The Representation Power of Probabilistic Knowledge by Undirected Graphs and Directed Acyclic Graphs: A Comparison

by

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Abstract

Two modes of representation of probabilistic knowledge are considered: by undirected graphs (UG's) and by directed acyclic graphs (DAG's). It is shown that there is a UG such that the knowledge represented in it requires exponentially many (in the number of vertices of the UG) DAG's for its representation and there is a DAG such that the knowledge represented in it requires exponentially many UG's for its representation. It is thus shown that the two modes of representation are uncomparable as to their representation power.

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1 Introduction

Graphoids are ternary relations over a finite domain that satisfy a finite set of axioms enabling symbolic derivations. They are intended as models for representing irrelevance relations consisting of statements of the form: given that the values of the variables in $Z$ are known, the values of the variables in $Y$ can add no further information about the values of the variables in $X$. Graphoids may have applications to knowledge representation and may also provide a means for discovering independence relations between random variables by symbolic manipulation methods (rather than tedious calculations).

They can be represented in graphs in several ways, under a semantics to be specified in the text. The main goal of this paper is to compare two ways of representation, by directed acyclic graphs (DAG's) and by undirected graphs (UG's). A more comprehensive exposition of this subject can be found in the book of Pearl [6], chapter 3.

2 Preliminaries and Basic Definitions

For the sake of completeness we reproduce here the basic definition needed in the sequel. As mentioned above a more comprehensive exposition can be found in Pearl [6]. Readers familiar with the subject can skip this section.

2.1 Probabilistic Distributions

Probabilistic distributions (denoted PDs) are defined over a finite set $U$ of random variables. $X,Y,Z,$ etc. denote disjoint subsets of $U$ and $x,y,z,$ etc.
denote individual random variables. Every random variable is defined over a finite domain.

Let \(a, b, c\) be the vectors of possible values of the sets of random variables \(X, Y, Z\), correspondingly. We shall use the notation (the concatenation of \(X\) and \(Y\) stands for the union of the two sets):

\[
P(XY) = P(X|Y)P(Y)
\]

for the statement

\[
(\forall a)(\forall b)P(X = a, Y = b) = P(X = a|Y = b)P(Y = b)
\]

which is an identity over PDs. Under the above notation the equality

\[
P(XY|Z) = P(X|Z)P(Y|Z)
\]

(1)

means that the set of variables \(X\) is independent on the set of variables \(Y\) given the set of variables \(Z\) (i.e., if the values of the variables in \(Z\) are known then the values of the variables in \(X\) may get are independent of the values that the variables in \(Y\) may get), for the given PD.

2.2 Graphoids

Definition: A graphoid is a ternary relation \(I = \{(X, Z, Y)\}\) where \(X, Y\) and \(Z\) are disjoint subsets of elements from a finite collection \(U = \{a, b, \ldots\}\) of elements (attributes, random variables).

Denote \((X, Z, Y) \in I\) by \(I(X, Z, Y)\) and let the notation \(YW\) stand for the union \(Y \cup W\). Graphoids satisfy the following five properties:
The relation of conditional irrelevance was defined with respect to probability theory by the equality (1). One can interpret conditional irrelevance as conditional independence. Given a joint probability distribution $P(O)$ over a (finite) set of random variables $U$, the random variables $X$ and $Y$ are irrelevant when $Z$ is known if

$$P(XY|Z) = P(X|Z) \cdot P(Y|Z).$$

We will say that the relation $I$ is induced by the distribution $P$ if a triplet $(X, Z, Y)$ is in $I$ if and only if $X$, $Z$ and $Y$ satisfy the above equation.

It has been shown [5] that any relation $I$ induced by a probability distribution is closed under the first 4 axioms, and if the function $P$ is strictly positive then the induced relation is closed under the fifth axiom (intersection). It has been shown however by Studney [7] that the above set of axioms is not complete for relations induced by probabilistic distributions.

