ON SEYMOUR'S AND LOMONOSOV'S PLANE INTEGRAL
TWO COMMODITY FLOW RESULTS

by

E. Korach and M. Penn

Technical Report #617
March 1990
ON SEYMOUR’S AND LOMONOSOV’S PLANE INTEGRAL TWO COMMODITY FLOW RESULTS

Ephraim Korach
Computer Science Department
Technion–Israel Institute of Technology
Haifa, Israel
32000

and

Michal Penn
Faculty of Commerce and Business Administration
The University of British Columbia
Vancouver, BC, Canada
V6T 1Y8

* This work was done as part of the author’s D. Sc. Thesis in the Faculty of Industrial and Management Engineering, Technion–Israel Institute of Technology, Haifa.
Abstract

We consider the maximum integral two-commodity flow problem in augmented planar graphs (i.e., with both source-sink edges added) and provide an $O(|V|^2 \log |V|)$ algorithm for that problem. Let $G = (V, E)$ be a graph and $w : E \rightarrow \mathbb{Z}^+$ a weight function. Let $T \subseteq V$ be an even subset of the vertices of $G$. A $T$-cut is an edge-cutset of the graph which divides $T$ into two odd sets. Lomonosov gave a good characterization of augmented planar graphs for which the maximum two-commodity flow is integral. We derive Lomonosov's characterization by using results of Seymour on integral packing of $T$-cuts in the case $|T| = 4$, and on the correspondence between plane integral multicommodity flow and integral packing of $T$-cuts.
1 Introduction

Let $G = (V, E)$ be an undirected graph and let $w : E \rightarrow Z^+$ be a weight (capacity) function. For a given graph $G$ with $q$ source-sink pairs $(s_k, t_k) = f_k$ the integral multicommodity flow problem can be defined as follows. For $k = 1, \ldots, q$ find a flow of value $r_k$ between $s_k$ and $t_k$ presenting the flow of commodity $k$, if one exists such that the total flow for every edge does not exceed its capacity. Even Itai and Shamir [4] have shown that this problem is NP-complete, even when $F, F = \{f_i : i = 1, \ldots, q\}$, consists of two nonadjacent edges. For the event $G$ is an augmented planar graph (i.e., with both source-sink edges added) In [12] an $O(|V|^2)$ algorithm is presented for the half integral multicommodity flow problem, that is where the flows can obtain either integral or half integral values. Barahana showed $O(|V|/\log |V|)$ algorithm for that case [1]. For $V' \subseteq V$, $\delta(V')$ denotes the set of edges with one end in $V'$ and the other in $V \setminus V'$. Let $T \subseteq V$ with $|T|$ even then a $T$-cut is a set of edges of the form $\delta(S)$ with $S \subseteq V$ and $|S \cap T|$ odd.

In this paper we provide a new characterization of certain planar graphs for which maximum packing of $T$-cuts is integral (theorem 5). We derived this characterization by using the following: (i) Seymour's theorems on integral packing of $T$-cuts in the case $|T| = 4$. (ii) The connection between maximum integral weighted packing of $T$-cuts and integral plane multicommodity flow. Theorem 5 enable us to present a simpler proof for Lomonosov's characterization of these planar graphs for which maximum two commodity flow is integral. A main contribution of this paper is an $O(|V|/\log |V|)$ simple algorithm for the maximum integral two-commodity flow problem in augmented planar graphs that is derived from Theorem 5.

2 Definitions and Notations

Let $(G, w)$ be a pair of an undirected graph $G = (V, E)$ and $w : E \rightarrow Z^+$ a weight function. Let $E' \subseteq E$ then $w(E') = \sum \{w(e) : e \in E'\}$. Without loss of generality we may assume that $G$ is loopless and without parallel edges.

2.1 Definitions

2.1.1 A flow with a set of sources $X$ and a set of sinks $Y$, where $X, Y \subset V$ and $X \cap Y = \emptyset$
is a function $f : V \times V \rightarrow R^+$ where $f(v, u) = 0$ and the following conditions hold:
\[
d_f(v) = 0 \text{ for } \forall v \in \{X \cup Y\}, \quad d_f(v) \geq 0 \text{ for } \forall v \in X \text{ and } d_f(v) \leq 0 \text{ for } \forall v \in Y,
\]
where
\[
d_f(v) = \sum (f(v, u) - f(u, v)) : (u, v) \in E
\]

The quantity $d_f(X) = -d_f(Y)$ is denoted by $\| f \|$, where $d_f(X) = \sum (d_f(v) : v \in X)$

### 2.1.2 A Z-flow is a family of flows $\{ f_z : z \in \mathcal{Z} \}$ such that:

(i) $f_z$ is a flow between one end of $z$ to the other end.

(ii) (the joint capacity constraints)
\[
\sum (f_z(x, y) + f_z(y, x)) : z \in \mathcal{Z} \leq w(x, y) \quad \forall (x, y) \in E.
\]

### 2.1.3 Let $\max f^{(i)}(Z; w) = \max \sum (\| f_z \| : z \in \mathcal{Z}, \{ f_z \} \in Z$-flow in $G$)
\[
\max f^{(i)}(Z; w) = \max \sum (\| f_z \| : z \in \mathcal{Z}, \{ f_z \} \text{ an integral } Z$-flow in $G$).
\]

### 2.1.4 Let $S \subseteq V$, $Z \subseteq E$, $\delta(S)$ be a cut, then $\min \delta(Z; w) = \min \{ w(\delta(S)) : S \subseteq V, Z \subseteq \delta(S) \}$ if at least one such $S$ exists.

### 2.1.5 $(G, w)$ is called $Z$-special for a given $Z = \{z_1, z_2\}$ if there exists a 6-partition $(R_1, R_2, S_1, S_2, T_1, T_2)$ of $V$ such that:

(i) $z_i = (s_i, t_i)$ where $s_i \in S_i$ and $t_i \in T_i$; $i = 1, 2$.

(ii) $w(\delta(S_1 \cup S_2)) = w(\delta(T_1 \cup T_2)) = w(\delta(S_1 \cup R_1 \cup T_2)) = \min \delta(Z; w)$.

(iii) both $w(\delta(R_1))$ and $w(\delta(R_2))$ are odd.

### 3 Plane Integral Multicommodity Flows and T-Cuts

Seymour [13] pointed out on an interesting correspondence between the problem of integral optimum packing of $T$-cuts and the integral multicommodity flow problem.

The integral multicommodity flow problem can be formulated as follows: Let $G = (V, E)$ be a graph, $F \subseteq E$ and let $w : E \rightarrow Z^+$. Does there exist a collection of circuits $C$ and a function $\phi : C \rightarrow Z^+$ such that:

### 3.1 (i) $\forall C_i \in C : |C_i \cap F| = 1$.

(ii) $\forall f \in F : \sum (\phi(C_i : f \in C_i) = w(f)$ and
(iii) \( \forall e \in E \setminus F : \sum \{ \phi(C_i) : e \in C_i \} \leq w(e) \)

where \( F \) is the set of edges such that \( f_i \in F \) connects the source \( s_i \) to the sink \( t_i \) and \( w(f_i) \) is the demand for the \( i^{th} \) commodity.

