ONE-PAGE BOOK EMBEDDING UNDER VERTEX-NEIGHBORHOOD CONSTRAINTS

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One-Page Book Embedding Under Vertex-Neighborhood Constraints

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Abstract

We study the VLSI-related problem of embedding graphs in books. A book embedding of a graph \(G=(V,E)\) consists of two parts, namely, (1) an ordering of \(V\) along the spine of the book, and (2) an assignment of each \(e \in E\) to a page of the book, so that edges assigned to the same page do not intersect. In devising an embedding, one seeks to minimize the number of pages used.

This paper addresses a generalization of the book embedding problem, called the black/white (blw) book embedding problem, where a vertex-neighborhood constraint is imposed on the ordering of the vertices. A blw graph is a pair \((G,U)\), where \(G=(V,E)\) is a graph and \(U \subseteq V\) is a set of distinguished black vertices (the vertices in \(V-U\) are called white). A blw book embedding of \((G,U)\) is a book embedding of \(G\), where the black vertices are arranged consecutively along the spine. Given a blw graph \((G,U)\), the blw book embedding problem is that of finding a blw book embedding of \((G,U)\), such that number of pages is minimized. The need for blw embeddings may arise, for example, in applications where the input ports of a VLSI chip are to be separated from the output ports.

The main result of this paper is a characterization of the blw graphs that admit a one-page blw embedding. The characterization is given in terms of a set of forbidden blw subgraphs, the absence of which is necessary and sufficient for one-page blw embedding. For a blw graph with none of these forbidden subgraphs, a one-page blw embedding is constructible in linear time. The construction utilizes a technique called blw unfolding, which is a feature of independent interest.

Key words: book embedding, outerplanar graphs, algorithm

AMS(MOS) subject classifications: 05C45, 68Q35, 94C15

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1. Introduction

The embedding of a graph \( G=(V,E) \) in a book consists of two parts. The first part is an ordering of \( V \) along the spine of the book. The second part is an assignment of each edge \( e \in E \) to a page of the book, such that if two edges are assigned to the same page, then they do not intersect. Given a graph \( G \), the book embedding problem is that of finding a book embedding of \( G \) with the minimum possible number of pages.

The notion of book embedding is useful in modeling layout problems arising in VLSI design (cf. [CLR]). Many of the basic properties of book embeddings were established in [BK1]. In subsequent papers, several authors have investigated various aspects of book embeddings, establishing lower and upper bounds on the minimum number of pages required to embed some classes of graphs (cf. [CLR,H,R,Y1]). We note that it is NP-complete to decide whether a given graph admits a 2-page book embedding [W,CLR]. Also, given a graph \( G=(V,E) \), an ordering of \( V \), and an integer \( k \), it is NP-complete to decide whether \( G \) admits a \( k \)-page book embedding with the vertices placed on the spine in the specified order [GJMP]. On the other hand, a graph admits a one-page book embedding iff it is outerplanar [BK1], and this property is decidable in linear time [M].

This paper addresses a generalization of the book embedding problem, called the black/white (b/w) book embedding problem, where a vertex-neighborhood constraint is imposed on the ordering of the vertices. A b/w graph is a pair \( (G,U) \), where \( G=(V,E) \) is a graph and \( U \subseteq V \) is a set of distinguished black vertices (the vertices in \( V-U \) are called white). A b/w book embedding of \( (G,U) \) is a book embedding of \( G \), where the black vertices are arranged consecutively along the spine. Given a b/w graph \( (G,U) \), the b/w book embedding problem is that of finding a b/w book embedding of \( (G,U) \), such that the number of pages is minimized. The need for b/w embeddings may arise, for example, when the input ports of a VLSI chip are to be separated from the output ports. Also, in certain applications external wires are used to connect some vertices in order to, say, meet some connectivity property (cf. [ET]). Putting those vertices consecutively serves to reduce the total wire length, an important consideration in VLSI design. The b/w book embedding problem resembles a problem previously considered in the context of planar VLSI embedding [BK2,PRS,Y2]. There, the authors considered the problem of embedding a binary tree on a chip such that all the leaves are on the boundary.

The main result of this paper is a characterization of the b/w graphs that admit a one-page b/w embedding. The characterization is given in terms of a set of forbidden b/w subgraphs, the absence of which is necessary and sufficient for one-page b/w embedding. We note that our characterization can be described in terms of forbidden b/w minors, in a sense similar to that of [RS]. For a b/w graph with none of the forbidden b/w subgraphs, a one-page b/w embedding is constructible in linear time. The construction utilizes a technique called b/w unfolding. We believe that this technique is a feature of independent interest, and its power is further demonstrated in a companion paper [MW].
The paper is organized as follows. The formal framework is set in Section 2. In Section 3 we present the one-page conditions, and prove that they are necessary conditions for one-page b/w embedding of a b/w graph. The b/w unfolding technique is presented in section 4, and is thereafter used to constructively prove the sufficiency of the one-page conditions. A summary is given in Section 5.

2. The Formal Framework

The following definitions provide a formal framework for the b/w book embedding problem.

A book consists of a spine and a number of pages. The spine of the book is a line. For simple exposition, view the spine as being horizontal. Each page of the book is a half-plane that has the spine as its boundary. Thus, any half-plane is a 1-page book, and a plane with a distinguished horizontal line is a 2-page book.

Let \( G = (V, E) \) be a graph, and let \( \rho \) be a linear ordering of \( V \). For each \( v \in V \), denote the rank of \( v \) with respect to \( \rho \) by \( p(v) \). Let \( (v_1, v_j) \) and \( (v_k, v_i) \) be two edges of \( G \) where \( p(v_1) < p(v_j) \) and \( p(v_k) < p(v_i) \). If either \( p(v_1) < p(v_k) < p(v_i) < p(v_j) \) or \( p(v_k) < p(v_1) < p(v_i) < p(v_j) \), we say that \( (v_1, v_j) \) and \( (v_k, v_i) \) intersect, or are incompatible (w.r.t. \( \rho \)). Otherwise, we say that \( (v_1, v_j) \) and \( (v_k, v_i) \) are compatible. A set \( E' \subseteq E \) is compatible if any two edges \( e_1, e_2 \in E' \) are compatible.

A \( k \)-page book embedding of \( G = (V, E) \) may be formally defined as a the pair \( E = (\rho, \pi) \), where \( \rho \) is an ordering of \( V \) along the spine and \( \pi \) is a partition of \( E \) into \( k \) compatible sets. Each page of the book is said to accommodate one of these compatible sets. An edge accommodated by some page is also said to be assigned to that page. A \( k \)-page b/w book embedding of a b/w graph \( (G, U) \) is a \( k \)-page book embedding of \( G \), say \( E = (\rho, \pi) \), where the black vertices are consecutive in \( \rho \).