### 2.3 Representation by Undirected Graphs (UG's)

Consider an undirected graph as a model of representation for an independence relation $I$. The vertices of the graph represent the variables in $U$. A triplet $(X, Z, Y)(X, Y$ and $Z$ are subsets of the vertex set of the graph) is
represented by the graph if the set of vertices $Z$ is a cutset between the sets of vertices $X$ and $Y$.

Consider for example the graph shown in figure 1.

![Figure 1](image)

The two vertices $z$ and $z'$ separate between $y$ and $w$ and therefore the triplet $(y, zx, w)$ is represented. In addition, the two vertices $y$ and $w$ separate between $z$ and $z'$ and therefore the triplet $(z, yw, z')$ is represented. No other triplet (except for symmetrical images of these two triplets) is represented by the graph.

Pearl and Paz [5] have shown a characterization of the properties of cutset separation in graphs by a set of axioms (below) similar to the Graphoid axioms. A relation can be represented by an undirected graph if and only if it is closed under the following independent axioms:

1. $(X, Z, Y) \rightarrow (Y, Z, X)$  
   \textbf{Symmetry}

2. $(X, Z, YW) \rightarrow (X, Z, Y) \land (X, Z, W)$  
   \textbf{Decomposition}

3. $(X, Z, YW, Y') \rightarrow (X, Z, YW)$  
   \textbf{Intersection}

4. $(X, Z, Y, W) \rightarrow (X, Z, W, Y), W \subseteq U$  
   \textbf{Strong Union}

5. $(X, Z, Y) \rightarrow (X, Z, \gamma) \lor (\gamma, Z, Y), \gamma \in U$  
   \textbf{Transitivity}
Remarks:

1. $\gamma$ is a singleton element of $U$ and all three arguments of $I()$ must represent disjoint subsets.

2. The axioms are clearly satisfied for vertex separation in graphs. Ax. (7) is the contrapositive form of connectedness transitivity, stating that if $X$ is connected to some vertex $\gamma$ and $\gamma$ is connected to $Y$ then $X$ is connected to $Y$. Ax. (6) states that if $Z$ is a vertex cutset separating $X$ from $Y$ then adding more vertices $W$ from the graph to $Z$ leaves $X$ and $Y$ still separated. Ax. (5) states that if $X$ is separated from $Y$ with $W$ removed and $X$ is separated from $W$ with $Y$ removed then $X$ must be separated from both $Y$ and $W$.

3. Axs. (5) and (6) imply the contraction axiom (no. (4)), and also the converse of Ax. (2) which is:

$$(X, Z, Y) \land (X, Z, W) \rightarrow (X, Z, YW)$$

meaning that $I$ is completely defined by the set of triplets $(a, Z, b)$ in which $a$ and $b$ are individual elements of $U$. Note that the strong union axiom, (6) is unconditional and therefore strictly stronger than Ax. (3) which is required for probabilistic dependencies.

While axioms 1, 2, 5, 6 and 7 imply the set of axioms 1, 2, 3, 4 and 5 as is easy to see, the two sets of axioms are logically inequivalent. Therefore, not every graphoid can be represented by an undirected graph.

Consider for example an experiment in which one tosses two coins, and if the outcomes of the coins are equal, then he rings a bell. The outcomes
of the two coins are independent of each other, therefore we have the triplet \( t_1 = (c_1, \emptyset, c_2) \) induced. But once the outcome of the bell is known, then the two outcomes of the coins are no longer independent, and therefore the triplet \( t_2 = (c_1, b, c_2) \) is not induced. No undirected graph can represent the triplet \( t_1 \) and at the same time, not represent the triplet \( t_2 \).

The advantage of graphs for graphoid representation is evident. The size of a graph is polynomial in the size of its vertices, but the number of triplets which can be represented by it is usually exponential.