A necessary condition for the existence of such a flow is known as the cut condition, i.e., for every cut \( D = \delta(X), X \subseteq V : w(D \cap F) \leq w(D \setminus F) \).

Even Itai and Shamir [4] have shown that the integral multicommodity problem is \( NP \)-complete, even when \( F \) consists of two nonadjacent edges. Few cases have been solved; a partial list in given in [7].

If \( G = (V, E) \) is a plane graph and \( G^* = (V^*, E^*) \) is its dual, the following dual relations are well known (see Bondy and Murty [2]):

3.2 (i) Faces of \( G \) correspond to vertices of \( G^* \).
(ii) Edges of \( G \) correspond to edges of \( G^* \).
(iii) Circuits of \( G \) correspond to coboundaries of \( G^* \).

It is easy to see that the multicommodity flow problem in a plane graph \( G = (V, E) \) where \( F \subseteq E \) as expressed in terms of \( G^* \) is: Let \( F^* \subseteq E^* \) and \( w^* : E^* \rightarrow \mathbb{Z}^+ \) be given.

Does there exist a collection of coboundaries \( D^* \) in \( G^* \) and a function \( \phi^* : D^* \rightarrow \mathbb{Z}^+ \) such that:

3.3 (i) \( \forall D^* \in D : |D^* \cap F^*| = 1 ; \)
(ii) \( \forall f^* \in F^* : \sum \{ \phi^*(D^*) : f \in D^* \} = w^*(f^*) \) and
(iii) \( \forall e^* \in E^* \setminus F^* : \sum \{ \phi^*(D^*) : e^* \in D^* \} \leq w^*(e^*) \).

Let \( T^* = \{ v^* \in V^* : |\delta(v^*) \cap F^*| = \text{odd} \} \).

Let \( T \subseteq V, |T| \) even \( T^- \)-join, \( F \), is a minimal set of edges so that \( T \) is exactly the set of all vertices in \( (V, F) \) with odd valency. Hence \( F^* \) is a union of a \( T^- \)-join and circuits.

It is well known that \( F^* \) is a minimum weight \( T^- \)-join if and only if for every circuit \( C^* \) in \( G^* \) we have:

\[ w^*(C^* \cap F^*) \leq w^*(C^* \setminus F^*) \]

If \( w(e) = w^*(e^*) \) for every \( e \in E \) then the last condition is satisfied if and only if the cut-condition is satisfied.
Clearly, every coboundary \( D^* \in D^* \) is a \( T^* \)-cut and the collection \( D^* \) is a packing of \( T^* \)-cuts in \( (G^*, T^*) \) with value \( w'(F^*) \). So,

3.4 (Seymour [13]) We can solve the integral multicommodity flow problem in planar graphs by solving the problem of maximum weight integral packing of \( T \)-cuts.

4 Lomonosov’s and Seymour’s Results

Below we present Lomonosov’s and Seymour’s Theorems. In the sequel based on Seymour’s results we will derive Lomonosov’s theorems.

Theorem 4.1 (Lomonosov [8]): Let \((G, w)\) be a graph and \( Z = (z_1, z_2) \subset E \). Suppose that \( G \) is planar. Then either

4.1.1 \[ \max f(Z; w) = \min \delta(Z; w), \]

4.1.2 \[ \max f(Z; w) = \min \delta(Z; w) - 1. \]

Moreover, 4.1.2 holds if and only if \((G, w)\) is \( Z \)-special.

Corollary 4.2 (Lomonosov [8]): Let \( G \) be a planar graph with \( Z = (z_1, z_2) \subset E \), if:

4.2.1 \[ \min \delta(z_1; w) + \min \delta(z_2; w) \neq \min \delta(Z; w) \] then 4.1.1 holds.

Remark 4.3: Let \( Z = (z_1, z_2) \). If \( z_2 \in \delta(z_1; w) \), where \( \delta(z_1; w) \) is the cut for which the minimum obtained then, \( \min \delta(z_1; w) = \min \delta(Z; w) \), so the problem can be reduced to maximum flow from \( s_1 \) to \( t_1 \) and obviously 4.1.1 holds.

4.4 Let \( T \subseteq V, T = \{t_1, t_2, t_3, t_4\} \). Let \( A_{ij} \) be a shortest path from \( t_i \) to \( t_j \) \( 1 \leq i < j \leq 4 \) and \( w(A_{ij}) \) the length (weight) of the path.

Theorem 4.5 (Seymour, [19]): For \( k \in \mathbb{Z}^+ \), the following are equivalent:

4.5.1 Maximum integral packing of \( T \)-cuts is greater or equal to \( k \).

4.5.2

\[ w(A_{12}) + w(A_{34}) \geq k \]
\[ w(A_{13}) + w(A_{24}) \geq k \]
\[ w(A_{14}) + w(A_{23}) \geq k \]
and if equality holds in all three inequalities of 4.5.2, then \( w(A_{ij}) + w(A_{jk}) + w(A_{ik}) \) is even for each choice of \( i, j, k \) in \( T \).

**Theorem 4.6 (Seymour, [18])** If \( |T| = 4 \) then, (maximum integral weighted packing of \( T \)-cuts) \( \geq (\text{min}\{w(F) : F \text{ a } T \text{-join }\} - 1) \)

**Theorem 4.7 (Seymour, [18])** Suppose \( \{t_1, \ldots, t_4\} \subseteq V \) are distinct vertices, \( k \in Z^+ \) is such that:

\[
\begin{align*}
w(A_{12}) + w(A_{34}) &= k \\
w(A_{13}) + w(A_{24}) &= k \\
w(A_{14}) + w(A_{23}) &= k,
\end{align*}
\]

and \( w(A_{12}) + w(A_{34}) + w(A_{1k}) \) is odd. Then, \( A_{12}, A_{34} \) are vertex disjoint. (Similarly so are \( A_{31}, A_{24} \) and \( A_{14}, A_{12} \)).

5 Analogous Results on Integral Packing of \( T \)-cuts to Lomonosov’s Results

In what follows we present and prove Theorem 5.9 which is the analogue result to Theorem 4.1 for the integral packing of \( T \)-cuts problem, and Lemma 5.10 which corresponds to Corollary 4.2. The aim of this section is to prove Theorem 5.9 and Lemma 5.10 which in the next section will lead us to an alternative simpler proofs for Lomonosov’s results, and to the maximum plane integral two commodity flow algorithm.

In order to achieve this goal we first present and prove some other results.

