TIGHT INTEGRAL DUALITY GAP IN THE CHINESE POSTMAN PROBLEM
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by

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ABSTRACT

Let $G = (V, E)$ be a graph and a weight function $w : E \to \mathbb{Z}^+$. Let $T \subseteq V$ be an even subset of the vertices of $G$. A $T$-cut is an edge-cutset of the graph which divides $T$ into two odd sets. A $T$-join is a minimal subset of edges that meets every $T$-cut (a generalization of solutions to the Chinese postman problem). The main theorem of this paper gives a tight upper bound on the integral duality gap. That is to say, a bound on the difference between the minimum weight $T$-join and the maximum weighted integral packing of $T$-cuts. This theorem is proved algorithmically. Let $n_F$ be the number of components in the optimal $T$-join, $\tau_w = \text{minimum weight } T$-join and $\nu_w = \text{max weight integral packing of } T$-cuts then we have $\tau_w - \nu_w \leq n_F - 1$.

This result unifies and generalizes Fulkerson's result for $|T| = 2$ and Seymour's result for $|T| = 4$.

For a certain integral multicommodity flow problem in the plane, the above algorithmic result gives a solution such that for every commodity the flow is less than the demand by at most one unit.

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1. INTRODUCTION

Let $G = (V, E)$ be an undirected graph and let $T \subseteq V$ be such that $|T|$ is even. A $T$-join $F$ is a minimal set of edges so that $T$ is exactly the set of all vertices in $(V, F)$ with odd valency. Every $T$-join is a forest. For $V' \subseteq V$, $\gamma(V')$ denotes the set of edges with both ends in $V'$ and $\delta(V')$ denotes the set of edges with one end in $V'$ and the other in $V \setminus V'$. A $T$-cut is a set of edges of the form $\delta(S)$ with $S \subseteq V$ and $|S \cap T|$ odd. In this paper we assume that $G$ is connected. The results are naturally extended to graphs which are not connected but each connected component contains an even number of vertices from $T$. In case there is a connected component contains an odd number of vertices of $T$, there is no $T$-join, and there is no bound on the packing of $T$-cuts.

Minimal $T$-joins are exactly the minimal sets of edges intersecting all $T$-cuts. That is to say, each $T$-join, $F$, is a minimal cover of $T$-cuts, and vice versa, each minimal cover of $T$-cuts is a $T$-join.

If $|T| = 2$ then minimal $T$-joins are exactly the minimal paths that connected the two vertices of $T$. If $T$ is the set of all vertices with odd valency in $G$, then $T$-joins are exactly the set of edges which make $G$ Eulerian by "doubling" them. Therefore, a minimum weighted $T$-join is a solution for the Chinese Postman problem.

Edmonds and Johnson [4] showed (in an algorithmic way) that if $T$ is the set of all vertices with odd valency in $G$ then there is a minimum weighted cover of $T$-cuts which is integral, and this cover is a $T$-join. With a simple transformation it is possible to implement this algorithm for each even set of vertices $T \subseteq V(G)$. Moreover, they proved [3] that there exists a maximum weight packing of $T$-cuts which is half integral. Another presentation of the last two problems is as linear programming problems. Let $A$ be the incidence matrix of $T$-cuts and let $w$ be a weight function on the edges. Then we shall define the following problems:

The minimum weighted cover problem is:

$$\tau^*_w(A) = \min \{ wx : Ax \geq 1, x \geq 0 \} .$$

(1.1)

The minimum integral weighted cover problem is:

$$\tau_w(A) = \min \{ wx : Ax \geq 1, x \geq 0, x \text{ integral} \} .$$

(1.2)

The maximum weight packing problem is:
The maximum integral weight packing problem is:

\[ \nu_w(A) = \max\{1 : y : yA \leq w, y \geq 0, y \text{ integral}\}. \quad (1.3) \]

(1.3) is the dual problem of (1.1).

\[ \Delta_w = \tau_w(A) - \nu_w(A) = \tau_w^*(A) - \nu_w(A) \]

is called the integral duality gap (or duality gap, for short).

Seymour [17] proved that if \( G \) is bipartite and \( w = 1 \) then there is a maximum weighted packing that is integral. He also proved that if for each circuit \( c, w(c) = \sum_i w(e) : e \in c \) is even, then there exists an optimal integral packing.

We shall prove, in an algorithmic way, that for each \( G \) there exists an integral feasible packing \( y \), so that if \( y \) is a maximum weight packing then \( \Delta_w = y \cdot 1 - \bar{y} \cdot 1 \leq n_F - 1 \), where \( n_F \) is the number of components in the minimum weight \( T \)-join, \( F^* \). Note that it is possible that there exists an optimal \( T \)-join, \( \hat{F}^* \), with \( \hat{n}_F < n_F \) and if we knew it we could get a better integral dual solution. Moreover, we shall show that this upper bound is tight, in the following sense: for each \( n_F \) there exists \((G,T,w)\) such that \( \Delta_w = y \cdot 1 - \bar{y} \cdot 1 = n_F - 1 \), which trivially implies that \( \Delta_w < \frac{|T|}{2} \leq \frac{|V|}{2} \).

Our main theorem unifies and generalizes Fulkerson's result [7] and Seymour's result [17]. Fulkerson proved that for \( |T| = 2 \) there is a maximum weighted packing which is integral. Clearly for \( |T| = 2 \), \( n_F = 1 \) so by our result we get \( y \cdot 1 - \bar{y} \cdot 1 \leq 0 \), and since \( \Delta_w \geq 0 \) Fulkerson's result follows.

Seymour showed that if \( |T| = 4 \) then \( \Delta_w = y \cdot 1 - \bar{y} \cdot 1 \leq 1 \). It is easy to see that for \( |T| = 4 \), \( n_F \leq 2 \). Therefore, Seymour's result follows from our main theorem. (Seymour strengthens this result by showing a necessary and sufficient conditions for \( \Delta_w = 0 \).) Seymour [17] pointed out an interesting connection between maximum weighted packing of \( T \)-cuts which is integral and integral multicommodity flow in the plane. In Section 4 we show the meaning of our result to that integral multicommodity flow.