### 2.4 Representation by Directed Acyclic Graphs (DAG's)

A second way of representing graphoids is by Directed Acyclic Graphs (DAG's). Again the vertices represent the variables in \( U \). The definition of the representation is more complex and it takes into consideration the possibility of directing the arcs. There are three ways that a pair of arrows may meet at a vertex:

1. tail to tail, \( X < -Z - > Y \)
2. head to tail, \( X - > Z - > Y \)
3. head to head, \( X - > Z < -Y \)

**Definitions:**

1. Two arrows meeting head to tail or tail to tail at a node \( \alpha \) are said to be **blocked** by a set \( S \) of vertices if \( \alpha \) is in \( S \).

2. Two arrows meeting head to head at node \( \alpha \) are **blocked** by \( S \) if neither \( \alpha \) nor any of its descendents is in \( S \).
3. An undirected path $P$ in a DAG $G$ is said to be $d$-separated by a subset $S$ of the vertices if at least one pair of successive arrows along $P$ is blocked by $S$.

4. Let $X, Y$ and $S$ be three disjoint sets of vertices in a DAG $G$. $S$ is said to $d$-separate $X$ from $Y$ if all paths between $X$ and $Y$ are $d$-separated by $S$.

For example, in the graph shown in figure 2 below, the triplet $(2,1,3)$ is represented, as the set $\{1\}$ $d$-separates between the vertices 2 and 3. On the other hand, the triplets $(2,4,3)$ and $(2,\{1, 5\},3)$ are not represented.

![Figure 2](image)

This model of DAG representation has limitations as well. It has been shown [6] that a necessary condition for a graphoid to be representable by a DAG is that it satisfies the following independent axioms:
Lower case letters stand for single elements of $U$. These axioms imply but are not equivalent to the graphoid axioms. Therefore there are graphoids which are not representable by DAGs.

### 2.5 Representation by sets of Graphs

One can also represent graphoids by a set of graphs, Directed or Undirected. A triplet is represented in a set of graphs if it is represented in one of the graphs in the set.

While every graphoid can be represented in a set of graphs, some graphoids may require exponentially many graphs for their representation.

### 2.6 Testing Membership in UG’s and DAG’s

It is easy to check whether a triplet $(X, Z, Y)$ is represented in a given UG. It is represented iff the vertices corresponding to $Z$ are a cutset between the vertices representing $X$ and the vertices representing $Y$ in the given UG. The following is an algorithm given by Lauritzen et al. [2] for testing the representation of triplets in DAG’s. For the sake of simplicity we shall
identify the vertices of the DAG with their corresponding variables.

To check whether a given triplet \((X, Z, Y)\) is represented in a DAG \(D\) do the following:

1. Remove all vertices of \(D\) which are not ancestors of \(X \cup Y \cup X\), together with incident edges.

2. Connect ("Marry") all common parents of remaining vertices (i.e., connect vertices with a common sibling).

3. Remove directions from the resulting graph thus creating a UG \(G'\).

4. \((X, Z, Y)\) is represented in \(G\) iff \((X, Z, Y)\) is represented in \(G'\).

3 Comparison between UG representation and DAG representation

As we have seen in the previous two sections, both models of representation, the undirected graph and the DAG, have limitations. There are graphoids which cannot be represented by either one of them.

There are graphoids which can be represented in one model and cannot be represented in the other. The graphoid induced by the experiment with the two coins and the bell can be represented by a DAG with three vertices corresponding to the three variables and arrows from each vertex representing a coin to the vertex representing the bell. Thus, the triplet \((c_1, 0, c_2)\) is represented, and the triplet \((c_1, b, c_2)\) is not.

As mentioned before, this graphoid cannot be represented by an undirected graph.
On the other hand, the graphoid represented by the diamond shaped graph in figure 1 cannot be represented by a DAG. The graph is non-chordal, and the represented graphoid does not satisfy the chordality axiom (10). If we try to direct the arrow in the graph, then there must be a pair of nonadjacent parents sharing a common child, a configuration which yields conditional independence in undirected graphs but conditional dependence in DAGs.

Regardless of the above considerations it might still be possible that (a): one mode of representation is more powerful than the other - this would be the case if every DAG can be represented by a polynomial set of UG’s but not the other way around, or if every UG can be represented by a polynomial set of DAG’s but not the other way around; or (b): the two modes of representation are polynomially equivalent - this would be the case if every UG can be represented by polynomially many DAG’s and the other way around.