Let \( G + F \) denote the graph obtained from \( G \) by adding a set of edges \( F \).

**5.1 Definition:** Let \( (G, w) \) be a weighted graph, (possibly with parallel edges), \( T \subseteq V \), \( T = \{t_1, t_2, t_3, t_4\} \), \( f_1 = \{t_1, t_2\}, f_2 = \{t_3, t_4\}, F = \{f_1, f_2\} \) and \( G' = G + F \). Let \( k = \text{min}\{w(C \setminus F) : F \subseteq C, C \subseteq C\} \) where \( C \) is the family of all cycles in \( G' \). Let \( w'(f_1) = w(A_{12}) \), \( w'(f_2) = k - w(A_{12}) \), \( A_{12} \) as in Definition 4.4 then \( (G', w') \) is the extended graph of \( (G, w) \) where \( w'(e) = w(e) \) if \( e \notin F \) and for \( f \in F \) \( w'(f) \) as defined above.

By 3.4, we can solve an integral a multicommodity flow problem in an augmented planar graph by solving a maximum integral packing of \( T \)-cut problem in the planar dual.
Example: The following graph $G^*$ is what we call a connected planar graph.

5.3 Definition Let $G'$ be the extended graph of $(G, w)$ such that $w'(f_1) \geq 0$, then $F$ is an optimal $T$-join in $(G', w)$.

Proof: Let $C_1 = A_{12} \cup \{f_1\}$ then $w'(C_1) = 2w'(f_1)$. Clearly $w'(C \cap F) \leq w'(C \setminus F)$ for each cycle $C$ such that $f_1 \not\in C$. Let $C$ be a cycle such that $f_2 \not\in C$ and $f_1 \not\in C$ then:

5.2.1 $w(C) \geq 2w'(f_1)$. If not, then

$$w'(C) < 2w'(f_1) \Rightarrow w'(C \Delta C_1) \leq w'(C_1) < 2w'(f_1) + 2w'(f_1) = 2k$$

5.2.2 Since $F \subset (C \Delta C_1)$ and $w'(F) = w'(A_{12}) + k - w'(A_{12}) = k$ we have that $w'(C \Delta C_1 \setminus F) < k$, a contradiction to the definition of $k$.

Obviously for each cycle $C$ such that $F \subset C$ the inequality $k = w'(C \cap F) \leq w'(C \setminus F)$ holds. From the definition of $w'(f_1)$ every cycle $C$ such that $C \cap F = f_1$ is not a negative cycle (we say that $C$ is a negative cycle relative to $F$ if $w(C \cap F) > w(C \setminus F)$). From 5.2.1 every cycle $C$ such that $C \cap F = f_2$ is also not a negative cycle. Therefore there is no negative cycle in $(G', w')$ and from a theorem by Meigu [9] $F$ is an optimal $T$-join.

Let $G^*$ be the planar dual graph of $G$. It is known that cuts in $G$ correspond to cycles in $G^*$, vertices in $G$ correspond to faces in $G^*$. Analogously, a dual expression to $Z$-special graph is what we call $F$-special graph defined below:

5.3 Definition Let $(G', w')$ be the extended graph of $(G, w)$ such that $G'$ is a 2-edge connected planar graph. $(G', w')$ is called $F$-special if the following holds:

5.3.0 There exists a 6-partition $(S_1, S_2, T_1, T_2, R_1, R_2)$ of the faces of $G'$ with boundaries $C(X)$, $X \in \{S_1, S_2, T_1, T_2, R_1, R_2\}$, satisfying:

5.3.1 $f_1 \in C(S_i) \cap C(T_i)$ \quad $i = 1, 2$

5.3.2 $w(C(S_1) \Delta C(S_2)) = w(C(T_1) \Delta C(T_2)) = w(C(S_1) \Delta C(T_2) \Delta C(R_1)) = 2k = k + w(F)$

5.3.3 both $w(C(R_1))$ and $w(C(R_2))$ are odd.

Example: The following graph $\Gamma^*$ (the dual graph of $\Gamma$) is $F$ special while $\Gamma$ is $Z$-special.
Lemma 5.6 Let \((G, w)\) be a planar \(2\)-edge-connected graph such that \(Z \subseteq E\), and let \(w\) be a weight function on \(E\) such that \(w(e) = w(z) \quad \forall e \in E \setminus Z\) and \(w(z) = 0 \quad \forall z \in Z\). Then it is easily implied from the definition of \(Z\)-special graphs that \((G, w)\) is \(Z\)-special if and only if \((G, w)\) is \(Z\)-special. Moreover if \((G, w)\) is \(Z\)-special then \(\delta(Z; w) = \min \delta(Z; w) + w(Z)\). Therefore, from now on we will assume that \(w(Z) = 0\). If \((G', w')\) is the extended graph of \((G, w)\) then we can assume that \(w'(f_1) \geq 0\), because otherwise there is an integral maximum two-flow from the following reason. If \(w'(f_1) < 0\) then \(\delta(Z; w') < \min \delta(z_i, w')\). Hence, there is an integral flow of value \(\min \delta(Z; w)\) from \(s_1\) to \(t_1\), and this flow is a max two commodity flow.

Lemma 5.6 Let \((G, w)\) be a planar \(2\)-edge-connected graph such that \(Z \subseteq E\). Let \(G^*\) be the dual graph of \(G\) with \(F = Z^* \subseteq E^*\). Let \(G = G^* \setminus F\) and \(w(z) = w(e) \quad \forall e \in E\). Let \((G^*, w^*)\) be an extended graph of \((G, w)\), then \((G, w)\) is \(Z\)-special if and only if \((G^*, w^*)\) is \(F\)-special.

Proof: It is not difficult to see from the dual relations 3.2, i.e. edges of \(G\) correspond to edges of \(G^*\) and circuits of \(G\) correspond to coboundaries of \(G^*\), that

\[
5.6.1 \quad k = \min \{w'(C^* \setminus F) : F \subseteq C^*, C^* \subseteq C^* \} = \min \delta(Z; w) \quad \text{where } C^* \text{ is a collection of cycles of } G^*.
\]

Also, since vertices of \(G\) correspond to faces of \(G^*\) we have that \((S_1, S_2, R_1, R_2, T_1, T_2)\)

\[
w(e) = 1 \quad \forall e \in E - Z
\]
\[
w(z_i) = 0 \quad i = 1, 2
\]
\[
w(e^*) = 1 \quad \forall e^* \in E^*
\]
is a $\delta$-partition of $V$ if and only if $(S'_1, S'_2, R'_1, R'_2, T'_1, T'_2)$ is a 6-partition of the faces of $G^*$, with $(C^*(S'_1), C^*(S'_2), C^*(R'_1), C^*(R'_2), C^*(T'_1), C^*(T'_2))$ as the respective boundaries.