We say that a set of vertices \( S \subseteq V \) is equally-colored if the vertices of \( S \) are either all black, or all white. Vertices that are not equally-colored are said to be oppositely-colored.

A basic property of book embeddings is that the order of the vertices on the spine can be viewed as a circular order. That is, if \( E = (\rho, \pi) \) is a \( k \)-page book embedding of \( G \), and \( \rho' \) is a circular permutation of \( \rho \), then \( E' = (\rho', \pi) \) is also a \( k \)-page book embedding of \( G \). Thus, if \( (G, U) \) admits a \( k \)-page b/w embedding, then it admits a \( k \)-page b/w embedding in which the black vertices are placed to the left of the white vertices. Such an embedding is called a regular b/w embedding.

A path in \( G = (V, E) \) is either a single vertex or a sequence of distinct vertices \( (v_1, v_2, \ldots, v_k) \) where \( (v_i, v_{i+1}) \in E \) for \( 1 \leq k < -1 \). Let \( p_1 = (v_1, \ldots, v_k) \) and \( p_2 = (w_1, \ldots, w_l) \) be two paths in \( G \), where \( (v_k, w_1) \in E \). Then \( p_1 \cdot p_2 \) is the path \( (v_1, \ldots, v_k, w_1, \ldots, w_l) \). A path in a b/w graph is mixed if it contains oppositely-colored vertices. A mixed edge is defined similarly. Note that in a regular one-page b/w embedding, mixed edges must nest.
3. One-Page b/w Embedding: A Characterization Theorem

As mentioned in the introduction, deciding whether there is a 2-page book embedding of a given graph is NP-complete. It immediately follows that the b/w book embedding problem is also NP-complete. This negative result motivates the study of one-page b/w embeddings. In particular, a question of primary interest is that of characterizing the b/w graphs which admit a one-page b/w embedding. Such a characterization, in terms of a set of forbidden b/w subgraphs, is developed in the rest of this paper. In this section we present the one-page conditions, and prove that they are necessary conditions for one-page b/w embedding of a b/w graph. The sufficiency of the conditions will be proved in the following section.

Definition: An outerplanar representation of a graph G is a planar embedding of G where the vertices are arranged in an imaginary circle, and they all lie on the external face. A graph G=(V,E) is outerplanar if it admits an outerplanar representation. We hereafter identify an outerplanar representation of G with the circular order of V along the circle associated with the representation.

A basic property of outerplanar graphs is the following:

P₁ A graph G is outerplanar if and only if it admits a one-page book embedding [BK1]. In fact, an outerplanar representation of G defines a one-page book embedding of G.

Definition: A biconnected component in a graph G is a maximal component where for every three vertices x,y,z there is a path from x to z that avoids y. Note that by this definition, an edge not on a cycle is a biconnected component.

A biconnected outerplanar graph has a unique Hamiltonian cycle [S1], hence a unique outerplanar representation (up to reflection). Given a biconnected outerplanar graph G, let Φ(G) denote the circular vertex order identified with the unique outerplanar representation of G. Combining the said above with P₁, we have the following important property of biconnected outerplanar graphs:

P₂ A biconnected outerplanar graph G has a unique one-page book embedding (up to reflection and circular permutation of the vertices), which is defined by Φ(G).

Definitions: Let (G,U) be a b/w graph. A vertex v ∈ V is a U-fork if G contains three vertex-disjoint mixed paths, each of which does not contain v and has its first vertex adjacent to v. Figure 1 depicts a U-fork. Given an outerplanar graph G, a loop in G is a biconnected component which is not an edge. A loop L=(V_L,E_L) is an equally-colored loop if V_L is equally-colored, and it is a mixed loop otherwise. Vertices consecutive on Φ(L) are also said to be consecutive on L. A block of a mixed loop L is a maximal set of consecutive equally-colored

†Traditionally, an outerplanar graph is defined as a one that can be embedded in the plane with all the vertices on the external face; however, one easily verifies that the two definitions are equivalent. The reason for our alternative definition is that it more useful in the context of book embeddings.
vertices on $L$. Similarly, a block of a mixed path $p$ is a maximal set of consecutive equally-colored vertices of $p$. A trail of a loop $L$ is a path $p=(v_1, \ldots, v_k)$ where only the first vertex, $v_1$, is on $L$. The vertex $v_1$ is called the base of $p$. We also say that $p$ emanates from $v_1$. Note that by definition of a loop, trails emanating from different vertices of a loop are vertex-disjoint.

Insert Figure 1 here

The following conditions will be proved necessary and sufficient for one-page b/w embedding.

Definition: A b/w graph $(G,U)$ satisfies the one-page conditions if

1. $G$ admits a one-page embedding (i.e., $G$ is outerplanar).
2. There is no $U$-fork in $G$.
3. In each equally-colored loop, $L$, there are at most two bases of mixed trails. If $L$ has two such bases, then they are consecutive on $L$. (see Figure 2.a).
4. In each mixed loop, $L$,
   (1) there are exactly two blocks,
   (2) each base of a mixed trail is an end-point of a block of $L$,
   (3) oppositely-colored bases of mixed trails are not consecutive, unless one of them is a single-vertex block, and
   (4) there are at most two bases of mixed trails.
   (see Figure 2.b.)

Insert Figure 2 here

The main result of this paper is the following.

Characterization Theorem for One-Page b/w Embedding: A b/w graph $(G,U)$ admits a one-page b/w embedding if and only if it satisfies the one-page conditions.

The proof of the Characterization Theorem is established in two parts, namely, necessity and sufficiency of the one-page conditions. The necessity part is proved in Theorem 1 below. The sufficiency part is deferred to section 4.

Theorem 1: Assume that $(G,U)$ admits a one-page b/w embedding. Then $(G,U)$ satisfies the one-page con-
Proof: The one-page conditions consist of four conditions, namely $C_1-C_4$. We show that if $(G,U)$ admits a one-page b/w embedding, then each of these conditions must hold. The proof of $C_1$ is immediate.