Our main result is that the representation power of the two modes of representation are not comparable, i.e., there is a UG which requires exponentially many DAG’s for its representation and there is a DAG which requires exponentially many UG’s for its representation. This is shown next.

3.1 The UG’s \( W(n) \)

For \( n = 1,2,... \) define the UG \( W(n) \) as below

\[
W(n) = (V, E)
\]

\[
V = \{A_1, ..., A_n, B_1, ..., B_n, C_1, ..., C_n, D_1, ..., D_n\}, |V| = 4n
\]

\[
E = \{(A_i, B_i), (A_i, C_i), (D_i, B_i), (D_i, C_i) : 1 \leq i \leq n\}, |E| = 4n.
\]

e.g. \( W(3) \) is the graph shown in Fig 3.
Let $S$ be any subset of $\{1, \ldots, n\}$, let $CS$ be the complementary set i.e., $CS = \{1, \ldots, n\} \setminus S$, let $A_S, B_S, C_S, D_S$ be the sets of vertices $\{A_i : i \in S\}, \{B_i : i \in S\}, \{C_i : i \in S\}, \{D_i : i \in S\}$ and denote the sets $A_{cs}, B_{cs}, C_{cs}, D_{cs}$ in a similar way. The following lemma is straightforward.

**Lemma 1:** All $2^n$ triplets of the form $t(s) = (A_S B_{cs}, A_{cs} B_S C, D_{cs} C_{cs} D_S), S \subseteq \{1, \ldots, n\}$, are represented in $W(n)$.

**Proof:** By the definition of $W(n)$ the vertices in $A_S$ are disconnected from the vertices in $C_{cs}$ and the vertices in $B_S \cup C$, are a cutset between the vertices in $A_S$ and the vertices in $D_{cs}$. A similar argument holds for the vertices in $B_{cs}$. □
Let $K$ be any set of DAG's representing the same set of triplets as $W(n)$. We show now that $K$ must satisfy certain properties.

**Lemma 2:** $K$ must contain at least one DAG whose set of vertices is the same as the set of vertices of $W(n)$.

**Proof:** The triplets $t(s)$ defined in lemma 1 involve all the vertices of $W(n)$. Any DAG with a different set of vertices cannot represent such a triplet. □

**Lemma 3:** Let $H$ be a DAG in $K$ over all the vertices of $W(n)$. If $(v_1, v_2)$ is an edge in $W(n)$ then either $v_1 \rightarrow v_2$ or $v_2 \rightarrow v_1$ is a directed arc in $H$.

**Proof:** Assume to the contrary that $(v_1, v_2)$ is an edge in $W(n)$ but neither $v_1 \rightarrow v_2$ nor $v_2 \rightarrow v_1$ is an arc in $H$.

Let $S$ be the set of all vertices of $H$ excluding $v_1$ and $v_2$ and excluding all the vertices which are not ancestors of $v_1$ or of $v_2$. ($S$ may be empty). Then $(v_1, S, v_2)$ is represented by $H$, by Lauritzen's et al. test (see section 2.6), since any common sibling of $v_1$ and $v_2$ is not in $S$. But $(v_1, S, v_2)$ is not represented in $W(n)$, since $S$ is an edge in $W(n)$, a contradiction. □

