Some Aspects of the Membership Problem for Graphoids

by

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Abstract

Given a set of triplets \( B = \{(X, Z, Y)\} \) of polynomial size over a finite domain (random variables, attributes), and a triplet \( t = (P, Q, R) \) over the same domain, the membership problem is to ascertain whether \( t \) is in the closure of \( B \) under the graphoid axioms defined in the text. The closure of such \( B \)'s under the graphoid axioms (graphoids) are intended as models for representing irrelevance relations. The complexity of the membership problem is not known. In this paper we reduce the general membership problem to a polynomially equivalent simpler membership problem and we define an elementary membership problem, whose complexity is also not known, and which is a particular case of the simple membership problem.

1 Introduction

Graphoids are ternary relations over a finite domain that satisfy a finite set of axioms enabling symbolic derivations. They are intended as models for representing irrelevance relations consisting of statements of the form: given that the values of the variables in \( Z \) are known, the values of the variables \( Y \) can add no further information about the values of the variables \( X \). Graphoids may have applications to knowledge representation and may also provide a means for discovering independence relations between random variables by symbolic manipulation methods (rather than tedious calculations).

One of the important problems whose complexity is still not known for graphoids, is the membership problem: given a set of triplets \( B \) (elements of the relation) of polynomial size (in the

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number of elements of the domain), and a particular triplet \( t \), can \( t \) be derived from \( B \) via the graphoid axioms.

The main purpose of this paper is to introduce some modifications and simplifications of Graphoids. The modifications and simplifications are intended to provide some new insights and techniques which may help solve the membership problem, or prove that it is not solvable, in polynomial time.

2 Preliminaries and Basic Definitions

For the sake of completeness we reproduce here the basic definitions needed in the sequel. A more comprehensive exposition can be found in [3]. Readers familiar with the subject can skip this section.

Probabilistic distributions (denoted PDs) are defined over a finite set \( U \) of random variables \( X, Y, Z, \) etc. denote disjoint subsets of \( U \) and \( x, y, z, \) etc. denote individual random variables. Every random variable is defined over a finite domain.

Let \( a, b, c \) be vectors of possible value of the sets of random variables \( X, Y, Z, \) correspondingly.

We shall use the notation (the concatenation of \( X \) and \( Y \) stands for the union of the two sets):

\[ P(XY) = P(X|Y)P(Y) \]

for the statement

\[
(\forall a)(\forall b)P(X = a, Y = b) = P(X = a|Y = b)P(Y = b)
\]

which is an identity over PDs.

Under the above notation the equality

\[ P(XY|Z) = P(X|Z)P(Y|Z) \]

means that the set of variables \( X \) is independent on the set of variables \( Y \) given the set of variables \( Z \) (i.e., if the values of the variables in \( Z \) are known then the values of the variables in \( X \) may get are independent of the values of the variables in \( Y \) may get), for the given PD.

Definition: A graphoid is a ternary relation \( I = \{(X, Z, Y)\} \) where \( X, Y \) and \( Z \) are disjoint subsets of elements from a finite collection \( U = \{a, b \cdots\} \) of elements (attributes, random variables).

Denote \( (X, Z, Y) \in I \) by \( I(X, Z, Y) \) and let the notation \( YW \) stand for the union \( Y \cup W \).

Graphoids satisfy the following five properties:
The relation of conditional irrelevance was defined with respect to probability theory in [1]. One can interpret conditional irrelevance as conditional independence. Given a joint probability distribution \( P(\cdot) \) over a (finite) set of random variables \( U \), the random variables \( X \) and \( Y \) are irrelevant when \( Z \) is known if

\[
P(X \mid Y \mid Z) = P(X \mid Z)P(Y \mid Z).
\]

We will say that the relation \( I \) is induced by the distribution \( P \) if a triplet \( (X,Y,Z) \) is in \( I \) if and only if \( X, Z \) and \( Y \) satisfy the above equation.

It has been shown [2] that any relation \( I \) induced by a probability distribution is closed under the first 4 axioms, and if the function \( P \) is strictly positive then the induced relation is closed under the fifth axiom (intersection). It has been shown however in [4] that the above set of axioms is not complete for relations induced by probabilistic distributions.

## 3 Simple Graphoid

A simple graphoid is a ternary relation \( I = \{(x,Z,y)\} \) where \( Z \) is a subset of a finite set of elements \( U = \{a,b,\cdots\} \) (attributes, random variables) and \( x, y \) are elements in \( U \setminus Z \). As for graphoids, \( (x,Z,y) \in I \) is denoted by \( I(x,Z,y) \). Capital letters denote subsets of \( U \) and lower case letters denote elements of \( U \). The notation \( Zx \) stands for the union \( Z \cup \{x\} \). Simple graphoids satisfy the following 3 axioms (the \( I \) is omitted from \( I(X,Z,Y) \), to simplify the notation).