**Proof of $C_2$.** Suppose that there is a $U$-fork, $v$, in $(G,U)$, and assume w.l.o.g. that $v$ is black. Then there are three vertex-disjoint mixed paths, each of which does not contain $v$ and has its first vertex adjacent to $v$. Choose $p_i, 1 \leq i \leq 3$, to be three such paths, where in each $p_i$ the last two vertices are the only two oppositely-colored vertices. Let $e_i=(x_i,y_i), 1 \leq i \leq 3$ be the mixed edge induced by the last two vertices of $p_i$, where $x_i$ is black and $y_i$ is white. If $(G,U)$ admits a one-page b/w embedding, then it admits a regular one. In this embedding the mixed edges $e_i, 1 \leq i \leq 3$, must nest; w.l.o.g assume that their end-points appear on the spine in the order $(x_1,x_2,x_3,y_3,y_2,y_1)$. The $U$-fork, $v$, lies either either (a) to the left of $x_1$, or (b) between $x_1$ and $x_2$, or (c) between $x_2$ and $x_3$, or (d) to the right of $x_3$. In the first two cases, $e_2$ intersects with any path from $v$ to an end-point of $e_3$. In the other two cases, $e_2$ intersects with any path from $v$ to an end-point of $e_1$. Since there are paths from $v$ to end-points of both $e_1$ and $e_3$, it follows that $(G,U)$ does not admit a one-page b/w embedding.

In the sequel, if $S$ is a subgraph of $G$, then $U/S$ denotes the set of black vertices of $S$.

**Proof of $C_3$.** Suppose that in some equally-colored loop, $L$, there are three bases of mixed trails, or two non-consecutive ones. Let $(S,U/S)$ be the subgraph of $(G,U)$ induced by the vertices of $L$ and the mixed trails emanating from these bases. Our aim is to show that $(S,U/S)$ does not admit a one-page b/w embedding. If $(S,U/S)$ admits a one-page b/w embedding, then it admits a regular one. In this embedding, by $P_3$ the vertices of $L$ appear on the spine according to $\Phi(L)$, up to reflection and circular permutation. It follows that the leftmost and rightmost vertices of $L$ on the spine are consecutive on $L$, and are thus connected by an edge. Call these vertices the $L$-extremal vertices. Now, since $L$ has three bases of mixed trails, or two non-consecutive ones, one of these bases must be placed somewhere between the $L$-extremal edges. The mixed trail emanating from this base must intersect with the edge connecting the $L$-extremal vertices. It follows that $(S,U/S)$, hence $(G,U)$, does not admit a one-page b/w embedding.

**Proof of $C_4$.** Hereafter, let $L$ be a mixed loop of $G$.

1. By $P_4$, $L$ has a unique one-page book embedding, up to reflection and circular permutation of the vertices. If there are more than two blocks in $L$, then this unique one-page book embedding of $L$ is not a b/w embedding of $(L,U/L)$, and hence $(G,U)$ does not admit a one-page b/w embedding.

2. Suppose that $L$ has a mixed trail $p=(x,...,y)$ emanating from a vertex $x$, such that $x$ is not an end-point of a block of $L$, and $y$ is the first vertex on $p$ whose color is opposite to that of $x$. Let $(S,U/S)$ be the sub-
graph of \((G, U)\) induced by the vertices of \(L\) and \(p\). Our aim is to show that \((S, U/S)\) does not admit a one-page b/w embedding. Assume the converse, and let \(E=(p, \pi)\) be a regular one-page b/w embedding of \((S, U/S)\). Let \(u_1\) and \(u_n\) be the leftmost and rightmost black vertices of \(L\) in \(p\), respectively. Similarly, let \(w_1\) and \(w_n\) be the leftmost and rightmost white vertices of \(L\) in \(p\), respectively. (If \(L\) has a single-vertex block, then either \(u_1 = u_n\) or \(w_1 = w_n\).) By \(P_2\), the vertices of \(L\) appear in \(p\) according to \(\Phi(L)\), up to reflection and circular permutation. It follows that \(u_1\) and \(w_n\) are connected by an edge, and so are \(u_n\) and \(w_1\).

Without loss of generality, assume that \(x\) is black, so \(y\) is white. By the regularity of \(E\), \(y\) must be placed either (a) between \(u_m\) and \(w_1\), or (b) between two white vertices \(w_i\) and \(w_{i+1}\) of \(L\), or (c) to the right of \(w_n\). However, if \(y\) is placed according to (a), (b) or (c) above, then the mixed trail \(p\) intersects with \((u_m, w_1)\), \((w_i, w_{i+1})\) or \((u_1, w_n)\), respectively. It follows that \((S, U/S)\), hence \((G, U)\), does not admit a one-page b/w embedding.

(3) Suppose that \(L\) has no single-vertex block, and that it has two mixed trails, \(p_1=(u, \ldots, x)\) and \(p_2=(w, \ldots, y)\), that emanate from oppositely-colored consecutive vertices of \(L\). Without loss of generality, assume that \(u\) is black and \(w\) is white. Assume also that in each of those two trails, the color of the last vertex is opposite to that of the first vertex. Let \((S, U/S)\) be the subgraph of \((G, U)\) induced by the vertices of \(L\) and the trails \(p_i, i=1,2\). If \((S, U/S)\) admits a one-page b/w embedding, then it admits a regular one, \(E=(p, \pi)\). By \(P_2\), the vertices of \(L\) appear in \(p\) according to \(\Phi(L)\), up to reflection and circular permutation. Thus, either (a) \(u\) is the rightmost black vertex of \(L\) on the spine and \(w\) is the leftmost white one, or (b) \(u\) is the leftmost black vertex of \(L\) on the spine and \(w\) is the rightmost white one. If case (a) holds, then by the regularity of \(E\), \(y\) must be placed to the left of \(x\), so \(p_1\) and \(p_2\) intersect. Suppose then that case (b) holds. In this case, \(y\) must be placed to the left of the leftmost white vertex of \(L\), by the regularity of \(E\). Thus, \(y\) must be placed to the left of \(u\), for otherwise \(p_2\) intersects some edge of \(L\). By a similar argument, \(x\) must be placed to the right of \(w\). It now follows that \(p_1\) and \(p_2\) must intersect. We therefore conclude that \((S, U/S)\), hence \((G, U)\), does not admit a one-page b/w embedding.

