MANAGING THE SHAPE OF PLANAR SPLINES BY THEIR CONTROL POLYGONS

by

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Managing the Shape of Planar Splines by their Control Polygons

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A practical formulation of the shape preservation theorem is presented. It states how to position the control points to obtain a prescribed curve shape and avoid shape anomalies. The concept of hyper convex polygons is introduced. These polygons may be intersected by straight lines at three points, and are therefore not covered by previous versions of the theorem. The extended theorem addresses all possible control point configurations, and is proved for important classes of planar B- and Beta2-splines.

Introduction

This paper discusses planar spline curves whose shape may be defined by their control points, e.g. B-splines. These curves are specially useful on graphical workstations, where a designer can shape a curve simply by moving the control points by a pointing device, such as a mouse. A designer can develop a feeling for how much to displace a single control point in order to achieve a small modification of a given curve. Larger curve modifications require a coordinated movement of several control points. This can be quite tricky, and a number of trials may be needed to achieve a required shape. The

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number of trials may be reduced by exploiting the shape preservation theorem as formulated in \(^1\) page 80, \(^2\) page 92 and \(^3\) page 266. The essence of these different versions of the theorem is that a convex control point polygon generates a convex B-spline (\(^1\) covers also Bézier curves). The theorem may for instance be exploited by a designer that has to design a curve that is convex and has to meet some further requirements. If the designer only tries convex control polygons, iteration steps involving non convex curves are avoided. It is noted that while the theorems in \(^2\) and \(^3\) consider only convex polygons, the theorem given in \(^1\) also covers inflected polygons, which have both convex and concave parts. Control polygons may however attain some further kinds of shapes, which were not covered by these theorems. It was felt that practitioners need a shape preservation theorem that addresses all the possible polygon shapes. Such a theorem is introduced and proved in this paper.

Previous shape preservation theorems consider a polygon to be convex if it may be intersected by a straight line at two points at the most. There are however cases, where some practitioners regard polygons that intersect a straight line at three points as convex. These practitioners will regard both the convex polygon shown in Figure 1a and the polygon that intersects a straight line shown in Figure 1b as convex polygons. For these practitioners the usual definition of polygon convexity is not natural, and the need to check that the polygon is not intersected by any straight line at three points is felt as a superfluous burden. In order to help these users we extend the shape preservation theorem to include the class of polygons illustrated in Figure 1b, which we define precisely and denote as hyper convex polygons. It will be shown that in some important cases the distinction between convex and hyper convex polygons is not necessary.

Cubic parametric splines may have undesired shape anomalies i.e. cusps, loops, two inflections (Figure 2). A cusp is a point on the cubic spline where the unit tangent vector vanishes. If a given looped spline segment is gradually changed into a segment with two inflection points (by moving some of the control points), then a cusp will appear
at the moment of transition between these two different kinds of shapes. The cases when these shape anomalies occur were not considered in the previous shape preservation theorems, but will be addressed in the theorem presented in this paper. A thorough analysis of these shape anomalies for cubic parametric curves was made by Wang\(^4\) who proposed a combination of algebraic and geometric criteria for their detection. An improved method for the analysis of anomalies was introduced by Su and Liu\(^5\). A further simplified method for detection of the anomalies in cubic curves was developed by Stone and Derose\(^6\), and demonstrated on Bézier curves. This simplified method analyzes a single spline segment defined by four control points. The polygon is mapped into a "canonical form", where it may be determined whether the corresponding spline segment has a loop, cusp or inflections. However, this kind of criteria is not ideal for designers at workstations who think in terms of geometrical shapes rather than in their "canonical" representations. The criteria developed in this paper only involve the shape of the control polygon, and may therefore be checked visually on the computer screen without any kind of transformation.