2. DEFINITIONS AND NOTATIONS

Let us give some definitions and notations that are used later on.

2.1. Let \( V \) be a set of elements and \( A, B \subseteq V \). We say that \( A \) and \( B \) are laminar if \( A \cap B = \emptyset \) or \( A \subseteq B \).
2.2. Let $S$ be a family of subsets of $V$. We say that $S$ is a laminar (or nested) family of subsets of $V$ if all the members of $S$ are pairwise laminar.

2.3. For a graph $G = (V, E)$ with an even subset of vertices $T$, we define the function $Y$ to be $Y: \{S \subseteq V: |S \cap T| \text{ is odd} \} \rightarrow R^+$ and call $Y(S)$ the dual variable that corresponds to $S$.

2.4. $S$ induces the value $Y(S, e)$ on the edge $e$ where

$$Y(S, e) = \sum_{S': S' \subseteq S, e \in \delta(S')} Y(S').$$

2.5. The graph $G \times S$ obtained from $G$ by contracting $S$ is defined by: $V(G \times S) = (V \setminus S) \cup \{s\}$, $E(G \times S) = E \setminus \gamma(S) \cup \{(u, s): u \in S, v \in S\}$ where $(u, v) \in E$. Given a laminar family $S$ of subset of $V$ having maximal members $S_1, \ldots, S_k$, the graph $G \times S$ is defined to be $(G \times S_1) \times \cdots \times S_k$.

It is easy to see that the order in which these disjoint sets are contracted is irrelevant. $G \times S$ is called the surface graph relative to $S$.

2.6. The equality subgraph relative to $Y$, denoted by $G^Y$, is the spanning subgraph with

$$E^Y = \{e \in E: \exists[Y(S): e \in \delta(S), |S \cap T| \text{ odd}] - w(e) = 0\}.$$

2.7 Let $G = (V, E)$ be a graph, $F$ a set of edges and let $V(F) = \{v \in V: \text{there exist } u \in V \text{ such that } (u, v) \in F\}$, then $G \cup F$ is the graph with $E(G \cup F) = E \cup F$ and $V(G \cup F) = V \cup V(F)$. Let $S \subseteq V$ then $G[S] = (V', E')$ is the subgraph of $G$ induced by $S$ where $V' = S$ and $E' = \gamma(S)$.

Similarly, if $H \subseteq E$ then $G[H] = (V', E')$ is the subgraph of $G$ induced by $H$ where $V' = V(H)$ and $E' = H$.

A level set, which a recursive definition will be given in 2.8 plays an important role in the proof of the main theorem. In order to define a level set, the following definition is given:

Let $S$ be a laminar family of subsets of $V$ and $S \subseteq S$, then $S(S) = \{S' \in S: S' \subseteq S\}$. 
2.8 Let $F \subseteq E$ be a minimal $T$-join, $S$ laminar family of subsets of $V$ such that $\forall S \in S$, $\delta(S)$ is a $T$-cut.

Let $Y : S \rightarrow \mathbb{R}^+$ be a dual function and $G_T^*$ be the equality subgraph relative to $Y$. Let $R$ be a subset of edges such that $R \subseteq E^*$. A subset $S \in S$ is a level set if the following conditions hold:

(i) $|\delta(S) \cap F| = 1$

(ii) $(G[S])[R] = (V', E')$ is the subgraph with $V' = S$ and $E' = \gamma(S) \cap R$.

(iii) $(G[S])[R] \times S(S) \cup (\delta(S) \cap F)$ is one of the configurations in Figure 1.

(iv) Every $S' \in S$, $S' \subseteq S$ is a level set.

The meaning of conditions (ii) and (iii) is in fact that by deletion some of the edges of $G[S] \times S(S) \cup (\delta(S) \cap F)$ and keeping all the vertices we get one of the configuration of Figure 1. Then, $R$ is the union of all the edges that remain in all the level sets inside $S$. Roughly speaking, the meaning of $R$ is the collection of edges that are used to obtain the configurations for $S$ and for all the recursive sets inside $S$.
(a) where $k \in \mathbb{Z}^+$ and $v \in V(G)$ and if $k \geq 1$, $\{v\} \in S$.

(b) where $k \in \mathbb{Z}^+$ and $v \in V(G)$ and $\{v\} \notin S$.

(c) where $k \in \mathbb{N}$.

Figure 1: The wiggly edges represent edges in $F$, and all edges in this figure are in $R \subseteq E^*$. 
3. DUALITY GAP

Our proof is based on the following algorithm [10] (algorithm K) for minimum $T$-join, therefore we shall first give a brief description of that algorithm.

Algorithm K (from [10])

The algorithm is a primal algorithm which is a slight variant of Edmonds-Johnson's primal dual algorithm for the Chinese Postman Problem. The basic approach of this algorithm is analogous to the Cunningham March primal algorithm for optimum matching [2]. However, algorithm K starts from a feasible $T$-join, $F$, and then it either

(i) improves this $T$-join by finding a circuit $C$ such that $w(C \cap F) > w(C \setminus F)$ (to be called negative circuit) or

(ii) show that $F$ is optimal by presenting a half integral optimal dual solution.

More precisely, the algorithm starts with a $T$-join (i.e. a feasible solution to the minimum weighted cover problem $(1,1)$, $S$, a laminar family of subsets of $V$ where for each $S \in S$ $\delta(S)$ is a $T$-cut. A dual function $Y$ on $S$ and a subset of edges $R \subseteq E_T$ such that every $S \in S$ is a level set in the subgraph induced by $R \cup F$ and for every edge $e \in F$ we have $w(e) = \sum_{(Y(S), e \in \delta(S), S \in S)}$. The dual solution $y$ is given by

$$y(\delta(S)) = \begin{cases} Y(S) + Y(V \setminus S) & \text{if } S \in S \\ 0 & \text{otherwise} \end{cases}$$

Under these assumptions $F$ is feasible primal solution but $y$ is not necessarily feasible. However, the complementary slackness conditions for these dual linear programs hold.