(4) If \(L\) has no single-vertex block, then it cannot have more than two bases of mixed trails, for then a contradiction would arise to (1)-(3) above. Thus, one only has to prove the case where \(L\) has a single vertex block. Assume then that \(L\) has three bases of mixed trails, and that it has a vertex \(w_0\), w.l.o.g. white, that is a single-vertex block. By (1-2) above, we find that \(w_0\) is a base of a mixed trail, and so are the two endpoints of the black block, called \(u_1\) and \(u_2\). Let the corresponding mixed trails be \(p_0=(w_0, \ldots, u_0)\), \(q_1=(u_1, \ldots, w_1)\) and \(q_2=(u_2, \ldots, w_2)\), where the color of the last vertex in each trail is opposite to the color of the first vertex. Let \((S, U/S)\) be the subgraph of \((G; U)\) induced by the vertices of \(L\) and the trails \(p_0, q_1\) and \(q_2\). Our aim is to show that \((S, U/S)\) does not admit a one-page b/w embedding. Assume the converse, and let \(E=(p, \pi)\) be a regular one-page b/w embedding of \((S, U/S)\). By \(P_2\), the vertices of \(L\) appear in \(p\) according to \(\Phi(L)\), up to reflection and circular permutation. It follows that the leftmost and
rightmost black vertices of $L$ on the spine are $u_1$ and $u_2$; w.l.o.g. let $u_1$ be the leftmost one. Now, by the regularity of $E$, $w_2$ must be placed to the right of $u_2$. Similarly, $u_0$ must be placed to the left of $w_0$. Observe that $u_0$ must be placed to the left of $u_1$, for otherwise $p_0$ and $q_2$ intersect. Furthermore, $u_0$ must be placed to the left of $u_1$, for otherwise $p_0$ intersects some edge of $L$. But then $p_0$ intersects with $q_1$, since $w_1$ must be placed to the right of $u_1$. It follows that $(S', U', S)$, hence $(G, U)$, does not admit a one-page b/w embedding. □

4. Sufficiency of the One-Page Condition

In order to complete the proof of the Characterization Theorem, we yet have to prove the sufficiency of the one-page conditions. For this purpose we introduce a technique for constructing good b/w embeddings, called b/w unfolding. Using this technique, we will constructively prove that any b/w graph which satisfies the one-page conditions admits a one-page b/w embedding. Further applications of the b/w unfolding technique are demonstrated in a companion paper [MW].

To describe the b/w unfolding technique (abbr. unfolding), we next exemplify the unfolding of a b/w graph consisting of a single mixed path. Recall that a block of a mixed path is a maximal set of consecutive equally-colored vertices. Let $(G, U)$ be a b/w graph consisting of a single mixed path $p = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n$, where each $V_k = (v_{k1}, \ldots, v_{kn})$ is a block. Define $V_k, 1 \leq k \leq n$, to be the path obtained by reversing $V_k$. That is, $V_k = (v_{kn}, \ldots, v_{k1})$. In the sequel, the paths $V_1, V_n$ are viewed as linear orders of their respective vertices. Given two linear orders $\rho_1 = (v_1, \ldots, v_n)$ and $\rho_2 = (w_1, \ldots, w_n)$, define $\rho_1 \rho_2$ to be the linear order $\rho_1 \rho_2 = (v_1, \ldots, v_n, w_1, \ldots, w_n)$. A b/w unfolding of $p = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n$ is the ordering $\rho(p) = v_1 \rightarrow v_2 \rightarrow v_4 \rightarrow v_2 \rightarrow v_1 \rightarrow v_3 \rightarrow v_2 \rightarrow \cdots$. Two immediate properties of $\rho(p)$ are the following. First, equally-colored vertices of $p$ are consecutive in $\rho(p)$. Second, the edges on $p$ are all compatible w.r.t. $\rho(p)$. It follows that $\rho(p)$ underlies a one-page b/w embedding of $(G, U)$. Figure 3 depicts the unfolding.

Insert Figure 3 here

Given a b/w graph, its structure is typically more complicated than that of the graph described above. In such cases, the unfolding technique essentially involves a careful choice of the path(s) to unfold, such that the rest of the graph can be embedded within a small number of pages. The choice of the specific path(s) to unfold depends on the graph in hand, and is a key to the efficient use of unfolding. In particular, the sufficiency part of the Characterization Theorem is proved through the unfolding of an ultimate road, defined below. b/w
embedding algorithms that demonstrate different choices of paths to unfold are presented in [MW].

Definitions: Let $(G, U)$ be a b/w graph. Extend the definition of a trail as follows. Let $p$ and $q$ be paths in $G$. Say that $q$ is a trail w.r.t. $p$ if $q = (v, v_1, \ldots, v_e)$, where $v$ is the only vertex which is both in $p$ and $q$. The trail $q$ is said to emanate from $v$, and is strongly mixed if $q' = (v_1, \ldots, v_e)$ is mixed. If there are no strongly mixed trails w.r.t. $p$, then $p$ is ultimate. A vertex $v \in V$ is a $p$-vertex if it is a vertex on $p$. A loop $L$ is a $p$-loop if some vertex on $L$ is a $p$-vertex. Say that $p$ is a road if the $p$-vertices of every $p$-loop, $L$, appear consecutively on $p$, according to $\Phi(L)$.

The concept of ultimate road plays a key role in proving the sufficiency of the one-page conditions. In fact, the proof is established by means of an algorithm that unfolds an ultimate road. In the sequel, we develop a scheme for finding an ultimate road in a b/w graph $(G, U)$ that satisfies the one-page conditions. Rather than searching for an ultimate road in $(G, U)$, we find an ultimate path in a tree called the hypertree of $(G, U)$, and then transform that path to an ultimate road in $(G, U)$. This ensures that the obtained road has some additional properties that are desirable by the algorithm. As a first step, we show below that our attention can be restricted to trimmed graphs.

Definitions: Let $(G, U)$ be a b/w graph where $G = (V_o, E_o)$ is connected. A vertex $v \in V_o$ separates $x$ from $y$, where $x, y \in V_o$, if all the paths connecting $x$ and $y$ pass through $v$. In this case, we also say that $v$ is a separation vertex. If $X$ and $Y$ are disjoint subsets of $V_o$, and for each $x \in X$ and $y \in Y$, $v$ separates $x$ from $y$, we say that $v$ separates $X$ from $Y$. A separable subgraph of $G$ is a (possibly disconnected) subgraph $A = (V_A, E_A)$, such that some separating vertex $v \in V_o - V_A$ separates $V_A$ from $(V_o - V_A) - \{v\}$. If, in addition, $V_A \cup \{v\}$ is equally-colored, we say that $A$ is a $U$-separable subgraph of $G$, and that $v$ $U$-separates it from $G$. If $G$ contains no $U$-separable subgraphs, then $(G, U)$ is said to be trimmed. We note that trimming $(G, U)$, i.e. bringing it to a state where no $U$-separable subgraphs exist, can be done in $O(1)$ time, using a DFS-based algorithm for finding biconnected components [E].

Definition: Let $G = (V, E)$ be an outerplanar graph, let $v_1 \in V$ be a vertex of $G$ and let $\rho=(v_2, v_3, \ldots, v_e)$ be a linear order of $V - \{v_1\}$. We say that $\rho$ is canonical w.r.t. $v_1$ if $\rho'=(v_1, v_2, \ldots, v_e)$ underlies a one-page book embedding of $G$ (or, equivalently, if $\rho''=(v_2, \ldots, v_e, v_1)$ underlies a one-page book embedding of $G$).