Clearly $z_i \in \delta(S_i) \cap \delta(T_i)$ $i = 1, 2$ if and only if $j_i \in C^*(S'_i) \cap C^*(T'_i)$ $i = 1, 2$.

From the 6-partition of $V$, Property 5.4, the definition of $w^*(f_i)$ and 5.6.1 we have that, $w(\delta(S_i \cup S_t)) = w(\delta(T_i \cup T_t)) = w(\delta(S_i \cup R_i \cup T_t)) = \min \delta(Z; w)$ if and only if $w(C^*(S'_i) \Delta C^*(S'_t)) = w(C^*(T'_i) \Delta C^*(T'_t)) = w(C^*(S'_i) \Delta C^*(R'_t) \Delta C^*(T'_t)) = 2k = k + w(F)$.

Obviously both $w(\delta(R_i))$ and $w(\delta(R_t))$ are odd if and only if both $w(C^*(R'_i))$ and $w(C^*(R'_t))$ are odd. Hence $(G, w)$ is $Z$-special if and only if $(G^*, w^*)$ is $F$-special. $\square$

Procedure 5.7 and Lemma 5.8 are needed for the proof of Theorem 5.9; hence these two results are presented below.

Let $(G, w)$ be a graph $T \subseteq V, T = \{t_1, t_2, t_3, t_4\}, A_{ij}$ a shortest path from $t_i$ to $t_j$.

Procedure 5.7 provides us with a constructive way for building the collection $A$ of shortest paths, $A = \{A_{ij} : 1 \leq i, j \leq 4\}$, such that $A_{ij} \Delta A_{ik} k \neq i \neq j, 1 \leq i, j, k \leq 4$ is a simple path.

Procedure 5.7: Choose a collection of shortest paths $A_{ij}, 1 \leq i, j \leq 4$. If each $A_{ij} \Delta A_{ik} k \neq i \neq j, 1 \leq i, j, k \leq 4$ is a simple path, we are done.

If not, let us construct the members of the collection $A$ in the following order: $A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}$, by the following method. Let $A_{ij}$ be a shortest path from $t_i$ to $t_j$ in $(G, w^{(i)})$, where $w^{(i)}$, defined as follows.

$$w^{(i)}(e) = w(e) \quad \forall e \in E \quad \text{and} \quad w^{(k+1)}(e) = \begin{cases} w^{(i)}(e) - \varepsilon & \text{if } e \in A_{ij} \\ w^{(k)}(e) & \text{otherwise} \end{cases}$$

for $\varepsilon < \frac{1}{10k+1}$. Then $A_{ij}$ is the unique shortest path from $t_i$ to $t_j$ in $(G, w^{(k+1)})$, and $A$ is the collection of the shortest paths in $(G, w^{(i)})$. Moreover, since there are only 6 paths and $\varepsilon < \frac{1}{10k+1}$ we have that $A$ is a collection of shortest paths in $(G, w)$ such that $A_{ij} \Delta A_{ik} k \neq i \neq j, 1 \leq i, j, k \leq 4$ is a simple path. (Note that $E(A_i) = \{\cup E(A_{ij}) : j \in \{1, 2, 3, 4\} - \{i\}\}$ form a tree, $A_i$, of shortest paths from $t_i$ to $t_j$; $j = 1, 2, 3, 4$). $\square$

If $G = (V, E)$ is a graph, $E' \subseteq E$ then $G/E'$ denote the graph we obtain by contracting $E'$.

**Lemma 5.8** Let $(G, w)$ be a weighted graph, $T = \{t_1, t_2, t_3, t_4\} \subseteq V$ and let $(G', w')$ be the extended graph of $(G, w)$. Assume that there is no optimal packing of $T$-cuts which is integral. Then there exists a set of 6 edges $E_0 \subseteq E(A)$ ($A$ the collection of 6 shortest
paths as defined previously in Procedure 5.7), \( E_\theta = \{ e_{ij} : 1 \leq i < j \leq 4 \} \) such that 
\( e_{ij} \not\in \cup \{ A_nk : 1 \leq n < k \leq 4, n \neq i,j \} \), \( E(A)/(E(A)\setminus E_\theta) = K'_4 \) with \( V(K'_4) = T \) and \( K'_4 \) is isomorphic to \( K'_4 \).

**Proof:** First let us show that there exists \( e_{12} \in A_{11} \setminus \cup_{n \neq 12} A_{nk} \). If \( A_{12} \subset A_{13} \) then \( w'(A_{12}) + w'(A_{12}) + w'(A_{13}) \) is even contradiction to Theorem 4.5. Hence, \( A_{12} \nsubseteq A_{13} \).

In a similar way it can be shown that \( A_{12} \nsubseteq A_{14} \). Since \( A_1 \) is a tree we have that \( A_{12} \nsubseteq (A_{13} \cup A_{14}) \).

Let us define \( E_{12} = A_{12} \setminus (A_{13} \cup A_{14}) \), clearly \( E_{12} \neq \phi \), and \( E_{12} \) is a path which is a subpath of \( A_{12} \) from 2 to some vertex \( v \) which we denote by \( A_{2v} \).

Assume that \( E_{12} \subset A_{24} \). If \( (A_{12} \cap A_{14}) \subset A_{14} \) then \( A_{24} \subset (A_{12} \cup A_{14}) \) and \( w'(A_{12}) + w'(A_{12}) + w'(A_{14}) \) is even, a contradiction to Theorem 4.5. If \( (A_{12} \cap A_{14}) \subset A_{12} \), then \( v \in A_{13} \) contradicting the fact that \( A_{13} \) and \( A_{24} \) are vertex disjoint. Hence \( E_{12} \nsubseteq A_{24} \) and we denote \( E_{12} = A_{12} \setminus (A_{13} \cup A_{14} \cup A_{24}) \) and \( E_{12} \) is not empty. \( E_{12} \) is a path which is a subpath of \( A_{24} \) from \( v \) to some vertex \( u \). With similar arguments it can be shown that \( E_{12} \subseteq A_{33} \), and \( E_{12} \) is defined as \( E_{12} = A_{12} \setminus (A_{13} \cup A_{14} \cup A_{24} \cup A_{23}) \). It follows from Theorem 4.7 that \( E_{12} \) and \( A_{24} \) are vertex disjoint, hence \( E_{12} = A_{12} \setminus \cup_{n \neq 12} A_{nk} \) is not empty, and \( e_{12} \) could be any edge belong to \( E_{12} \).