**Lemma 4:** Let $H$ be a DAG in $K$. At most one triplet of the form $t(s)$ can be represented in $H$.

**Proof:** Assume, by way of contradiction, that two triplets $t(s_1)$ and $t(s_2)$ are represented in $H$. Either $S_1 \cap CS_2$ or $S_2 \cap CS_1$ is not empty. Without
loss of generality assume that $S_1 \cap CS_2$ is not empty and let $i$ be an index such that $i \in S_1 \cap CS_2$. Since $\mathfrak{t}(S_1)$ and $\mathfrak{t}(S_2)$ are triplets over all the vertices of $W(n)$, $H$ must contain all the vertices of $W(n)$. Denote this set of vertices by $V$. From $\mathfrak{t}(S_1)$, by weak union (which is satisfied on DAG's) we get that $t_1 = (A_i, V - \{A_i, D_i\}, D_i)$ is represented in $H$. Similarly, from $\mathfrak{t}(S_2)$ and weak union we get that $t_2 = (B_i, V - \{B_i, C_i\}, C_i)$ is represented in $H$. On the other hand, since $(A_i, B_i), (B_i, D_i), (D_i, C_i), (C_i, A_i)$ form a cycle in $W(n)$, by the definition of $W(n)$, we have, by lemma 3, that the above arcs exist in $H$ with some superimposed orientation. To avoid cycles in $H$, this orientation must direct two of the above arcs into the same vertex, say $A_i$, i.e., the orientation of the corresponding edges in $H$ is $B_i \rightarrow A_i$ and $C_i \rightarrow A_i$. We shall apply now the algorithm of Lauritzen et al (see section 2.6) in order to verify whether $t_2$ is represented in $H$.

Step 1. $t_2$ is over all the vertices $V$ so that no vertex is removed.

Step 2. $B_i$ and $C_i$ are common parents of $A_i$.

Therefore the edge $(B_i, C_i)$ is added to $H$.

Step 3. Remove orientations from the transformed $H$ resulting in the $\cup G H'$.

Step 4. $t_2' = (B_i, V - \{B_i, C_i\}, C_i)$ is not represented in $H'$, since $(B_i, C_i)$ is an edge in $H'$, and therefore it is not represented in $H$, a contradiction.

A similar contradiction can be derived if the vertex $A_i$ is replaced by another vertex in the cycle $A_i, B_i, D_i, C_i$. □

Corollary 5: $K$ must contain at least $2^n$ different DAG's in order to represent all the triplets represented in $W(n)$. 

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3.2 The DAG $J(n)$

For $n = 1, 2, \ldots$ define the DAG $J(n)$ as below

$J(n) = (V, E)$ where

$V = \{A_1, \ldots, A_n, B_1, \ldots, B_n, V_1, \ldots, V_n, W_1, \ldots, W_n, P\}, |V| = 4n + 1.$

$E = \{A_n \leftarrow A_1 \leftarrow P \rightarrow B_1 \rightarrow B_n, \ldots, A_n \leftarrow A_{n-1} \leftarrow P \rightarrow B_{n-1} \rightarrow B_n, \}

A_1 \rightarrow U_1 \leftarrow V_1 \rightarrow W_1 \leftarrow B_1, \ldots, A_n \rightarrow U_n \leftarrow V_n \rightarrow W_n \leftarrow B_n\}, |E| = 8n - 4.$

e.g. $J(3)$ is the DAG shown in fig.4.

Fig. 4: The DAG $J(3)$
In the proof of the following lemma we shall use the axiom below which holds for UG's

$$CT : (X, Y, Z) \rightarrow (X\gamma, Y, Z) \lor (X, Y, Z\gamma)$$

where $\gamma$ is any vertex not in $X \cup Y \cup Z$.

The CT axiom is a combination of the Transitivity Symmetry and Composition axioms (axioms 1, 7 and 8) which hold for UG's.

The notation used in this section is the same as the notation used in the previous section.

**Lemma 6**: Let $S$ be any subset of $\{1 \cdots n\}$. The triplet $t(s) = (A_s, A_{ca} P, B V_{ca} U_{ca} W_{ca})$ is represented in $J(n)$ where $B = \{B_i : 1 \leq i \leq n\}$.

**Proof**: We use the algorithm of Lauritzen et al (section 2.6), to verify that $t_s$ is represented.

Step 1. The vertices $V_s, U_s, W_s$ which are not in $t(S)$ and are not ancestors of the vertices in $t(S)$ are removed from $J(n)$.