Given an outerplanar graph $G = (V, E)$ and $v \in V$, a canonical ordering of $V$ w.r.t. $v$ always exists, by property $P_1$. Such an ordering can be found using the algorithm of [M] for recognizing outerplanar graphs.

Proposition 1: Let $(G, U)$ be a b/w graph, where $G = (V, E)$ is outerplanar. Let $(G', U')$ be the b/w graph obtained by trimming $(G, U)$, where $G' = (V', E')$ and $U' = U \cap V'$. Then, given a one-page b/w embedding of $(G', U')$, one can construct a one-page b/w embedding of $(G, U)$ in linear time.
Proof: For each vertex $v \in V'$, define $A(v)$ to be the largest subgraph which $v$ $U$-separates from $G$, such that $A(v)$ contains only vertices of $V' - V'$ (if such subgraph exists). Also, if $A(v)$ is the subgraph $(V_A, E_A)$, then $E_A^*(v)$ is the edge set of the subgraph induced on $\{v\} \cup V_A$. The one-page b/w embedding of $(G, U)$ is constructed as follows. The vertices of $V'$ are ordered as in the one-page embedding of $(G', U')$. The vertices of each subgraph $A(v)$ are placed immediately to the left of $v$, in a canonical ordering w.r.t. $v$. We next show that the edges of $G$ are all compatible.

- By definition of canonical ordering, the edges of each set $E_A^*(v)$ are compatible.
- The vertices of each $A(v)$ are arranged consecutively near $v$, so the edges of $E_A^*(v)$ are compatible with those of $G'$.
- For different $v$ and $w$, the subgraphs $A(v)$ and $A(w)$ are vertex-disjoint, and the vertices of each such subgraph are placed consecutively. Thus, the edges of any two sets $E_A^*(v)$ and $E_A^*(w)$ are compatible.

Finally, the subgraphs $A(v)$ can all be found in $O(|E| \cdot O(|V|)$ time, using the DFS-based algorithm for finding separating vertices and biconnected components [E]. A canonical ordering of the vertices of each $A(v)=(V_A, E_A)$ can be found in $O(|V_A|)$ time, using the the algorithm of [M]. The linearity of the construction follows.

Recall that our main goal in this chapter is to prove that the one-page conditions are sufficient for one-page b/w embedding. By Proposition 1, we need only consider trimmed graphs for that matter.

Definitions: Let $(G, U)$ be trimmed b/w graph that satisfies the one-page conditions. Let $A(G) = \{L_1, \ldots, L_k\}$ be the set of the loops of $G$, and let $S(G) = \{s_1, \ldots, s_p\}$ be the set of the separation vertices and leaves of $G$. An edge $e \in E$ is simple if it is not an edge of a loop.

The hypertree of $(G, U)$ is the pair $H(G, U) = (T(G), f)$, defined as follows (see Figure 4):

1. $T(G)$ is the tree $T(G) = (V_T, E_T)$, where $V_T = V_A \cup V_S$, $V_A = \{v_s | L_s \in A(G)\}$, $V_S = \{v_s | s \in S(G)\}$, and $E_T = \{(v_s, v_j) | v_s, v_j \in S(G), (s_i, s_j) \text{ is a simple edge in } E \} \cup \{(v_s, v_j) | s \in S(G), L_j \in A(G), s_i \text{ is a vertex of } L_j\}$.

2. $f : V_T \to \{B, W, B\&W\}$ is an assignment of hypercolors (h-colors) to $V_T$, where
   - For each black [white] $s \in S(G)$, $f(v_s) = B$ [$f(v_s) = W$].
   - For each loop $L_i = (V_i, E_i)$, let $W_i = V_i - \{s_j | s_j \in S(G)\}$. If $W_i$ is equally-colored, say black [white], then $f(v_s) = B$ [$f(v_s) = W$]. Otherwise, $f(v_s) = B\&W$.

Note that each separation vertex on a loop is a base of a mixed trail, since $(G, U)$ is trimmed. Moreover, since $(G, U)$ satisfies the one-page conditions, each loop has at most two separation vertices (see $C_3 - C_4$). Thus $f$ is well-defined, as $W_i \neq \emptyset$ for each loop $L_i$. Two vertices $v_i, v_j \in V_T$, are said to be oppositely h-colored if $f(v_i) = B$.
Lemma 1: Let \((G, U)\) be a trimmed b/w graph that satisfies the one-page conditions. Then a \(h\)-ultimate path in \(H(G, U)\) can be found in linear time.

Proof: An algorithm for finding the desired path is described (see Figure 5).

Step 1. Find the loops of \(G\), the separation vertices, and the leaves. Then, construct the hypertree \(H(G, U)\). If \(H(G, U)\) contains B&W vertices, let \(p_{B&W}\) be the path in \(H(G, U)\) that starts and ends in B&W vertices, and contains all the other B&W vertices (that such a path exists is proved below).

Step 2. If \(H(G, U)\) contains B&W vertices, extend \(p_{B&W}\) to obtain a path \(p = p_1 \cdot p_{B&W} \cdot p_2\) of the maximum possible length. Otherwise, find a path \(p\) of maximum length in \(H(G, U)\). The resulting path is \(h\)-ultimate.

Correctness. In Step 1, if \(H(G, U)\) contains B&W vertices, then there is indeed a path \(p_{B&W}\) as defined. For otherwise, the minimal subtree containing all B&W vertices has a vertex of degree at least 3, \(v_{s_j}\), which corresponds to a separation vertex \(s_j\) in \(G\). The latter separation vertex is easily seen to be \(U\)-fork in \(G\), contradicting the fact that \((G, U)\) satisfies the one-page conditions. Consider the path \(p\) obtained in Step 2. Note that if \(p\) starts [ends] in a non-B&W vertex, than this vertex is followed [preceded] in \(p\) by an oppositely \(h\)-colored vertex, since \((G, U)\) is trimmed. One can thus view \(p\) as the path \(p = (v_1, \ldots, v_n)\), where (a) \(v_1, v_2\) are oppositely \(h\)-colored or \(f(v_1) = B&W\), and (b) \(v_n, v_{n-1}\) are oppositely \(h\)-colored or \(f(v_n) = B&W\). We next show, using the assumption that \((G, U)\) satisfies the one-page conditions, that there is no strongly \(h\)-mixed trail w.r.t. \(p\). Observe first that no such trail emanates from \(v \in (v_1, v_2, v_{n-1}, v_n)\), for then the length of \(p\) can be increased. Suppose that a strongly \(h\)-mixed trail emanates from \(v \in (v_3, \ldots, v_{n-2})\). Then \(v\) has in \(p\) two oppositely \(h\)-colored predecessors, or a B&W predecessor. Similarly, \(v\) has in \(p\) two oppositely \(h\)-colored successors, or a B&W successor. Note that \(v\) either corresponds to a loop \(v_{s_j}\) in \(G\), or to a separation vertex \(v_{s_j}\) in \(G\). One can verify that in the first case \(L_j\) contains a \(U\)-fork, and in the second case \(s_j\) is itself a \(U\)-fork. Either way, a contradiction arises to the fact that \((G, U)\) satisfies the one-page conditions. It follows that the path \(p\) is indeed a \(h\)-ultimate path.

