THE TWO-CARDINALS TRANSFER PROPERTY AND RESURRECTION OF SUPERCOMPACTNESS

by

S. Ben-David and S. Shelah

Technical Report #598
December 1989
THE TWO-CARDINALS TRANSFER PROPERTY AND RESURRECTION OF SUPERCOMPACTNESS

Shai Ben-David
Department of Computer Science
Technion-Israel Institute of Technology
Haifa, Israel

Saharon Shelah
Department of Mathematics
The Hebrew University
Jerusalem, Israel

ABSTRACT

We show that the transfer property $<\mathcal{N}_1, \mathcal{N}_0> \rightarrow <\lambda^+, \lambda>$ for singular $\lambda$, does not imply (even) the existence of a non-reflecting stationary subset of $\lambda^+$. The result assumes the consistency of ZFC with the existence of infinitely many super compact cardinals. We employ a technique of "resurrection of supercompactness". Our forcing extention destroys the supercompactness of some cardinals, to show that in the extended model they still carry some of their compactness properties (such as reflection of stationary sets), we show that their supercompactness can be resurrected via a tame forcing extention.
1. INTRODUCTION

The results presented in this paper extend our previous work on the relative strength of combinatorial properties of successors of singular cardinals.

In a seminal paper [J-72] Jensen has presented a collection of combinatorial properties that hold in the constructable universe $L$. From the point of view of applications of set theory to other branches of mathematics, these properties are 'all you have to show about $L$'. Ever since that paper these properties were applied to a wide spectrum of questions to provide consistency results inside set theory as well as in other branches of mathematics ([S-74], [E-80], [F-83], to mention just a few).

It seems natural to ask to what degree can these properties replace the axiom $V = L$? Is there any combinatorial principle that implies all these properties? What is the relative strength of these properties? What are the implication relations among them?

The picture seems to be basically settled for limit cardinals and for successors of regular cardinals, [Mi-72], [G-76]. Our investigations has focused on successors of singular cardinals. Essentially we have been able to prove, assuming the consistency of the existence of large cardinals, that all the nontrivial implications among these properties are not provable in ZFC (see [BS-85], [BS-86], [BM-86]).

Here we examine the strength of the model theoretic two-cardinals-transfer-property $<\kappa_1, \kappa_0> \rightarrow <\lambda^+, \lambda>$. Jensen [J72] has shown that it is implied by $\square$. A quite straightforward argument can show that it implies the weaker $\square^+$ principle.

We show that $<\kappa_1, \kappa_0> \rightarrow <\lambda^+, \lambda>$ does not imply the existence of a non-reflecting stationary subset of $\lambda^+$ for any singular $\lambda$ (as long as ZFC is consistent with the existence of infinitely many super-compact cardinals). It follows that the implications from $\square$ to $<\kappa_1, \kappa_0> \rightarrow <\lambda^+, \lambda>$ is strict and that $\square^+$ (or, equivalently, the existence of a special $\lambda^+$ - Aronszajn tree) does not imply the existence of a non-reflecting stationary subset of $\lambda^+$.

We use a novel technique which we call "resurrection of super compactness". We start with a model $V$ in which $\lambda$ is a limit of super compact cardinals and therefore all stationary subsets of $\lambda^+$ reflect. We extend it through forcing to a model $V[G]$ in which the two cardinal transfer property holds.
Now we have to argue why we still have reflection of all stationary subsets of $\lambda^+$ (although our forcing has inevitably destroyed the supercompactness of a final segment of cardinals below $\lambda$). Instead of applying the commonly used combinatorial analysis to our forcing partial order, we demonstrate the reflection property by showing that we could "resurrect" the supercompactness of any cardinal $\rho$ below $\lambda$ by a further forcing extension $Q_\rho$ that preserves reflection of appropriate subsets of $\lambda^+$.

2. PROOF OUTLINE

The proof is based on a translation of the transfer property to a combinatorial principle $S_\lambda$. We show how $S_\lambda$ (and therefore $<\kappa_1, \kappa_0> \rightarrow <\lambda^+, \lambda>$) can be forced using a "mild" forcing notion. The mildness of the forcing notion guarantees that over certain models where every stationary subset of $\lambda^+$ reflects, such forcing extension would not destroy the reflection.

The natural candidate for exhibiting reflection of all stationary subsets of $\lambda^+$ is a model in which $\lambda$ is a limit of supercompact cardinals. Let $V$ be such a model, standard compactness arguments show that $\square^*_\lambda$ fails and therefore $<\kappa_1, \kappa_0> \rightarrow <\lambda^+, \lambda>$ fails in $V$. It follows that if we extend $V$ to a model $V[G]$ of $<\kappa_1, \kappa_0> \rightarrow <\lambda^+, \lambda>$ the super compactness of a final segment of the cardinals below $\lambda$ will be destroyed. We wish to show that our extension was mild enough to retain some of the supercompactness consequences - namely the reflection of all stationary subsets of $\lambda^+$.