Figure 1. A convex (a) and hyper convex (b) polygon.
The shape preservation theorem presented in this paper is proved to be correct for important kinds of splines, i.e. planar quadratic and cubic uniform B-splines and for planar Beta2-splines for which $\beta_2 \geq 0$. The quadratic B-splines are assumed to be $C^1$ (continuous first derivative), while the corresponding cubic splines are assumed to be $C^2$.

We included the simple quadratic B-splines although most designers seem to prefer cubic splines. The $C^2$ cubic splines are necessary when smooth motion at high speed is required, e.g. rail roads and cam shafts of fast engines. The Beta2-spline is defined to have $C^2$ continuity and has proven to be useful in the modeling of complex shapes. We assume further that $\beta_2 \geq 0$. This kind of Beta2-splines obeys the variation diminishing rule (see page 135 of ), and this will be exploited for controlling the shape. It is noted that Beta2-spline for which $\beta_2 = 0$ is a uniform B-spline (see page 49 in ). The Beta2 spline may thus be considered as a generalization of the uniform cubic $C^2$ B-splines. In the following the notion spline means a spline of one of types specified in this section.
Characterization of the Shape of Splines

The term *shape* is usually employed to characterize the convexity/concavity properties of splines. In this paper we consider only this property. Other characterizations of the form of the curves, such as their *smoothness* and *fairness* are discussed elsewhere in the literature, e.g. page 3 in [1]. The convexity/concavity property of a spline segment can be determined from the sign of its curvature \( \kappa \), which can be computed from its parametric representation \( (x(t), y(t)) \):

\[
\kappa(t) = \frac{\ddot{y}x - \ddot{x}y}{(x'^2 + y'^2)^{3/2}}.
\]

We are only interested in the sign of the curvature, which is equal to the sign of the numerator \( s(t) \) of equation (1) (the singular case, where both \( y=0 \) and \( x=0 \) corresponds to a cusp, which will be treated separately):

\[
s(t) = \ddot{x}y - \ddot{y}x
\]

The shape of a spline segment between two consecutive knots is said to be *convex*, *concave* or *straight*, when for all points in the segment \( \kappa < 0 \), \( \kappa > 0 \) or \( \kappa = 0 \) respectively (see page 23 in [10]). For convex and concave curve segments it is further required that a segment may be intersected at two points at most, by any given straight line. A cubic spline segment may have both convex \( (\kappa<0) \) and concave parts \( (\kappa>0) \) connected at a point of inflection, where \( \kappa = 0 \). Such a spline segment will be called *inflected* if it has only one inflection point. These definitions are illustrated by the curve shown in Figure 3. It is assumed that we walk along the curve in a direction that corresponds to increasing \( t \) value. We are on a convex region when we move clockwise (CW) because \( \kappa < 0 \). We are on a concave segment when we move counterclockwise (CCW) since \( \kappa > 0 \). For cubic spline segments three further kinds of shapes are possible: *loop*, *cusp* and *two inflections*. These shapes have already been discussed in connection with Figure 2, and are usually regarded as undesirable.
Each one of the segments in a quadratic uniform B-spline is a parabola, and therefore has no inflection points (see page 120). However, a knot in a quadratic spline, where a convex segment happens to meet a concave segment will be considered as an inflection point. It is noted that since the quadratic spline is only \( C^1 \) the second derivative at such a point of inflection is probably discontinuous. A designer who employs quadratic splines is however not expecting a curve that is \( C^2 \).

**Figure 3. Example of a curve.** Following the curve from left to right we encounter a convex, a concave, a straight and a convex region.

