $C$ be a collection of $T$-cuts and let $y$ be a solution to (1.3) then the pair $(C, y)$ is a packing of $T$-cuts if $C = \{ C_i : y(C_i) > 0 \}$.

Let $S \subseteq 2^V$ be a collection of subsets of vertices such that $\forall S \in S$, $\delta(S)$ is a $T$-cut. $\delta(S) = \{ \delta(S) : S \in S \}$, is a collection $C$ of $T$-cuts. We say that $S$ represents the collection $C$.

Denote $\cup S = \{ \cup S : S \in S \}$ and $\lambda(e, S) = \sum \{ y(S) : S \in S, e \in \delta(S) \}$.

If $(C, y)$ is a packing of $T$-cuts and $S$ represents $C$ then we say that a set $S_i \in S$ has value $y_i$ and we mean that the $T$-cut $c_i = \delta(S_i)$ has $y(c_i) = y_i$.

Let $V$ be a set and $A, B \subseteq V$. We say that $A$ and $B$ are laminar if $A \cap B = \emptyset$ or $A \subseteq B$ or $B \subseteq A$. Let $E' \subseteq E$, $V(E')$ denotes the set of end vertices of the edges in $E'$.

Let $S \subseteq 2^V$ be a collection of subsets of vertices. We say that $S$ is a laminar family of subsets of $V$ if all the members of $S$ are pairwise laminar.

Let $y$ be a packing of $T$-cuts and let $S$ be a collection of subsets of $V$. We say that $S$ represents $y$ if $S = \{ S_i : y(\delta(S_i)) > 0 \}$ and for each $T$-cut $c_i$ with $y(c_i) > 0$, there exists exactly one set $S_i \in S$ such that $c_i = \delta(S_i)$. Let $(y, S)$ be an optimal packing of $T$-cuts and let $(y', S')$ be another optimal packing of $T$-cuts. We say that $(y', S')$ is a condensed version of $(y, S)$ if the following conditions...
hold:

(i) There is a mapping \( h : S \rightarrow S' \) such that \( \forall S \in S, h(S) \subseteq S' \).

(ii) \( \forall S' \in S', y(S') = \Sigma \{ y(S) : S \in S, h(S) = S' \} \)

(iii) \( \forall e \in \delta(\cup S), y(e, S') \leq y(e, S) \).

We say that \( (y, S) \) can be condensed to \( (y', S') \).

Seymour [S81b] proved that for every \((G, T)\) there exists an optimal packing of \( T \)-cuts \( y \) which is \( \frac{3}{4} \) integral and is represented by a laminar \( S \).

Without loss of generality, we may assume that the graph \( G \) is connected.

3. The Main Result.

In this section we present the main theorem and show that this theorem generalizes a theorem of Seymour on packing of \( T \)-cuts where \( |T| = 4 \) and a theorem of Frank on disjoint paths in a planar graph where all the demand edges are across 2 faces. In addition, we also show that the main theorem solves positively a conjecture of Frank.

Theorem 1 (The Main Theorem): Let \( G = (V, E) \) be an undirected graph, \( T \subseteq V \), \( |T| \) even and \( F \subseteq E \) be an optimal T-join which is a forest with two trees. Then the following are equivalent:

(i) There is an optimal packing of \( T \)-cuts which is integral.

(ii) The union of every two zero-circuits is a bipartite graph.

(iii) There is no optimal packing of \( T \)-cuts \( y \), represented by a laminar collection of level sets \( S \) with the following properties:

(a) \( S \) has exactly 4 maximal level sets \( S_1, S_2, S_3, S_4 \) with \( y \)-value \( \frac{3}{4} \)

(b) all other level sets in \( S \) are with \( y \)-value 1

(c) there are six edges: \( f_1, f_2 \in F, e_1, e_2, e_3, e_4 \in E \setminus F \) such that
Corollary 2: (Seymour's theorem [S81b]) Let \( T = \{t_1, t_2, t_3, t_4\} \subseteq V \). \( A_{ij} \) be a shortest path from \( t_i \) to \( t_j \), \( 1 \leq i < j \leq 4 \) and \( w(A_{ij}) \) be the length of the path \( A_{ij} \) for \( k \in \mathbb{Z}^+ \).