To exhibit this we use a technique we call "resurrection of supercompactness". We further extend $V[G]$ to a model $V[G][Q]$. We show that $V[G][Q]$ the supercompactness of certain cardinals is resurrected. Consequently, in $V[G][Q]$ we do have the desired reflection principle. All that is left to do is make sure that reflection of some stationary $S \subseteq \lambda^+$ in $V[G][Q]$ can only occur if $S$ was already a reflecting stationary subset of $\lambda^+$ in $V[G]$.

More precisely, for every super compact cardinal $\rho$ below $\lambda$ we exhibit the existence of a forcing notion $Q_\rho$ such that: (i) $Q_\rho$ preserves stationarity of subsets of $S^*_\rho = \{ \alpha < \lambda^+ : \text{cof} \alpha < \rho \}$, (ii) $[G][Q_\rho]$ preserves the supercompactness of $\rho$. As $\lambda$ is a singular limit of super compact cardinals, given any stationary subset $S$ of $\lambda^+$ (in $V[G]$), there is some super compact $\rho$ for which $S \cap S^*_\rho$ is stationary in $\lambda^+$. In
$V[G][\mathcal{Q}_\rho]$, $\rho$ is a super compact cardinal and, by property (i), $S \cap S^S_\rho$ is still stationary so it reflects, i.e. for some $\alpha<\lambda^+$, $S \cap S^S_\rho \cap \alpha$ is stationary in $\alpha$. It follows that $S$ reflects in $V[G]$.

3. THE PRINCIPLE $S_\lambda$

In [S 87] Shelah introduced a box-like principle $S_\lambda$. This principle captures in a combinatorial formulation the model theoretic transfer property $<\kappa_1, \kappa_0> \rightarrow <\lambda^+, \lambda>$.

Definition 1: $S_\lambda$ asserts the existence of a sequence $\langle C^i_\alpha : \alpha<\lambda^+, i<\text{cof}\lambda \rangle$ such that:

1. For every $\alpha<\lambda^+$
   \[ \alpha = \bigcup_{i<\text{cof}\lambda} C^i_\alpha \text{ and } i<j \text{ implies } C^i_\alpha \subseteq C^j_\alpha. \]

2. For every $i<\text{cof}\lambda$, $\text{sup}\{ |C^i_\alpha| : \alpha<\lambda^+\} < \lambda$

3. For $\alpha<\beta$, $\alpha \in C^i_\beta$ implies $C^i_\alpha = C^i_\beta \cap \alpha$.

Theorem (Shelah): For a strong limit singular $\lambda$, $S_\lambda$ is equivalent to $<\kappa_1, \kappa_0> \rightarrow <\lambda^+, \lambda>$. (Actually both properties are equivalent to a seemingly stronger transfer property.)

We refer the reader to [S-87] for the full theorem and its proof. To gain a feeling for the content of the new $S_\lambda$ principle let us demonstrate its strength by proving the following corollary of the above theorem directly.

Corollary 1: For a strong limit singular cardinal $\lambda$, $S_\lambda$ implies $\Box_\lambda^\delta$.

Proof: Let $\langle C^i_\alpha : \alpha<\lambda^+, i<\text{cof}\lambda \rangle$ be a $S_\lambda$ sequence. Define $A_\alpha$ to be $\bigcup_{i<\text{cof}\lambda} \{ s : s \text{ is an increasing sequence } t \subseteq C^i_\alpha, \text{card } t < \lambda \text{ and } S = t \text{ (the closure of } t \text{ is } \alpha) \cup \{ s \subseteq \alpha : |s| < \text{cof}\lambda \}$. As each $C^i_\alpha$ has cardinality less than $\lambda$ and $\lambda$ is a strong limit cardinal $|A_\alpha| \leq \lambda$ for all $\alpha$. Let us see that the sequence $\langle A_\alpha : \alpha<\lambda^+ \rangle$ is a $\Box^\delta_\lambda$ sequence. For any $\delta(\lambda^+)$, if $\text{cof}\lambda < \text{cof}\delta$ then for some $i$, $C^i_\delta$ is an unbounded subset of $\delta$. Let $t_\delta$ be any increasing sequence of members of such a $C^i_\delta$ such that $\overline{t_\delta}$ - the closure of $t_\delta$ in $\delta$, has order type $\text{cof}\delta$ and is unbounded in $\delta$. For any limit point $\beta$ of $t_\delta$, for some $j \geq i \beta \in C^i_\delta$ therefore $t_\delta \cap \beta \subseteq C^j_\delta$ (as $C^j_\delta = C^j_\delta \cap \beta$) so $t_\delta \cap \beta$ is a member of $A_\beta$ but $t_\delta \cap \beta = t_\delta \cap \beta$. We still have to handle ordinals $\delta$ of cofinality $\leq \text{cof}\lambda$ but for such an ordinal we can pick any continuous sequence increasing to $\delta$, $t_\delta$, such
5

that $otp(t_\delta) \leq cof\lambda$ and then for any $\beta < \delta t_\delta \cap \beta$ is a subset of $\beta$ of cardinality less than $cof\delta$ so it is a member of $A_\beta$. □

4. THE MAIN THEOREM

Theorem 2: Assuming ZFC is consistent with the existence of infinitely many super compact cardinals, there is a model of SFC with a singular strong limit cardinal $\lambda$ for which $S_\lambda$ holds and every stationary subset of $\lambda^+$ reflects.