If $y$ is feasible we are done, otherwise we choose an infeasible edge $e$ which is either in $G \times S$ or else is contained in some maximal $S$ in $S$ and accordingly we choose a maximal set in $S$ to be the root of an alternating tree that we grow. Then we either find a negative circuit and decrease the value of the primal solution or else we make a dual change that decreases the infeasibility of $e$. Eventually $e$ becomes feasible and once an edge is feasible it remains feasible. The algorithm terminates after $O(1|E|^2|V|^3)$ steps with a $T$-join and feasible $y$ that satisfy the complementary slackness and therefore are optimal.
It is shown at [10] that at the end of the execution of the algorithm we get:

(i) A minimum $T$-join, $F^*$, which is a forest.
(ii) A laminar family $S$ such that for every $S \in S$, $\delta(S)$ is a $T$-cut.
(iii) A dual function $Y$ such that if $Y(S) > 0$ then $S \in S$.
(iv) Subset of edges $R \subseteq E_1^*$ such that every $S \in S$ is a level set in the subgraph induced by $R \cup F^*$, and as a result $|F^* \cap \delta(S)| = 1$.
(v) For every $f \in F^*$ we have: $w(f) = \sum\{Y(S): f \in \delta(S), S \in S\}$ and for every $e \in E$ we have: $w(e) \geq \sum\{Y(S): e \in \delta(S), S \in S\}$.
(vi) We assume that $G$ is connected hence the dual solution (solution of 1.3) is given by $y(\delta(S)) = Y(S) + Y(V \setminus S)$.

Therefore $y$ and $F^*$ are feasible solutions, such that the complementary slackness conditions hold for them, and hence they are optimal solutions.

**Postoptimality Method (from [10])**

In addition it is shown in [10] that using postoptimality method it is possible to change the optimal solution by changing $Y$ to $Y''$ and $S$ to $S''$ such that the following conditions hold:

(i) $Y''$ implies an optimal solution $y''$.
(ii) Every vertex in the surface graph $G \times S''$ induces an integral value.
(iii) For every $e \in E$ it is possible to change $Y$ to $Y''_e$ and $S$ to $S''_e$ such that (i) and (ii) are satisfied and $e$ is in the surface graph, $G \times S''_e$.

**Theorem 3.1:** Let $G = (V, E)$ be an undirected connected graph, $T$ an even non-empty subset of $V$ and $w$ an integral non-negative weight function on $E$. Let $F^*$ be a minimum weight $T$-join, $y$ maximum weighted packing of $T$-cuts, then there exists an integral packing of $T$-cuts, $\tilde{y}$, such that:

(a) $\Delta_w \leq n_F - 1$ ($n_F$ denotes the number of components of $F^*$);
(b) For each component $\tilde{F}$ of $F^*$ it is true that

$$\Sigma(w(f); f \in \tilde{F}) - \Sigma(\Sigma(\tilde{y}(S); f \in \delta(S), |S \cap T| \text{ odd})f \in \tilde{F}) \leq 1.$$

That is to say, the total value of the $T$-cuts that intersect $\tilde{F}$ is either $w(\tilde{F})$ or $w(\tilde{F}) - 1$;
(c) For each component $F_i^*$ of $F^*$ there exists an integral packing of $T$-cuts $\tilde{y}(F_i^*)$ satisfying (a) and (b) and the total value of $T$-cuts that intersect $F_i^*$ is $w(F_i^*)$.

Note: The proof implicitly contains an algorithm which is presented in [14].

Proof: We prove by induction on $|V|$. It is straightforward to see that for $|V| = 2$ we have integral solution $y$. We assume that the statement of the theorem is true for $|V| \leq n$ and we shall show its correctness for $|V| = n + 1$.

At the termination of algorithm $K$ we have a minimum weight $T$-join, $F^*$, a dual function $Y$, and a laminar nested family $S$ (as described at the beginning of this section).

There are two possibilities:

3.1 There are no contracted vertices in the surface graph.

3.2 There is at least one contracted vertex in the surface graph.

3.1 There are no contracted vertices in the surface graph. Using postoptimality method [10], as described at the beginning of the section, we are going to change the dual solution and $S$, to an optimal solution, such that each vertex in the surface graph induces an integral value. If after the changes there are no contracted vertices then we have an integral solution and the duality gap is zero. Otherwise, 3.2 holds.

3.2 There is at least one contracted vertex in the surface graph.

The idea is to create a new graph $G'$ out of $G$ with $|V(G')| < |V(G)|$ in the following way. Let $S$ be a subset of vertices such that:

(i) $S$ is a level set, therefore $\delta(S)$ is a $T$-cut.

(ii) $|S| > 1$, and if $S' \subset S$ and $Y(S') > 0$ then $|S'| = 1$. We get $G'$ by contradicting $S$, i.e., $G' = G \times S$. In addition, we change the weight function $w$, the dual function $Y$ and $F^*$ to new feasible solutions such that the complementary slackness conditions hold. By the induction hypothesis the theorem holds for $G'$. Then we extend $G'$ back to $G$ and extend the dual integral solution of $G'$ to a dual integral solution of $G$ which satisfies the theorem. We extend the integral
dual with \[ Y'(S') \] for each \( S' \subseteq S \).

Note that the idea of the proof is easy to grasp although the formal proof, given below, might be more difficult to follow.

At the beginning we prove (a) and (b), and at the end (c).

Let \( \bar{S} = \{s_1, \ldots, s_i\} \), \( |\bar{S}| \geq 2 \), \( \bar{S} \in S \), \( s_1, \ldots, s_i \in V(G) \).

\( s \) - is the vertex that we get after contracting \( \bar{S} \).

\( G' = G \times \bar{S} \),

\[ w'(e) = \begin{cases} w(e) - Y(S_j) & e \in \delta(\bar{S}) \cap \delta(S_j), S_j = \{s_j\}, s_j \in \bar{S} \\ w(e) & \text{otherwise} \end{cases} \]

and

\[ F' = F^* \cap E(G') \]

Let \( I(\bar{S}) \equiv Y(S_1)(\text{mod } 1) \)

\[ S' = \{S \in S : \bar{S} \cap S = \emptyset\} \cup \{S \setminus \bar{S} \cup \{s\} : S \cap \bar{S} \neq \emptyset, S \in S\}. \]

Note that for every \( S' \in S' \), there is a unique set \( S \in S \) from which \( S' \) was obtained, namely \( S' = S \) if \( s \notin S' \) and if \( s \in S' \), then the corresponding \( S \) in \( S \) is \( S \setminus \{s\} \cup \bar{S} \). Therefore, we might use the same symbol \( S \) for both \( S \) and \( S' \).

\[ T' = T \cap V(G') \cup \{s\}. \]

First we show that \( F' \) and \( Y' \) are feasible solutions in \((G', T')\). \( \bar{S} \) is a \( T \)-cut in \( G \) so \( |T \cap \bar{S}| \) is odd.

Therefore, \( |T'| \) is even. \( F^* \) is a feasible solution. Therefore, for every \( \delta(S) \) we have \( |S \cap T| = |F^* \cap \delta(S)| (\text{mod } 2) \). Since \( F' = F^* \cap E(G') \), for every \( S \subseteq V(G') \) we have \( |F' \cap \delta(S)| = |F^* \cap \delta(S)| \), and we have the feasibility of \( F' \).

\[ \lfloor x \rfloor \text{ denote the largest integer smaller than or equal to } x. \]
In order to prove the feasibility of $Y'$ we have to show that: for every $e \in E(G')$ we have that

$$\Sigma(Y'(S): e \in \delta(S)) \leq w'(e).$$

For each $S \subseteq V(G'), S \neq \{s\}, Y'(S) = Y(S)$ and for each $e \notin \delta(s), w'(e) = w(e)$, therefore $Y'$ is feasible relative to every $e \in E(G') \delta(s)$.

For $e \in \delta(s)$:

$$\Sigma(Y'(S): e \in \delta(S)) = \Sigma(Y'(S): e \in \delta(S), s \in S) + \Sigma(Y'(S): e \in \delta(S), s \cap S = \emptyset) =$$

$$= \Sigma(Y(S): e \in \delta(S), \overline{S} \subseteq S) + Y'(s) + \Sigma(Y(S): e \in \delta(S), \overline{S} \cap S = \emptyset) =$$

$$= \Sigma(Y(S): e \in \delta(S)) - \left( Y(S) \right) \leq w(e) = \left( Y(S) \right) = w'(e).$$

Now let us show that the complementary slackness conditions hold for $F'$ and $Y'$ and therefore their optimality in $(G', T')$ follows. In fact, we show that:

(i) $f \in F' \Rightarrow \Sigma(Y'(S): f \in \delta(S)) = w'(f)$

(ii) $Y'(S) > 0 \Rightarrow |F' \cap \delta(S)| = 1$ and $|S \cap T'|$ is odd.

Clearly, if (ii) holds then also $Y'(S) > 0 \Rightarrow |F' \cap \delta(S)| = 1$.

Let $e \in \delta(s_j) \cap \delta(S)$ for $s_j \in S$, and let $S_{s}$ and $S_{s}'$ be the following sets:

$S_{s}$ be the maximal subset such that:

$$S_{s} = \{ S : e \in \delta(S), \overline{S} \subseteq S, S \in S \}$$

and $S_{s}'$ be the maximal subset such that

$$S_{s}' = \{ S : e \in \delta(S), \overline{S} \cap S = \emptyset, S \in S \}$$
\[ Y(S'', f) + Y(S', f) - \left[ Y(S_j) \right] = w(e) - \left[ Y(S_j) \right] = w'(e). \]

(ii) For every \( S \subseteq V(G') \), \( S \neq \{s\} \) we have: \( Y'(S) = Y(S) \) and \( |F' \cap \delta(S)| = |F^* \cap \delta(S)| \). \( F^* \) and \( Y \) are a pair of optimal solutions, so: \( Y'(S) > 0 \implies |S \cap T| \) is odd and \( |F' \cap \delta(S)| = 1 \). 

\( s \in T \) is a contracted vertex. Therefore \( |F' \cap \delta(s)| = |F^* \cap \delta(s)| = 1 \). So \( F' \) and \( Y' \) are optimal in \((G', T')\). 

Since \(|V'| < |V|\) we have by the induction hypothesis that there exists an integral feasible solution \( \tilde{Y}' \), such that: \( \Delta w' = \Sigma(Y'(S) - \tilde{Y}'(S); S \in S') \leq n_F - 1 \), and that for each component \( \tilde{F}' \) in \( F' \) we have that

\[ w'(\tilde{F}') - \Sigma(\Sigma(\tilde{Y}'(S); f \in \delta(S), |S \cap T| \text{ is odd}) f \in \tilde{F}') \leq 1. \]

Let us extend \( S' \) to a new nested family \( S'' \) in the following way:

\[ S'' = \{ S \cup \tilde{S} \setminus \{s\}; s \in S \in \mathcal{S}' \} \cup \{ S: s \notin S \in \mathcal{S}' \} \cup \{ S: S \in \mathcal{S}, S \subseteq \tilde{S} \}. \]

and consider it as the new family \( S \).

Let us define:

\[ \forall S \in \mathcal{S}: \quad \tilde{Y}(S) = \begin{cases} Y'(S) & \tilde{S} \cap S = \emptyset \\ Y'(\tilde{S} \setminus \{s\} \cup \{s\}) & \tilde{S} \subseteq S \\ Y(S_j) & S_j \in \tilde{S} \end{cases} \]

It is obvious that \( \tilde{Y}(S) \) is an integer solution. We will show that \( \tilde{Y}(S) \) is feasible, then that

\[ \Sigma(\tilde{Y}(S) - \tilde{Y}'(S); S \in \mathcal{S}) \leq n_F - 1, \]

and at least that for each component \( F \) of \( F^* \) we have:

\[ w(F) - \Sigma(\Sigma(\tilde{Y}'(S); f \in \delta(S), |S \cap T| \text{ is odd}) f \in \tilde{F}) \leq 1. \]

For every \( e \notin \delta(\tilde{S}), e \notin \gamma(\tilde{S}) \) we have:

(i) \( w(e) = w'(e) \), and

(ii) since \( Y(\tilde{S}, e) = 0 \) and using the definition of \( \tilde{Y} \) we have:

\[ \Sigma(\tilde{Y}'(S); e \in \delta(S), S \subseteq \mathcal{S}) = \Sigma(\tilde{Y}'(S); e \in \delta(S), S \cap \tilde{S}, S \subseteq \mathcal{S}) \]

\[ = \Sigma(\tilde{Y}'(S); e \in \delta(S), S \subseteq \mathcal{S}') \leq w'(e) = w(e). \]

Let \( S_k \) denote \( \{s_k\} \).
For $e \in \gamma(S)$: $e = (s_i, s_j); S_i, S_j \subseteq S$:

\[
\widetilde{Y}(S, e) = \widetilde{Y}(S_i) + \widetilde{Y}(S_j) = \left[ Y(S_i) \right] + \left[ Y(S_j) \right] \leq Y(S_i) + Y(S_j) \leq w(e).
\]

For $e \in \delta(S)$:

Let $s_j \in \tilde{S}, e \in \delta(S) \cap \delta(S_j)$ then $w(e) = w'(e) + \left[ Y(S_j) \right]$.

\[
\Sigma(\tilde{Y}(S); e \in \delta(S)) = \tilde{Y}(S''_e, e) + \Sigma(\tilde{Y}(S); e \in \delta(S), S \subseteq \tilde{S}) + \tilde{Y}(S_j) =
\]

\[
= \tilde{Y}'(S''_e, e) + (\tilde{Y}(S); e \in \delta(S), s \in S) + \left[ Y(S_j) \right] \leq w'(e) + \left[ Y(S_j) \right] = w(e).
\]

We are going to prove now that $\Sigma(\gamma(S) - \tilde{Y}(S); S \in S) \leq n_F - 1$.

From algorithm K-[10] $\tilde{S}$ is a level set. We are going to check each one of the possible configurations (Figure 1).

3.2.1: Consider $\tilde{S}$ is of the form a of Figure 1.

\[
v \notin S \Rightarrow Y(v) = 0 \Rightarrow Y(S_i) = \left[ Y(S_i) \right] = \gamma(s_i, v), i = 1, \ldots, k
\]

therefore:

\[
\Sigma(\gamma(S) - \tilde{Y}(S); S \in S) = \Sigma(\gamma(S) - \tilde{Y}(S); S \subseteq \tilde{S}, S \in S) + \gamma(S_i) - \tilde{Y}(S)
\]

\[+ \sum_{i=1}^{k} \left[ Y(S_i) - \tilde{Y}(S_i) \right] = \Sigma(\gamma'(S) - \tilde{Y}'(S); S \in S) \leq n_{F'} - 1 = n_F - 1.
\]

(Note in this case $n_{F'} = n_F$.)

3.2.2: Consider $\tilde{S}$ is of the form b of Figure 1.

\[
v \notin S \Rightarrow Y(v) = 0 \Rightarrow Y(S_i) = \left[ Y(S_i) \right], i = 1, \ldots, k.
\]
Proposition 3.1: There are graphs for which \( \Delta_w < n_F - 1 \).

Proof: For example graphs for which \( \tau_w = v_w \) and \( n_F > 1 \), such as bipartite graphs with \( |w| = 1 \), \(|T| = |V| \geq 4 \) with perfect matching.

Q.E.D.

Proposition 3.2: The upper bound on the integral duality gap is tight even for planar graphs.

Proof: By Example 1.

Q.E.D.

Example 1:
\[
\forall e \in E \quad w(e) = 1
\]
\[
\{T\} = \begin{cases} 
  V & \text{if } |V| \text{ is even} \\
  V \setminus \{v\} & \text{if } |V| \text{ is odd}
\end{cases}
\]
\[
m = \frac{|V| - 1}{3}
\]
\[
n_F = m + 1
\]

\[
\Delta_w = \nu(F^*) - 1 \cdot \bar{y} = 2 \cdot m - m = m = (m + 1) - 1 = n_F - 1
\]
(The numbers in squares represent the integral dual solution and wiggly edges represent edges in \( F \).)

Let \( Y \) be the dual solution that we get at the termination of algorithm K. The naive integral approximation is the solution that we get by rounding down \( Y \), denoted by \( \lfloor Y \rfloor \), i.e. subtracting 1/2 from each component of \( Y \) which is a half integral.
= \Sigma (Y'(S) - \tilde{Y}'(S)) : S \neq s, S \in S') + Y'(s) - \frac{1}{2^S} \tilde{Y}'(s) + k + \frac{1}{2} =

= \Sigma (Y'(S) - \tilde{Y}'(S)) : s \in S') + k \leq n_F - 1 + k = n_F - 1.

We shall prove that for every component \( \bar{F} \) in \( F^* \) we have that:

\[ w(\bar{F}) - \Sigma(\Sigma(\tilde{Y}(S)) : f \in \delta(S), |S \cap T| \text{ is odd}) f \in \bar{F}) \leq 1. \]

Cases (1), (2), (3i): Since \( Y(S_i) = \left[ Y(S_i) \right] = \tilde{Y}(S_i), i=1,...,k \) we have that for each \( f \in F^* \cap [\chi(\tilde{S}) \cup \delta(\tilde{S})] \)

\[ w(f) - \Sigma(\tilde{Y}(S)) : f \in \delta(S), |S \cap T| \text{ is odd}) = 0, \]

this follows from the induction hypothesis.

In case (3ii), for each \( f \in F^* \cap \chi(\tilde{S}) \) we have:

\[ w(f) - \Sigma(\tilde{Y}(S)) : f \in \delta(S), |S \cap T| \text{ is odd}) = 1 \]

and \( Y(\tilde{S}, f) = \tilde{Y}(\tilde{S}, f) \) and again it follows from the induction hypothesis.

It remains to be proved that for each component \( F^*_i \) of \( F^* \) there exists an integral packing of \( T \)-cuts \( \bar{y}(F^*_i) \) satisfying (a) and (b) and the total value of \( T \)-cuts that intersect \( F^*_i \) is \( w(F^*_i) \). Let \( f \in F^*_i \). Using the postoptimality method we change \( Y \) to an optimal solution \( Y''_f \) and \( S \) to \( S''_f \) such that \( f \) is in the surface graph \( G \times S''_f \), and every vertex in the surface graph induces an integral value. Let \( \bar{y}''_f \) be an integral dual function that was obtained by the method of this proof, then \( \bar{y}''_f \) satisfies (a) and (b) and \( f \in F^*_i \) is in the surface graph \( G \times S''_f \). Under the above assumptions it was shown in [14] that the following holds:

\[ w(F^*_i) - \Sigma(\Sigma(\bar{y}''_f(S)) : f \in \delta(S), |S \cap T| \text{ odd}) f \in F^*_i) = 0 \]

and we are done.

Q.E.D.

It is easy to see that for each \( K_{2n} \), (complete graph with \( 2n \) vertices) when \(|T| = V(K_{2n})\) and \( w=1 \), \( n_F = n \) and the maximum number of disjoint \( T \)-cuts is 1. Hence for every \( n_F \) there exist \((G,T,w)\) for which \( \Delta_w = n_F - 1 \).

Corollary 3.1: The upper bound on the duality gap is tight. That is to say that for every \( n_F \geq 1 \) there exist \((G,T,w)\) such that \( \Delta_w = n_F - 1 \).
Proposition 3.1: There are graphs for which $\Delta_w < n_F - 1$.

Proof: For example graphs for which $\tau_w = v_w$ and $n_F > 1$, such as bipartite graphs with $|w| = 1$, $|T| = |V| \geq 4$ with perfect matching.

Q.E.D.

Proposition 3.2: The upper bound on the integral duality gap is tight even for planar graphs.

Proof: By Example 1.

Q.E.D.

Example 1:

$\forall e \in E \cdot w(e) = 1$

$|T| = \begin{cases} V & \text{if } |V| \text{ is even} \\ V - \{v\} & \text{if } |V| \text{ is odd} \end{cases}$

$m = \frac{|V| - 1}{3}$

$n_F = m + 1$

$\Delta_w = w(F^*) - 1 \cdot \bar{y} = 2 \cdot m - m = m - 1 = (m + 1) - 1 = n_F - 1$

(The numbers in squares represent the integral dual solution and wiggly edges represent edges in $F$.)

Let $Y$ be the dual solution that we get at the termination of algorithm $K$. The naive integral approximation is the solution that we get by rounding down $Y$, denoted by $\lfloor Y \rfloor$, i.e. subtracting $1/2$ from each component of $Y$ which is a half integral.
Theorem 3.2: a) $2(n_F - 1)$ is a tight upper bound on the integral duality gap for the naive integral solution $\lfloor Y \rfloor$.

b) In the worst case, the total value of $T$-cuts that intersect an edge $f \in F$ is zero while $w(f) = \Omega(n_F)$.

c) Let $Y^0$ be any fractional dual solution and let $Y^0_j = \lfloor Y^0 \rfloor$. The duality gap for $Y^0_j$ is exponential in $V$ in the worst case. In fact, it is true for every $f \in F^*$, simultaneously for every $f \in F^*$.

Proof: The proof of part (a) is very similar to the proof of Theorem 3.1 so we just outline the proof.

There are two cases to consider

Case (i): There are no contracted vertices in the surface graph. Since $Y(S_j)$ can be fractional only for a single vertex set $S_j$ which belongs to an odd circuit $C_0$ such that all the edges of $F$ in $C_0$ are a single edge component in $F$. So, on each such an edge component in $F$, $f = (s_i, s_j)$ we have

$$Y(S_j)\lfloor Y(S_j)\rfloor + Y(S_i)\lfloor Y(S_i)\rfloor \leq 1.$$  

Hence:

$$\Sigma(Y(S)\lfloor Y(S)\rfloor : S \in S) \leq n_F < 2(n_F - 1) \quad \text{for } n_F > 1$$

and we are done.

Case (ii): There is at least one contracted vertex in the surface graph.

We shall prove this case by induction on $|V|$. For $|V| = 2$, $\Delta_w = 0$ since $|T| = 2$ or $|T| = 0$, so we are done.

Assume that the argument is true for $|V| \leq n$ and we shall show its correctness for $|V| \leq n + 1$. We shall create the same graph $G'$ with the dual function $Y'$ as we created in the proof of Theorem 3.1. By the induction hypothesis the statement of the theorem hold for $G'$ so we have

$$\Sigma(Y'(S)\lfloor Y'(S)\rfloor : S \in S') \leq 2(n_F' - 1)$$

Assume that we contracted $\bar{S}$, and $n_F'$ denotes the number of components of $\overline{F^*}$ in $\gamma(\bar{S})$.

(3.3) If $\bar{S}$ is a level set of kind (a), (b), or (c) with $w(c)$ even (see Figure 1),

then $Y(S_j) \equiv (mod 1) \quad \forall S_j \subseteq \bar{S}$. So, we extend $\lfloor Y' \rfloor$ to $Y$ in the following way:

$$\hat{Y}(S) = \begin{cases} 
\lfloor Y'(S) \rfloor & S \cap \bar{S} = \emptyset \\
\lfloor Y'(S \setminus (\bar{S})) \rfloor & \bar{S} \subset S \\
Y(S_j) & S_j \in \bar{S}
\end{cases}$$
and we have:

\[ \Sigma(Y(S) - \tilde{Y}^* S): S \in S_0) \leq 2(n_{\bar{F}} - 1) \leq 2(n_{\bar{F}} - 1). \]

(3.4) Let \( \bar{S} \) be a level set of kind (c) with \( w(c) \) odd, then \( Y(S_j) = 1/2 (mod \ 1) \) for \( S_j \in \bar{S} \), and either,

(3.4.1) \( Y(\bar{S}) = 0 (mod \ 1) \) or

(3.4.2) \( Y(\bar{S}) = 1/2 (mod \ 1) \).

(3.4.1) In this case we extend \( \lfloor Y^* \rfloor \) to \( \bar{Y} \) in the following way:

\[
\bar{Y}(S) = \begin{cases} 
Y(S) & \text{if } \bar{S} \cap S = \emptyset \\
Y(S \setminus \bar{S} \cup \{S_j\}) & \text{if } \bar{S} \subset S \\
Y(S_j) = Y(S_j) - 1/2 & \text{if } S_j \in \bar{S}
\end{cases}
\]

By the induction hypothesis we have:

\[ \Sigma(Y(S) - \bar{Y}(S): S \in S_0) \leq 2(n_{\bar{F}} - 1) + \bar{n}_{\bar{F}} + 1/2 < 2(n_{\bar{F}} - 1). \]

(3.4.2) We defined \( \bar{Y} \) as follows:

\[
\bar{Y}(S) = \begin{cases} 
\lfloor Y(S) \rfloor & \text{if } \bar{S} \cap S = \emptyset \\
\lfloor Y(S \setminus \bar{S} \cup \{S_j\}) \rfloor & \text{if } \bar{S} \subset S \\
\lfloor Y(S_j) \rfloor - 1 & \text{if } \bar{S} \subset S \\
Y(S_j) = Y(S_j) - 1/2 & \text{if } S_j \in \bar{S}
\end{cases}
\]

Again, by the induction hypothesis, the fact that the value of the dual solution is integral, \( \bar{n}_{\bar{F}} \geq 1 \), and that \( \bar{Y} \) is integral we have:

\[ \Sigma(Y(S) - \bar{Y}(S): S \in S_0) \leq \left\lfloor 2(n_{\bar{F}} - 1) + \bar{n}_{\bar{F}} + 1 \frac{1}{2} \right\rfloor \leq 2(n_{\bar{F}} - 1). \]

Claim: In all above cases \( \bar{Y}(S) = \lfloor Y(S) \rfloor \) \( S \subseteq \bar{V} \). This claim completes the proof that the upper bound is \( 2(n_{\bar{F}} - 1) \).

The tightness of the bound is given by Example 1.

The proof of part b is by Example 2.
Example 2:

Let \( w(e) = 1 \), unless otherwise mentioned.

\[ T = V - \{ v_1, v_2 \} \]

\[ \Delta_w = w(f) \]

Wiggly edges represent edges in \( F \).

The proof of part (c) by Example 3.

Example 3:

\[ |V| = |T| \text{ even} \]

\[ w(e) = 2^{\frac{|V|}{2} - 2} \]

Wiggly edges represent edges in \( F \).

For each edge \( f \in F, f = (u, v) \), contract all other edges in \( F \) to get \( V' \), and consider all the subsets of \( V' \) such that \( X \subseteq V' - \{ u \}, v \in X \) and give a dual value of \( 1/2 \) to each such \( T \)-cut. Hence for each \( f \) we have \( 2^{\frac{|V|}{2} - 2} \) different \( T \)-cuts (with dual value of \( 1/2 \)) which implies exponential integral duality gap on each \( f \in F \).

Q.E.D.

Remark: After we have communicated our result (Theorem 3.1) to Andras Sebo, he has proved part (a) of it by using his structure theorem [20]:

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4. PLANE INTEGRAL MULTICOMMODITY FLOWS AND T-CUTS

Seymour [18] pointed out an interesting correspondence between the problem of integral optimum packing of T-cuts and the integral multicommodity flow problem.

The integral multicommodity flow problem can be formulated as follows: Let $G(V, E)$ be a graph, $F \subseteq E$ and let $w : E \to \mathbb{Z}^+$. Does there exist a collection of circuits $C$ and a function $\phi : C \to \mathbb{Z}^+$ such that:

(i) $\forall C_i \in C : |C_i \cap F| = 1$.

(ii) $\forall f \in F : \Sigma(\phi(C_i) : f \in C_i) = w(f)$ and

(iii) $\forall e \in E \setminus F : \Sigma(\phi(C_i) : e \in C_i) \leq w(e)$

where $F$ is the set of edges such that $f_i \in F$ connects the source $s_i$ to the sink $t_i$ and $w(f_i)$ is the demand for the $i$-th commodity. Even, Itai and Shamir [5] have shown that this problem is NP-complete, even when $F$ consists of two non-adjacent edges.

A necessary condition for the existence of such a flow is known as the cut condition, i.e. for every cut $D = \delta(X), X \subseteq V : w(D \cap F) \leq w(D \setminus F)$.

The following cases are among the few cases that have been solved:

(i) $F$ is a set of parallel edges (Ford and Fulkerson's max-flow min-cut theorem), Ford and Fulkerson [6].

(ii) $G \setminus F$ is planar and can be drawn in the plane so that the ends of edges in $F$ are all on the boundary of the infinite region. In addition

$\forall X \subseteq V, w(\delta(X) \setminus F) = w(\delta(X) \cap F)$ is even (H. Okamura and P.D. Seymour [13]).

(iii) $G$ is planar and a plane representation of $G$ is given such that

$\forall v \in V, w(\delta(v))$ is even (Seymour [17]) and algorithmic solution in Korach [10].

(iv) $|F| = 2$ and $w(f) = 1, \forall f \in F$ (Seymour [16], Shiloach [19]).

(v) $|F| = 2, (G \cup F)$ planar graph, Lomonobov [11], an algoirthmic solution in Penn [14].

(vi) $F \subseteq E(G[S])$ for some $S \subseteq V, |S| \leq 3$. (An algorithmic solution in Seymour [16]).

(vii) $G$ is a plane graph such that there are at most 4 faces with odd number of edges of $F$ in their boundary (Seymour [18]).