**Control Polygon Shape Characterization**

The knots of a spline curve divide it into a number of consecutive segments. Each one of these segments is in the case of a quadratic spline defined by three consecutive control points, and in the case of a cubic spline by four control points. In the following we consider four consecutive control points \( P_1, P_2, P_3 \) and \( P_4 \), which define either one cubic segment or two quadratic segments. It is assumed that the four control points \( P_1 \) to \( P_4 \) are ordered in the direction that corresponds to increasing parameter (t) values.
A four points polygon $P_1P_2P_3P_4$ has one of the shape characterizations listed below. These shapes are illustrated in left column of Figure 4, which employ the same numbering as in the list below:

1. **Convex/concave polygon**: a straight line can at most intersect the polygon at two points (disregarding the three lines which coincide with the polygon legs). In a convex polygon (Figure 4 case 1.1a) the movement along the polygon from $P_1$ to $P_4$ represents a CW direction. In a concave polygon the movement from $P_1$ to $P_4$ represents a CCW direction. Whether a given polygon is considered as convex or concave depends thus only on the direction in which we decide to enumerate the nodes. We shall therefore assume in the following that the points have been enumerated such that all the polygons are convex. Note the special case where three of the four points are collinear (Figure 4 case 1.1b). The polygon is also in this case considered as convex.

2. **Hyper Convex Polygon**: the two straight line segments $P_1P_2$ and $P_3P_4$ are on the same side of the middle line $P_2P_3$, and do not intersect each other. Furthermore, there exists a straight line which intersects the polygon $P_1P_2P_3P_4$ at three points.

3. **Straight polygon**: all the four points are collinear. (are on the same straight line).

4. **Inflected polygon**: the two end points $P_1$ and $P_4$ are on opposite sides of the middle line $P_2P_3$.

5. **Looped polygon**: The straight line segments $P_1P_2$ and $P_3P_4$ intersect each other.

To determine the characterization of a given four points polygon in accordance with the above classification one can either inspect the polygon visually or employ the algorithms given in [11] pages 349-351.
<table>
<thead>
<tr>
<th>Column 1</th>
<th>Column 2</th>
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<tbody>
<tr>
<td>1.1 a, convex</td>
<td>![U-shaped curve]</td>
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<tr>
<td>1.1 b, convex with three collinear control points</td>
<td>![U-shaped curve]</td>
</tr>
<tr>
<td>1.2, hyper convex</td>
<td>![U-shaped curve]</td>
</tr>
<tr>
<td>2, straight</td>
<td>![Straight line]</td>
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<tr>
<td>3, inflected or looped</td>
<td>![Inflected line]</td>
</tr>
<tr>
<td>4, looped</td>
<td>![Looped line]</td>
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**Figure 4.** Visual presentation of the shape preservation theorem. The left column shows the different polygon shapes, while the right column shows the shapes of the curves that they generate. The polygon shapes shown in the left column are: (1.1a) convex, (1.1b) convex with three collinear control points, (1.2) hyper convex, (2) straight, (3) inflected or (4) looped.
The Shape Preservation Theorem.

The theorem regards planar splines. We consider uniform B-splines that are either quadratic and \( C^1 \) or cubic and \( C^2 \). We consider also Beta2-splines (Beta splines for which \( \beta_1 = 1 \)), and assume furthermore that \( \beta_2 \geq 0 \). The theorem states that every spline curve segment has essentially the same shape characterization as the corresponding four points control polygon. A specification of the correspondence between the shape of the polygon and the shape of the spline segment that it generates is given below.

A visual presentation of the theorem is shown in Figure 4. It is noted that in the case of a quadratic spline the four point polygon defines two spline segments, while in the case of cubic spline only one segment is generated.

1. Convex and hyper convex polygons generate convex curve segments. Similarly concave and hyper concave polygons generate concave curve segments. The generated convex and concave curve segments have no inflection points and no anomalies. The term "arch" was employed in 6 for these kinds of curve segments.

2. A linear polygon generates a straight line curve segment. In the case of a quadratic B-spline two curve segments are generated from a four point polygon. Therefore when three of the four points are collinear one straight line segment and one curved segment are generated.

3. An inflected polygon generates an inflected curve segment with only one inflection point and with no curve anomalies. In the case of a quadratic B-spline the point of inflection is at the knot between the two segments produced by the four control points.