Proof: Let $\lambda$ be a singular limit of super compact cardinals. It follows that $\lambda$ is a strong limit cardinal and that every stationary subset of $\lambda^+$ reflects. We define a forcing notion $P$ such that if $G$ is generic for $P$ then $S_\lambda$ holds in $V[G]$, we will show that in $V[G]$, $\lambda$ is still strong limit and every stationary subset of $\lambda^+$ reflects. By iterating Laver's indestructibly forcing [L 78] we may assume that $\lambda$ is a limit of an increasing sequence of super compact cardinals $[\lambda_i; i < cof\lambda]$ such that for each $i < cof\lambda$ if $Q$ is a $\lambda_i$-directed-closed forcing notion than $\lambda_i$ remains supercompact after forcing with $Q$.

Definition of $P$: A condition in $P$ is an initial segment of a $S_\lambda$ sequence, i.e., for some $\beta < \lambda^*$, $P = \{C_\alpha: \alpha \leq \beta, i < cof\lambda\}$ where the $C_\alpha$'s satisfy the demands (i) and (iii) from the definition of a $S_\lambda$ sequence and demand (ii) is replaced by $|C_\alpha| \leq \lambda_4$. We call $\beta dom(p)$. For $p, q \in P, p \leq q$ iff $dom(p) \leq dom(q)$ and $p = q|dom(p)$ (where $q|dom(p)$ denotes the restriction of $q$ to $dom(p)$).

The forcing notion $P$ is the natural candidate for introducing a $S_\lambda$ sequence. We do not know how to guarantee reflection of stationary sets in the model obtained by forcing with $P$, to obtain the model we are aiming at we shall later apply a further forcing extension.

Lemma 3: For $p \in P, \gamma = dom(p)+1$ there is some $q \in P, p \leq q$ such that $dom(q) = \gamma$.

Proof: As $q$ extends $p$, its $\langle C_\alpha; i < cof\lambda, \alpha < \gamma \rangle$ is already defined and we have to define only $\langle C_\alpha; i < cof\lambda \rangle$. Let $C_\gamma = \{\beta\} \cup C_\beta$ where $\beta = dom(p)$ (so $\gamma = \beta+1$). As for all $i < cof\lambda$ we get $\beta \cap C_\gamma = C_\beta$ it is trivial to check that $q$ is a condition in $P$. □

We would like to have some closure properties for $P$. The next Lemma shows that under some circumstances an increasing chain of $P$ conditions is guaranteed to have an upper bound.
Lemma 4: Let \( \langle p_j: j < \delta \rangle \) be an increasing sequence of conditions in \( P \). \( \beta = \text{dom}(p_j) \) and \( \beta = \lim_{j < \delta} \beta_j \).

For each \( \alpha < \beta \) let \( \langle C^\alpha_i: i < \text{cof}(\lambda) \rangle \) be such that whenever \( \alpha \in \text{dom}(p_j) \) this is the \( \alpha \)'s sequence in \( p_j \) (as the \( p_j \)'s form an increasing chain this is well defined).

If there is an unbounded \( C \subseteq \beta \), such that for every \( \alpha < \gamma \) both in \( C \), for some \( i \) \( C^\alpha_i = C^\gamma_i \cap \alpha \), (or for every \( \gamma \) in \( C, C \cap C^\gamma_i \) for some \( i \)) then there is some \( q \in P \) such that \( p_j < q \) for all \( j < \delta \).

Proof: Let \( \text{dom}(q) = \beta \) as \( q \) extends all the \( p_j \)'s

\[ q = \bigcup_{j < \delta} \{ \langle C^\alpha_i: \alpha < p_j, i < \text{cof}(\lambda) \rangle \} \cup \{ \langle C^\beta_i: i < \text{cof}(\lambda) \rangle \} \]