Complexity. The loops and the separation vertices of \(G\) can be found in linear time, using a DFS-based algorithm for finding biconnected components [E]. Finding the leaves, constructing \(H(G, U)\), and constructing \(p_{B&W}\) (by initiating DFS from some B&W vertex) is easily seen to be also implementable within the same time bound. If \(p_{B&W}\) is not empty, then it can extended to a path of the maximum possible length using two applica-
tions of BFS \([E]\), one from each of its end-points. Otherwise, if \(p_{raw}\) is empty, a longest path in \(H(G,U)\) can also be found using two applications of BFS - one from an arbitrary vertex \(v\), and the second from a leaf furthest from \(v\). As the time complexity of BFS is linear, the lemma follows.

\[\square\]

Insert Figure 5 here

**Lemma 1:** Assume that \((G,U)\) is trimmed and satisfies the one-page conditions. Given a \(h\)-ultimate path in \(H(G,V)\), an ultimate road in \((G,V)\) is constructible in linear time.

**Proof:** Some definitions are first required. Recall that \(\Phi(L)\) denotes the circular vertex order identified with the unique outerplanar representation of a loop \(L\). Let \(L\) be an *equally-colored* loop of \(G\) with two bases of mixed trails, say \(x\) and \(y\). By \(C_3\), these bases are consecutive on \(L\). For such \(L\), define \(\Phi_{x-y}(L)\) to be the road in \(G\) that starts in \(x\), contains all the vertices of \(L\), and ends in \(y\). Next, let \(L\) be a *mixed* loop of \(G\). If \(L\) has two bases of mixed trails, say \(x\) and \(y\), then they are end-points of blocks, by \(C_4\). For such \(L\), let \(\Phi_{x-y}(L)\) denote the road that starts in \(x\), proceeds through all the vertices of the oppositely-colored block, and ends in \(y\). If a mixed loop \(L\) has a single base, \(x\), of a mixed trail, let \(\Phi_{x-y}(L)\) denote the road that starts in \(x\), proceeds through all the vertices of the oppositely-colored block (in clockwise direction, if \(x\) is a single-vertex block), and ends in the end-point of that block. Define \(\Phi_{x-y}(L)\) to be the road obtained by reversing \(\Phi_{x-y}(L)\).

We can now present the construction. Let \(p\) be the given \(h\)-ultimate path in \(H(G,U)\). An ultimate road in \((G,U)\), called \(q\), is constructed in the following manner. If \(p\) consists of a single vertex, then \(G\) is biconnected, or, equivalently, consists of a single loop \(L\). In this case, choose \(x\) and \(y\) to be two equally-colored end-points of a block of \(\Phi(L)\), and set \(q=\Phi_{x-y}(L)\). Otherwise, proceed as follows. Set \(q=p\). For each vertex \(v_L\) in \(q\) that corresponds to a loop \(L\) in \(G\), replace \(v_L\) and its neighbors, if such exist, by a road \(r\) defined as follows. If \(v_L\) has one neighbor to its left [right], say \(x\), then \(r=\Phi_{x-y}(L)\) \([r=\Phi_{x-y}(L)\]. If \(v_L\) has two neighbors, say \(x\) and \(y\) to its left and right, respectively, then \(r=\Phi_{x-y}(L)\). That \(q\) is an ultimate road in \((G,U)\) follows directly from the fact that \(p\) is a \(h\)-ultimate path in \(H(G,U)\). The construction can clearly be done in \(O(|V|)\) time. It is exemplified in Figure 6.

\[\square\]

Insert Figure 6 here

**Definitions.** Let \((G,U)\) be a b/w graph where \(G=(V,E)\) is outerplanar, and let \(p\) be a road in \((G,U)\). Recall that a \(p\)-vertex is a vertex on \(p\). Say that a vertex \(v\in V\) is *free* (w.r.t. \(p\)) if \(v\) is not a \(p\)-vertex. A \(p\)-vertex \(v\) of a loop \(L\) is a *\(p\)-terminal* if \(v\) is the first, or last, vertex of \(L\) that appears in \(p\). If \(v\in V\) is a \(p\)-vertex of a loop
L, but not a p-terminal of L, then v is called an internal p-vertex of L. A loop with two p-terminals is called a 2-terminal loop. Note that a vertex v can be a p-terminal of at most two 2-terminal loops. In this case, v is the first p-terminal of one loop and the second p-terminal of the other. A normalized ultimate road is an ultimate road p that has the following two properties:

N₁ there is no 2-terminal equally-colored loop with free vertices, and

N₂ there is no 2-terminal mixed loop with oppositely-colored internal p-vertices.

Lemma 3: Assume that (G, U) is trimmed and satisfies the one-page conditions. Let p be an ultimate road in (G, U), as constructed in Lemma 2 from a h-ultimate path of H(G, U). Then q is normalized.

Proof: Follows from the definition of the roads Φx,y(L) in Lemma 2.

Lemma 4: Assume that (G, U) satisfies the one-page conditions, and let p be a normalized ultimate road in (G, U), as constructed in Lemma 2 from a h-ultimate path in H(G, U). Let ρ(p) be an unfolding of p, and let L be a 2-terminal loop. Then,

1. the p-vertices of L are arranged in ρ(p) according to Φ(L), such that equally-colored vertices are arranged consecutively.

2. the free vertices of L (if any) can be added to ρ(p), in such a that all the vertices of L are arranged according to Φ(L), and equally-colored vertices are arranged consecutively.

Proof: Recall that an unfolding of p = V₁, V₂, V₃, ... is the order ρ(p) = ... V₂ₖ₋₁ V₂ₖ V₂ₖ₊₁ V₂ₖ₊₂ ... Note that ρ(p) preserves the order of equally-colored vertices, a property hereafter referred to as the order preserving property of ρ(p). This property together with N₁ directly imply the lemma when L is equally-colored, so assume that L is mixed. Let Φ(L) = p₁, p₂, where p₁ = (v₁, v₂, ..., vₘ) contains the p-vertices, starting from the first p-terminal, and p₂ = (vₘ₊₁, ..., vₙ) contains the free vertices. We first consider the case where v₁ belongs to an odd-numbered block of p.