The following are equivalent:

(i) The maximum packing of \( T \)-cuts \( \geq k \)
(ii) \( w(A_{12}) + w(A_{34}) \geq k \)
\( w(A_{13}) + w(A_{24}) \geq k \)
\( w(A_{14}) + w(A_{23}) \geq k \)

and if equality holds in all three cases then:
\[ w(A_{ij}) + w(A_{jk}) + w(A_{ki}) \text{ is even for each choice of } 1 \leq i, j, k \leq 4, i \neq j \neq k, i \neq k \]

Proof: Follows from the equivalence of (i) and (ii) in Theorem 1 and the fact that in this case, an optimum \( T \)-join is a forest consisting of at most 2 trees.

Corollary 3: (Frank’s theorem [F89a]) Let \( H \) be the demand graph of an edge disjoint paths problem in a graph \( G \). Suppose that \( G + H \) is planar and the demand edges are on at most two faces of \( G \). The edge-disjoint paths problem has a solution if and only if the cut criterion and the intersection criterion hold.

**Cut criterion:** \( d_G(X) \geq d_H(X) \) \( \forall X \subseteq V \).

**Intersection criterion:** \( d_{G+H}(S \cap T) \) is even for any tight sets \( S, T \subseteq V \)

\( (X \text{ is tight if } d_G(X) = d_H(X)) \).

Proof: Follows from the equivalence of (i) and (ii) in Theorem 1, and by using planar duality.

Note that the demand edges across one face of \( G \) form a tree in the planar dual graph.
Frank's Conjecture ([F89b]): Let $G = (V,E)$ be an undirected graph $T \subseteq V$, $|T|$ even and $F \subseteq E$ is a minimum $T$-join with 2 components. Then there is an optimal packing of $T$-cuts which is integral if and only if the union of all "zero-circuits" is a bipartite graph.

Corollary 4: Frank's conjecture is true.

Proof: Follows from the equivalence of (i) and (ii) in Theorem 1.

4. Proof of the Main Result.

Proof of Theorem 1: We shall show the following implications: (i) $\rightarrow$ (ii) $\rightarrow$ (iii) $\rightarrow$ (i).

(i) $\rightarrow$ (ii):

Clearly, every edge in a zero-circuit is covered in an optimal packing of $T$-cuts, by cuts with total value = 1. Also, every integral packing of $T$-cuts leaves in every odd circuit, at least one edge which is not covered by it. Therefore, if there is an optimal packing of $T$-cuts which is integral, the union of any two zero-circuit cannot contain an odd circuit, and the above implication is proved.

(ii) $\rightarrow$ (iii):

Since $S_1, S_2, S_3, S_4$ are level sets, there are eight paths of zero length $P_{i,j}, 1 \leq i \leq 2, 1 \leq j \leq 4$ such that $P_{i,j}$ connects $f_i$ with $e_j$ inside one of the above appropriate level set. Hence, $C_1 = (f_1, P_{1,2, e_2}, P_{2, 2, f_2}, P_{2, 3, e_3}, P_{1, 3})$ and $C_2 = (f_1, f_{1, 1}, e_1, P_{2, 1, f_2}, P_{2, 4}, e_4, P_{1, 4})$ are two zero-circuit.

Claim: We may assume that each $P_{i,j}$ can be partitioned into 2 sub-paths such that the first sub-path (starting from the end of $f_i$) consists of only $T$-join edges, and the other sub-path consists of only non-$T$-join edges.