all we have to define is \( \langle C^\beta_i: i < \text{cof}(\lambda) \rangle \). We may assume \( \text{otp}(C) = \text{cof}(\beta) \), otherwise replace \( C \) with such an unbounded closed subset. Let \( \lambda_i \) be the first in \( \langle \lambda_i: i < \text{cof}(\lambda) \rangle \) above \( \text{cof}(\beta) \). For \( i < i_0 \) let \( C^\alpha_i = \emptyset \), for \( i_0 \leq i \) let \( C^\beta_i = \bigcup_{\alpha \in C} C^\alpha_i \). As \( \bigcup_{\alpha \in C} C^\alpha_i = \alpha \) for all \( \alpha \in C \) and \( \bigcup_{\alpha \in C} \alpha = \beta \) we get \( \bigcup_{i < \text{cof}(\lambda)} C^\beta_i = \beta \). As for each \( \alpha \) \( \langle C^\alpha_i: i < \text{cof}(\lambda) \rangle \) is an increasing sequence, so is \( \langle C^\beta_i: i < \text{cof}(\lambda) \rangle \). As each \( \lambda_i \) is a regular cardinal, \( |C^\alpha_i| \leq \lambda_i \) and \( \text{otp}(C) \leq \lambda_{i_0} \) we get \( |C^\beta_i| \leq \lambda_i \) for all \( i < \text{cof}(\lambda) \).

We are left with the coherence demand (iii). Assume \( \delta \in C^\beta \). Let \( y \) be the first member of \( C \) above \( \delta \). \( C^\delta_i = \bigcup_{\alpha \in C} C^\alpha_i \) so \( C^\delta_i \cap \gamma = \bigcup_{\alpha \in C} C^\alpha_i \cap \gamma \) for \( \alpha > \gamma \) \( C^\alpha_i \cap \gamma = C^\gamma_i \) and for \( \alpha < \gamma \) \( C^\alpha_i \subseteq C^\gamma_i \) (as \( C^\gamma_i \cap \alpha = C^\alpha_i \)).

We may conclude that \( C^\beta_i \cap \gamma = C^\gamma_i \) and \( \delta \in C^\gamma_i \). Pick some \( p_\rho \) in the sequence of conditions such that \( \text{dom}(p_\rho) \geq \gamma \) so \( p_\rho(\gamma) = \langle C^\gamma_i: i < \text{cof}(\lambda) \rangle \) as \( \delta \in C^\gamma_i \), \( p_\rho \models "C^\gamma_i \cap \delta = C^\gamma_i" \), for all \( p_j \) with \( j > \rho \) have \( p_j(\delta) = p_\rho(\delta) \) and this is \( q(\delta) = \langle C^\beta_i: i < \text{cof}(\lambda) \rangle \). It follows that \( q \models "C^\beta_i \cap \delta = C^\beta_i" \), as needed.

Lemma 5: \( P \) is \( \mu \)-strategically closed for all \( \mu < \lambda \).

Proof: Given any such \( \mu \) we have to define a strategy for player I such that if an increasing sequence of conditions \( \langle p_i: i < \delta \leq \mu \rangle \) is constructed and for any even and limit \( < \mu \), \( p_\rho \) is defined by applying our strategy to \( \langle q_i: i < \rho \rangle \) then there is a condition \( p_\delta \) above all members of the sequence. Denote by \( i_0 \) the first \( i \) such that \( \lambda_i > \mu \).

Our strategy will have the property

\[(*) \quad \text{For } \rho_1 < \rho_2 \text{ if } p_{\rho_1}, p_{\rho_2} \text{ are both defined by the strategy, then for all } i < \text{cof}(\lambda) \]

\[ C(\text{dom}(p_{\rho_2})) = C(\text{dom}(p_{\rho_2}) \cap \text{dom}(p_{\rho_1})). \]

Let us define the strategy.
Case (i): $p$ is a limit ordinal. $E_p = \{\text{dom}(p_i): i \text{ is even or limit}\}$ is an unbounded increasing subset of $\text{dom}(p)$, along which (*) holds. We use Lemma 4 to define $p$. Note that the definition of Lemma 4 does satisfy (*) for $p_1, p_2 \in E_p \cup \{p\}$.

Case (ii): $p = 2$. Let $p_1$ be any one-level extension of $p$. By Lemma 3 such an extension exists.

Case (iii): $p$ is a successor ordinal. Let $\gamma$ be $\text{dom}(p_{p-1})$ and let $i^*$ be the maximal $i < p$ for which $p_i$ is defined by the strategy (of Player I). Such an $i^*$ always exists as Player I gets to play at limit stages. Let $\beta = \text{dom}(p_i)$. As $p_{p-1}$ extends $p_{i^*}$, for all $\alpha \leq \beta$, $p_{p-1}(\alpha) = p_{i^*}(\alpha)$ and we denote it by $\langle C^{i^*}_i: i < \text{cof}(\lambda) \rangle$ for $\beta < \alpha < \gamma$ let $\langle C^{i^*}_i: i < \text{cof}(\lambda) \rangle$ enumerate $p_{p-1}(\alpha)$.