Step 2. Since $A_i \rightarrow A_n \leftarrow A_j$ and $B_i \leftarrow B_n \leftarrow B_j$ are arcs in $J(n)$ for every $i,j < n$, all the edges $(A_i, A_j), (B_i, B_j), i,j < n$, are added to $J(n)$; Since $(A_i \rightarrow U_i \leftarrow V_i \rightarrow W_i \leftarrow B_i$ are arcs in $J(n)$ for all $i$, all the edges $(A_i, V_i)$ and $(B_i, V_i)$ are added to $J(n)$, for the remaining $V_i$'s.

Step 3. Orientation is removed resulting in the UG $J'$.

Step 4. It is easy to check now that $t_s$ is represented in $J'$ and therefore in $J$. $\square$

Consider as an example the triplet $t(s)$ with $S = \{1, 2\}$ for $J(3)$ i.e.,

$t(s) = (A_1 A_2, A_3 P, B_1 B_2 B_3 V_3 U_3 W_3)$:

Steps 1,2 and 3. Remove from $J(3)$ the vertices $V_1, V_2, U_1, U_2, W_1, W_2$, then...
add the edges \((A_1, A_2)(B_1, B_2), (A_3, V_3), (B_3, V_3)\), then remove orientations.

The resulting graph \(J'\) is shown in Fig. 5 below.

It is easy to verify now that the set \(\{A_3, P\}\) is a cutset between \(\{A_1, A_2\}\) and \(\{B, U_3, V_3, W_3\}\) so that \(t(s)\) is represented in \(J'\) and therefore in \(J\).

Let \(K\) be a set of UG's representing all the triplets represented by \(J(n)\) and only such triplets. Let \(G\) be a graph in \(K\).

**Lemma 7:** At most one of the triplets \(t(s)\) can be represented in \(G\).

**Proof:** Assume, by way of contradiction, that two triplets \(t(s_1)\) and \(t(s_2)\) are represented in \(G\). Either \(CS_1 \cap S_2\) or \(CS_2 \cap S_1\) is not empty and wlog
we may assume the former. Let \( i \in CS_1 \cap S_2 \). From \( t(s_2) \), by the CT axiom we can insert \( W_i \) and derive the representation in \( G \) of either 
\[
\begin{align*}
t_1 &= (A_{s_2}, A_{CS_1} P, BU_{CS_1} V_{CS_1} W_{CS_1} W_i) \\
t_2 &= (A_{s_2} W_i, A_{CS_1} P, BU_{CS_1} V_{CS_1} W_{CS_1} W_i).
\end{align*}
\]

or
\[
\begin{align*}
t_1 &= (A_{s_2}, A_{CS_1} P, BU_{CS_1} V_{CS_1} W_{CS_1} W_i) \\
t_2 &= (A_{s_2} W_i, A_{CS_1} P, BU_{CS_1} V_{CS_1} W_{CS_1} W_i).
\end{align*}
\]

\( t_2 \) is not represented in \( J(n) \) since \( B_i \rightarrow W_i \) is an arc in \( J(n) \). We conclude therefore that \( t_1 \) is represented in \( G \). From \( t_1 \) and using again the CT axiom we can insert \( V_i \) and derive the representation in \( G \) of either 
\[
\begin{align*}
t_{11} &= (A_{s_1}, A_{CS_2} P, BU_{CS_2} V_{CS_2} W_{CS_2} V_i W_i) \\
t_{12} &= (A_{s_2} V_i, A_{CS_2} P, BU_{CS_2} V_{CS_2} W_{CS_2} W_i).
\end{align*}
\]

\( t_{12} \) is not represented in \( J(n) \) since \( V_i \rightarrow W_i \) is an arc in \( J(n) \). We conclude therefore that \( t_{11} \) is represented in \( G \). From \( t_{11} \) by the CT axiom we can insert \( U_i \) and derive, in the same way, the representation in \( G \) of either 
\[
\begin{align*}
t_{111} &= (A_{s_1}, A_{CS_2} P, BU_{CS_2} V_{CS_2} W_{CS_2} V_i W_i U_i) \\
t_{112} &= (A_{s_2} V_i, A_{CS_2} P, BU_{CS_2} V_{CS_2} W_{CS_2} V_i W_i).
\end{align*}
\]