The existence of the other edges in \( E_\theta \) could be derived in a similar way. It is not difficult to verify that \( E(A)/(E(A)\setminus E_\theta) = K'_4 \) with \( V(K'_4) = T \) and \( K'_4 \) is isomorphic to \( K'_4 \). \( \square \)

We shall present and prove Theorem 5.9 which is the analogue result of Theorem 4.1 (Lomonosov) for the integral packing of \( T \)-cuts problem. Theorem 4.1 gives necessity and sufficiency of the conditions for the existence of maximum two commodity flow which is integral in \( (G,w) \). Since Lomonosov gave a short and simple proof for the sufficiency of the condition (i.e., if \( (G,w) \) is \( Z \)-special, then \( \max f(0)(Z,w) < \min \delta(Z,w) \)); this condition will be used for proving the sufficiency of the condition in Theorem 5.9.

On the other hand, Lomonosov's proof for the necessity of the condition in Theorem 4.1 is very complicated. Our proof for the necessity of the condition of Theorem 5.9, implies a simpler proof for the necessity of the condition of Theorem 4.1.

Seymour [13] characterized the graphs with \( |T| = 4 \) for which there exists an optimal packing of \( T \)-cuts which is integral. Based on Seymour's results Theorem 5.9 presents a different characterization of planar extended graphs with \( |T| = 4 \) for which there exists an optimal packing of \( T \)-cuts which is integral.
Theorem 5.9  Let \((G, w)\) be a connected weighted planar graph, and let \((G', w')\) be its extended planar graph. The following are equivalent:

5.9.1 There is no optimal packing of \(T\)-cuts which is integral.

5.9.2 \((G', w')\) is \(F\)-special.

Proof: A. Assume first that \(G'\) is 2-edge connected graph.

A.1. Let us first prove the necessity of the condition i.e. 5.9.1 \(\Rightarrow\) 5.9.2.

Let it be a collection of shortest paths that we get as a result of Procedure 5.7. Let us define:

5.9.3 \(C(S_1) = A_{12} \cup \{f_1\} = A_{12} \Delta \{f_1\}\)

\(C(S_2) = A_{34} \cup \{f_2\} = A_{34} \Delta \{f_2\}\)

\(C(R_1) = A_{14} \Delta A_{12}\)

\(C(R_2) = A_{24} \Delta A_{23} A_{34}\)

\(C(T_1) = A_{12} \Delta A_{34} \cup \{f_1\} = A_{12} \Delta A_{34} \Delta \{f_1\}\)

\(C(T_2) = A_{12} A_{34} \cup \{f_1\} = A_{12} A_{34} \Delta \{f_2\}\)

where, in this subsection we use \(\sim j\) to denote the edge-set of \(\sim j'\).

From 5.9.3 and procedure 5.7 one can see that these circuits are 6 simple circuits in \(G\).

In 5.9.7 it will be shown that these circuits define a 6-partition of the faces of \(G'\).

From 5.9.3 we have that \(f_i \in C(S_i) \cap C(T_i)\), i = 1, 2. I.e., \((G', w')\) satisfies 5.3.1.

In order to prove that \((G', w')\) satisfy 5.3.2, 5.3.3 and that there exists a 6-partition of the faces of \(G'\) as required, the following claims are needed.

It follows from Lemma 5.2 that \(F = \{f_1, f_2\}\) is an optimal \(T\)-join in \((G', w')\), therefore there is no negative circuit relative to \(F\), in particularly:

\[ w'(A_{14}) + w'(A_{32}) \geq w'(A_{14} \Delta A_{32}) \geq w'(f_1) + w'(f_2) = k \]

\[ w'(A_{34}) + w'(A_{12}) \geq w'(A_{34} \Delta A_{12}) \geq w'(f_1) + w'(f_2) = k \]

\[ w'(A_{12}) + w'(A_{34}) \geq w'(A_{12} \Delta A_{34}) \geq w'(f_1) + w'(f_2) = k \]

Since \(w'(F) = k\) (the value of the optimal primal solution) then the value of the optimal packing of \(T\)-cuts (the value of dual solution) have to be less than or equal to \(k\).

Using Seymour's results (Theorem 4.5 and Theorem 4.7) and our assumption 5.9.1, we get:
5.9.4 \( w'(A_{14}) + w'(A_{32}) = k \)
\( w'(A_{14}) + w'(A_{12}) = k \)
\( w'(A_{12}) + w'(A_{34}) = k \)

5.9.5 \( w'(A_{ij}) + w'(A_{ij}) + w'(A_{ij}) \) is odd for each choice of \( i, j, k \) in \( T \).

5.9.6 The following pairs of paths are each vertex disjoint \((A_{14}, A_{32}), (A_{24}, A_{12}) \) and \((A_{12}, A_{34})\).

Using 5.9.4, 5.9.5 and 5.9.6 we shall prove that properties 5.3.2 and 5.3.3 hold for \((G', w')\).

From 5.9.5 we get that \( w'(A_{14}) + w'(A_{24}) + w'(A_{12}) \) is odd therefore \( w'(A_{14} \Delta A_{24} \Delta A_{12}) \)
is odd hence \( w'(C(R_1)) \) is odd. In a similar way we can show that \( w'(C(R_2)) \) is odd, and 5.3.3 holds for \((G', w')\).

From 5.9.4 and 5.9.6 we get:

\[(i) \ C(S_1) \Delta C(S_2) = A_{12} \Delta A_{34} \Delta \{f_1\} \Delta \{f_2\} \Rightarrow w'(C(S_1) \Delta C(S_2)) = 2k.\]

\[(ii) \ C(T_1) \Delta C(T_2) = (A_{12} \Delta A_{32} \{f_1\}) \Delta (A_{14} \Delta A_{24} \{f_2\}) = A_{32} \Delta A_{14} \{f_1\} \Delta \{f_2\} \Rightarrow w'(C(T_1) \Delta C(T_2)) = 2k.\]

\[(iii) \ C(S_1) \Delta C(T_1) \Delta C(R_1) = (A_{12} \Delta \{f_1\}) \Delta (A_{14} \Delta A_{24} \Delta A_{12}) \Rightarrow A_{14} \Delta \{f_1\} \Delta \{f_2\} \Rightarrow w'(C(S_1) \Delta C(T_1) \Delta C(R_1)) = 2k.\]

5.9.7 It remains to be shown that these six circuits define a 6-partition of the plane. Clearly \( K_4 \cup F \) which is isomorphic to \( \Gamma^* \) has 6 faces \( F_1, \ldots, F_6 \) with the circuits \( C_1, \ldots, C_6 \) surrounding them, respectively. Our aim is to show that \( C_1, \ldots, C_6 \) correspond to the 6 circuits in 5.9.3 in the obvious way and analogously \( F_1, \ldots, F_6 \) correspond to 6 partition of the faces to regions in \( G' \) with boundaries as listed in 5.9.3 (a subset of faces defined what we call a region, which is the union of the faces in the region). We shall show it by proving the fact that each region in \( E(\mathcal{A}) \cup F \) has one corresponding region in \( K_4 \cup F \). At the first stage, before the contraction, each region is defined by three different but not necessarily disjoint paths from \( \mathcal{A} \). From Lemma 5.8 each \( A_{ij} \) path has an edge \( e_{ij} \) contained only in that path. Therefore each circuit in \( A \) remains a circuit in \( E(\mathcal{A}) \cup E_0 = K_4 \) with \( V(K_4) = T \). Hence \( \mathcal{A} \) defines a 4-partition of the plane and \( \mathcal{A} \cup F \) defines the require 6-partition.
A.2. Based on Lomonosov's result we shall prove the sufficiency of the condition i.e. 