$p_\omega$ will be a one-level extension of $p_{p-1}$ so we have to define only its last level $p_\omega(\mu+1) = \langle C^{i^*}_i: i < \text{cof}(\lambda) \rangle$. Let $j$ be the first such that $\beta \in C^{i^*}_j$. For $i < i_0$ let $C^{i^*+1}_i = \emptyset$ for $i_0 \leq i < j$, $C^{i^*+1}_i = C^{i^*}_i$ for $j \leq i$, $C^{i^*+1}_i = \{\gamma\} \cup C^{i^*}_i$. If $j < i_0$, for $i < i_0$ $C^{i^*}_i = \emptyset$ as $\langle C^{i^*}_i: i < \text{cof}(\lambda) \rangle$ was defined using the same strategy so the definition remains unchanged. It is easy to check that (*) is satisfied and the the $p_\omega$ thus defined is a condition in $P$ extending $p_{p-1}$. To show that this strategy works we just have to invoke Lemma 4 and by (*) $E_p$ contains a set $C$ as assumed by the lemma.

Lemma 6: For any $p \in P$ and $\text{dom}(p) < \alpha < \lambda^+$ there is an extension $q$ of $p$ such that $\text{dom}(q) \geq \alpha$.

Proof: Assume, by way of contradiction that there is a $\beta < \lambda^+$ so that there is some $p$ with no extension $q$ of $p$ satisfying $\text{dom}(q) \geq \beta$. Let $\beta_0$ be the first such $\beta$ and $p$ such a condition. If $\beta_0$ is a successor apply Lemma 3 to a $q'$ extending $p$ with $\text{dom}(q') = \beta-1$. If $\beta$ is a limit ordinal pick an increasing sequence $\langle a_i: i < \mu < \lambda \rangle$ unbounded in $\beta_0$ ($\lambda$ is singular so $\text{cof}(\beta) < \lambda$). Now play a game of length $\mu$ such that $p_{\beta_0} = p$. Player I uses the strategy and player II picks at state $i+1$ an extension of $p_i$ with Domain at least $a_i$, such an extension exists as $a_i < \beta_0$. Now $\langle p_i: i < \mu \rangle$ has some $q$ extending all $p_i$'s so necessary $\text{dom}(p) \geq \beta_0$ contradicting the choice of $\beta_0$.

Lemma 7: If $G$ is a generic set for $P$ then

(i) $S_\lambda$ holds in $V[G]$.

(ii) $V$ and $V[G]$ share the same cardinals, power function and cofinalities.

Proof: (i) Naturally we define in $V[G]$ the sequence $\langle C^{i^*}_i: \alpha < \lambda^*, i < \text{cof}(\lambda) \rangle$ as the union of all $p(\alpha)'s$ for conditions $p$ in $G$ and $\alpha$'s in their domain. Clearly it is a $S_\lambda$ sequence.
(ii) By Lemma 5 no subsets of size < \lambda are added to V by G, as \lambda is singular no subsets of size \lambda are added to V. Therefore cardinals \leq \lambda^+ are not collapsed and cofinalities \leq \lambda^+ are not changed. As \lambda is a strong limit cardinal |P| = \lambda^+ so it trivially satisfies the \lambda^{++}. c.c. so cardinals and cofinalities above \lambda^+ are preserved.

In V[G] we introduce a further forcing notion R. Let \langle C^\beta_\alpha : \alpha < \lambda^+, i < \text{cof} \lambda \rangle be the P generic S_\lambda sequence. A condition r \in R is a closed bounded subset of \lambda^+ such that \alpha \in r implies that for some i < \text{cof} \lambda, C^\alpha_i is not bounded in \alpha. R is ordered by end extensions. R is designed to introduce a closed unbounded subset in \lambda^+, along which, for each \alpha some C^\alpha contains an unbounded subset of \alpha.

The model in which our theorem is realized is V[\mathbb{P}*R] so we will study the properties of P*R (rather than those of R).

Let us work in the ground model V. P*R can be represented as the set of all pairs (p, r) such that p \in P, and p \not\models "r \in R". Note that as P does not introduce any new sets of size \leq \lambda, each member of R is in V. It is easy to see that p \not\models "r \in R" iff r \subseteq \text{dom}(p)+1, for every \alpha \in r there is some C^\alpha in p(\alpha) that is unbounded in \alpha, and of course r is closed (as a subset of \text{dom}(p)+1).

Lemma 8: \{(p, r) : (p, r) \in P*R and sup(r) = \text{dom}(p)\} is dense in P*R.

Proof: Given any (p, r) \in P*R define a one-level-extension q of p as in the proof of Lemma 3. As \beta(=\text{dom}(p)+1) = \text{dom}(q), is a member of each C^\beta we may define r' = r \cup \{\beta\} to get (q, r') in P*R above (p, r).  

From now on let us assume that all the members of P*R have this property (the second coordinate is a closed cofinal subset of the domain of the first).