4. A looped polygon defines a curve segment that may possess one of the anomalies discussed in the introduction and illustrated in Figure 2. It is therefore suggested to avoid looped polygons. Anomalies are however not possible when quadratic B-splines are employed. The existence and the kind of the anomaly in a cubic curve may be determined with the methods of Wang 4 and Stone et Derose 6.
Application of the Theorem

Usually a control polygon is constructed from a large (i.e. > 4) number of control points. In order to determine the shape of a spline the theorem is employed separately on each set of four consecutive control points. This is illustrated in the example of Figure 5, where the shape of curve (a) is determined by analyzing all the four points polygons:

(1) The convex polygon \( P_1P_2P_3P_4 \) generates a convex spline segment.
(2 and 3) The inflected polygons \( P_2P_3P_4P_5 \) and \( P_3P_4P_5P_6 \) generate two inflected spline segments.
(4) The convex polygon \( P_4P_5P_6P_7 \) generates a convex curve segment.

The complexity of the process of determining the shape of the curve is only \( O(n) \), where \( n \) is the number of legs in the control polygon. The curves (a), (b) and (c) in Figure 5 illustrate how the designer can change the shape of a curve by moving a control point. Polygon (b) is derived from polygon (a) by moving \( P_4 \) upward to achieve an entirely convex polygon and curve. Polygon (c) is derived from polygon (b) by moving \( P_4 \) further upward. This causes the introduction of four inflections in the polygon and in the corresponding spline curve.

According to the theorem a looped four point polygon may under certain circumstances (detailed in 4 and 6) produce a loop on the curve. We feel therefore that a designer who wishes a loop on the curve, should avoid these intricacies, and instead simply specify the curve by a large (i.e. > 4) number of control points. This approach is illustrated in figure 6.
Figure 5. Applications of the shape preservation theorem on a cubic B-spline curve.

Figure 6. A looped curve designed by a large number (> 4) of control points.
Proof of the Shape Preservation Theorem

We consider first the case of quadratic B-splines.

Claim: Quadratic B-splines are shape preserving.

Proof: The parametric representation of a quadratic B-spline segment is:

\[ q(t) = (x(t), y(t)) = \sum_{i=1}^{3} B_i P_i, \]  \hspace{1cm} (3)

where the \( P_i \) are the control points and the \( B_i \) are the B-spline base (blending) functions (see for instance page 120):

\[ B_1(t) = \frac{1}{2} (t-1)^2 \]
\[ B_2(t) = \frac{1}{2} (-2t^2+2t+1) \]  \hspace{1cm} (4)
\[ B_3(t) = \frac{1}{2} t^2 \]

Each one of the segments of the spline is a parabola whose two end points coincide with the mid points of the corresponding polygon legs. At these points the parabola is also tangent to the two legs. (see for instance page 120). There are therefore no inflection points and no anomalies of the kinds illustrated in Figure 2. With these observations the shape preservation theorem may be proved by a simple analysis of its four cases.

Cubic splines

The proof is made only with Beta2-splines because cubic uniform B-splines are a special case of the Beta2-spline (\( \beta_2=0 \)). The cubic Beta2-splines may be defined as:

\[ q(t) = (x(t), y(t)) = \sum_{i=1}^{4} B_i P_i, \]  \hspace{1cm} (5)

where \( P_i \) is a control point and \( B_i \) is a Beta2 base function (page 49 in [2]):

\[ B_1(t) = 2\gamma(1-t)^3 \]
\[ B_2(t) = \gamma(\beta_2 + 8 + t^2(-3(\beta_2 + 4) + 2t(\beta_2 + 3))) \]
\[ B_3(t) = \gamma(2t(6 + t(3(\beta_2 + 2) - 2(\beta_2 + 3)))\]
\[ B_4(t) = 2\gamma^3 \]

where

\[ \gamma = \frac{1}{\beta_2 + 12} \]

To facilitate the reading, the same numbering is employed for the four different cases of the theorem and for the corresponding claims in the proof. The proof of case one is further divided into three sub cases.