Proof of the claim: By the optimality of $F$ and the fact that in each $S_k, k \in \{1, 2, 3, 4\}$, the $T$-join edges form a tree, one can see that every zero path from $f_i$ to $e_j$ in $S_k$ can be transformed into a...
Consider the paths $P_{2,2}$ and $P_{2,4}$; it follows from the claim that, by starting from $f_2$, there must be a vertex $v$ which is the last common vertex of these two paths. Let $P'_{2,2}$ be the part of $P_{2,2}$ from $v$ to $e_2$ and $P'_{2,4}$ be the part of $P_{2,4}$ from $v$ to $e_4$. One can see that $P'_{2,2}$ and $P'_{2,4}$ have the same parity and therefore $C_3 = (f_1,P_{1,2},e_2,P'_{2,2},P'_{2,4},e_4,P_{1,4})$ is an odd circuit which is contained in the union of $C_1$ and $C_2$. That is to say, if (ii) is not true then (i) is not true which implies that (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i):

To prove this part we need the following lemmas:

**Lemma 1:** Let $F$ be an optimal $T$-join such that $F$ is a tree. Let $(y,S)$ be an optimal $\frac{1}{2}$ integral packing of $T$-cuts where $S$ is laminar. Then $(y,S)$ can be condensed into an optimal packing of $T$-cuts $(y',S')$ such that every set in $S'$ is a level set. The packing is either integral, or else the value of each set in $S'$ is 1 except for exactly 2 maximal sets with $y$ value $= \frac{1}{2}$.

**Proof:** By induction on the size of $F$. We leave the details for the reader. $\square$(Lemma-1)

**Lemma 2:** Let $F$ be an optimal $T$-join such that $F$ is a forest consisting of 2 trees, $F_1$ and $F_2$. Let $(y,S)$ be an optimal $\frac{1}{2}$ integral packing of $T$-cuts where $S$ is laminar. Then there exists an optimal $\frac{1}{2}$ integral packing of $T$-cuts, $(y',S')$ where $S'$ is laminar, such that $\forall S^* \in S', S^* \cap F_1 = \emptyset$ or $S^* \cap F_2 = \emptyset$.

**Proof:** Let $(y,S)$ be an optimal $\frac{1}{2}$ integral packing of $T$-cuts where $S$ is laminar and let $S' \subset S$ be the subset of all sets in $S$ that contain one of the trees $F_1$ or $F_2$. (By the laminarity of $S$, at most one of the trees may be contained in any set of $S$.) Let $S'' = \{ \bigvee S' \in S' \}$ and $S^* = \{ S \setminus S' \} \cup S''$. Change $y$ to $y^*$ in the following way: $y^*(S') = y(S')$ if $S^* \cap S' = \emptyset$ for a set $S' \in S'$ and $y^*(S*) = 0$ if $S^* \in S'$. Otherwise, $y^*(S*) = y(S*)$. One can see that $(y^*, S^*)$ is the
The following lemma is well known:

**Lemma 3**: Let $F$ be an optimal $T$-join such that $F$ is a tree. Then there is a packing of $T$-cuts which is integral. □[Lemma-3]

**Lemma 4**: Let $F$ be an optimal $T$-join such that $F$ is a forest consisting of 2 trees, $F_1$ and $F_2$. Let $(y,S)$ be an optimal $\frac{1}{2}$ integral packing of $T$-cuts where $S$ is laminar and can be partitioned into 2 collections: $S'$ and $S''$ such that $\forall S' \in S'$, $S' \cap V(F_2) = \emptyset$ and $\forall S'' \in S''$, $S'' \cap V(F_1) = \emptyset$. Assume that the $y$-value of each set in $S''$ is 1. Then there is an optimal packing of $T$-cuts which is integral.

**Proof**: If the assumption is true, then by contracting all the sets in $S''$ and all the edges in their coboundaries, we get a graph $G'$ and set $T' = T \cap F_1$. one can show that $G', T'$ satisfies the conditions of Lemma 3. By Lemma 3, there is an optimal packing in $G', T'$ which is integral and it can be extended to an integral packing in $G, T$ by using the part that is represented by $S''$, and the proof is completed. □[Lemma-4]

Seymour [S81b] proved that for every graph $G$ and every even $T \subseteq V$, there exists an optimal packing of $T$-cuts $y$ which is $\frac{1}{2}$-integral and which is represented by a laminar $S$. By using Lemma 2 and then Lemma 1, we may assume that $(y, S)$ has the following properties:

(a') $S$ can be partitioned into 2 collections: $S'$ and $S''$ such that $\forall S' \in S'$, $S' \cap V(F_2) = \emptyset$ and $\forall S'' \in S''$, $S'' \cap V(F_1) = \emptyset$. Moreover, $S'$ has exactly 2 maximal sets $S_1$ and $S_2$ with $y$-value = $\frac{1}{4}$ and $S''$ has exactly 2 maximal sets $S_3$ and $S_4$ with $y$-value = $\frac{1}{4}$

(b') All the sets in $S$ which are not maximal have $y$-value = 1

(c') there are two edges $f_1 \in F_1$ and $f_2 \in F_2$ such that $f_1 \in \delta(S_1) \cap \delta(S_2)$ and $f_2 \in \delta(S_3) \cap \delta(S_4)$. 

□
Now, assume that (i) of the theorem does not hold. The only sets with a non integral value are $S_1, S_2, S_3$ and $S_4$. We shall try to change their $y$-value so that we shall get an integral $y$ (or an integral packing of $T$-cuts) with the same total value, i.e. optimal. Clearly, by the above assumption we shall not succeed. However, we shall be able to show that (iii) of the theorem does not hold.

Case 1: Try to reduce the $y$-value of $S_1$ by $\frac{1}{4}$ and increase the $y$-value of $S_2$ by $\frac{1}{4}$. If this was possible the conditions of Lemma 4 would hold and this contradicts the assumption that (i) does not hold. Therefore, there must be an edge in the coboundary of $S_2$ that is also in the coboundary of $S_3$ or $S_4$. W.l.o.g., assume there is an edge $e_3 \in \delta(S_2) \cap \delta(S_3)$. Then we must reduce the $y$-value of $S_3$ by $\frac{1}{4}$ and increase the $y$-value of $S_4$ by $\frac{1}{4}$. If we could do this, we would have an integral packing contradicting our assumption. Hence, there must be also an edge, say $e_4$, in the coboundary of $S_4$ that is also in the coboundary of $S_2$.

Case 2: Try to reduce the $y$-value of $S_2$ by $\frac{1}{4}$ and increase the $y$-value of $S_1$ by $\frac{1}{4}$. By repeating the argument of case 1 to this case, we have that there must be two edges: $e_1 \in \delta(S_1) \cap \delta(S_3)$ and $e_2 \in \delta(S_1) \cap \delta(S_4)$.

Now we have that the optimal packing $(y, z)$ has the properties (a'), (b') and (c') and also the 4 edges mentioned in case 1 and case 2. This shows that (iii) of the theorem does not hold, and hence $(iii) \rightarrow (i)$.

This complete the proof of the last implication and the proof of the Theorem. □[Theorem 1]

5. Algorithms and Extensions.

(1) The proof of the main theorem implies a polynomial algorithm for optimal integral packing of $T$-cuts for the case where the optimal $T$-join consists of 2 trees. This algorithm is in fact a
simple post optimality method (similar to the one in [K82], but the presentation here is self-contained) that can be applied for every existing algorithm for \( \frac{1}{2} \) integral packing of T-cuts. Note that if there is no optimal packing of T-cuts which is integral, then by [KP85], there is an integral one which is one less than the optimal. In this case, the algorithm will give the collection of sets as in (iii) of Theorem 1 and from that point, it is very easy to get an optimal integral packing of T-cuts.

(2) Extention of the main theorem to the case where the optimal T-join consists of 3 trees or more, is not true. This can be seen by the example, shown in the Figure where there is no zero circuit and there is no optimal packing of T-cuts which is integral. However a different characterization for the 3-trees case can be proved and will be given elsewhere.

![Figure](image)

\[ T = \{a, b, c, d, e, f\}, \quad F = \{(a, b), (c, d), (e, f)\} \]

(3) The main theorem can be naturally extended to the weighted case. Also in the weighted case, a polynomial algorithm is implied. This algorithm can be used to solve the planar integral multicommodity flow problem ([S81]), where the demand edges are across 2 faces of the planar graph \( G \).

Final remark: The results presented here can be obtained, algorithmically, from the results in [K82]. However this paper is self contained.
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References


[F89b] A. Frank, Private communication.