Lemma 9: For any (p, r) \in P*R and any \alpha < \lambda^+ there is some (p', r') \geq (p, r) such that \alpha \in \text{dom}(p') = \text{sup}(r').

Proof: This is an easy consequence of Lemmas 8 and 6.

Lemma 10: P*R is \mu-strategically-closed for any regular \mu < \lambda.

Proof: The strategy for player I will be an adaptation of the strategy presented in the proof of Lemma 5. Let \langle (p_i, r_i) : i < \rho \rangle be the sequence played so far and define (p_\rho, r_\rho) - the next condition picked by player I.
We start with the successor stages. For \( p = 2 \) we pick any level extension of \((\mathcal{P}_1, \mathcal{R}_1)\). For \( p \) successor bigger than 2, we modify the definition of the \( C^i_{\mathcal{P}_1} \) from Lemma 5 by defining for \( i < i_0 \), define \( C^i_{\mathcal{P}_1} = \emptyset \) for \( i_0 \leq i < j \) let \( C^i_{\mathcal{P}_1} = \{ \beta \} \cup C^j_{\mathcal{P}_1} \) and for \( j \leq i \) \( C^i_{\mathcal{P}_1} = \{ \gamma \} \cup C^j_{\mathcal{P}_1} \). As \( \gamma \in C^i_{\mathcal{P}_1} \) for \( i \geq j \) we may define \( r_\gamma = r_{\mathcal{P}_1} \cup \{ \gamma \} \). Now we are left with the limit stages. Let \( \gamma \) be \( \bigcup_{i \in \langle \mathcal{P}_1 \rangle} \text{dom}(p_i)\) (= \( \bigcup_{i \in \langle \mathcal{P}_1 \rangle} \text{sup}(r_i) \)). We repeat the definition of the \( \mathcal{P} \) part of Lemma 5 \( C^i_\alpha = \bigcup_{\alpha \in E_\gamma} C^i_\alpha \) (the union of the \( C^i_\alpha \) for all \( \alpha \)'s where \( \alpha = \text{dom}(p_i) \) for \( p_i \)'s played by player \( i \)).

The \( \mathcal{R} \) part can only be \( r_\gamma = \bigcup_{i \in \langle \mathcal{P}_1 \rangle} \{ \gamma \} \). We have to verify that indeed \( \langle p_\gamma, r_\gamma \rangle \in P^*R \). The only potential problem is that maybe there is no \( C^i_\gamma \) is unbounded in \( \gamma \), but as for \( \alpha \in E_\gamma \text{dom}(p_i) \in C^i_\alpha \) where \( i \) is any even, such that \( \text{dom}(p_i) < \alpha \), we get \( E_\gamma \subseteq C^i_\gamma \) so \( C^i_\gamma \) is unbounded in \( \gamma \).

**Lemma 11:** Forcing with \( P^*R \) does not add sets of size \( \leq \lambda \) to the ground model, does not collapse cardinals or chang cofinalities and introduces a \( \kappa \)-sequence \( \langle C^i_\alpha : i < \text{cof}(\lambda), \alpha < \lambda^+ \rangle \) and a closed unbounded subset \( C \subseteq \lambda^+ \) such that for \( \alpha \in C \) some \( C^i_\alpha \) is unbounded in \( \alpha \).

**Proof:** The proof is just a straight forward adaptation of the proof of lemma 7.

The next step is, of course, to prove that in \( V[\dot{P}^*\dot{R}] \) every stationary subset of \( \lambda^+ \) reflects.

Let \( S \subseteq \lambda^+ \) be stationary. As \( \lambda \) is singular, \( S_\mu = \{ \alpha \in S : \text{cof}(\alpha) = \mu \} \) is stationary in \( \lambda^+ \) for some regular \( \mu < \lambda \). As \( S_{\text{max}} \subseteq S \) if \( S_{\text{max}} \) reflects then so does \( S \).

For a supercompact \( \kappa \) and an ordinal \( \rho \) if \( \text{cof}(\rho) > \kappa \) then every stationary subset of \( S_\kappa = \{ \alpha : \alpha < \rho, \text{cof}(\alpha) < \kappa \} \) reflects. Working in \( V[\dot{P}^*\dot{R}] \) we define partial orders \( Q^i_\lambda \) for every regular \( \lambda_i \), \( i < \text{cof} \lambda \) and every stationary subset \( S \subseteq \{ \alpha < \lambda^+ : \text{cof}(\alpha) < \lambda_3 \} \).

Each \( Q^i_\lambda \) satisfies:

(i) \( P^*R \cdot Q^i_\lambda \) is a \( \lambda_i \)-directed closed forcing notion (in \( V \)).