(1) \( (x,Z,y) \rightarrow (y,Z,x) \) \hspace{1cm} S-Symmetry
(2) \( (x,Z,y) \land (x,Zy,w) \rightarrow (x,Zw,y) \) \hspace{1cm} S-Contraction
(3) \( (x,Zy,w) \land (x,Zw,y) \rightarrow (x,Z,y) \land (x,Z,w) \) \hspace{1cm} S-Intersection.
4 Membership Problems

General graphoid membership problems (GGM):

Given a (polynomial) set of general triplets \( A = \{(X, Z, Y)\} \) over a finite domain and a triplet \( t \). Decide whether \( t \) can be derived from \( A \) via the 5 graphoid axioms.

Simple graphoid membership problems (SGM):

Same as above only \( A \) is a set of simple triplets, \( t \) is a simple triplet and the axioms are the three simple graphoid axioms.

In the sequel we will show that the two problems are polynomially equivalent. We notice first the following property.

Let \( t = (X, Z, Y) \) be a general triplet and denote by \( S(t) \) the set of simple triplets \( S(t) = \{(z, Z(X - z)(Y - y), y) : z \in X, y \in Y\} \). It is easy to see that \( t \) belongs to a general graphoid \( G \) iff all the elements of \( S(t) \) belong to \( G \), since if \( t \) belongs to \( G \) then all the elements of \( S(t) \) can be derived from \( t \) by weak union, while if all the elements of \( S(t) \) belong to \( G \) then \( t \) can be derived from those elements by intersection. The details are left to the reader.

5 A polynomial reduction from GGM to SGM

Given a set \( A = \{t_i\} \) of general triplets and a general triplet \( t \). Define the set of simple triplets \( B = \{US(t_i) : t_i \in A\} \) and the set of simple triplets \( S(t) \). Since the number of triplets in \( S(t_i) \) is polynomial in the size of the domain (actually quadratic), this reduction is polynomial.

Theorem 1 \( t \) can be derived from \( A \) via the 5 general graphoid axioms iff all the simple triplets in \( S(t) \) can be derived from \( B \) via the 3 simple graphoid axioms. Moreover, the length of the \( S \)-derivation chains for the elements in \( S(t) \) is polynomial in the length of the derivation chain for \( t \), and vice versa.

Proof: Let \( t_1, \ldots, t_k \) be a derivation chain for \( t = t_k \) in the general graphoid model. We prove by induction that the sequence \( S(t_1), \ldots, S(t_k) \) is a ‘derivation chain’ in the simple model, in the following sense: \( S(t_i) \subseteq B \) and all the simple triplets in \( S(t_i) \) such that \( t_i \not\in A(S(t_i) \not\subseteq B) \), can be derived via the \( S \)-axioms from the triplets in \( S(t_j) \) and \( S(t_r) \), for some \( j, r < i \). The proof is split into 5 cases, according to the 5 graphoid axioms.

Basis: If $t_i$ is in $A$ then, by construction, $S(t_i)$ is in $B$. $t_1 \in A$ necessarily so that $s(t_1) \subseteq B$.

Step Assume that the sequence $S(t_1), \ldots, S(t_{i-1})$ has the required property (it is a 'derivation chain' in the simple model, as defined above).

a. Assume that $t_i$ is derived from $t_j$, $j < i$, by symmetry. Then every triplet in $S(t_i)$ can be derived by S-symmetry from its corresponding triplet in $S(t_j)$.

b. Let $t_i = (X, Z, Y)$ and assume that it was derived from $t_j = (X, Z, YW)$ by decomposition, $j < i$. By induction $S(t_j)$ is already in the derivation chain of the simple model. Using S-intersection $|S(t_j)|$ times ($|Z|$ denotes the number of elements in a set of $Z$) we proceed with the following derivations for all $x \in X, y \in Y, w' \in W$ and an arbitrary $w \in W$. From $(x, Z(X - x)(Y - y)(W - w)w', y)$ and $(x, Z(X - x)(Y - y)(W - w)y, w)$ in $S(t_j)$ derive $(x, Z(X - x)(Y - y)(W - w), y)$. From $(x, Z(X - x)Y(W - w - w')w', w)$ and $(x, Z(X - x)Y(W - w - w')w, w')$ in $S(t_j)$ derive $(x, Z(X - x)(Y - y)(W - w - w'), w')$. The set of triplets thus derived for all $x \in X, y \in Y$ and $w' \in W - w$ form the set $S((X, Z, Y)(W - w))$. The process is repeated until all $w \in W$ are eliminated resulting in the derivation of $S((X, Z, Y)) = S(t_i)$. The total number of simple derivations involved is $|W| * |S(t_j)|$.

c. Let $t_i = (X, ZY, W)$ and assume it was derived by weak union from $t_j = (X, Z, YW), j < i$. Since in this case $S(t_i) \subseteq S(t_j)$ (as follows from the definitions), $S(t_i)$ can be added to the derivation chain of the simple model and no simple derivations are involved in this step.

d. Let $t_i = (X, Z, YW)$ and assume it was derived from $t_j = (X, Z, Y)$ and $t_r = (X, ZY, W)$, $j, r < i$ by contraction. By induction $S(t_j)$ and $S(t_r)$ are already in the derivation chain of the simple model. Now $S(t_j) \subseteq S(t_i)$ with $S(t_i) \setminus S(t_j) = \{(x, Z(X - x)(Y - y)w, y) : x \in X, y \in Y\}$ and we must show that all the elements in this set can be derived in the simple model. Let $x \in X, y \in Y$ and $w_1, w_2 \in W$ be arbitrary variables.

It follows from the definitions that

$t_1 = (x, Z(X - x)(Y - y), y) \subseteq S(t_j)$ while $t_2 = (x, Z(X - x)(Y - y)y, w_1)$ and $t_3 = (x, Z(X - x)(Y - y)yw_1, w_2)$ can be derived from $S(t_r)$ as shown in the proof of part b. above, and the length of the derivations is polynomial, in the number of variables. From $t_1$ and $t_2$ we get, by S-contraction $t_4 = (x, Z(X - x)(Y - y)w_1, y)$ and from $t_4$ and $t_3$ using S-contraction again we get $t_5 = (x, Z(X - x)(Y - y)w_1w_2, y)$. 

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Continuing this way we can construct, after $|W|$ such iteration, $t_0 = (z, Z(X - z)(Y - y)W, y)$ which is any arbitrary element in $S(t_i) \setminus S(t_r)$. The total number of derivations required for constructing the whole set $S(t_i) \setminus S(t_r)$ from $S(t_j)$ and $S(t_r)$ is thus shown to be $O(|W| * |X| * |Y|)$, with every iteration involving $O(|W| * |S(t_j)|)$ operations.

e. Let $t_i = (X, Z, YW)$ and assume it was derived from $t_j = (X, ZW, Y)$ and $t_r = (X, ZY, W)$ by general intersection. By induction $S(t_j)$ and $S(t_r)$ are already in the derivation chain in the simple model. It follows easily from the definitions that $S(t_j) \cup S(t_r) = S(t_i)$ and we can add $S(t_i)$ to the derivation chain in the simple model at no extra cost.

We have thus shown that a derivation chain for $S(t_k)$, starting from $S(t_i) \subseteq B$, can be constructed in the simple model, if a derivation chain for $t_k$ starting from $t_i$ is given in the regular model. The length of the simple derivation chain is polynomial in the length of the given chain.

Assume now that all the triplets in $S(t_k)$ can be derived from the triplets in $B$ via the $S$-axioms. Let $s_j$ be any triplet in $S(t_k)$ and let $s_1, ..., s_j$ be a derivation chain for $s_j$ from $B$ via the $S$-axioms. We will construct a derivation chain for $s_j$ from $s_1$ via the general axioms.

The construction is inductive and is split into 3 cases according to the three $S$-axioms.

**Basis:** If $s_i \in B$, then $s_i \in S(t_i)$ for some $t_i \in A$ so that $s_i$ can be derived from $t_i$ by regular weak union. The entry $s_i$ in the simple derivation chain is simulated by the sequence $t_i, s_i$ in the regular chain. $s_i$ is in $B$ necessarily. So $t_i, s_i$ are the first entries in the regular derivation chain.

**Step** Assume that the sequence $s_1, ..., s_{r-1}$ in the simple derivation chain has already been simulated and assume that $s_r$ is derived from $s_p$ and $s_q$ by one of the 3 $S$-axioms. By induction $s_p$ and $s_q$ exist in the simulating regular derivation chain.

**a.** $s_r$ is derived from $s_p$ by $S$-symmetry then $s_r$ is derived from $s_p$ by regular symmetry and we can add $s_r$ to the regular derivation chain.