\((G', w')\) is \(F\)-special \(\Rightarrow\) There does not exist an optimal packing of \(T\)-cuts which is integral.

Let \((G', w')\) be the dual graph of \((G', w')\) where \(w'(f') = w'(f)\) for each \(f' \in E' \setminus F\) and \(w'(f_i) = w'(f_i) = 0\) for \(i = 1, 2\). It follows from Lemma 5.6 that \((G', w')\) is \(Z\)-special. From Theorem 4.1 we have that there does not exist an optimal flow which is integral (i.e. 

\[\max f_i(Z; w') \leq \min \delta(Z; w')\]). We shall now show that \(\max f_i(Z; w') \leq \min \delta(Z; w')\) implies that there does not exist an optimal packing of \(T\)-cuts in \((G', w')\) which is integral.

Assume that there exists an optimal packing of \(T\)-cuts in \((G', V')\) which is integral. From 5.6.1 and Lemma 5.2 we have that \(F\) is an optimal \(T\)-join with \(w(F) = k = \min \delta(Z; w')\). Using Seymour's relations to integral multicommodity flow, 3.4, we have that there exists an optimal flow in \((G', w')\) of value \(\min \delta(Z; w')\). Hence \(\max f_i(Z; w') \geq \min \delta(Z; w')\). A contradiction and we have proved that 5.9.2 \(\Rightarrow\) 5.9.1.

B. Assume \((G', w')\) is not 2-edge connected graph. Let \(\hat{E} = (e_1, \ldots, e_k)\) be the set of its cut-edges (for \(G, a\) connected graph, \(e \in E\) is a cut-edge if \(G[e] \) is not connected). \(\hat{E}\) might be empty. Let \(G_1, \ldots, G_k\) be the components of \(G \setminus \hat{E}\). Let \(V(G_j)\) denote the set of vertices of \(G_j\). If \(|T \cap V(G_i)| = 4\) for some \(1 \leq i \leq k\), then by using part A the theorem holds. If not, then \((G', w')\) is not \(F\)-special. It remains to show that in that case either there is no finite optimal packing of \(T\)-cuts or 5.9.1 does not hold.

Assume \((G, w)\) is not connected then either \(|T \cap V(G_i)| = 1\) or \(|T \cap V(G_i)| = 2\) for some \(1 \leq i \leq k\). If \(|T \cap V(G_i)| = 1\), then no \(T\)-join exists and hence there is no finite optimal packing of \(T\)-cuts. If \(|T \cap V(G_i)| = 2\), then it is not difficult to see that 5.9.1 does not hold. Now, assume \((G', w')\) is connected and let \(e_i\) be a cut-edge such that \(|T \cap V(G_i)| \neq 0\) where \(G_i\) is one of the two components of \(G \setminus e_i\). Then either \(|T \cap V(G_i)| = odd or \(|T \cap V(G_i)| = 2\), in both cases it is easy to see that 5.9.1 does not hold. □

**Lemma 5.10** Let \((G, w)\) be a weighted planar graph and \((G', w')\) its extended planar graph. If

\[
\min\{w(C) : f_1 \in C, C \in \mathcal{C}\} + \min\{w(C) : f_2 \in C, C \in \mathcal{C}\} \neq \min\{w(C) : (f_1, f_2) \subset C, C \in \mathcal{C}\} = 2k
\]

then there exists an optimal packing of \(T\)-cuts which is integral.

**Proof:** From Lemma 5.2 \(F\) is an optimal \(T\)-join; hence there are no negative circuits relative to \(F\), especially

\[w(A_{12}) + w(A_{4a}) \geq k\]
6 Alternative Proofs of Lomonosov's Results.

In previous sections we saw the correspondence between maximum two-commodity flow problem and packing of T-cuts. Also we saw some integrality results for packing of T-cuts problem (Theorem 5.9 and Lemma 5.10) which are analogous to Lomonosov's results. Based on these results and Lemma 5.6, we shall present alternative simpler proofs for Theorem 4.1 and Corollary 4.2.

6.1 Alternative Proof of Theorem 4.1: From Remark 5.5 we may assume w.l.o.g. that \( w(x_i) = 0 \), \( i = 1, 2 \). Let \( G' \) be a planar dual graph of \( G \) such that \( F = Z^* \subseteq E' \). Let \( G = G' \setminus F \) and \( \bar{w}(e) = w(e), \bar{e} \) is the dual edge of \( e \), \( \forall e \in E \) and \( \mathcal{T} = \{ \bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4 \} \subseteq \bar{V} \) where \( f_1 = (\bar{f}_1, \bar{f}_2) \) and \( f_2 = (\bar{f}_3, \bar{f}_4) \). Let \( (G', w') \) be the extended graph of \( (G, w) \).

From 5.6.1 and Lemma 5.2, \( k = w^*(F) = \min \delta(Z; w) \) and \( F \) is an optimal T-join. From 3.4 there exists a flow of value \( \min \delta(Z; w) \) and hence an optimal one. Using 3.4 and Theorem 4.6 we conclude that there exists an integral flow of value \( \min \delta(Z; w) - 1 \) in \( (G, w) \), hence \( \max f^{(l)}(Z; w) \geq \min \delta(Z; w) - 1 \).

The second part of the theorem is derived as follows:

By 3.4 we have \( \max f^{(l)}(Z; w) = \min \delta(Z; w) - 1 \) if and only if there is no optimal packing of T-cuts which is integral in \( (G', w^*) \). By Theorem 5.9 there is no optimal packing of T-cuts which is integral in \( (G', w^*) \) if and only if \( (G', w^*) \) is F-special. By Lemma 5.6 \( (G', w^*) \) is F-special if and only if \( (G, w) \) is Z-special. \( \square \)

Remark 6.2: Theorem 6.3 below which was proved by us in [7] is an extention of Theorem 4.6 and hence could replace Theorem 4.6 in the alternative proof of Theorem 4.1.

**Theorem 6.3** Let \( T \subseteq V \) be an even subset of vertices and \( n_F \) the number of components in an optimal T-join, \( F^* \). Let \( \nu_w(G) \) be the value of an optimal integral packing of T-cuts then \( w(F^*) - \nu_w(G) \leq n_F - 1 \).