(ii) If \( Q^i_\lambda \) is generic for \( Q^i_\gamma \) over \( V[\dot{P}^*\dot{R}] \) then in \( V[\dot{P}^*\dot{R} \cdot Q^i_\lambda] \) \( S \) is a stationary subset of \( \lambda^+ \).
If such $Q^\lambda_\xi$ exist then, working in $[\dot{P}\times\dot{R}]$, for every stationary $S \subseteq \lambda^+$, pick $i < \text{cof}(\lambda)$ such that $S_{\lambda_i} = \{\alpha \in S : \text{cof}(\alpha) < \lambda_i\}$ is stationary then force with the appropriate $Q^\lambda_{\xi_i}$. In this forcing extension $S_{\lambda_i}$ is stationary in $\lambda^{\text{Hyp}}$ and $\lambda_i$ is a supercompact cardinal (as in $V \lambda_i$ was an indistructible supercompact cardinal and $P^*R^*Q^\lambda_{\xi_i}$ is $\lambda_i$-directed-closed). It follows that $V[\dot{P}\times\dot{R}] = \{S_{\lambda_i} : \text{reflects}\}$. I.e. for some $\alpha < \lambda^{\text{Hyp}}$, $S_{\lambda_i} \cap \alpha$ is stationary in $\alpha$. It follows that in $V[\dot{P}\dot{Q}]$, $S_{\lambda_i} \cap \alpha$, is stationary in $\alpha$, hence $S \cap \alpha$ is stationary and therefore $S$ reflects.

We are left with the task of constructing the $Q^\lambda_{\xi_i}$'s.

**Definition of the $Q^\lambda_{\xi_i}$:** We work in $V[\dot{P}\times\dot{R}]$. Let $\langle \alpha < \lambda^+, i < \text{cof}(\lambda) \rangle$ be the $S_{\lambda_i}$ sequence generated by the $P$ generic set $\dot{P}$ and $C$ the closed unbounded subset of $\lambda^+$ generated by $\dot{R}$.

For every $i < \text{cof}(\lambda)$ and stationary $S \subseteq \{\alpha < \lambda^+ : \text{cof}(\alpha) < \lambda_i\}$ let $j_0$ be such that $S_{j_0} = \{\alpha \in C \cap S : C^\xi_{\alpha} \}$ is unbounded in $\alpha$ is stationary. (Recall that by the definition of $C$ for each $\alpha \in S \cap C$ there is such a $j$.) Without loss of generality we may assume $S_{j_0} = S$.

A condition $q \in Q^\lambda_{\xi_i}$ is a bounded subset of $\lambda^+$ such that: (i) $\text{otp}(q) < \lambda_i$, (ii) for all $\delta \in q$, $q \cap \delta \subseteq C^\delta_{\xi_i}$. The order on $Q^\lambda_{\xi_i}$ is that of end-extensions.

Note that the only role of $S$ is determining $j_0$.

**Lemma 12:** For every $S$ and every $\lambda_i$, $Q^\lambda_{\xi_i}$ is (less than) $\lambda_i$ closed.

**Proof:** This is trivial as the definition of a condition is closed under unions of size $< \lambda_i$.

**Definition:** A condition $\langle p, r, q \rangle \in P^*R^*Q^\lambda_{\xi_i}$ is leveled if $\text{dom}(p) = \text{sup}(r) = \text{sup}(q)$.

**Lemma 13:** The following holds in the ground model $V$:

(a) The set of leveled conditions is dense in $P^*R^*Q^\lambda_{\xi_i}$.

(b) The set of leveled conditions of $P^*R^*Q^\lambda_{\xi_i}$ is $\lambda_i$-closed.

**Proof:** We may assume that the minimal condition of $P^*R$ forces that $S$ is a stationary subset of $\{\alpha < \lambda^+ : \text{cof}(\alpha) < \lambda_i\}$ and the value of $j_0$. 

Technion - Computer Science Department - Technical Report CS0598 - 1989
(a) By Lemma 8 we may assume \( \text{dom}(p) = \text{sup}(r) \) as \( q \) is forced by \( (p,r) \) to be a member of \( Q^\lambda_\gamma \) sup \( q \) cannot exceed \( \text{dom}(p) \). Denote \( \text{dom}(p) = \delta + 1 \) and \( \sup q = \gamma \) and \( i^* \) the first \( i \), such that \( \gamma \in C^\gamma_\delta \) and let \( p' \) be the one level extension of \( p \) defined by \( p'(\delta + 1) = \langle C^i_{\delta + 1} : i < \text{cof} \langle \lambda \rangle \rangle \) where

\[
C^i_{\delta + 1} = \begin{cases} 
\emptyset & \text{if } i < j_0 \\
C^i_\gamma & \text{if } j_0 \leq i < i^* \\
C^i_\delta \cup \{\delta + 1\} & \text{if } i^* < i < \text{cof} \lambda
\end{cases}
\]

Let \( r' \) be \( r(\delta + 1) \) and \( q = q \cup \{\delta + 1\} \) now it is clear that \( (p',r',q') \in P'^*R'^*Q^\lambda_\gamma \).