**Claim 1.1:** A convex polygon generates a convex spline segment with no inflection point and no anomalies.

**Proof:** A convex polygon may at the most be intersected at two points by any given straight line. Since a Beta2-spline for which \( \beta_2 \geq 0 \) obeys the variation diminishing rule (e.g. page 135 of \(^9\)), the generated spline may at the most be intersected at two points by a straight line. Inflections and loops permit three intersections, and are therefore not possible. A cusp is also not possible, because it may be considered as a degenerate loop which permits three intersections.

**Claim 1.2.1:** A hyper convex polygon creates a spline curve segment without loops or cusps.

**Proof:** If there is a loop then the angle between the tangent at \( t = 0 \) and \( t = 1 \) must be greater than \( \pi \) (Figure 7a). There must therefore be a point where the tangent is parallel with the \( x \) axis, i.e. \( y(t) = 0 \) for some \( t \in [0,1] \). It will be shown that such a point does not exist and hence no loop is present. In order to simplify the computations we perform an affine mapping of the given polygon to a canonical form for hyper convex polygons,
as illustrated in Figure 7b. It is quite simple to show that any given hyper convex polygon may be transformed to this canonical form through a number of elementary mappings.

These transformations map the three points \( P_1, P_2 \) and \( P_3 \) to the locations shown in Figure 7b. The position of \( P_4 \) is determined by the same transformation. The investigations can be conducted with the canonical spline because the characteristics of the original curve (inflections, loops and cusps) are preserved in the mapping (see page 148 in \(^6\)). A detailed proof for this preservation is found in \(^7\) page 3. Our task is therefore reduced to prove that for the canonical spline the equation \( \dot{y}(t) = 0 \) has no solutions in \([0,1]\). The equation of the canonical spline curve can be obtained quite easily by inserting the coordinates of the control points from Figure 7b into equations (5) and (6). This yields the same result as a transformation of the equation of the original spline into the canonical space. This is due to the affine invariance property, which is for instance discussed in \(^13\).

For the canonical spline we thus obtain after some simplifications:

\[
y(t) = \frac{-2(\beta_2 + 3 - y_4) t^3 + 3(\beta_2 + 2) t^2 + 6 t + 2}{\beta_2 + 12} \tag{7}
\]

The quadratic equation \( \dot{y}(t) = 0 \) produces the two solutions:

\[
t_1 = \frac{\beta_2 + 2 - \sqrt{(\beta_2 + 4)^2 - 4y_4}}{2(\beta_2 + 3 - y_4)} \tag{8}
\]

\[
t_2 = \frac{\beta_2 + 2 + \sqrt{(\beta_2 + 4)^2 - 4y_4}}{2(\beta_2 + 3 - y_4)} \tag{9}
\]

It is observed that \( t_1 < 0 \), since \( \beta_2 \geq 0 \) and since \( 0 < y_4 < 1 \) (see Figure 7b). \( t_1 < 0 \) means that \( t_1 \notin [0,1] \). What is left is to show that also \( t_2 \) is not in \([0,1]\). This is done by showing that the assumption that \( t_2 \leq 1 \) results in an untrue inequality \( y_4 \geq 3 + \beta_2 \). We have shown that \( \dot{y}(t) \neq 0 \) for any \( t \in [0,1] \), i.e. there is neither a loop nor a cusp (which is also characterized by \( \dot{y}(t) = 0 \)) in the generated spline.

* A general discussion of mapping techniques is found for instance in \(^{14}\) page 16-18.
Claim 1.2.2: A hyper convex polygon generates a spline segment without inflection points. The generated spline is convex (not hyper convex).