6.4 Alternative Proof of Corollary 4.2: Let \( (G^*, w^*) \) be the extended graph as defined in 6.1. Clearly, based on 3.2 and 5.6.1 we have, \( \min \delta(x; w) + \min \ell(x; w) \neq \min \delta(Z; w) \).
if and only if \[ \min\{w(C) : f_1 \in C, C \subseteq \mathcal{C}\} + \min\{w(C) : f_2 \in C, C \subseteq \mathcal{C}\} \neq \min\{w(C) : (f_1, f_2) \subseteq C; C \subseteq \mathcal{C}\} \]

From Lemma 5.10 we get:

There exists an optimal packing of \( T \)-cuts which is integral and 3.4 implies that

\[ \max f^{(0)}(Z; w) = \min \delta(Z; w). \]

7 Algorithms

As a consequence of the previous results we present three polynomial time algorithms. Algorithm 1 is a recognition algorithm. Given \((G', w')\) an augmented planar graph, with \( Z \subseteq E, |Z| = 2 \) find whether there exists an integral two-commodity flow \( f \) such that:

\[ \|f\| \geq \min \delta(Z; w), \]

i.e., find whether there exists an optimal two-commodity flow which is integral in \((G', w')\).

Given \((G', w')\) an augmented planar graph with \( Z \subseteq E, |Z| = 2 \), Algorithm 2 computes the maximum integral two-commodity flow in \((G', w')\).

Algorithms 1 and 2 are consequences of Seymour's theorems and algorithms and are not based on Lomonosov's results.

Given \((G, w)\) an augmented planar graph with \( Z \subseteq E, |Z| = 2 \), \((G, w)\) is a \( Z \)-special graph, then Algorithm 3 provides us with a simple method for defining the 6-partition of the vertices of the graph as described in Definition 2.1.5.

First we shall present this three algorithms and discuss their complexity in the sequel. Since these three algorithms start with identical four steps we shall present these four steps under the preprocessing algorithm.

Preprocessing Algorithm:

Input: planar graph \((G, w), Z \subseteq E, |Z| = 2, w : E \rightarrow Z^+, w(Z) = 0\).

Output: planar graph \((G^*, w^*)\), such that \( G^* \) is the dual graph of \( G \) with \( w^* \) as weight function and if \( |V(Z^*)| = 4 \) then, \( \delta \), a collection of six shortest paths between the ends of \( Z^* \), as described in Procedure 5.7. (Note that in Algorithms 1 and 2 any collection of six shortest paths is suffices).
Step 1: Using max-flow algorithm calculate $\min \delta(Z; w)$.

Step 2: Using Booth and Lueker's algorithm find $G'$, the dual graph of $G$. Let $Z' = F$ and define $w'(e') = w(e)$ for all $e' \in G' \setminus F$ where $e'$ is the dual edge of $e$. $w'(F)$ will be defined in Step 4.

Step 3: If $|V'^*(F)| \leq 3$ stop. Otherwise let $V'^*(F) = T'$ where $V'^*(f_i) = (t_i, t_j)$. Otherwise let $T' = \{t_i, t_i, t_j, t_j\}$. Find $A = \{A_{ij} : 1 \leq i < j \leq 4\}$ a collection of shortest paths in $G' \setminus F$ from $t_i$ to $t_j$ and their length for each $1 \leq i < j \leq 4$ by using Frederickson's algorithm and Procedure 5.7.

Step 4: Let $w^*(f_i) = w^*(A_{12})$, $w^*(f_3) = \min \delta(Z; w) - w(A_{12})$.

End.

Algorithm 1:

Input: $(G, w)$, $Z \subseteq E$, $|Z| = 2$, $w : E \to \mathbb{Z}^+$, $w(Z) = 0$.

Output: An answer to the question: "Does there exist an optimal two-commodity flow which is integral?"

Step 0: Perform the Preprocessing Algorithm.

Step 1: If $|V'^*(F)| \leq 3$ go to Step 5.

Step 2: If:

\[ w^*(A_{12}) + w^*(A_{34}) \geq \min \delta(Z; w) \]
\[ w^*(A_{13}) + w^*(A_{4}^*) \geq \min \delta(Z; w) \]
\[ w^*(A_{4}^*) + w^*(A_{2}^*) \geq \min \delta(Z; w) \]

go to Step 3, otherwise go to Step 6.

Step 3: If there is any strict inequality in Step 2 go to Step 5.

Step 4: If $w^*(A_{12}) + w^*(A_{14}) + w^*(A_{13})$ is odd go to Step 6.

Step 5: There exists an optimal two-commodity flow which is integral. Stop.

Step 6: There does not exist an optimal 2-commodity flow which is integral.

End.
Algorithm 2:

Input: \((G, w)\), \(Z \subset E\), \(|Z| = 2\), \(w : E \rightarrow \mathbb{Z}^+\), \(w(Z) = 0\)

Output: Maximum integral two-commodity flow

Step 0: Perform the Preprocessing Algorithm.

Step 1: If \(|V^*(F)| \leq 3\) there is always an integral packing of \(T^*\)-cuts and a direct way to find the optimal integral flow.

Step 2: If \(|T^*| = 4\) using Seymour's technique find \(P^*\) — an optimal integral packing of \(T^*\)-cuts in \((G^*, w^*)\).

Step 3: Calculate the optimal integral two-commodity flow in \((G, w)\) by the following method: \(\forall e \in E : f(e) = m\) where \(e = (e^*)^*\) and there are \(m\) \(T^*\)-cuts in \(P^*\) that intersect \(e^*\).

End.

As was mentioned before, Algorithm 3 provides a simple method to define a 6-partition of the vertices of a \(Z\)-special graph. This simplifies the method for finding the 6-partition as is implicitly suggested by Lomonosov's long proof of Theorem 4.1.
Algorithm 3:

Input: \((G, w)\)

Output: If the graph is \(Z\)-special then the output is a 6-partition of \(V\) as in the definition of \(Z\)-special graphs.

If not, the output is: "The graph is not \(Z\)-special."

Step 0: Perform Algorithm 1 - If there is an optimal flow which is integral go to the End with the answer: "The graph is not \(Z\)-special."

Step 1: Define the following 6-partition (as defined in the proof of Theorem 5.9) in the following way:

Let \(K_4\) be a subdivision of \(K_4\) with \(T = \{t_1, t_2, t_3, t_4\}\) as \(V(K_4)\), and \(A_{ij}, 1 \leq i < j \leq 4\) (the shortest path from \(t_i\) to \(t_j\) as can be obtained by Procedure 5.7) as the subdivision of \(e = (t_i, t_j), e \in E(K_4), \in K_4\).

\(K_4\) is the following graph:

Let us add \(f_1 = (t_1, t_2)\) and \(f_2 = (t_3, t_4)\) to \(K_4\).