(b) First we note that in \( P'^*R'^*Q^\lambda_\gamma \) if two conditions are compatible then they are comparable (i.e. if \( (p,r,q) \) and \( (p',r',q') \) have a common extension, then one of them is above the other), therefore, the notions of being \( \mu \)-closed and \( \mu \)-directed-closed coincide for this partial order.

Let \( \langle (p_j,r_j,q_j) : j < \rho < \lambda \rangle \) be an increasing sequence of leveled conditions. Let \( \delta = \text{sup}(\text{dom}(p_i) : i < \rho) \) and define \( p_\rho \) by \( \bigcup_{i < \rho} p_\rho(\delta) \) when \( p_\rho(\delta) \) is the sequence \( (C^i_\delta : i < \text{cof} \lambda) \) defined by

\[
C^i_\delta = \begin{cases} 
\emptyset & \text{if } i < j_0 \\
\bigcup_{a \in \mathcal{A}_i} C^i_a & \text{if } j_0 \leq i < \text{cof} \lambda
\end{cases}
\]

Let \( r_\rho = \bigcup_{i < \rho} r_i \cup \{\delta\} \) and \( q = \bigcup_{i < \rho} q_i \). Applying Lemma 4, it is straightforward to check that \( (p_\rho,r_\rho,q_\rho) \) is a condition and extends each \( (p_i,r_i,p_i) \).

\[ \square \]

**Lemma 14:** If in \( V[\check{\mathcal{P}}\check{\mathcal{R}}] \) \( S \) is a stationary subset of \( \{\alpha < \lambda^+ : \text{cof}(\alpha) < \lambda_i \} \) then in \( V[\check{\mathcal{P}}\check{\mathcal{R}}*\check{Q}^\lambda_\gamma] \) \( S \) is stationary in \( \lambda^+(\check{\mathcal{P}}\check{\mathcal{R}}) \) (for every \( \check{Q}^\lambda_\gamma \) generic for \( \check{Q}^\lambda_\gamma \) over \( V[\check{\mathcal{P}}\check{\mathcal{R}}] \)).

**Proof:** We work in \( V[\check{\mathcal{P}}\check{\mathcal{R}}] \). Assume, by way of contradiction, \( q_0 \in \check{Q}^\lambda_\gamma \) forces some \( \check{Q}^\lambda_\gamma \) name \( \check{\tau} \) to be realized as a closed unbounded subset of \( \lambda^+(\check{\mathcal{P}}\check{\mathcal{R}}) \) disjoint from \( S \). Apply the usual elementary submodels chain technique to construct, for some \( \delta \in S \), an increasing sequence of conditions \( \langle q_\rho : i < \rho < \lambda_i \rangle \) in \( \check{Q}^\lambda_\gamma \), all of them above \( q_0 \), each forcing more members of \( C^i_\delta \) to belong to \( \check{\tau} \). As \( \check{Q}^\lambda_\gamma \) is \( \lambda_i \)-closed, there is a condition on top of this sequence. Such a condition forces "\( \delta \in S \cup \tau" \) (for any realization of \( \tau \) of \( \check{\tau} \)) contradicting the assumption that \( q_0 \models \lnot "S \cup \tau = \emptyset" \).  

\[ \square \]
5. A GENERALIZATION

Our main theorem is stated in terms of existence of some cardinal $\lambda$ with the desired properties. Using results from [BD-86] we get a generalization of the theorem to every singular $\lambda$.

Theorem 15 (Ben-David):

1) If ZFC is consistent with the existence of a proper class of super compact cardinals then it is consistent with the statement "For every regular cardinal $\lambda$ every stationary subset of $\mathcal{S}_{\lambda}^{\lambda}$ reflects and reflection of subsets of $\mathcal{S}_{\lambda}^{\lambda}$ is retained after forcing with $\mu$-directed-closed forcing notions".

2) Assuming the consistency of ZFC with the existence of $\rho$ many super compact cardinals, the statement of 1) holds for all cardinals $\lambda^+$ such that $cof\lambda \leq \rho$.

**Proof:** This is the content of theorems 4.1, 4.5 and remark 4.7 of [BD-86].

Theorem 16: For every singular cardinal $\lambda$, if ZFC is consistent with the existence of $cof\lambda$ many super compact cardinals then it cannot prove that $<\mathcal{N}_1,\mathcal{N}_0> \rightarrow <\lambda^+,\lambda>$ implies the existence of a non-reflecting stationary subset of $\lambda^+$.

**Proof:** Just note that the conclusion of Theorem 15 is all we need to make our proof of the main theorem go through.
REFERENCES


[F-83] Fleissner, W.G., "If all Moore spaces are metrizable, then there is an inner model with a measurable cardinal". Trans. of the American Mathematical Society, 273, pp. 365-373, 1983.