**b.** $s_r$ is derived from $s_p$ and $s_q$ by $S$-contraction. Let $s_r = (x, Zw, y), s_p = (x, Z, y), s_q = (x, Z, w)$ then $s_p$ and $s_q$ imply $t = (x, Z, wy)$ by regular contraction, and $t$ implies $s_r$ by (regular) weak union. The entry $s_r$ in the simple derivation chain can be simulated by the sequence $t, s_r$ in the regular derivative chain.

**c.** $s_r$ is derived from $s_p$ and $s_q$ by $S$-intersection. Let $s_r = (x, Z, y), s_p = (x, Z, w), s_q = (x, Zw, y)$ then $s_p$ and $s_q$ imply, by regular intersection, $t = (x, Z, wy)$ which, by regular decomposition,
implies \( s_r \). The entry \( s_r \) in the simple derivation chain can be simulated by the sequence \( t, s_r \) in the regular derivation chain.

It follows from the above that every triplet in \( S(t_k) \) can be derived from \( A \) in the regular model given that it can be derived in the simple model from \( B \). The length of the derivation chain for any such triplet in the regular model is at most double the length of the derivation chain in the simple model. Let \( t_k = (X, Z, Y) \). Now \( t_k \) can be derived in the regular model from \( S(t_k) \) by at most \( |X| \cdot |Y| \) applications of the intersection axiom. The length of the regular derivation chain for \( t_k \) is thus shown to be polynomial in the length of the derivation chain in the simple model for any triplet in \( S(t_k) \). The proof is now completed.

### 6 A polynomial reduction from SGM to GGM

Since every simple triplet is also a general triplet, we define the reduction from a SGM problem to a GGM problem to be the identity reduction which is trivially polynomial. We prove now

**Theorem 2:** Given a set of simple triplets \( A \) and a simple triplet \( t \), \( t \) can be derived from \( A \) by the S-axioms iff \( t \) can be derived from \( A \) by the general axioms. The length of the derivation chain for \( t \) in the simple model is polynomial in the length of the derivation chain for \( t \) in the general model and vice versa.

**Proof:** As shown in the proof of theorem 1, simple symmetry can be simulated by general symmetry, simple contraction can be simulated by general contraction followed by (general) weak union and simple intersection can be simulated by general intersection followed by (general) decomposition. So we can simulate any simple derivation chain by a general derivation chain whose length is at most twice the length of the simple chain.

Let \( t_1, \ldots, t_k = t \) be a derivation chain for \( t \) in the general model. We use the notation \( S(t_1) \) as in the previous section. Since \( t_1 \) and \( t_k \) are simple we have that \( S(t_1) = t_1 \) and \( S(t_k) = t_k \). As shown in the proof of theorem 1, from the derivation chain for \( t \) in the general model, we can construct a 'derivation chain' in the simple model of the form \( t_1 = S(t_1), \ldots, S(t_k) = t_k \) whose length is polynomial in the length of the derivation in the general model. This completes the proof. □

Combining theorem 1 and theorem 2, we have

**Theorem 3:** The problems SGM and GGM are polynomially equivalent (in the size of the variable set).
7 An Elementary Membership Problem

It is not known whether the GGM problem has polynomial complexity or whether the problem is NP-hard. It is not even known whether it is in NP and, since the GGM problem is polynomially equivalent to the SGM problem (as shown in theorem 3 above), finding the complexity of one problem will provide an answer to the complexity assessment of the other. We define in this section a simple problem whose complexity is not known either. If the simpler problem, which is a particular case of the SGM problem, will be shown to be NP-hard, that would imply that the SGM (and the GGM problem) is also NP-hard - but not the other way around.

Definition: An Elementary Graphoid is a set of triplets of the form \{(γ, Z, y)\}, where γ is a constant singleton, closed under the two axioms below.

(1) \((γ, Z, y) ∨ (γ, Z y, w) → (γ, Z w, y)\): E-contraction

(2) \((γ, Z w, y) ∨ (γ, Z y, w) → (γ, Z, y) ∨ (γ, Z, y)\): E-intersection

Remark 1: Since γ is a constant and fixed singleton one can simplify the notation replacing triplets by pairs, with the singleton γ assumed. The simplified notation is provided below.

(1) \((Z, y) ∨ (Z y, w) → (Z w, y)\): E-contraction

(2) \((Z w, y) ∨ (Z y, w) → (Z, w) ∨ (Z, y)\): E-intersection.

Remark 2: As mentioned above the complexity of the membership problem for elementary graphoids is not known. An elementary graphoid represents the irrelevance relation induced by some fixed random variable and may be of interest in some applications.
References