First add \(f_1\) to \(R_4\) so that \(f_1\) divides \(F_1\) into two regions \(S_1^*\) and \(T_1^*\), then add \(f_2\) such that \(F_4\) is divided to two regions \(S_2^*\) and \(T_2^*\). Eventually we have the following graph.
Clearly, \{S_1, S_2, R_1, R_2, T_1, T_2\} is a 6-partition of the plane, with the following circuits, accordingly, as their boundaries. (These circuits were defined previously in 5.9.3.)

\[ C'(S_1) = A_{12} \cup \{f_1\} \]
\[ C'(S_2) = A_{14} \cup \{f_2\} \]
\[ C'(R_1) = A_{14} \Delta A_{24} \Delta A_{12} \]
\[ C'(R_2) = A_{14} \Delta A_{24} \Delta A_{54} \]
\[ C'(T_1) = A_{14} \Delta A_{24} \Delta \{f_1\} \]
\[ C'(T_2) = A_{14} \Delta A_{14} \Delta \{f_2\} \]

Step 2: Let \((S_1, S_2, R_1, R_2, T_1, T_2)\) be the sets of vertices that correspond to the regions in \((G^*, w^*)\) via duality. Then \((S_1, S_2, R_1, R_2, T_1, T_2)\) is the required 6-partition.

End.

Lemma 7.1 The time complexity of the Preprocessing Algorithm is \(O(|V|^2 \log |V|)\).

Proof: By checking each step separately, we shall show that the complexity of the algorithm determined by Step 1.

Step 1: By using a simple transformation we can calculate \(\min \delta(Z; w)\) by calculating max-flow in the following way: Let \(Z = \{x_1 = (t_1, t_2), x_2 = (t_3, t_4)\}\), where \((t_1, t_2)\) be the sources and \((t_3, t_4)\) the sinks. Let us add two vertices, \(s_1\), a super source and \(s_2\),
a super sink. Join \( s_1 \) to the two source \( t_1 \) and \( t_3 \), and \( s_2 \) to the two sinks \( t_2 \) and \( t_4 \).

Let the weight of each of the new added edges be \( M = \sum_{e \in \mathcal{E}} w(e) \).

\[
\begin{align*}
\text{Let the weight of each of the new added edges be } M = \sum_{e \in \mathcal{E}} w(e). \\
\text{Let the weight of each of the new added edges be } M = \sum_{e \in \mathcal{E}} w(e).
\end{align*}
\]

It is not difficult to see that the maximum flow from \( s_1 \) to \( s_2 \) is equal to \( \min \delta(Z; w) \).

We can find the maximum flow from \( s_1 \) to \( s_2 \) by using Sleator flow algorithm \([14]\) in \( O(\sqrt{V} \cdot |E| \log |V|) \). Since in planar graphs \( O(|V|) = O(|E|) \) we have that the complexity of the first step is \( O(|V|^2 \log |V|) \).

The original graph is planar, but after adding the four edges the graph might be nonplanar and therefore we cannot use flow algorithms for planar graphs which are more efficient.

**Step 2:** By using Booth and Lueker's algorithm \([3]\) we can find \( G^* \) the dual graph of \( G \) in \( O(|V|) \).

**Step 3:** For constructing \( A \), a collection of 6 shortest paths from \( t_i^* \) to \( t_j^* \) \( 1 \leq i < j \leq 4 \), we follow Procedure 5.7 and use Frederickson's \([5]\) \( O(|V| \log |V|) \) algorithm for shortest paths.

**Step 4:** Constant.

Hence, the complexity of the algorithm is determined by Step 1 and is \( O(|V|^2 \log |V|) \). \( \square \)

**Theorem 7.2** The complexity of Algorithms 1, 2, and 3 is \( O(|V|^2 \log |V|) \).

**Proof:** Algorithm 1: It is easy to see that Step 1 -- Step 5 take \( O(1) \), hence the complexity of the algorithm is determined by Step 0.

Algorithm 2: The complexity of Step 1 is \( O(|E|) \) and the same holds for Step 2. Since \( G \) is planar, we have that \( O(|E|) = O(|V|) \) and again the complexity of the algorithm is determined by Step 0.
Algorithm 3:

Step 1: $O(1)$.

Step 2: $O(n)$, since from the Preprocessing Algorithm we have the planar representation and at that stage we have to construct its dual. Hence, again, the complexity of the Algorithm is determined by Step 0. □

8 Summary

In this paper we presented alternative simpler proofs for Lomonosov's characterization of these augmented planar graphs $(G, w)$ for which maximum two-commodity flow is integral. These proofs are based on Seymour's results on integral packing of $T$-cuts.

Based on our alternative proofs we presented an $O(|V|^2 \log |V|)$ algorithm for the maximum integral two-commodity flow problem.

Lomonosov in his paper has proved an additional result, namely:

Theorem 8.1 (Lomonosov) [8]: Let $(G, w)$ be an augmented $Z$-special planar graph (with $Z \subseteq E$). Then there exists a subgraph $(H, w_H)$ of $(G, w)$ with $Z \subseteq H$, $w_H(e) = 1$ for $e \in H$ and 0 otherwise, such that

(a) $H$ is a subdivision of $K_4$ with $z_1, z_2$ lying on nonadjacent (subdivided) edges of $K_4$ (shortly $H \setminus Z$ is a subdivision of $\Gamma \setminus Z$ as defined in Figure 1).

(b) $\min_e \delta(Z; w - w_H) = \min_e \delta(Z; w) - 2$.

(c) $(G, w - w_H)$ is not $Z$-special.

We do not have an alternative proof for this result, but it seems that such a proof can be derived by using the results of Section 5 with Seymour's [12] following characterization:

Let $(G, w)$ be a graph, $Z \subseteq E$, $|Z| = 2$. There exists an optimal two-commodity flow which is integral for every $w : E \to Z^+$ if and only if $G \setminus Z$ contains no subdivision of $\Gamma \setminus Z$.

Korach [6] characterized these graphs for which maximum packing of $T$-cuts is integral for $|T| = 6$ and Newman [10] characterized these graphs for the case $|T| = 8$. It would be...
an interesting target to characterize these augmented planar graphs for which there exists
an optimal 3-commodity flow or 4-commodity flow which is integral by using Korach and
Newman results respectively. The characterization of the graph with maximum packing
of T-cuts which is integral for $|T| = 6$ and $|T| = 8$ are much more complicated then for
the case that $|T| = 4$; hence it seems that the characterizations for the 3-commodity or
4-commodity would be much more complicated then for 2-commodity flow case.

ACKNOWLEDGEMENT

Research of the first author was partially supported by Technion V.P.R. Fund - E. and
J. Bishop Research Fund, and - Argentinian Research Fund. Research of the second author
was partially supported by Technion V.P.R. Fund – the Baltimore Fund for Industrial
Engineering and Management Research.
References:


