MAXIMAL RANK OF $m \times n \times (m \ n-k)$ TENSORS

by

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ABSTRACT

It is shown that the maximal rank of $m \times n \times (m \cdot n - k)$ tensors with $k \leq \min ((m - 1)\frac{3}{2}, (n - 1)\frac{3}{2})$ is greater than $m \cdot n - 4\sqrt{2} \cdot k + O(1)$.

1. INTRODUCTION

A classical problem in algebraic computational complexity is to determine the minimal number of non-scalar multiplications required to evaluate some set $\sum_{i,j,k} \alpha_{i,j,k} x_i y_j$, $k = 1, \ldots, p$, of bilinear forms in noncommuting variables $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ over a field $F$. This number is equal to the rank of the defining 3-tensor $(\alpha_{i,j,k}) \in F^m \otimes F^n \otimes F^p$, cf. [5].

An interesting problem, which does not depend on the coefficients $\alpha_{i,j,k}$, is the determination of

$$R_F(m, n, p) = \max_{T \in F^m \otimes F^n \otimes F^p} \text{rank of } T,$$

the maximal rank of tensors in $F^m \otimes F^n \otimes F^p$. This problem has been studied quite extensively in [A-1-AS, Gat-J]. Atkinson and Stephens [AS] gave the following general reduction of $R_F(m, n, m \cdot n - k)$ with $k \leq \min \{ m, n \}$ to $R_F(k, k, k^2 - k)$:

$$R_F(m, n, m \cdot n - k) = m \cdot n - k^2 + R_F(k, k, k^2 - k).$$

They conjectured the bound

$$R_F(k, k, k^2 - k) \geq k^2 - \left\lceil \frac{k}{2} \right\rceil,$$
that implies

\[ R_F(m, n, mn-k) \geq mn - \left\lceil \frac{k}{2} \right\rceil. \]

The rank of such type of tensors was discussed in [AL1], [AL2], [AS] and [Gat].

In this paper we shall give a constructive proof of Atkinson and Stephens conjecture and prove the following stronger bound:

If \( k \leq \min \{(m-1)^{3/2}, (n-1)^{3/2}\} \) then

\[ R_F(m, n, mn-k) \geq mn - \sqrt{2k} + O(1). \]  \hspace{1cm} (1)

For tensors of the size \((n, n, \alpha n^2)\), \(1 > \alpha > \frac{1 + \sqrt{5}}{4} = 0.809\), our bound is better than those known from the literature. (See (2) in the next section).

Atkinson and Stephens, cf. [AS], proved that

\[ R_F(m, n, mn-k) = mn - \left\lceil \frac{k}{2} \right\rceil, \]

for \( \min \{m, n\} \geq k, 0 \leq k \leq 4 \). We shall prove the identity for \( \max \{m, n\} \geq k, 0 \leq k \leq 4 \).

2. NOTATION AND AUXILIARY LEMMA

In this section we shall prove the major auxiliary lemma needed for the proof of our theorem. The technique we shall use in the proof was used in [B], [BD], [KB], and [W] to obtain lower bounds for the rank of tensors of certain shapes.

Definition 1. For \( m, n \geq k \) let

\[ \Delta_F(m, n, k) = mn - R_F(m, n, mn-k). \]

It is known, cf. [AS], that if \( k \leq \min \{m, n, r, s\} \) then

\[ \Delta_F(m, n, k) = \Delta_F(r, s, k). \] \hspace{1cm} (1)

The best lower bound known from the literature is, cf. [Ho],

\[ R_F(m, n, p) \geq \frac{mnp}{m+n+p-2}. \] \hspace{1cm} (2)

And the best lower bound for tensors in \( F^m \otimes F^s \otimes F^{n-k} \) is, cf. [Gat]...
Lemma. Let \( r \leq m, s \leq n \) and \( k \leq rs \). Then

\[ \Delta_F(m,n,k) \leq \Delta_F(r,s,k). \]  

Proof. Let \( x = (x_1, \ldots, x_r) \) and \( y = (y_1, \ldots, y_s) \) be vectors of indeterminates. Let \( \mathcal{A} = \{A_1, \ldots, A_t\} \) be a \( t \)-element set of \( r \times s \) \((t < rs)\) matrices such that, the rank of the bilinear forms defined by \( \mathcal{A} \), i.e. of \([x^T A_1 y, \ldots, x^T A_t y]\), is \( R_F(r, s, t) \). Define \( \mathcal{B} = \{B_1, \ldots, B_t, \hat{B}_1, \ldots, \hat{B}_{mn-rs}\} \) by

\[
B_i = \begin{bmatrix}
A_i & 0_{rs-rs} \\
0_{m-rs} & 0_{m-rs}
\end{bmatrix}, \quad i = 1, \ldots, t,
\]

and

\[
\hat{B}_j = \begin{bmatrix}
0_{rs} & C_j \\
D_j & E_j
\end{bmatrix}, \quad j = 1, \ldots, mn-rs,
\]

where \( 0_{kj} \) is the \( k \times l \) zero matrix and \( \{\hat{B}_j\}_{j=1}^{mn-rs} \) are linearly independent matrices.

For a set of matrices \( C \) let \( \delta(C) \) denote the multiplicative complexity of the bilinear forms defined by the matrices of \( C \). We have

\[ R_F(m,n,mn-rs+t) \geq \delta(C). \]

By [BD, Theorem 9] we have

\[
\delta(C) \geq mn-rs + \min_{\lambda_{ij} \neq 0} \delta\left(B_1 + \sum_{i=1}^{mn-rs} \lambda_{ij} \hat{B}_i, \ldots, B_t + \sum_{i=1}^{mn-rs} \lambda_{ij} \hat{B}_i\right).
\]

Because

\[
B_k + \sum_{i=1}^{mn-rs} \lambda_{ik} \hat{B}_i = \begin{bmatrix}
A_k & * \\
* & *
\end{bmatrix},
\]

it follows that

\[
\delta(C) \geq mn-rs + \delta(\mathcal{A}) = mn-rs + R_F(r, s, t).
\]

Now by (4), we obtain

\[ R_F(m,n,mn-rs+t) \geq mn-rs + R_F(r, s, t). \]

Let \( k = rs-t \). Then the lemma follows from the definition of \( \Delta_F \).

The corollary below is a generalization of (1).

Corollary. If \( k \leq \min\{m,n\} \) then for every integers \( r \) and \( s \) we have
Theorem 1. (Atkinson and Stephens conjecture) Let \( k \leq \max \{ m, n \} \). Then

\[
R_p(m, n, m - k) \geq m - \left\lceil \frac{k}{2} \right\rceil.
\]

Proof. Assume \( k \leq n \). By the lemma,

\[
\Delta_F(m, n, k) \leq \Delta_F(2, k, k) = 2k - R_F(2, k, k).
\]

Since

\[
R_F(2, k, k) = k + \left\lceil \frac{k}{2} \right\rceil,
\]

cf. [J, Theorem 3.5], we have

\[
R_F(m, n, m - k) \geq m - \left\lceil \frac{k}{2} \right\rceil.
\]

Remark. According to the proof of the lemma, we can construct a tensor in \( F^m \otimes F^n \otimes F^{m-k} \) of rank \( m - \left\lceil \frac{k}{2} \right\rceil \) as follows:

Let \( \mathcal{A} = \{A_1, \ldots, A_k\} \) be a set of \( k \times k \) matrices satisfying \( \delta(\mathcal{A}) = k + \lfloor k/2 \rfloor \) (cf. [J, Theorem 3.5]). Let

\[
\mathcal{B} = \{B_1, \ldots, B_k\} \cup \{E_{i,j}\}_{i=3,\ldots,m} \cup \{E_{i,j}\}_{i=1,2 \atop j=1,\ldots,n} \cup \{E_{i,j}\}_{i=k+1,\ldots,n} \]

where

\[
B_i = \begin{bmatrix}
A_i & 0_{2,k-k} \\
0_{m-2,k} & 0_{2,n-k}
\end{bmatrix},
\]

and \( E_{i,j} \) is the matrix with 1 in entry \((i,j)\) and zero in all other entries. Then, as in the proof of the lemma, we have
Theorem 2. If \( k \leq \min \left( (m-1)^2/2, (n-1)^2/2 \right) \) then

\[
R_F(m, n, m n - k) \geq m n - 4\sqrt{2k} + O(1).
\]

Proof. Let \( q = \left\lfloor \sqrt{k/2} \right\rfloor \). Since \( k \leq \min \left( (n-1)^2/2, (m-1)^2/2 \right) \) we have

\[
2q = 2\left\lfloor \sqrt{k/2} \right\rfloor \leq 2\left\lfloor \frac{\min(m, n)-1}{2} \right\rfloor \leq \min(m, n).
\]

Therefore, by the lemma, we have

\[
\Delta_F(m, n, k) \leq \Delta_F(m, n, 2q^2) \leq \Delta_F(2q, 2q, 2q^2).
\]

(2) implies

\[
\Delta_F(2q, 2q, 2q^2) = 4q^2 - R_F(2q, 2q, 2q^2) \leq 4q^2 - \frac{8q^4}{2q^2 + 4q - 2}
\]

\[
= 8q - 20 + \frac{48q - 20}{q^2 + 2q - 1} \leq 4\sqrt{2}k + O(1).
\]

For tensors in \( F^n \otimes F^n \otimes F^{\alpha n} \), \( \alpha < 1 \), bound (2) implies

\[
R_F(n, n, \alpha n^2) \geq n^2 - \frac{2}{\alpha} n + O(1).
\]

Whereas Theorem 2 gives the lower bound

\[
R_F(n, n, \alpha n^2) \geq n^2 - 4\sqrt{2}(1-\alpha)n + O(1).
\]

A simple calculation shows that this bound improves the previous lower bound in the case when

\[
\alpha > (1+\sqrt{5})/2 = 0.809.
\]

Theorem 3. Let \( 0 \leq k \leq 4 \), \( \max(m, n) \geq k \). Then

\[
R_F(m, n, m n - k) = mn - \left\lfloor \frac{k}{2} \right\rfloor.
\]

Proof. Let \( m \geq n \). By the corollary we have

\[
\Delta_F(m, n, k) \geq \Delta_F(m, m, k),
\]

and by [AS, Theorem 2], for \( 0 \leq k \leq 4 \),

\[
\Delta_F(m, m, k) = \left\lfloor \frac{k}{2} \right\rfloor.
\]
Therefore

\[ R_F (m, n, m n - k) \leq m n - \left\lceil \frac{k}{2} \right\rceil. \]

Now, the result follows from Theorem 1. 

REFERENCES


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