A PRODUCT INTEGRATION METHOD FOR VORTEX DYNAMICS IN TWO AND THREE DIMENSIONS

by

M. Israeli and M.J. Shelley

Technical Report #542*

March 1989

* This research was supported by the Technion V.P.R. Fund - Henry Gutwirth Fund for the Promotion of Research.
A PRODUCT INTEGRATION METHOD FOR VORTEX DYNAMICS IN TWO AND THREE DIMENSIONS

by

M. Israeli and M.J. Shelley

Technical Report #542

March 1989

*This research was supported by the Technion V.P.R. Fund - Henry Gutwirth Fund for the Promotion of Research.
A PRODUCT INTEGRATION METHOD FOR VORTEX DYNAMICS
IN TWO AND THREE DIMENSIONS

M. Israeli¹ and M.J. Shelley²

Abstract

A new, high order, spectral method for evaluating the multi-dimensional integrals appearing in the vorticity formulation of inviscid, incompressible fluid dynamics is presented. This formulation is most relevant for vortex dominated flows which are of major importance in science and engineering, and can be modified to account for viscous effects in high Reynolds number flows.

The present method, unlike discrete vortex methods, reduces the singularity in the (Biot-Savart) integrand analytically and does not require the introduction of a smoothing finite core function ("vortex blob"), and its associated nonphysical length scale.

The support of a general compact vorticity distribution in two or three dimensions is covered by a coordinate system periodic in (at least) one direction. By evaluating the integrals in an iterated fashion, the integrals over the periodic coordinate are evaluated with spectral accuracy (i.e. the error decays faster than any power of the mesh distance). Special analytic treatment is used for very close contours. The resulting scheme yields good accuracy even at moderate resolution and is expected to improve both short and long time solutions of time dependent flows.

¹ Technion-Israel Institute of Technology, 2 University of Chicago.
1. Introduction

The present paper focuses on the development, evaluation and analysis of a new high order method for computing the multi-dimensional integrals appearing in the vorticity formulation of inviscid, incompressible fluid dynamics. This formulation is most relevant for vortex dominated flows which are of major importance in science and engineering, and can be modified to account for viscous effects in high Reynolds number flows. Some of the computational advantages of these methods are:

a) Computational points are used only in regions with non zero vorticity and the algorithms follow the concentration of vorticity as they evolve in time.

b) Like in the vorticity stream function formulation it is not necessary to compute the pressure in order to advance the calculation in time as the incompressibility constraint is automatically satisfied in a discrete way by the formulation.

Current discrete vortex methods, are extensions of the point vortex method based on a particular interpretation of the Biot-Savart law. In these methods a smoothing cut-off function multiplies the singular kernel. Thus point vortices are replaced by "vortex blobs". (For a review of current vortex methods see Anthony Leonard, Annual Review of Fluid Mechanics, 17 (1985), pp. 523-559.)

Vortex blob methods are a class of quadrature methods first advanced by Chorin (JFM, 57 (1973), 785-796) for evaluating the two- and three-dimensional singular integrals which appear in the vorticity-stream formulation of incompressible fluid dynamics. For ease of presentation, we restrict ourselves to discussing the two-dimensional, inviscid case. For such flows the vorticity \( \omega \) satisfies the vorticity transport equation.
\[
\frac{D \omega}{Dt} = 0
\]  
(1)

where \( \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \). \( \mathbf{v} \) is given by the Biot-Savart integral

\[
\mathbf{v}(x,t) = \int \int_{\mathbb{R}^2} K(x-x') \omega(x',t) \, dx',
\]  
(2)

where the singular kernel \( K \) is given by:

\[
K(x) = \frac{1}{2\pi} \left( -y \hat{i} + x \hat{j} \right) / (x^2 + y^2).
\]  
(3)

In 2-d, we use incompressibility and the fact that the vorticity is conserved along particle paths to give the vorticity-stream formulation in Lagrangian or particle trajectory form. Let \( X(\alpha,t) \) be the location of the fluid particle at time \( t \) which began at the point \( \alpha \in \mathbb{R}^2 \). Then (1) implies \( \omega(X(\alpha,t),t) = \omega(\alpha,t=0) \), while from incompressibility we also have that \( \frac{\partial (x,y)}{\partial (\alpha_1,\alpha_2)} = 1 \). These identities then give the Lagrangian formulation of 2-d inviscid fluid mechanics:

\[
\frac{d}{dt} X(\alpha,t) = \int \int_{\mathbb{R}^2} K(X(\alpha,t)-X(\alpha',t)) \omega(\alpha',t=0) \, d\alpha',
\]  
(4a)

with

\[
X(\alpha,t=0) = \alpha.
\]  
(4b)

The main advantages of the Lagrangian formulation (4) is that by following the fluid particles, the vorticity naturally remains resolved, and only those portions of the fluid containing vorticity need be tracked. The disadvantage lies in the fact that the underlying Lagrangian coordinate system can become severely distorted, and that one must evaluate integrals containing strongly singular kernels.
Vortex blob methods attack the last problem of evaluating the singular integrals in equation set (4). This is done by smoothing the singular kernel $K(x)$, leaving an integral with a smooth integrand to which standard quadrature methods may be applied. The smoothing is accomplished by simply convolving $K$ with an appropriate radially symmetric approximate $\delta$-function. In particular, for $\delta > 0$ let

$$K_\delta(x) = \int_{\mathbb{R}^2} K(x-x') \phi_\delta(x') dx',$$

where among some other constraints $\phi_\delta$ satisfies

$$\phi_\delta(x) = \delta^{-2} \phi(x/\delta)$$

with

$$\int_{\mathbb{R}^2} \phi(x) dx = 1.$$

Equation set (4) is then replaced by

$$\frac{d}{dt} X^\delta(\alpha,t) = \iint_{\mathbb{R}^2} K_\delta(X^\delta(\alpha,t)-X^\delta(\alpha',t)) \omega(\alpha',t=0) d\alpha',$$

with

$$X^\delta(\alpha,t=0) = \alpha,$$

where the integrands in equation set (5) are now smooth, and may be evaluated by standard quadrature formulae. Choices for the function $\phi$ are discussed by Beale and Majda (JCP, v.58, no.2, (1985), p. 188). There is also now a rigorous convergence theory for the vortex blob method (see, for example, O. Hald, SIAM J. Num. Anal., 16, (1979), p. 726).

The most unsatisfactory aspect of the the vortex blob method is the introduction of the unphysical length-scale $\delta$ into the problem. The presence of $\delta$ is felt in the convergence theory by the require-
ment that in order to obtain convergence, a proper relation must be chosen between $\delta$ and the quadrature discretization parameter $h$. At the least, it is required that $h/\delta \to 0$ as $h, \delta \to 0$. From this statement, the vortex blob method can be viewed as a two-tiered process. One first examines consistent discretizations (in $h$) to the $\delta$-equation set (5), and then allows $\delta \to 0$ for this series of resolved calculations. However the scientific questions of current interest are related precisely to the behavior of the solutions as the limits are approached. Thus the extra degree of freedom introduced by the size and shape of the smoothing function complicate considerably the investigation of problems where large gradients or complex behavior develop and casts doubts on the relevance of the conclusions.

On a more practical level there are indications from numerical experiments (Beale and Majda, op. cit.) that the accuracy of the vortex blob methods decays with time and also that the higher order methods of this type deteriorate faster than the lower order methods so that after time of order of a few revolutions there is no motivation to use the more complicated kernels. For longer times the accuracy drops considerably probably due to the neglect of the finite core distortion (op. cit.). Other experiments (Rokhlin private communication) indicate that these vortex methods suffer from limited accuracy at moderate resolutions when compared with more conventional high order (spectral) methods especially for smooth flows.

Since the essential part of the vortex method involves the evaluation of singular integrals in two or three dimensions and since the point vortex method and its modifications seem to be of limited accuracy for short integration times and of low accuracy for longer times, it is important to look for methods which are based on different methodology and which are expected to give superior accuracy. In addition these methods should not involve the finite-core approximation and should require a simpler one parameter limiting process.

We develop a family of integration methods for the present class of problems which exploit properties of the geometry and physics of the flow. These methods are described in the next section and are expected to perform particularly well for high resolution. Preliminary numerical experiments show good performance even at moderate resolution.

In the future we will develop and test several versions of the new methods for problems in two and three dimensions and then apply them to a number of cases of scientific and engineering interest.

Since the vortex methods are now an important competitor in the field of numerical methods for incompressible inviscid and slightly viscous flows a major improvement in their accuracy will be of considerable value for computational fluid dynamics.

One alternative to the vortex blob method is based on a recent work of Sidi and Israeli (Numerical quadrature methods for integrals of singular periodic functions and their application to singular and weakly singular integral equations, Icase report, 1985, submitted to S.I.A.M.J. Num. Anal.), there the periodicity in angle which appears naturally in problems of practical interest is coupled with extrapolation techniques to get very accurate and inexpensive integration methods.
In the present problems periodicity is obtained by introducing coordinate systems which are periodic in at least one direction. The use of the techniques of Sidi and Israeli while improving the accuracy on the singular contours was not sufficiently powerful to handle the case of non singular contours where the denominator becomes as small as a mesh distance. An alternative approach is described below.

The two-dimensional Biot-Savart integral of incompressible fluid mechanics is given by

\[ V(x) = \frac{1}{2\pi \supp(\omega)} \int \frac{-(y-y') \hat{i} + (x-x') \hat{j}}{(x-x')^2 + (y-y')^2} \omega(x', y') dA, \]  

(1)

Where \( x = x \hat{i} + y \hat{j} \). Consider \( \supp(\omega) \) to be compact, and cover it with a generalized polar coordinate system \((\rho, \xi)\), where \( \xi \) is such that the coordinate system is periodic in the \( \xi \) direction. This new coordinate system may be the Lagrangian coordinate system which has evolved from an initially periodic coordinate system. Assuming the period to be always 2\( \pi \), we now write (1) as the iterated integral

\[ V(x) = \int_0^P d\rho \left[ \frac{1}{2\pi} \int_0^{2\pi} \frac{-(y-y(\rho, \xi)) \hat{i} + (x-x(\rho, \xi)) \hat{j}}{(x-x(\rho, \xi))^2 + (y-y(\rho, \xi))^2} \omega(\rho, \xi) \frac{\partial(x', y')}{\partial(\rho, \xi)} \, d\xi \right] \]  

(2)

The approach is to evaluate the \( \xi \) integrals accurately for each \( \rho \) and then integrate in the \( \rho \) direction. The advantage is that after the \( \xi \) integration, we are left with a piece-wise smooth function, with known jump discontinuities, to integrate in the \( \rho \) direction. This may be done with standard quadrature formulae. The \( \xi \) integrals are in general of the form:

\[ I_x = I_x \hat{i} + I_y \hat{j} = \frac{1}{2\pi} \int_0^{2\pi} \frac{-(y-y(\xi)) \hat{i} + (x-x(\xi)) \hat{j}}{R^2} \hat{\omega}(\xi) d\xi, \]  

(3)

where \( R^2 = (y-y(\xi))^2 + (x-x(\xi))^2 \), and \( \hat{\omega} \) is the product of the vorticity and the Jacobian in equation.
(2) Let \( \Gamma \) be the contour traced out by \( x(\xi) \hat{i} + y(\xi) \hat{j} \). Then for \( x \notin \Gamma \), the integrand in equation (3) is seen as a smooth, periodic function of \( \xi \), which may then be evaluated with formal spectral accuracy by the trapezoidal rule. In practice however, the trapezoidal rule can be very inaccurate when \( x \) is close to \( \Gamma \). This inaccuracy arises from the presence in the kernel \( (-(y-y(\xi)) \hat{i} + (x-x(\xi)) \hat{j})/R^2 \) of a narrow spike which can be ill-resolved over the quadrature points. This spike is due to the denominator \( R^2 \) becoming small, and is centered where \( x \) is closest to \( \Gamma \), say at \( \xi^* \).

Instead of selectively interpolating in new quadrature points so as to resolve the spike (see Baker and Shelley, JCP, v. 64, no. 1), we seek to handle spike in an analytic fashion which requires no expensive global interpolation. This may be done using a product integration method, where the integrand is rewritten over a new kernel, and the new weight function (formerly \( \hat{\omega} \)) expanded over the trigonometric polynomials. Over this set of expansion functions, the new kernel may be integrated exactly. For convenience, we let \( \eta = x + iy \), and now write the \( I_x \) in complex form as

\[
I_\eta = I_y + iI_x = \frac{1}{2\pi} \int_0^{2\pi} \frac{(x-x(\xi)) - i(y-y(\xi))}{R^2} \hat{\omega}(\xi) d\xi.
\]

where \( \nu(\xi) = x(\xi) + iy(\xi) \). We now rewrite (4) as

\[
I_\eta = \frac{1}{2\pi} \int_0^{2\pi} \left[ \hat{\omega}(\xi) \frac{\eta - w(z(\xi))}{\eta - \nu(\xi)} \right] d\xi.
\]

The periodic function \( \hat{\omega}(\xi) \) is now expanded in a Fourier series as
\[ \tilde{\omega}(\xi) = \sum_{k=0}^{\infty} a_k e^{ik\xi}. \]  

and substituted into (5) to get

\[ I_\eta = \sum_{k=0}^{\infty} a_k I_k, \]  

where

\[ I_k = \frac{1}{2\pi} \int_0^{2\pi} e^{ik\xi} \frac{d\xi}{\eta - \nu'(z(\xi))}. \]  

The new kernel \( 1/(\eta - \nu'(z(\xi))) \) should be chosen so that the \( I_k \) s can be evaluated in closed form, and so that \( \tilde{\omega}(\xi) \), which involves the ratio of the two kernels, is smooth (no spikes) and has a rapidly converging Fourier transform. As noted previously, for \( \eta \) close to the contour \( \Gamma \), the denominator \( \eta - \nu'(\xi) \) of \( \tilde{\omega} \) becomes small at \( \xi \). Therefore, \( \eta - \nu'(z(\xi)) \) is chosen so that \( |\eta - \nu'(z(\xi))| = |\eta - \nu'(\xi)| \). This eliminates the spike. In addition, we require that the parameterizations of the \( \Gamma \) contour, and the contour traced by \( \nu'(z(\xi)) \) match locally. This is done as follows: Let \( z(\xi) \) trace out the osculating circle to \( \nu'(\xi) \) at \( \xi \). Match the parameterizations at \( \xi \) by choosing \( \nu'(z) \) as the bilinear map which maps the osculating circle onto itself, and satisfies the equalities

\[ \nu'(z(\xi)) = \nu'(\xi) \quad \text{and} \quad \frac{d}{d\xi} \nu'(z(\xi)) |_{\xi} = \frac{d}{d\xi} \nu'(\xi) |_{\xi}. \]  

This matches both the height and width of the spike in \( \frac{1}{\eta - \nu'(\xi)} \), and we are now left with a smooth function \( \tilde{\omega}(\xi) \), with a rapidly converging Fourier series. Lastly, each integral \( I_k \) can be done in closed form using contour integration.

The above techniques replaces the vortex blob integration in the fluid dynamic computation and the results compared with the presently existing computations. It is expected that further improvements will be motivated by the behavior and quality of the results.
3. Results; steady configurations.

In order to assess the accuracy and convergence rate of the method it was decided to test it first on a series of numerical problems of increasing complexity. The simplest problem is that of finding the velocity induced by predefined patch of vorticity of radial symmetry.

Let the vorticity \( \omega = \omega(r) \) be given for \( 0 < r < 1 \) by:

\[
\omega(r) = A e^{-\frac{1}{1-r^2}}
\]  

(1)

This function is smooth but vanishes together with all its derivatives at \( r = 1 \). The velocity was evaluated on three coordinate systems:

a) concentric circles having the same origin \( r = 0 \) as the vorticity patch.

b) offset circles, having a different origin but still covering the patch.

c) concentric ellipses where the eccentricity is specified.

all points on all contours covering the finite support of the vorticity The accuracy and resolution of numerical methods usually increases as the mesh spacing decreases, so one has to increase the number of contours or the number of points on them or both. In the present case however increasing the number of contours may increase the numerical error since the smallest distance between integration points decreases and the integrand becomes more singular. To resolve the singular behavior one has to increase the number of points per contour. It was hoped that this effect will be mitigated or eliminated by the introduction of our improved kernels. As our product integration method involves a Discrete Fourier transform and its convergence rate depends on the rate of decay of the tail of the spectrum, it is expected that the form of the spectrum will characterize the success of the method.

The Discrete Fourier Transform of the improved integrand is plotted in Figure 1, for \( \delta = 0.1 \). We observe a very fast exponential decay of the amplitude of the Fourier modes for six decades before
the straight line breaks to a much slower exponential decay for larger k. The break point is at an amplitude of $10^{-5}$ corresponding to $\delta^3 - \delta^4$.

In Figure 2 a similar graph is reproduced for $\delta$ of 0.1, 0.05, 0.0025. the break point reappears at amplitudes which are about eight times smaller for each halving of $\delta$ thus establishing a $O(\delta^3)$ estimate. We can conclude that the present method gives a cubic factor of improvement over and beyond the direct treatment of the integrand by spectral methods.

In order to compare the efficiency of the present method with the "blob" methods, we used the Beale-Majda mollifiers of order 2, 4, 6 in order to evaluate the Biot-Savare integrals on the same periodic contours that were used with the spectral (Fourier Transform) code. We would like to point out that the advantages of periodicity will enhance the "blob" method in the same way as it enhances the spectral method, since it involves a trapezoidal rule integration on a periodic domain.

The first experiment (case (a)) appears in Figure 3. We find that the higher order mollifiers are not much better than the second order mollifier. The spectral method gives about four orders of magnitude improvement. Case (b) in Figure 4 leads to similar conclusions, again the six's order mollifier is disappointing, all the errors are bigger now and the spectral method is still much better. Similar conclusions for case (c) in figure 5 show that the circular contours are not essential for the success of the spectral method.

In Figures 6 and 7 we show the angular dependence is illustrated for the second and fourth order mollifiers, it can be observed that in regions of large error both scheme are about the same. One can conclude that for low resolution there is no advantage to a high order smoothing, only when the singular behavior is resolved it pays to make the extra effort. In Figure 8, all four schemes are compared for the case of (b), "offset circles". All schemes incur large errors for the angle of 90 degrees but the spec-
The method is again about 10,000 times better at that point. The error in the result is dominated by the large local error thus explaining the global behavior in the previous Figures.

New results for time dependent periodic vortex layer code will be summarized in the next report in this series.
References

Figure 1 - The discrete Fourier transform spectrum, $a_k$, for the modified kernel. Logarithm of absolute value of the amplitude as a function of the wave number. Ellipse 2:1 Field point on symmetry line.
Figure 2 - Logarithm of absolute value of the amplitude as a function of the wave number for the modified kernel.

Ellipse 2:1
Field point on symmetry line
Figure 3 - Circles on Circular Patch
Logarithm of absolute value of the error.
Figure 4 - Offset circles 50% field point on line of symmetry
Logarithm of the absolute value of the error.
Figure 5 - Centered ellipses field point on line of symmetry
Figure 6 - Second order mollifier.
Velocity order as a function of $\theta$
for different $\Delta \theta = \frac{2\pi}{16}, \frac{2\pi}{32}, \frac{2\pi}{64}, \frac{2\pi}{128}$
Figure 7 - Fourth order mollifier
Velocity order as a function of $\theta$
for different $\Delta \theta = \frac{2\pi}{16}, \frac{2\pi}{32}, \frac{2\pi}{64}, \frac{2\pi}{128}$
Figure 8 - Error in Velocity for $r = 0.75$ as a function of $\theta$.
Offset circles $d = 0.5$
(128 contours) × (128 per contour)
"Blob" methods Beale Majdo mollifiers of order 2, 4, 6.