Now \( t_{112} \) is not represented in \( J(n) \) since \( V_i \rightarrow U_i \) is an arc in \( J(n) \). \( t_{111} \) is not represented in \( J(n) \) either since \( i \in CS_1 \cap S_2 \) implies that \( A_i \in A_{S_1} \) and therefore \( A_{S_1} \) cannot be separated from \( V_i \) since \( A_i \rightarrow V_i \) is an arc in \( J(n) \). We have thus reached a contradiction. 

\begin{comment}
\begin{align*}
\text{Corollary 8: & K must contain at least 2^m UG's in order to represent the same set of triplets as J(n).}
\end{align*}
\end{comment}

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\end{align*}

4 Remarks

**Remark 4.1** The results shown in section 3 prove that the two modes of representation of Graphoids, by DAG's or by UG's, are not comparable. Which mode to choose would depend therefore on the properties of the Graphoid.
we want to represent. The examples used in the previous section provide the following heuristic for a proper choice.

Definition: Two triplets in a Graphoid are crossing if the middle part of each triplet contains variables from both the right and the left side of the other triplet. A triplet \( t = (X, Z, Y) \) in a graphoid \( I \) is a common influence triplet if there is a variable \( \gamma \) which is not in \( X \cup Y \cup Z \) such that \( (X, Z \gamma, Y) \) is not in \( I \).

Heuristic of Representation: Given a graphoid \( I \), if \( I \) contains "many" crossing pairs of triplets then choose the UG mode of representation. If \( I \) contains "many" common influence triplets then choose the DAG mode of representation.

Remark 4.2 Let \( B \) be a set of triplets. \( cl_G(B) \) denotes the Graphoid closure of \( B \), i.e., the set of all triplets which can be derived from the set \( B \) by the graphoid axioms.

Let \( D \) be a DAG and let \( v_1, v_2, \ldots, v_n \) be an ordering of the vertices of \( D \) which agree with the orientation of \( D \) (i.e., if there is a directed path in \( D \) from \( v_i \) to \( v_j \) then \( v_i \) precedes \( v_j \) in the order). Denote by \( Pa(v_i) \) the set of parents of \( v_i \) in \( D \).

Let \( D \) be a DAG and let \( v_1, \ldots, v_n \) be an ordering of its vertices, as above. Define the set of \( n \) triplets \( S = \{(v_i, Pa(v_i), \{v_i \cdots v_{i-1}, Pa(v_i)\})\} \).

It has been shown by Pearl and Verma [4] that the graphoid represented by \( D \) is the same as \( cl_G(S) \). It follows from this result and from section 3 above that there is a set of \( n \) triplets \( S \) such that \( cl_G(S) \) contains exponentially many triplets and such that \( cl_G(S) \) require an exponential set of UG for its
representation. Just set $S$ to be the set defined as above for the DAG $J(n)$ relative to some proper ordering.

Remark 4.3 Let $G$ be a UG and let $V$ be the set of vertices of $G$. Let $S$ be the set of triplets represented by $G$ of the form $(a, V - c - b, b)$ where $a$ and $b$ are vertices. It has been shown by Pearl and Paz [5] that the graphoid $\text{cl}_G(S)$ is the same as the graphoid represented by $G$. Trivially, the number of triplets in $S$ cannot exceed the number of edges of $G$ and is therefore less than $n^2$.

It follows from this result and from section 3 that there exists a set $S$ containing less than $n^2$ triplets such that $\text{cl}_G(S)$ contains exponentially many triplets and such that any set of DAG's representing $\text{cl}_G(S)$ contains exponentially many DAG's. Just set $S$ to be the set of triplets defined by the UG $W$.

Remark 4.4 Another related result is the following. There are two UG's such that the graphoid closure of the graphoids represented by them requires exponentially many UG's for its representation [3]. A similar result can be proven for DAG's.

References