Proof: To simplify the proof we employ the same canonical affine mapping for hyper convex polygons as in the proof of claim 1.2.1 above. Our goal is thus reduced to show that there are no inflection points in the mapped curve. The location of possible inflection points may be found by solving the equation $s(t) = 0$, where $s(t)$ is given by the expression (2). In this equation we employ the expression of $x(t)$ and $y(t)$ from (5) and (6). After some simplifications we obtain a quadratic equation in $t$:

$$s'(t) = t^2(\beta_2(x_4+1)+2(x_4+2-y_4))-2t(\beta_2-(x_4+y_4-4))\beta_2+4 = 0 \quad (10)$$

The goal is now to show that the two possible real solutions $t_1$ and $t_2$ of equation (10) can not be in $[0, 1]$. First we compute...
From the affine mapping (Figure 7b) follows that:

$$0 < x_4, y_4 < 1$$  \hspace{1cm} (13)

Furthermore \( \beta_2 \geq 0 \). Therefore \( s(0) > 0 \) and \( s(1) > 0 \); so there must be an even number of inflection points. The quadratic equation (10) has thus either two or no solutions in \([0, 1]\).

If there are two solutions \( t_1 \) and \( t_2 \) in \([0, 1]\) then:

$$t_1 t_2 \leq 1$$  \hspace{1cm} (14)

\( t_1 t_2 \) may be computed from equation (10) as

$$t_1 t_2 = \frac{(\beta_2 + 4)}{\beta_2 (x_4 + 1) + 2(x_4 + 2 - y_4)}$$  \hspace{1cm} (15)

Inserting (15) into (14) yields

$$y_4 \leq \frac{\beta_2 + 2}{2} x_4$$  \hspace{1cm} (16)

We complete the proof by showing that equation (10) has no real solutions when (16) is satisfied. To do this we show that the discriminant of equation (10) (divided by 4):

$$(x_4 + y_4)^2 - (\beta_2 + 4)^2 x_4$$  \hspace{1cm} (17)

is negative. We substitute \( y_4 \) in (17) by the higher or equal value taken from (16)

$$(\beta_2 + 4)^2 x_4 - \frac{x_4}{4} - 1$$  \hspace{1cm} (18)

This expression is negative for \( 0 < x_4 < 4 \), and we know from (13) that \( 0 < x_4 < 1 \).

We can show now that the produced spline segment is a convex curve (not hyper convex). We have shown above that \( s(0) > 0, s(1) > 0 \) and that the equation \( s(t) = 0 \) has no solution for \( 0 \leq t \leq 1 \), which mean that \( s(t) > 0 \) in the curve segment. We must further show that a straight line may at the most intersect the curve at two points. In order for a straight line to intersect this curve at three or more points the angle between the tangents at the points corresponding to \( t=0 \) and to \( t=1 \) must be larger than \( \pi \). There must therefore
be a point, where \( y(t) = 0 \), \( 0 \leq t \leq 1 \). It has been shown in 1.2.1 that such a point does not exist.

Claim 2: A polygon where all the nodes are on the same straight line generates a straight line curve segment.

Proof: The convex hull of the polygon is the section of the straight line that contains the control points. The spline curve must therefore be a segment of the same straight line section.

Claim 3: An inflected polygon generates a spline segment with only one inflection point and with no loop or cusp.

Proof: The locations of possible inflection points may be found by solving (2). It will be shown that this equation has only one solution for \( 0 \leq t \leq 1 \). To simplify the calculations we employ a canonical affine mapping for inflected control polygon. In this mapping the control points are placed in the locations shown in Figure 8. The coordinates of these points are inserted in (5) and (6) to yield the equations of the spline, which are employed in (2). After some simplifications we obtain:

\[
S(t) = (\beta_2 (y_4 - 1) + 2 (x_4 + y_4 - 1)) t^2 + 2 (\beta_2 + 3 + y_4 - x_4) t - (\beta_2 + 4) = 0 \quad (20)
\]

First we compute \( S(t) \) at the two end points of the spline segment.

\[
S(0) = -(\beta_2+4) < 0 \quad (21)
\]

\[
S(1) = y_4 (\beta_2+4) > 0 \quad (22)
\]