(viii) $G$ is a series parallel graph (Seymour [17]), an algoirthmic solution in Korach [10].
(ix) \( G \) is a plane graph such that there are at most 6 faces with an odd number of edges of \( F \) in their boundary (Korach [10]).

For additional cases see H. Okamura [12] and A. Frank [8].

If \( G = (V, E) \) is a plane graph and \( G^* = (V^*, E^*) \) is its planar dual, the following dual relations are well known (see Bondy and Murty [1]):

(i) Faces of \( G \) correspond to vertices of \( G^* \).
(ii) Edges of \( G \) correspond to edges of \( G^* \).
(iii) Circuits of \( G \) correspond to coboundaries of \( G^* \).

It is easy to see that the multicommodity flow problem in the plane graph \( G = (V, E) \) where \( F \subseteq E \) is expressed in terms of \( G^* \) is:

Let \( F^* \subseteq E^* \) and \( w^* : E^* \to \mathbb{Z}^+ \) be given. Does there exist a collection of coboundaries \( D^* \) in \( G^* \) and a function \( \phi^* : D^* \to \mathbb{Z}^+ \) such that:

(i) \( \forall D^* \in D^* : |D^* \cap F^*| = 1 \);
(ii) \( \forall f^* \in F^* : \sum(\phi^*(D^*)) : f \in D^* = w^*(f^*) \) and
(iii) \( \forall e^* \in E^* \setminus F^* : \sum(\phi^*(D^*)) : e^* \in D^* \leq w^*(e^*) \).

Let \( T^* = \{ v^* \in V^* : |\delta(v^*) \cap F^*| \text{ is odd} \} \). Hence \( F^* \) is a union of \( T^* \)-join and circuits.

\( F^* \) is an optimal \( T^* \)-join if and only if for every circuit \( C^* \) in \( G^* \) we have:

\[ w^*(C^* \cap F^*) \leq w^*(C^* \setminus F^*). \]

If \( w(e) = w^*(e) \) for every \( e \in E \) then the last condition satisfied if and only if the cut-condition is satisfied.

Clearly, every coboundary \( D^* \in D^* \) is a \( T^* \)-cut and the collection \( D^* \) is a packing of \( T^* \)-cuts in \( (G^*, T^*) \) with value \( w^*(F^*) \). So, we can solve the integral multicommodity flow problem in planar graphs by solving the problem of maximum weight integral packing of \( T \)-cuts.

There is a way to find an approximate solution to the above multicommodity flow problem in the plane graph based on the theorem that we proved. In the first stage solve the problem without the integrality constraints. Then, if the flow is feasible you have an half integral solution. Then, find an
integral flow such that the difference between the total demand and the total integer flow will not exceed $n_F-1$, by changing the half integral solutions in the packing of $T$-cuts problem to integral by the method described in the proof of the theorem and then converting this solution of packing of $T$-cuts to flow. Starting from the most inner contracted vertices, after changing the dual in one contracted vertex, the result is such that in the coboundary of that vertex the new dual solution induces the same value and everywhere outside this vertex, nothing has been changed. In addition, the total flow of each commodity will be reduced by at most one unit.

In [10] a characterization of the triples $(G,T,w)$ where $|T| \leq 6$ for which duality gap is zero is given. By our result, the duality gap in these cases is at most 2. It remains open to characterize the cases when the duality gap is exactly 2 (the cases where the duality gap is one will follow).

Another interesting open problems are: (i) to find a minimum weighted $T$-join with minimum number of components important in virtue of the main theorem; (ii) to characterize the cases where the duality gap meets the bound.

CONCLUDING REMARKS

We have proved (algorithmically) an upper bound on the integral duality gap for every triple $(G,T,w)$ which is equal to $n_F-1 \leq \frac{|V|}{2}-1$. The gap depends on the size of the graph and is independent of the weight function $w$. In addition there is always integral solution with duality gap that does not exceed the bound and has certain even distribution property.

Let approximation $A$ be the approximation that was proposed in the proof of Theorem 3.1 and the naive approximation the one that was proposed in Theorem 3.2, and let the integral duality gap of the problem be the difference between the minimum weighted $T$-join and the maximum integral weight packing or $T$-cuts.

(a) There are cases in which approximation $A$ gives a solution with integral duality gap equal to $n_F-1$, and cases in which the naive approximation gives a solution with integral duality gap of $2(n_F-1)$. It should be noted that there are cases in which the $n_F-1$ is also the integral duality gap of the prob-
lem which implies that approximation \( A \) is optimal. In contrast, \( 2(n_F-1), n_F \geq 2 \), is never the integral value of the duality gap of the problem. This remark applies also for planar graphs. (Example 1.)

(b) Other important differences between these two approximations are:

(i) The size of the upper bound on the duality gap in the naive approximation is twice the size of the upper bound on the duality gap in approximation \( A \).

(ii) In approximation \( A \) the total gap is divided "evenly" between the components of \( F^* \). That is to say that the total value of \( T \)-cuts that intersects \( \tilde{F} \) (\( \tilde{F} \) is a component of \( F^* \)) is either \( w(\tilde{F}) \) or \( w(\tilde{F})-1 \). For each component of \( F \) either there is no gap or the gap is only one. While in the naive approximation the total gap is not divided "evenly". In this approximation it might be that the gap relative to \( \tilde{F} \) or even relative to a single edge in \( \tilde{F} \) will be as large as we wish (Example 2).

(iii) If we used these two approximations for solving integral multicommodity flow problems in certain planar graphs (i.e. graphs which are planar when the demand edges are also included) then we shall have that in approximation \( A \) the total flow achieved for each commodity will be less than the demand by at most one unit, while in the naive one it might be that the total flow of a certain commodity will be far from the demand (e.g. in the planar dual of Example 2, one of the commodities will have flow one where as the demand is \( w \) (arbitrary large)).

(c) These two approximations can be calculated in polynomial time. This follows from the fact that algorithm \( K \) [10] is polynomial and that after its termination we have to add \( O(|S|)=O(|V|) \) operations in order to get each one of these two approximations. So, we have that both approximations are polynomial.
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