1. Using N₂ and the fact that the free vertices are equally-colored (by the ultimateness of p), we find that the vertices (v₂, ..., vₘ₋₁), together perhaps with vₘ, constitute a block of L. Furthermore, by the definition of Φx,y(L) in Lemma 2, the color of that block is opposite to that of v₁. By the order preserving property of ρ(p) and the fact that v₁ belongs to an odd-numbered block of p, the order of v₁, ..., vₘ₋₁, vₘ in ρ(p) is either p₁ = (v₁, v₂, ..., vₘ₋₁, vₘ) or p₂ = (vₘ, vₘ₋₁, ..., v₁), depending on whether vₘ has the color of v₁ or the opposite color, respectively. Whichever the case is, the obtained order is as claimed.

2. Place the free vertices of L, in the order vₘ, vₘ₋₁, ..., vₙ₊₁, immediately to the right of v₁. Thus, if the p-vertices of L appear in ρ(p) in the order p₁ or p₂ defined above, then the obtained order for all the vertices of L is p₁ = (v₁, v₂, ..., vₘ, vₘ₋₁, ..., vₙ₊₁, vₘ) or p₂ = (vₘ, vₘ₋₁, ..., v₁, vₘ, vₘ₋₁, ..., vₙ₊₂, vₙ₊₁),
respectively. The free vertices of \( L \) are equally-colored, by the ultimateness of \( p \). Their color is equal to that of \( v_1 \), by the construction of the roads in Lemma 2. Thus the obtained order is as claimed. (As an example, consider loop \( L_1 \) of Figure 4.a. Here, \((v_1, \ldots, v_n) = (5, 4, 7, 8) \) and \( v_{n+1} = v_n = (6) \). The order obtained in the lemma for the vertices of \( L_1 \) is \( p_1 = (7, 4, 5, 6, 8) \), depicted in Figure 7).

Symmetrically, one can verify that if \( v_1 \) belongs to an even-numbered block of \( p \), then the obtained order is either \( p_1 = (v_m, v_{m+1}, \ldots, v_1, v_{n+1}, v_n, \ldots, v_m) \) or \( p_2 = (v_{n+1}, \ldots, v_m, v_m, v_{m+1}, \ldots, v_1, v_n) \), depending on whether \( v_n \) has the color of \( v_1 \) or the opposite color, respectively. Either way, the order is as claimed in the lemma.

The sufficiency of the one-page conditions is established in Theorem 2 below. The proof is constructive, and relies on the unfolding technique. Throughout the proof, the unfolding of a path \( p \) is viewed as an iterative procedure for placing the vertices of \( p \) on the spine according to \( \rho(p) \). This procedure scans the vertices of \( p \) one by one, placing each vertex on the spine as it is encountered. A vertex of an odd-numbered block is placed to the right of the rightmost vertex currently on the spine. Similarly, a vertex of an even-numbered block is placed to the left of the leftmost vertex currently on the spine.

**Theorem 2:** Given a trimmed b/w graph \((G, U)\) that satisfies the one-page conditions, a one-page b/w embedding of \((G, U)\) can be found in linear time. (If \((G, U)\) does not satisfy the one-page conditions, then one of the forbidden b/w subgraphs can be found within the same time bound.)

**Proof:** An linear time algorithm is developed below, which produces a one-page b/w embedding of any trimmed b/w graph \((G, U)\) that satisfies the one-page conditions. The algorithm is based on unfolding a normalized ultimate road, \( p \), in \((G, U)\). Informally, the algorithm scans the vertices of \( p \) one by one, placing each \( p \)-vertex \( v \) according to the order prescribed by unfolding \( p \) (recall the iterative unfolding procedure described above). Immediately after placing \( v \), the algorithm places the vertices of a subgraph associated with \( v \), called \( A(v) \) (to be defined). The free vertices of the 2-terminal mixed loops are not contained in any subgraph \( A(v) \).

For each such loop, \( L \), these vertices are placed after the second \( p \)-terminal of \( L \) is placed.

Some definitions are next required. Let \( p \) be an ultimate road in \((G, U)\). For each \( p \)-vertex \( v \), let \( A(v) \) to be the largest subgraph which \( v \) separates from \( G \), such that \( A(v) \) contains no vertices of \( p \). The facts that \((G, U)\) is trimmed and \( p \) is ultimate imply that for each \( p \)-vertex, \( v \), the vertices of \( A(v) \) are equally-colored, and their color is opposite to that of \( v \).
We are now ready to present the algorithm.

Algorithm one-page

Input: A trimmed b/w graph \((G, U)\), where \(G = (V, E)\).

Output: A 1-page b/w embedding of \((G, U)\), provided \((G, U)\) satisfies the one-page conditions.

Method:

1. Apply Lemma 1 to find a \(h\)-ultimate path \(q\) in \(H(G, U)\). Apply Lemma 2 to transform \(q\) to a normalized ultimate path in \((G, U)\).
2. While the vertices of \(p\) are not all placed on the spine,
   3. Let \(v^*_1\) be the leftmost vertex of \(p\) that is not yet placed on the spine.
   4. If \(k\) is odd [even], then place \(v^*_1\) on the spine such that it is the rightmost [leftmost] vertex.
   5. If \(k\) is odd [even], then place the vertices of \(A(v^*_1)\) to the left [right] of the currently leftmost [rightmost] vertex, in a canonical ordering w.r.t. \(v^*_1\).
   6. If \(v^*_1\) is the second \(p\)-terminal of a mixed 2-terminal loop \(L = (V_L, E_L)\), then place the (equally-colored) free vertices of \(L\), as prescribed in Lemma 4, such that \(V_L\) is ordered according to \(\Phi(L)\).
8. od

Analysis of the algorithm. First, note that the algorithm places each \(p\)-vertex immediately near an equally-colored vertex (provided such vertex exists on the spine). The vertices the subgraphs \(A(v)\) and the free vertices of the 2-terminal loops are also placed in such a way. It follows that on completion of the algorithm, equally-colored vertices are arranged consecutively. We next show that the edges can all be assigned to a single page.

By \(Old(v)\) we denote the set of edges \((x, y) \in E\) such that \(x\) and \(y\) were placed before the vertex \(v\). Let \(v_1\) and \(v_2\) be two consecutive vertices on \(p\), where \(v_1\) precedes \(v_2\). Then \(New(v_1) = E_1 \cup E_2\), where \(E_1\) is the set of edges incident to \(v_1\), and \(E_2\) is the set of edges \((x, y) \in E\) such that \(x\) and \(y\) were placed after \(v_1\), and before \(v_2\).