\( S(t) \) had opposite signs at its two ends because \( \beta_2 \geq 0 \) and \( y_4 > 0 \) (see Figure 8). The number of inflection points in the spline segment must therefore be odd, and there must be at least one such point. However since quadratic equation (20) has at the most two real roots we conclude that there is only one inflection point in the spline segment. A loop or
a cusp is not possible in this spline segment because it has exactly one inflection point (see page 200 and page 149).

\[ P_1(-1,-1), P_2(-1,0), P_3(0,0), P_4(x_4,y_4) \]

**Figure 8.** Affine mapping employed for inflected polygon and its associated curve.

**Claim 4:** A looped polygon generates a spline segment which is either a convex curve or a curve which has either a loop or a cusp or two inflection points.

**Proof:** Wang and Stone analyzed looped control polygons in depth and showed that the four curve types mentioned in the claim may occur.

**Discussion and Conclusions**

The shape preservation theorem presented in this paper extends earlier versions of the theorem by addressing the cases of hyper convex and looped polygons. With these additions, the presented theorem covers all possible polygon configurations. The concept
of hyper convex polygons, introduced in this paper, extends the concept of convex polygons in a natural and useful way. The hyper convex polygon extends the range of applicable convex splines which were not addressed by previous formulations of the theorem. Practitioners that are not familiar with the formal definition of convex polygons would probably also consider hyper convex polygons as being convex. Fortunately for these practitioners, the distinction between convex and hyper convex polygons is not required for the investigated kinds of B and Beta2 splines. It was shown that both kinds of polygons produce convex curves.

The extended shape preservation theorem presented in this paper has been proved for some important kinds of splines, i.e. quadratic $C^1$ and cubic $C^2$ planar uniform B-splines, as well as planar Beta2-splines for which $f_2 > 0$. Further research is required into the shape preservation properties of other kinds of splines, and especially of the important rational B and Beta-splines. Rational B-splines obey the variation diminishing rule (see for instance [14]) and are therefore shape preserving at least as regards convex polygons. The example shown in Figure 9 demonstrates, however, that rational B-splines are not necessarily shape preserving when it comes to hyper convex polygons. In figure 9a all weights are 1. A hyper concave polygon generates a concave curve, and is thus shape preserving. In figure 9b a high weight is put on $P_2$ and low weight on $P_3$. A hyper concave polygon produces this time a convex curve, and is thus not shape preserving. These examples illustrate that the magnitudes of the weights determine whether the curve is convex or concave, and whether the shape of the given hyper convex control polygon is preserved. Bézier splines preserve the shape of convex and inflected polygons (see [1] page 80). Hyper convex polygons may however generate Bézier spline segments with undesired anomalies. This possibility may be observed from the diagram in Figure 8 on page 156 of [4] and is demonstrated by the example shown in Figure 10, where (a) shows a convex polygon which generates a nice convex curve. (b) shows a hyper convex polygon which generates a curve with a loop.

The problem in (b) is because the end points of the Bézier curve are enforced to coincide with the end points of the control polygon. This is opposed to B and Beta2
spline, where the end points of the curves are sufficient free to attain positions that
preserve the shape of the polygon.

Figure 9  Example showing that a rational B spline may not preserve the shape of a hyper
convex polygon.

The shape preservation theorem presented in this paper may be employed in the
design of spiral and looped shaped curves that are composed of convex spline segments
only. An example of such a loop shaped curve is shown in Figure 6.

It is expected that the formulation of the shape preservation theorem presented in
this paper will be practical for the designers using workstations. As opposed to previous
versions it addresses all possible control polygon configurations. The theorem is intui-
tive, and its essence is presented in a single engineering handbook type diagram (Figure
4). It should therefore be relatively easy for a designer to configure the control points
such that a required curve shape is obtained, and such that undesired shape anomalies
are avoided.
Figure 10. Example of a hyper convex polygon creating a looped Bezier curve segment.

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