Hereafter, if \(A(v)\) is the subgraph \((V_A, E_A)\), then \(E_A^*(v)\) is the edge set of the subgraph induced on \(\{v\} \cup V_A\).

To prove that one page suffices for the embedding, we use induction on the number of vertices placed by the algorithm on the spine. Specifically, the induction hypothesis states that for each \(p\)-vertex \(v^*_1\), the set \(Old(v^*_1) \cup New(v^*_1)\) is compatible. As the induction basis can be easily verified, we proceed to the induction step. Suppose that the hypothesis holds for the first \(m-1\) \(p\)-vertices placed by the algorithm. Consider the \(m\)-th \(p\)-vertex, \(v^*_1\). Let \(w\) be the predecessor of \(v^*_1\) on \(p\). Since \(Old(v^*_1) = Old(w) \cup New(w)\), the hypothesis implies that \(Old(v^*_1)\) is compatible. As no vertex \(x\), where \((x, y) \in New(v^*_1)\), is ever placed between vertices of \(Old(v^*_1)\), it suffices to show that \(New(v^*_1)\) is compatible.

Case 1: \(v^*_1\) is not a vertex of a 2-terminal loop. Here, \(New(v^*_1) = ((w, v^*_1)) \cup E_A^*(v^*_1)\). The edge \((w, v^*_1)\) is nested within \(E_A^*(v^*_1)\), whose compatibility follows from the definition of canonical ordering. Thus, \(New(v^*_1)\) is
compatible.

Case 2: \( v_t \) is the first \( p \)-terminal of a 2-terminal loop. If \( v_t \) is also the second \( p \)-terminal of some 2-terminal loop, see cases 4-5 below. Otherwise, the proof is identical to that of Case 1.

Case 3: \( v_t \) is an internal vertex of a 2-terminal loop \( L=(V_L,E_L) \). Since \((G,U)\) is trimmed, the \( p \)-terminals of \( L \) are bases of mixed trails. Thus, \( v_t \) is not a base of a mixed trail, for otherwise \( L \) would have three such bases, contradicting the fact that \((G,U)\) satisfies the one-page conditions (see \( C_5-C_6)\). It follows that \( New(v_t) \) is just the set of the edges of \( L \) that are incident to \( v_t \). This set is compatible, since the vertices of \( L \) that are currently on the spine are arranged, by Lemma 4, according to \( \Phi(L) \).

Case 4: \( v_t \) is the second \( p \)-terminal of a mixed 2-terminal loop \( L=(V_L,E_L) \). Let \( F \) be the set of free vertices of \( L \). Observe that \( New(v_t) = E_L \cup E_{A}^*(v_t^t) \), where \( E_{L}^* \) is the set of edges of \( L \) incident to vertices of \( F \cup (v_t^t) \). By Lemma 4, \( F \cup (v_t^t) \) are placed such that \( V_L \) is ordered according to \( \Phi(L) \), so \( E_{L}^* \) is compatible. The set \( E_{L}^* \) is nested within \( E_{A}^*(v_t^t) \), whose compatibility follows from the definition of canonical ordering. Thus, \( New(v_t) \) is compatible.

Case 5: \( v_t \) is the second \( p \)-terminal of an equally-colored 2-terminal loop. This case is similar to Case 4.

We next consider the complexity of the algorithm. The road \( p \) can be found in \( O(|V|) \) time, by Lemmas 1-2. Assume that at any given time within the execution, the end-points of the spine are recorded. The overall time required for this operation is \( O(|V|) \). The embedding of each subgraph \( A(v) = (V_A,E_A) \) consists of (1) finding this subgraph, (2) finding canonical orderings of its vertices, and (3) placing the vertices of these subgraphs, using the stored information on the end-points of the spine. Each such operation requires \( O(|V_A|) \) time (see proof of Proposition 1). Since the subgraphs \( A(v) \) are vertex-disjoint, the overall embedding process (steps 2-8) requires \( O(|V|) \) time. Thus, the time complexity of the algorithm is \( O(|V|) \). Figure 7 depicts the resulting b/w embedding.

Finally, we note that the algorithm can be extended such that if \((G,U)\) does not satisfy the one-page conditions, then one of the forbidden b/w subgraphs is found in Lemma 1 or 2, without increasing the complexity.
5. Summary

This paper investigated a generalization of the VLSI-related book embedding problem, namely, the \textit{b/w book embedding problem}. In the generalized problem, a vertex-neighborhood constraint is imposed on the ordering of the vertices. The main result presented in this paper is a characterization of the b/w graphs that admit a one-page b/w embedding. The characterization is given in terms of the \textit{one-page conditions}, which take the form of a set of forbidden b/w subgraphs.

The sufficiency part of the result was established by means of a linear time algorithm that produces a one-page b/w embedding of a b/w graph \((G,U)\), provided \((G,U)\) satisfies the one-page conditions. The algorithm can be easily extended to discover one of the forbidden subgraphs in b/w graphs not satisfying the one-page conditions. Our algorithm utilizes a technique called \textit{b/w unfolding}, which, we believe, is a feature of independent interest. Further applications of the unfolding technique are demonstrated in the companion paper [MW]. These include (I) an algorithm for b/w embedding of any b/w tree in a 2-page book, and (b) a sufficient condition for b/w embedding of a b/w graph \((G,U)\), where \(G\) is a biconnected outerplanar graph, in a 2-page book.

The main direction for further research in the area of b/w embedding seems to be that of deriving lower and upper bounds on the width of the embedding. Of particular interest are the possible trade-offs between width and number of pages in b/w embeddings.

References


Fig. 1 A graph with a $U$-fork, where $U=\{1,9,13\}$. The three vertex-disjoint mixed paths are $(6,5,...,1)$, $(10,...,13)$ and $(8,9)$. Throughout the paper, black vertices are drawn as two concentric circles.
(a) An equally-colored loop satisfying $C_3$

(b) A mixed loop satisfying $C_4$

Figure 2. The one-page loop conditions
Fig. 3. Unfolding $p$
a. A trimmed outerplanar graph, $G$, where $U = \{2, 4, 7, 9, 12, 15, 16, 18, 20\}$.

b. The hypertree $H(G, U)$

Fig. 4 Definition of hypertree
Fig. 5 The \( h \)-ultimate path obtained by applying Lemma 1 to the hypertree in Fig. 4(b).

Fig. 6 The ultimate road obtained in Lemma 2 for the graph of Fig. 4(a).
Fig. 7 Applying Algorithm $\mathcal{A}$ to the graph of Fig.4(a) (unfolding the ultimate road of Fig. 6).